On the absolutely continuous and negative discrete spectra of Schrodinger operators on the line with locally integrable globally square summable potentials

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Abstract. For one-dimensional Schrodinger operators with potentials q subject to

$$\sum_{n=-\infty}^{\infty}\left(\int_n^{n+1}|q(x)|\,dx\right)^2<\infty$$

we prove that the absolutely continuous spectrum is $[0,\infty)$, extending the 1999 result due to Dieft-Killip. As a by-product we show that under the same condition the sequence of the negative eigenvalues is $3/2-$summable improving the relevant result by Lieb-Thirring.

1. Introduction

The main object of our consideration is the Schrodinger operator

$$H=-d^2/dx^2+q(x)\ \text{ on }\mathbb{R}:=(-\infty,\infty) \tag{1.1}$$

with a real slowly decaying potential q. For the background information, literature, and open problems we refer to the recent surveys [**11**], [**12**] by Simon. In this paper we will be concerned with the essential support of the absolutely continuous (a.c.) spectrum $\sigma_{ac}(H)$ of H for L_2-type potentials without smoothness assumptions. It was proved in 1999 by Dieft-Killip [**2**] that

$$q\in L_2(\mathbb{R})+L_1(\mathbb{R})\Longrightarrow\sigma_{ac}(H)=\mathbb{R}_+:=[0,\infty), \tag{1.2}$$

settling down one conjecture due to Kiselev-Last-Simon [**7**] which drew considerable attention. We are unable to review here the extensive literature devoted to it and refer the interested reader to e.g. Christ-Kiselev [**1**], Killip [**5**] and the literature cited therein. Relevant to our note, the best Christ-Kiselev result is

$$q\in l_p(L_1(\mathbb{R})),1<p<2\Longrightarrow\sigma_{ac}(H)=\mathbb{R}_+, \tag{1.3}$$

where $l_p(L_1)$ stands for the Birman-Solomyak class $(0<p\leq\infty)$

$$f\in l_p(L_1(\mathbb{R}))\Longleftrightarrow\left\{\int_n^{n+1}|f(x)|\,dx\right\}\in l_p(\mathbb{Z}),\mathbb{Z}=\{0,\pm1,\pm2,...\}.$$

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Since $L_2 + L_1 \subset l_2(L_1)$ there is a certain mismatch between (1.2) and (1.3). Killip [**6**] informed the author about his conjecture that (1.3) holds for $p = 2$. In the present note we give an affirmative answer to this question. Actually we prove even more.

THEOREM 1. *Let q be real valued and from $l_2(L_1(\mathbb{R}))$. Then*

$$\sigma_{ac}(H) = \mathbb{R}_+ \tag{1.4}$$

and the negative discrete spectrum $\left\{-\kappa_j^2\right\}$ satisfies

$$\sum_j \kappa_j^3 < \infty. \tag{1.5}$$

Note that (1.5) extends the Lieb-Thirring bounds [**8**]

$$\sum_j \kappa_j^3 \le C \int_{\mathbb{R}} q^2(x)\, dx,$$

to include $l_2(L_1)$ potentials. The improved bounds will be give in [**10**].

2. A Lemma on a WKB-type asymptotics

The following technical lemma will play a crucial role in our consideration.

LEMMA 1. *Let $q \in l_2(L_1(\mathbb{R}_+))$, then the equation*

$$-u'' + q(x)\, u = \lambda^2 u, x \in \mathbb{R}_+, \tag{2.1}$$

has a solution $\Psi(x,\lambda)$ which can be represented in the domain Λ of all points (x,λ), $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$ such that

$$(\operatorname{Im}\lambda)^{-1}\left(2 + (\operatorname{Im}\lambda)^{-1}\right)^4 \|q\chi_x\|^4_{l_p(L_1)} < 1/4, \tag{2.2}$$

(where χ_x is the indicator of (x,∞)) in the form

$$\Psi(x,\lambda) = y(x,\lambda)\exp\left\{i\lambda x + \int_0^x Q(s,\lambda)\, ds\right\}, \tag{2.3}$$

where

$$Q(x,\lambda) := -\int_x^\infty e^{2i\lambda(s-x)} q(s)\, ds \tag{2.4}$$

and $y(x,\lambda)$ is some continuous function satisfying ($(x,\lambda) \in \Lambda$)

$$|y(x,\lambda) - 1| \le 4(\operatorname{Im}\lambda)^{-1}\left(2 + (\operatorname{Im}\lambda)^{-1}\right)^4 \|q\chi_x\|^4_{l_p(L_1)}. \tag{2.5}$$

PROOF. Let us prove first that if $q \in l_2(L_1(\mathbb{R}_+))$ then $Q(\cdot,\lambda) \in L_2(\mathbb{R})$ for every $\operatorname{Im}\lambda > 0$. Indeed, by Jensen's inequality

$$\begin{aligned} |Q(x,\lambda)|^2 &= \left|\sum_{n\ge 0}\int_n^{n+1} e^{2i\lambda s} q(s+x)\, ds\right|^2 \\ &\le \left\{\sum_{n\ge 0} e^{-2n\operatorname{Im}\lambda}\left(\int_n^{n+1} |q(s+x)|\, ds\right)\right\}^2 \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\leq \sum_{n\geq 0} e^{-2n\operatorname{Im}\lambda}\cdot\sum_{n\geq 0} e^{-2\operatorname{Im}\lambda n}\left(\int_n^{n+1}|q(s+x)|\,ds\right)^2 \\ &= \frac{1}{1-e^{-2\operatorname{Im}\lambda}}\sum_{n\geq 0} e^{-2\operatorname{Im}\lambda n}\left(\int_n^{n+1}|q(s+x)|\,ds\right)^2. \end{aligned}$$

Integrating (2.6) with respect to x one has

$$\|Q(\cdot,\lambda)\|_2^2 \leq \left(1+\frac{1}{2\operatorname{Im}\lambda}\right)\sum_{n\geq 0} e^{-2\operatorname{Im}\lambda n}\int_0^{\infty}\left(\int_n^{n+1}|q(s+x)|\,ds\right)^2 dx. \tag{2.7}$$

Observe now that

$$\begin{aligned} &\int_0^{\infty}\left(\int_n^{n+1}|q(s+x)|\,ds\right)^2 dx = \sum_{m\geq 0}\int_m^{m+1}\left(\int_x^{x+1}|q|\right)^2 dx \\ &\leq \sum_{m\geq 0}\left(\int_m^{m+2}|q|\right)^2 \leq 2\sum_{m\geq 0}\left\{\left(\int_m^{m+1}|q|\right)^2+\left(\int_{m+1}^{m+2}|q|\right)^2\right\} \\ &= 4\sum_{m\geq 0}\left(\int_m^{m+1}|q|\right)^2 = 4\|q\|_{l_2(L_1)}^2. \end{aligned} \tag{2.8}$$

Plugging (2.8) into (2.7), one has

$$\begin{aligned} \|Q(\cdot,\lambda)\|_2^2 &\leq \left(1+\frac{1}{2\operatorname{Im}\lambda}\right)\sum_{n\geq 0} e^{-\operatorname{Im}\lambda n}\cdot 4\|q\|_{l_2(L_1)}^2 \\ &\leq \left(2+\frac{1}{\operatorname{Im}\lambda}\right)^2\|q\|_{l_2(L_1)}^2 \end{aligned}$$

that yields

$$\|Q(\cdot,\lambda)\|_2 \leq \left(2+\frac{1}{\operatorname{Im}\lambda}\right)\|q\|_{l_2(L_1)}. \tag{2.9}$$

By inspection

$$\Theta(x,\lambda) := \exp\left\{i\lambda x+\int_0^x Q(s,\lambda)\,ds\right\}$$

is a solution to

$$-u''+\left(q(x)+Q^2(x,\lambda)\right)u = \lambda^2 u. \tag{2.10}$$

If $u_1(x,\lambda)=\Theta(x,\lambda)$ is a solution to (2.10) then as the other solution we choose

$$u_2(x,\lambda) = \Theta(x,\lambda)\int_0^x \Theta^{-2}(s,\lambda)\,ds.$$

Rewriting the original equation (2.1) as

$$-u''+\left(q(x)+Q^2(x,\lambda)\right)u-\lambda^2 u = Q^2(x,\lambda)u, \tag{2.11}$$

by variation of parameters one easily verifies (formally) that (2.11) implies $\left(y=\Theta^{-1}u\right)$

$$y(x,\lambda) = 1+\int_x^{\infty} K(x,s,\lambda)\,y(s,\lambda)\,ds \tag{2.12}$$

with the kernel

$$K(x,s,\lambda) := Q^2(s,\lambda)\cdot\Theta^2(s,\lambda)\int_x^s \Theta^{-2}(t,\lambda)\,dt. \tag{2.13}$$

To justify this formal computation it is enough to show that $K(x,\cdot,\lambda)\in L_1$ for every $x\in\mathbb{R}_+, \lambda\in\mathbb{C}_+$. By (2.9), $Q^2(\cdot,\lambda)\in L_1$ and it is only left to demonstrate that $\Theta^2(s,\lambda)\int_x^s \Theta^{-2}(t,\lambda)\,dt$ is bounded. Indeed,

$$\left|\Theta^2(s,\lambda)\int_x^s \Theta^{-2}(t,\lambda)\,dt\right| = \left|\int_x^s \exp^2\left\{i\lambda(s-t)+\int_t^s Q(t,\lambda)\,dt\right\}dt\right|$$

$$\leq \left|\int_0^{s-x}\exp^2\left\{-\operatorname{Im}\lambda t\left(1-\frac{1}{\operatorname{Im}\lambda}\left\|Q(\cdot,\lambda)\right\|_\infty\right)\right\}\right| \leq \int_0^\infty \exp(-\operatorname{Im}\lambda t)\,dt = \frac{1}{\operatorname{Im}\lambda}$$

and we have

$$\left\|K(x,\cdot,\lambda)\right\|_1 \leq \frac{1}{\operatorname{Im}\lambda}\left\|Q^2(\cdot,\lambda)\right\|_1 \leq \frac{1}{\operatorname{Im}\lambda}\left(2+\frac{1}{\operatorname{Im}\lambda}\right)^4 \|q\|^4_{l_p(L_1)}.$$

The standard iteration procedure completes the proof. ∎

Remark 1. *One can easily see that, up to a factor, solution* (2.3) *has the WKB-type asymptotic behavior*

$$\Psi(x,\lambda)\sim\exp\left\{i\lambda x+\frac{1}{2i\lambda}\int_0^x q(s)\,ds\right\}, x\to\infty, \tag{2.14}$$

for every $\lambda\in\mathbb{C}_+$. *We, by no means, claim that Lemma 1 proves* (2.14) *for almost all real* λ *for every potential* q *from* $l_2(L_1(\mathbb{R}_+))$. *Theorem 1 would be immediately proven then. It is a very difficult open question and the best known result here belongs to Christ-Kiselev* [**1**]. *Using some very elaborate techniques, they showed that* (2.14) *holds for a.e. real* λ *for* $q\in l_p(L_1(\mathbb{R}_+)), 1<p<2$. *This was exactly their way to prove* (1.3). *Our approach requires* (2.14) *only for* $\lambda\in\mathbb{C}_+$.

3. Proof of Theorem 1

We employ the standard approach. We first get all necessary formulas for the truncated potentials $\widetilde{q}$ defined as

$$\widetilde{q}(x) = q(x)\,\chi_{[-a,a]}(x), a>0,$$

$\chi_{[-a,a]}$ denotes the characteristic function of $[-a,a]$, and then pass to the limit when $\widetilde{q}\to q$ (i.e.$a\to\infty$). In the sequel, we agree to put $\widetilde{}$ on top of every object related to $\widetilde{q}$.

Let us evaluate the transmission coefficient $\tilde{T}(\lambda)$ for $\lambda\in\mathbb{C}_+$ in two different ways. Following Faddeev-Zhakharov [**3**] (adopting the convention $\int := \int_{-\infty}^{\infty}$)

$$\tilde{T}(\lambda) = \prod_j \frac{\lambda+i\widetilde{\kappa}_j}{\lambda-i\widetilde{\kappa}_j}\exp\left\{i\int\frac{\tilde{f}(k)}{k-\lambda}dk\right\}, \tag{3.1}$$

where $-\widetilde{\kappa}_j^2$ are negative eigenvalues of $\tilde{H}$ and

$$\tilde{f}(k) := \frac{1}{\pi}\log\left|\tilde{T}(k)\right|^{-1}. \tag{3.2}$$

Due to the general properties of the transmission coefficient, the function $\tilde{f}$ is integrable, non-negative, and even and hence

$$\begin{aligned}\int \frac{\tilde{f}(k)}{k-\lambda}dk &= -\frac{1}{\lambda}\int \tilde{f}(k)\,dk + \frac{1}{\lambda^2}\int \frac{k^2\tilde{f}(k)}{k-\lambda}dk \\ &= -\frac{1}{\lambda}\int \tilde{f}(k)\,dk + \frac{2}{\lambda}\int_0^\infty \frac{k^2\tilde{f}(k)}{k^2-\lambda^2}dk. \end{aligned} \tag{3.3}$$

We have used here the obvious identity

$$\frac{1}{k-\lambda} = -\frac{k}{\lambda^2} - \frac{1}{\lambda} + \frac{1}{\lambda^2}\frac{k^2}{k-\lambda}.$$

By the first Buslaev-Faddeev-Zhakharov trace formula [**3**]

$$-2\sum_j \widetilde{\kappa}_j + \int \tilde{f}(k)\,dk = \frac{1}{2}\int \tilde{q}(x)\,dx, \tag{3.4}$$

and (3.3) can be continued

$$\begin{aligned}\int \frac{\tilde{f}(k)}{k-\lambda}dk &= -\frac{1}{\lambda}\int \tilde{f}(k)\,dk + \frac{2}{\lambda}\int_0^\infty \frac{k^2\tilde{f}(k)}{k^2-\lambda^2}dk \\ &= -\frac{2}{\lambda}\sum_j \widetilde{\kappa}_j - \frac{1}{2\lambda}\int \tilde{q}(x)\,dx + \frac{2}{\lambda}\int_0^\infty \frac{k^2\tilde{f}(k)}{k^2-\lambda^2}dk. \end{aligned} \tag{3.5}$$

Plugging (3.5) into (3.1) one has

$$\begin{aligned}&\tilde{T}(\lambda) \\ &= \prod_j \frac{\lambda+i\widetilde{\kappa}_j}{\lambda-i\widetilde{\kappa}_j}e^{\frac{2i}{\lambda}\widetilde{\kappa}_j}\cdot\exp\left\{\frac{2i}{\lambda}\int_0^\infty \frac{k^2\tilde{f}(k)}{k^2-\lambda^2}dk\right\}\exp\left\{\frac{1}{2i\lambda}\int \tilde{q}(x)\,dx\right\}. \end{aligned} \tag{3.6}$$

On the other hand, following Hinton-Klaus-Shaw [**4**]

$$\tilde{T}(\lambda) = \frac{2i\lambda}{\tilde{m}_+\left(\lambda^2\right)+\tilde{m}_-\left(\lambda^2\right)}\left(\tilde{F}_+(\lambda)\,\tilde{F}_-(\lambda)\right)^{-1}, \tag{3.7}$$

where $\tilde{m}_\pm$ are the Weyl $m-$functions associated with $-d^2/dx^2+\tilde{q}(x), u(\pm 0)=0$ on $L_2(\mathbb{R}_\pm)$ and $\tilde{F}_\pm$ are the Jost functions of $\tilde{H} = -d^2/dx^2+\tilde{q}(x)$ on $L_2(\mathbb{R})$ corresponding to $\pm\infty$, respectively. We recall that for $q\in L_1(\mathbb{R})$ the equation

$$-u''+q(x)\,u = \lambda^2 u, x\in\mathbb{R}, \operatorname{Im}\lambda^2\geq 0,$$

has two linearly independent solutions $u_\pm(x,\lambda)$, referred to as Jost, subject to

$$u_\pm(x,\lambda) = e^{\pm i\lambda x}(1+o(1)), x\to\pm\infty.$$

Functions $F_\pm$ and $m_\pm$ are related to $u_\pm$ by

$$F_\pm(\lambda) := u_\pm(0,\lambda),\ m_\pm\left(\lambda^2\right) := \pm\frac{d}{dx}\log u_\pm(x,\lambda)\,|_{x=0}\,. \tag{3.8}$$

Evaluate now $\tilde{F}_+(\lambda)\,\tilde{F}_-(\lambda)$ in (3.7). Since $\tilde{F}_+$ and $\tilde{F}_-$ are similar we handle only $\tilde{F}_+$. Directly by definition (2.4), one verifies that

$$\begin{aligned}\int_0^a \tilde{Q}(s,\lambda)ds &= \frac{1}{2i\lambda}\int_0^a \tilde{q}(s)\,ds - \frac{\tilde{Q}(a,\lambda)-\tilde{Q}(0,\lambda)}{2i\lambda} \\ &= \frac{1}{2i\lambda}\int_0^a \tilde{q}(s)\,ds + \frac{1}{2i\lambda}\tilde{Q}(0,\lambda), \end{aligned} \tag{3.9}$$

where we have observed that $\tilde{Q}(a,\lambda)=0$. Comparing u_+ with the solution Ψ from Lemma 1, one has

$$\begin{aligned}\tilde{u}_+(x,\lambda) &= \tilde{\Psi}(x,\lambda)\exp\left\{-\int_0^a \tilde{Q}(s,\lambda)ds\right\}\\ &= \tilde{\Psi}(x,\lambda)\exp\left\{-\frac{1}{2i\lambda}\int_0^\infty \tilde{q}(s)\,ds\right\}\exp\left\{\frac{1}{2i\lambda}\int_0^\infty e^{2i\lambda s}\tilde{q}(s)\,ds\right\}, \qquad (3.10)\end{aligned}$$

It follows from (3.8), (3.9) and (2.2) that

$$\begin{aligned}&\tilde{F}_+(\lambda)\\ &= \tilde{y}_+(\lambda)\exp\left\{-\frac{1}{2i\lambda}\int_0^\infty \tilde{q}(s)\,ds\right\}\exp\left\{\frac{1}{2i\lambda}\int_0^\infty e^{2i\lambda s}\tilde{q}(s)\,ds\right\}, \qquad (3.11)\end{aligned}$$

where $\tilde{y}_+(\lambda):=\tilde{y}(+0,\lambda)$. Analogously

$$\tilde{F}_-(\lambda)=\tilde{y}_-(\lambda)\exp\left\{-\frac{1}{2i\lambda}\int_{-\infty}^0 \tilde{q}(s)\,ds\right\}\exp\left\{\frac{1}{2i\lambda}\int_{-\infty}^0 e^{-2i\lambda s}\tilde{q}(s)\,ds\right\}$$

with some function $\tilde{y}_-$ admitting a similar to (2.3) estimate. Thus

$$\begin{aligned}&\left(\tilde{F}_+(\lambda)\,\tilde{F}_-(\lambda)\right)^{-1}\\ &= \tilde{y}^{-1}(\lambda)\exp\left\{\frac{1}{2i\lambda}\int\tilde{q}(x)\,dx\right\}\exp\left\{\frac{i}{2\lambda}\int e^{2i\lambda|x|}\tilde{q}(x)\,dx\right\}, \qquad (3.12)\end{aligned}$$

where $\tilde{y}:=\tilde{y}_+\tilde{y}_-$. Inserting (3.12) into (3.7), we finally arrive at

$$\begin{aligned}&\tilde{T}(\lambda)\\ &= \tilde{y}^{-1}\frac{2i\lambda}{\tilde{m}_+(\lambda^2)+\tilde{m}_-(\lambda^2)}\exp\left\{\frac{1}{2i\lambda}\int\tilde{q}(x)\,dx\right\}\exp\left\{\frac{i}{2\lambda}\int e^{2i\lambda|x|}\tilde{q}(x)\,dx\right\}. \qquad (3.13)\end{aligned}$$

Equate now the right hand sides of (3.6) and (3.13). Taking $\lambda=i$ and observing that the divergent factor $\exp\left\{-\frac{1}{2}\int\tilde{q}(x)\,dx\right\}$ drops out from both sides, we obtain

$$\begin{aligned}&\prod_j\frac{1+\widetilde{\kappa}_j}{1-\widetilde{\kappa}_j}e^{-2\widetilde{\kappa}_j}\cdot\exp 2\int_0^\infty\frac{k^2}{1+k^2}\tilde{f}(k)\,dk\\ &= \left\{\frac{-2\tilde{y}^{-1}(\lambda)}{\tilde{m}_+(-1)+\tilde{m}_-(-1)}\right\}\exp\left\{\frac{1}{2}\int e^{-2|x|}\tilde{q}(x)\,dx\right\}. \qquad (3.14)\end{aligned}$$

Taking notice that due to basic facts of perturbation theory a change of the potential q on a finite interval does not effect the conclusions of the theorem, we alter, if needed, our potential to satisfy condition (2.2) for $\lambda=i$ for all $x\geq 0$ and $\sup|\widetilde{\kappa}_j|<1$. This makes every factor in (3.14) positive and one can pass in (3.14) to the logarithms

$$\sum_j\left(\log\frac{1+\widetilde{\kappa}_j}{1-\widetilde{\kappa}_j}-2\widetilde{\kappa}_j\right)+\int_0^\infty\frac{2k^2}{1+k^2}\tilde{f}(k)\,dk=\frac{1}{2}\int e^{-2|x|}\tilde{q}(x)\,dx+2\tilde{\omega}, \qquad (3.15)$$

where

$$\tilde{\omega}:=-\tilde{y}^{-1}(\lambda)\left\{\tilde{m}_+(-1)+\tilde{m}_-(-1)\right\}^{-1}. \qquad (3.16)$$

The sum in (3.15) has the lower bound $\frac{2}{3}\sum_j \widetilde{\kappa}_j^3$. Indeed, setting for the time being $\alpha = \widetilde{\kappa}_j$ one has

$$\begin{aligned}\log\frac{1+\alpha}{1-\alpha} - 2\alpha &= \log(1+\alpha) - \log(1-\alpha) - 2\alpha \\ &= \sum_{n\geq 0}(-1)^n\frac{\alpha^n}{n+1} + \sum_{n\geq 0}\frac{\alpha^n}{n+1} - 2\alpha = 2\sum_{n\geq 1}\frac{\alpha^{2n+1}}{2n+1} \geq \frac{2}{3}\alpha^3\end{aligned}$$

and thus

$$\sum_j\left(\log\frac{1+\widetilde{\kappa}_j}{1-\widetilde{\kappa}_j} - 2\widetilde{\kappa}_j\right) \geq \frac{2}{3}\sum_j \widetilde{\kappa}_j^3.$$

It follows from (3.15) that

$$\frac{1}{3}\sum_j \widetilde{\kappa}_j^3 + \int_0^\infty \frac{k^2}{1+k^2}\tilde{f}(k)\,dk \leq \frac{1}{4}\int e^{-2|x|}\tilde{q}(x)\,dx + \tilde{\omega}. \tag{3.17}$$

Show now that the right hand side of (3.17) is uniformly bounded as $\tilde{q} \to q$. Indeed, by (2.5) and (2.2)

$$|\tilde{y}(i)|^{-1} = \left|\tilde{y}_+(i)\right|^{-1}\left|\tilde{y}_-(i)\right|^{-1} \leq 2\cdot 2 = 4.$$

The other factor in (3.16) goes to $(m_+(-1) + m_-(-1))^{-1}$ due to the general properties of the Weyl $m-$function [**13**] and hence $\tilde{\omega}$ is uniformly bounded as $\tilde{q} \to q$. Since

$$\int e^{-2|x|}\tilde{q}(x)\,dx \to \int e^{-2|x|}q(x)\,dx, \tilde{q} \to q,$$

we see the left hand side of (3.17) is uniformly bounded too. Since both terms on the left hand side of (3.17) are positive, recalling (3.2), we arrive at

$$\int_0^\infty \frac{k^2}{1+k^2}\log\left|\tilde{T}(k)\right|^{-1} dk \leq C, \tag{3.18}$$

$$\sum_j \widetilde{\kappa}_j^3 \leq C, \tag{3.19}$$

with some finite constant C dependant only on q. Estimate (3.18) allows one now to use a lemma by Deift-Killip [**2**] (see also [**9**]). Thus (1.4), i.e. $\sigma_{ac}(H) = \mathbb{R}_+$, is proven.

It is left to establish (1.5). We can actually pass to the limit in (3.19). To this end consider the operator $H_- := -d^2/dx^2 + q_-(x)$ where q_- is the negative part of q. In obvious notation, estimate (3.19) for $\tilde{H}_-$ reads (with possibly different constant)

$$\sum_j\left(\widetilde{\kappa}_j^{(-)}\right)^3 \leq C. \tag{3.20}$$

But $0 < a < b$ implies $q_-(x)\chi_{[-a,a]}(x) \geq q_-(x)\chi_{[-b,b]}(x)$, which, in turn, yields

$$-d^2/dx^2 + q_-(x)\chi_{[-a,a]}(x) \supseteq -d^2/dx^2 + q_-(x)\chi_{[-a,a]}(x),$$

and it follows from the standard fact of perturbation theory that $\widetilde{\kappa}_j^{(-)}$ is an non-decreasing function of the cut-off point a for every fixed j. The latter combined

with the uniform bound (3.20) immediately yields

$$\sum_j \left(\kappa_j^{(-)}\right)^3 \leq C. \tag{3.21}$$

What is left to observe that, by the same perturbation principal, $\kappa_j \leq \kappa_j^{(-)}$(since $H_- \subseteq H$) and (3.21) implies (1.5). The theorem is proven.

REMARK 2. *Identity* (3.15) *can actually be viewed as the first formula in a certain chain of sum rules. Our elementary arguments provide a short-cut to Theorem 1 but yet are not fine enough to derive those formulas in a compact form. The adequate approach is rather operator theoretical and will be discussed in* [**10**] *where we also plan to improve on the results of* [**9**] *in the spirit of Theorem 1. At this point we only notice that* (3.15) *resembles the second (sic) Buslaev-Faddeev-Zhakharov trace formula*

$$\frac{1}{3}\sum_j \kappa_j^3 + \int_0^\infty k^2 \tilde{f}(k)\, dk = \frac{1}{16}\int q^2(x)\, dx$$

which was the main new insight of [**2**] *proving* (1.2) *and which higher order analogs were used in* [**9**].

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