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Limite idrodinamico di un sistema disordinato

Hydrodynamic limit of a disordered system

Dott.ssa Alessandra Faggionato

Relatore: Prof. Fabio Martinelli

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Introduction

We consider the motion of free electrons in a doped crystal. When there is no impurity the crystal creates a periodic electromagnetic field which influences the electron transport. In order to model the presence of impurities, we think of the field as given randomly and the electrons as lattice-gas particles with site exclusion (in conformity to Pauli exclusion principle) and random walks whose jump rates depend locally on the field. We neglect the interactions between electrons while the random field (the disorder) is given by a random chemical potential field.

In order to analyze the hydrodynamic behavior of the system it is necessary to distinguish between the macroscopic scale and the microscopic one. The macroscopic space is given by $\mathbb{T}^d$, the $d-$dimensional torus, and each deterministic macroscopic state is described by a Lebesgue absolutely continuous measure on $\mathbb{T}^d$. The microscopic scale parameter is given by ϵ, inverse of an integer, and for any ϵ the microscopic space is given by $\mathbb{T}^d_\epsilon$, the discrete torus of size ϵ^{-1}. A statistical microscopic state is described by a probability measure on the particle configurations on $\mathbb{T}^d_\epsilon$.
The disorder configuration α is stochastically chosen by a translation invariant product probability measure $\mathbb{P}$ on $[-B,B]^{\mathbb{Z}^d}$, where B is a fixed positive number. For any ϵ, α induces a disorder configuration on $\mathbb{T}^d_\epsilon$ by identifying this with a cube of side ϵ^{-1} centered at the origin of $\mathbb{Z}^d$. With abuse of notation we denote the induced disorder configuration by α.
Let us describe the model at the microscopic scale ϵ for a given disorder configuration α. Given an external chemical potential λ, the Hamiltonian of the system is defined as $H = -\sum_x (\alpha_x + \lambda)\eta_x$ where η is the particle configuration and x varies among the sites of $\mathbb{T}^d_\epsilon$. The respective grand canonical Gibbs measure, $\mu^{\alpha,\lambda}$, can be characterized as the product measure such that

$$\mu^{\alpha,\lambda}(\eta_x) = \frac{e^{\alpha_x+\lambda}}{1+e^{\alpha_x+\lambda}}.$$

If $\lambda = 0$ we write simply μ^α.
Let us describe the dynamics. Each particle waits an exponential time and then attempts to jump to a neighbor site, if this site is occupied then the jump is aborted otherwise it is realized with a probabilistic rate depending on the disorder configuration in the initial and final sites. Given two sites x, y we set

$$\left(\eta^{x,y}\right)_z = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise.} \end{cases}$$

Then the dynamics is determined by the diffusively rescaled Markov generator $\epsilon^{-2}\mathcal{L}^\alpha$ where, for any function f,

$$(1) \qquad \left(\mathcal{L}^\alpha f\right)(\eta) = \sum_{\substack{\{x,y\}\subset\mathbb{T}^d_\epsilon \\ |x-y|=1}} c^\alpha_{x,y}(\eta)\nabla_{x,y} f(\eta)$$

and

$$(2) \qquad \nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta).$$

We require the transition rates $c^\alpha_{x,y}(\eta)$ to belong to some interval $[\delta, \delta']$ with $\delta > 0$, to depend on the particle and disorder configurations in x, y, to satisfy the condition of detailed balance for which the process is reversible w.r.t. the canonical Gibbs measures with Hamiltonian $H = -\sum_x \alpha_x \eta_x$ and to satisfy the identity

$$c^\alpha_{x,y}(\eta) = c^{\tau_z\alpha}_{x+z,y+z}(\tau_z\eta) \quad \forall z \in \mathbb{Z}^d,$$

where $\tau_z\eta$, $\tau_z\alpha$ are the configurations translated by vector z.

Our initial target was the study of the hydrodynamic limit of a different model for the electron transport in doped crystals given by the dilute Ising lattice gas with Kawasaki dynamics depending on the spatial distribution of the interaction bonds. This model allows to consider the electron-electron interactions and simulates more effective traps. In order to comment this last feature we introduce some notation. In general, the spectral gap of the infinitesimal generator $\mathcal{L}$ of a Kawasaki dynamics restricted to a box Λ w.r.t. a canonical measure μ on Λ is defined as

$$\operatorname{gap}_\mu(\mathcal{L}) = \inf_{f:\operatorname{Var}_\mu(f)\neq 0} \frac{\mu\big(f, -\mathcal{L}f\big)}{\operatorname{Var}_\mu(f)}.$$

The spectral gap is said to have a diffusive scaling if it scales like L^{-2} when referred to a box of side L.
In the dilute Ising lattice gas with Kawasaki dynamics, although with low probability in the disorder configuration, the spectral gap can considerably deviate from the diffusive scaling. In particular, at low temperatures the clusters of the bond percolation behave as ferromagnetic regions. If the cluster has size L then the spectral gap referred to canonical measures on the cluster with particular densities is of order $e^{-cL^{d-1}}$ as shown in [8]. The deviations from the diffusive scaling can be estimated by means of the recursive techniques of [6] or the martingale methods of [20]. For more details we refer to [7].
The main difficulties in the study of the hydrodynamic limit of the above model are the following: *a*) the system is not translation invariant, *b*) the system is not gradient (we will

explain this aspect later), *c*) the equilibrium measures are not product, *d*) the spectral gap can deviate from the diffusive scaling which, by the current technology, plays a key role in the study of the hydrodynamic behavior of non gradient systems.
The model we take under consideration presents the first two difficulties, while the third one is avoided. Let us comment the forth one. By applying the recursive techniques developed in [6] we have obtained estimates on the spectral gap and on its possible deviations from the diffusive scaling which are similar to those given in [7] for the dilute Ising lattice gas under the percolation threshold with Kawasaki dynamics. We have proved the hydrodynamic limit in such a way that only estimates similarly to [7] are needed. Afterwards, P. Caputo has proved for our disordered system the diffusive scaling of the spectral gap uniformly in the disorder configuration by means of a specific analysis for which it is crucial that the grand canonical measures are product. Since the method of [6] can be applied to a larger class of disordered system also with particle interactions, giving estimates as in [7], the method we develop has the advantage to be more generic since it can be extended to other disordered systems for which the diffusive scaling of the spectral gap fails.

The diffusion of lattice gases in systems with site disorder has been studied by physicists by simulations and rough approximations as in [12], [14], [24] and reference therein.
Rigorous results are obtained in [29] for a period two disorder configuration α in one dimension, in [10] for the one-dimensional Ginzburg-Landau model with conservation law in the presence of random conductivities, in [3] and [15] for nearest-neighbor zero-range processes with random jump rates.
The model we consider is taken from [21]. There the model is described and some general ideas for its solution are indicated, but the details of the proof have never been published. Moreover, some of the technical estimates suggested in [21] turned out to be troublesome already in the absence of disorder (symmetric simple exclusion process) as explained in chapter 6. Therefore we were forced to take a different route w.r.t. that indicated in [21]

Our main result is that in dimension larger than 2, for almost any disorder configuration, by a diffusive space-time rescaling the system has an hydrodynamic limit described by a non linear diffusive equation which does not depend on the particular disorder configuration but only on the probability measure $\mathbb{P}$. Let us describe with some more details what we mean.
As already explained, a macroscopic state is described by a Lebesgue absolutely continuous measure on $\mathbb{T}^d$ while a microscopic state is given by a probability measure on the particle configurations on $\mathbb{T}^d_\epsilon$.
A sequence of microscopic states $\{\nu_\epsilon\}_{\epsilon>0}$ is said to correspond to the macroscopic state $m(\theta)d\theta$ if for any continuous function H on $\mathbb{T}^d$ and any $\delta > 0$

$$\lim_{\epsilon\downarrow 0}\nu_\epsilon\Big(\Big|\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H(\epsilon x)\eta_x-\int_{\mathbb{T}^d}H(\theta)m(\theta)d\theta\Big|>\delta\Big)=0$$

where "$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}$" denotes the spatial average.
Then, in dimension larger than 2, almost any disorder configuration α has the following property.
Given a sequence of microscopic states $\{\nu_\epsilon\}_{\epsilon>0}$ corresponding to the macroscopic state $m_0(\theta)d\theta$ we define ν^t_ϵ as the evolution at time t of the initial measure ν_ϵ by the process

determined by the difussively rescaled Markov generator $\epsilon^{-2}\mathcal{L}^\alpha$. Then for any time t the sequence $\{\nu_\epsilon^t\}_{\epsilon>0}$ corresponds to a macroscopic state $m(t,\theta)d\theta$ and $m(t,\theta)$ satisfies the Cauchy problem

$$\begin{cases}\partial_t m(t,\theta) = \nabla_\theta\Big(D\big(m(t,\theta)\big)\nabla_\theta m(t,\theta)\Big)\\ m(0,\theta) = m_0(\theta)\end{cases}$$

where the diffusion matrix D is equivalent to the one given by the Green-Kubo formula as shown in [18]. It can be variationally characterized as the only symmetric matrix such that for any $a \in \mathbb{R}^d$

$$\big(a, D(m)a\big) = \frac{1}{\chi(m)}\inf_g \frac{1}{2}\mathbb{E}\Big[\,\mu^{\alpha,\lambda_0(m)}\Big(\sum_{e\in\mathcal{E}} c^\alpha_{0,e}(a_e\nabla_{0,e}\eta_0 + \nabla_{0,e}\underline{g})^2\Big)\Big]$$

where the infimum is taken over all local functions $g = g(\alpha,\eta)$.
Let us explain the above formula for D. $\mathcal{E}$ is the canonical basis of $\mathbb{Z}^d$, $\mathbb{E}$ is the expectation w.r.t. $\mathbb{P}$, $\lambda_0(m)$ and $\chi(m)$ are respectively the annealed chemical potential and the static compressibility defined by

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_0)\Big] = m \qquad \chi(m) = \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)\Big].$$

Moreover, $\underline{g}$ is formally defined as $\underline{g} = \sum_{x\in\mathbb{Z}^d}\tau_x g$. Though $\underline{g}$ is a formal sum the expression $\nabla_{0,e}\underline{g}$ is meaningful with $\nabla_{0,e}$ defined as in (2) also for functions depending on the disorder.

The main features of the system are the absence of translation invariance (for a given disorder configuration) and the non validity of the gradient condition. This condition is purely mathematical, having no physical content, and corresponds to the Fick's law of fluid mechanics according to which the current can be written as the gradient of some function.
In order to clarify this aspect let H be a smooth function on $\mathbb{T}^d$. Then by stochastic calculus

$$\text{(3)}\qquad \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H(\epsilon x)\eta_x(t) - \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H(\epsilon x)\eta_x(0) = \epsilon^{-2}\int_0^t \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H(\epsilon x)\mathcal{L}^\alpha\eta_x(s)ds + M_\epsilon$$

where M_ϵ is a martingale whose contribution goes to 0 by taking the limit $\epsilon \downarrow 0$.
Since $\mathcal{L}^\alpha\eta_x = \sum_{e\in\mathcal{E}} j^\alpha_{x-e,x} - j^\alpha_{x,x+e}$, where $j^\alpha_{x,x+e}$ is the instantaneous current (i.e. the difference between the rate at which a particle jumps from x to $x+e$ and the rate at which a particle jumps from $x+e$ to x), by integrating by parts we can write the r.h.s. of (3) as

$$\text{(4)}\qquad \epsilon^{-1}\int_0^t \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\, j^\alpha_{x,x+e}\big(\eta(s)\big)\frac{\nabla_e H(\epsilon x)}{\epsilon}ds$$

where $\nabla_e H(\epsilon x) = H(\epsilon x + \epsilon e) - H(\epsilon x)$.
By the presence of factor ϵ^{-1}, expression (4) appears divergent. This obstacle can be overcome if the system is gradient, i.e. there exists a local function $h = h(\alpha,\eta)$ such that

$$j^{\alpha}_{x,x+e} = \tau_x h - \tau_{x+e} h \quad \forall x, e, \alpha \tag{5}$$

where in general $\tau_y h(\alpha, \eta) = h(\tau_y \alpha, \tau_y \eta)$.
In this case, we can execute a second integration by parts by which (4) can be approximated as $\epsilon \downarrow 0$ by

$$\int_0^t \mathrm{Av}_{x \in \mathbb{T}^d_\epsilon} \tau_x h\big(\alpha, \eta(s)\big) \Delta H(\epsilon x) ds. \tag{6}$$

and at this point the investigation generally continues by local equilibrium argumentations. In particular, with reference to non disordered systems, we can introduce a mesoscopic scale l with $l \uparrow \infty$ after $\epsilon \downarrow 0$ and substitute in (6) $\tau_x h\big(\eta(s)\big)$ with the local average $\mathrm{Av}_{y:|y-x|\leq l} \tau_y h\big(\eta(s)\big)$ (we call *mesoscopic* a scale which is microscopically large and macroscopically small). If there is ergodicity enough to consider the system near to equilibrium in mesoscopic regions of size l, then we can substitute this last average with $\Phi\big(m_{x,l}(s)\big)$ where $m_{x,l}(s)$ is the density of configuration $\eta(s)$ in the cube of side l centered at x and $\Phi(m)$ is defined as the expectation of h with respect to the equilibrium measure of density m. Then the hydrodynamic equation is given by

$$\partial_t h = \Delta\big(\Phi(m)\big).$$

The only case in which our system is gradient is the trivial case, where $\mathbb{P}$ is concentrated on a translation invariant configuration disorder. Therefore, we cannot execute a second integration by parts as above.

Non gradient systems are usually solved by reproducing a generalized Fick's law. A key observation is that, by stochastic calculus, it can be proved that taking the limit $\epsilon \downarrow 0$, for any local function $g = g(\alpha, \eta)$,

$$\epsilon^{-1} \int_0^t \mathrm{Av}_{x \in \mathbb{T}^d_\epsilon} \tau_x \mathcal{L}^{\alpha} g\big(\eta(s)\big) \cdot \frac{\nabla_e H(\epsilon x)}{\epsilon} ds \tag{7}$$

is negligible. Therefore, a natural generalization of the gradient condition (5) is to require that there exist local functions $\{g_e\}_{e \in \mathcal{E}}$ and h such that

$$j^{\alpha}_{x,x+e} = \tau_x \mathcal{L}^{\alpha} g_e + \tau_x h - \tau_{x+e} h \quad \forall x, e, \alpha.$$

At this point, by a second integration by parts, we obtain again (6). In [29] the case of a period two disorder configuration α in one dimension is exactly solved by explicitly finding functions g_e and h as above. This is possible since the model is particularly simple.
Usually, the generalized Fick's law for non disordered non gradient systems is derived by the more sophisticated methods initially developed by Varadhan in [27] and Quastel in [22], where the above negligibility of fluctuations $\mathcal{L}g$ plays a crucial role. We refer to [13] for an extensive bibliography on non gradient systems.
This technology can be generalized to our disordered model, but some features intimately

connected to the disorder, as in particular the fluctuations in the gradient of the density field that we will comment later, have to be controlled by new methods.
Since we prove a generalized Fick's law which, apart the negligible fluctuation term $\mathcal{L}^\alpha g$, does not depend on the particular disorder configuration, the hydrodynamic equation is independent of the disorder configuration.

We briefly describe some methods developed for non disordered non gradient systems that we use in our investigation. For more details we refer to [13], [28] and reference therein.
Let ν_ϵ be the initial probability distribution on $\mathbb{T}_\epsilon^d$ and let $\{f_x\}_{x\in\mathbb{T}_\epsilon^d}$ be a family of functions depending on the disorder and particle configurations. We fix a disorder α and we write μ_ϵ^α for the grand canonical Gibbs measure on $\mathbb{T}_\epsilon^d$ with Hamiltonian $H = -\sum_x \alpha_x \eta_x$. Since the Hamiltonian is bounded, it is simple to check that

$$H[\nu_\epsilon|\mu_\epsilon^\alpha] \le \kappa\epsilon^{-d}$$

where $H[\cdot|\cdot]$ denotes the relative entropy.
By the Chebyshev inequality and the *entropy inequality*, for any $\delta > 0$ and $\gamma > 0$

(8)
$$\mathbb{P}^{\nu_\epsilon}\Big(\epsilon^{-1}\int_0^t \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} f_x\big(\eta(s)\big) > \delta\Big) \le \delta^{-1}\gamma\kappa + \delta^{-1}\gamma\epsilon^d \ln \mathbb{E}^{\mu_\epsilon^\alpha}\Big(\exp\{\gamma^{-1}\epsilon^{-d-1}\int_0^t \mathrm{Av}_x f_x\}\Big)$$

where $\mathbb{P}^{\nu_\epsilon}$ is the probability measure on the space of paths determined by the Markov generator $\epsilon^{-2}\mathcal{L}^\alpha$ and the initial distribution ν_ϵ. The expectation $\mathbb{E}^{\mu_\epsilon^\alpha}$ is defined in a similar way.
By an application of the *Feynman-Kac formula*,

$$\gamma\epsilon^d \ln \mathbb{E}^{\mu_\epsilon^\alpha}\Big(\exp\{\gamma^{-1}\epsilon^{-d-1}\int_0^t \mathrm{Av}_x f_x\}\Big) \le t \sup spec_{L^2(\mu_\epsilon^\alpha)}\{\epsilon^{-1}\mathrm{Av}_x f_x + \gamma\epsilon^{d-2}\mathcal{L}^\alpha\}$$

where "$\sup spec_{L^2(\mu_\epsilon^\alpha)}$" denotes the supremum of the spectrum of the operator on $L^2(\mu_\epsilon^\alpha)$. Therefore, in order to prove that the l.h.s. of (8) goes to 0 by taking the limit $\epsilon \downarrow 0$, it's enough to show that for any $\gamma > 0$

$$\overline{\lim_{\epsilon\downarrow 0}} \sup spec_{L^2(\mu_\epsilon^\alpha)}\{\epsilon^{-1}\mathrm{Av}_x f_x + \gamma\epsilon^{d-2}\mathcal{L}^\alpha\} \le 0. \tag{9}$$

The problem is reduced to an eigenvalue estimate. The operator function "$\sup spec$" has a useful sub-addivity property which allows to benefit of the ergodicity of the random field α . For example, if every f_x has support in $\Lambda_{x,l}$, the cube of side $2l+1$ centered at x, the supremum of the spectrum appearing in (9) is bounded by

$$\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_{\nu_x^\alpha} \sup spec_{L^2(\nu_x^\alpha)}\{\epsilon^{-1}f_x + \gamma'\epsilon^{-2}(2l)^{-d}\mathcal{L}_{x,l}^\alpha\} \tag{10}$$

where ν_x^α is taken among the canonical Gibbs measures on $\Lambda_{x,l}$ with disorder α, $\mathcal{L}_{x,l}^\alpha$ is the infinitesimal generator obtained by considering jumps only in the cube $\Lambda_{x,l}$ and

$\gamma' = C(d)\gamma$.
At this point a fundamental tool is perturbation theory. For any Gibbs canonical measure ν_x^α the process determined by $\mathcal{L}_{x,l}^\alpha$ is irreducible and reversible. Therefore there exists a constant C such that if

$$\|f_x\|_\infty \epsilon l^d \,\mathrm{gap}_{\nu_x^\alpha}\big(\mathcal{L}_{x,l}^\alpha\big)^{-1}(\gamma')^{-1} \leq C \tag{11}$$

then

$$\sup spec_{L^2(\nu_x^\alpha)}\{\epsilon^{-1} f_x + \gamma'\epsilon^{-2}(2l)^{-d}\mathcal{L}_{x,l}^\alpha\} \leq \epsilon^{-1}\nu_x^\alpha(f_x) + 2(2l)^d \nu_x^\alpha\big(\bar{f}_x, (-\mathcal{L}_{x,l}^\alpha)^{-1}\bar{f}_x\big)$$

where $\bar{f}_x = f_x - \nu_x^\alpha(f_x)$.
It is clear that if any canonical expectation $\nu_x^\alpha(f_x)$ is null the situation is simpler.
In this case, if l does not depend on ϵ and $f_x = \tau_x f$ for some local function f, then we can benefit of the ergodicity and translation invariance of the random field α and obtain that for almost any disorder configuration the $\limsup$ as $\epsilon \downarrow 0$ of expression (10) can be bounded by

$$2(2l)^d \mathbb{E}\Big[\sup_{\nu_0^\alpha}\big(f, (-\mathcal{L}_{0,l}^\alpha)^{-1} f\big)\Big]. \tag{12}$$

By proceeding in this direction, and thanks to the ergodicy and translation invariance of the random field α, we obtain the following construction which is standard for non disordered non gradient systems.
Let $\mathcal{G}$ be the set of bounded local functions $h = h(\alpha, \eta)$ such that for some cube Λ, for any given disorder α, all canonical expectations on Λ of the function $h(\alpha, \cdot)$ are null.
Then, for any $m \in [0,1]$, there is a well-defined semi-inner product $V_m(\cdot,\cdot)$ on $\mathcal{G}$ such that

$$V_m(h,g) = \lim_{l\uparrow\infty}(2l)^d \mathbb{E}\Big[\mu^{\alpha,\lambda_0}\Big(\mathrm{Av}_{|x|\leq l-\sqrt{l}}\tau_x h, (-\mathcal{L}_{0,l}^\alpha)^{-1}\mathrm{Av}_{|x|\leq l-\sqrt{l}}\tau_x g\Big)\Big]. \tag{13}$$

Moreover, if there exists a family of functions $h_{e,m}$ in the completion $\bar{\mathcal{G}}$ of $\mathcal{G}$, parameterized by the vector $e \in \mathcal{E}$ and by the density m, such that for any m

$$V_m(j_{0,e} - h_{e,m}, j_{0,e} - h_{e,m}) = 0$$

then, for almost any disorder configuration α, in (4) we can substitute $j_{x,x+e}^\alpha\big(\eta(s)\big)$ with $\tau_x(h_{e,m})\big(\eta(s)\big)$ where m is the density of $\eta(s)$ in a particular mesoscopic region centered at x.

The infinite dimensional space $\mathcal{G}$ endowed of the scalar product $V_m(\cdot,\cdot)$ is a basic tool to prove a generalized Fick's law for non disordered non gradient systems. In this case, $\mathcal{G}$ is the set of functions h with canonical expectations null on some box and the semi-inner product V_m is defined similarly to (13).
For example, in [13] it is studied a non disordered non gradient system such that its

equilibrium measures are product and translation invariant and therefore $\nabla_{0,e}\eta_0 \in \mathcal{G}$. It can be proved that the subspace

$$\{\sum_{e\in\mathcal{E}} a_e j_{0,e} + \mathcal{L}g \,:\, a \in \mathbb{R}^d,\ g \text{ local}\}$$

is dense in $\mathcal{G}$ endowed of the semi-inner product V_m and that $\{\nabla_{0,e}\eta_0\}_{e\in\mathcal{E}}$ is a basis of

$$\{\mathcal{L}g \,:\, g \text{ local}\}^{\perp}.$$

Therefore, for any density m, we have the orthogonal decompositions of currents

$$j_{0,e} = -\sum_{e'\in\mathcal{E}} D_{e,e'}(m)\nabla_{0,e'}\eta_0 + \xi_{e,m} \quad \forall e \in \mathcal{E}$$

where $\xi_{e,m}$ belongs the closure of $\{\mathcal{L}g \,:\, g \text{ local}\}$ in the completion $\bar{\mathcal{G}}$ of $\mathcal{G}$.
Because of the entropy inequality, Feynman-Kac formula and perturbation theory and since fluctuations $\mathcal{L}g$ are negligible, by taking the limit $\epsilon \downarrow 0$ we can substitute in the formula analogous of (4) $j_{x,x+e}\big(\eta(s)\big)$ with

$$-\sum_{e'\in\mathcal{E}} D_{e,e'}(m)\nabla_{x,x+e'}\eta_x$$

where m is the density of $\eta(s)$ in a particular mesoscopic region centered at x. Therefore the hydrodynamic equation is given by

$$\partial_t m = \nabla_\theta(D(m)\nabla_\theta m).$$

In general, in order to obtain a generalized Fick's law it is crucial to exhibit a basis of $\{\mathcal{L}g \,:\, g \text{ local}\}^{\perp}$ correlated to gradients of density.

We come back to our disordered system. We want to emphasize that in the presence of disorder there is a major problem w.r.t. non disordered systems connected to the fluctuations in the gradient of the density field. In fact by the absence of translation invariance $\nabla_{0,e}\eta_0 \notin \mathcal{G}$. This aspect is present also in non disordered systems with interactions between particles as in [28], where the canonical measures are not spatially homogeneous because of the boundary conditions. But in this case the canonical expectations of $\nabla_{0,e}\eta_0$ in a cube of size n decay very fast when $n \uparrow \infty$. The situation changes drastically in our system because of the disorder. Let us give more details.
In [28] Varadhan and Yau study the hydrodynamic behavior of a generic lattice gas with translation invariant Hamiltonian of finite range satisfying some mixing conditions, which are always valid at high temperature, and with stochastic dynamics reversible w.r.t. Gibbs measures. Then for any $\rho > 0$ there exists $C_\rho > 0$ such that for any integer n

$$\sup_\nu |\nu(\nabla_{0,e}\eta_0)| \leq C_\rho n^{-\rho} \tag{14}$$

where the supremum is taken over all canonical Gibbs measures ν on the cube Λ_n of side n centered at the origin.
Differently, in our disordered system, for almost any disorder configuration α

$$\|\mu^\alpha[\nabla_{0,e}\eta_0 \,|\, m_n]\|_\infty = O(1) \quad \forall n,$$

where m_n is the density on the cube Λ_n. In fact, by the equivalence of ensembles, for a positive constant C and for any density m on Λ_n

$$|\mu^\alpha[\nabla_{0,e}\eta_0 \,|\, m_n = m] - \mu^{\alpha,\lambda}(\nabla_{0,e}\eta_0)| \leq C n^{-d}$$

where λ is the empirical chemical potential on Λ_n with density m, i.e. $\mu^{\alpha,\lambda}(m_n) = m$. Since

$$\mu^{\alpha,\lambda}(\nabla_{0,e}\eta_0) = \frac{e^\lambda(e^{\alpha_e} - e^{\alpha_0})}{(1+e^{\alpha_e+\lambda})(1+e^{\alpha_0+\lambda})},$$

for densities m such that $\lambda \approx 0$ we have $\|\mu^\alpha[\nabla_{0,e}\eta_0 \,|\, m_n = m]\|_\infty = O(1)$ (if $\alpha_e \neq \alpha_0$).
If we consider the gradient of the density in two large mesoscopic regions then, because of the ergodicity of the random field α, we can hope to have small fluctuations. Let us be more precise. Given $e \in \mathcal{E}$ and an integer s, let $\Lambda^e_{1,s}$ and $\Lambda^e_{2,s}$ be a couple of adjacent cubes of size s centered at the origin with $\Lambda^e_{2,s}$ obtained by moving $\Lambda^e_{1,s}$ along the direction determined by the vector e of the canonical basis of $\mathbb{Z}^d$.
We write $m^{1,e}_s$ and $m^{2,e}_s$ for the density respectively in $\Lambda^e_{1,s}$ and $\Lambda^e_{2,s}$, while m^e_s is the density in $\Lambda^e_{1,s} \cup \Lambda^e_{2,s}$.
Then, given integers $s \leq s'$, we set

$$\phi^e_{s,s'} = \mu^\alpha[m^{1,e}_s - m^{2,e}_s \,|\, m^e_{s'}] \quad \text{and} \quad \psi^e_{s,s'} = m^{1,e}_s - m^{2,e}_s - \phi^e_{s,s'}. \tag{15}$$

By simple large deviations estimates it can be proved that, if $s' \geq s = O(\epsilon^{-\beta})$ with $\beta > 0$, then given $\gamma > 0$, for almost any disorder configuration α and for $\epsilon \leq \epsilon_0(\alpha, \gamma)$,

$$\|\tau_x \phi_{s,s'}\|_\infty \leq C s^{-\frac{d}{2}+\gamma} \qquad \forall x \in \mathbb{T}^d_\epsilon. \tag{16}$$

We come back to [28], where $\nabla_{0,e} \notin \mathcal{G}$ as in our disordered model, and indicate how a generalized Fick's law is obtained. By (14) it can be proved, that for any density m the sequence

$$\nabla_{0,e}\eta_0 - \mu^\eta_{\Lambda_n, m_n(\eta)}\big(\nabla_{0,e}\eta_0\big) \quad n \in \mathbb{N}, \tag{17}$$

where $\mu^\eta_{\Lambda_n, m_n(\eta)}$ is the canonical Gibbs measure on Λ_n with density $m_n(\eta)$ and exterior configuration η, is Cauchy in $\mathcal{G}$ for the semi-norm $V_m^{-1/2}$. Moreover, the limit points of the sequences (17) with e varying in $\mathcal{E}$ form a basis of $\{\mathcal{L}g \,:\, g \text{ local}\}^\perp$. Therefore, by taking the limit $n \uparrow \infty$ after $\epsilon \downarrow 0$, in the expression analogous to (4) the current $j_{x,x+e}$ can be substituted with some negligible fluctuation $\tau_x \mathcal{L} g$ and the linear combination

$$-\sum_{e' \in \mathcal{E}} D_{e,e'}(m) \tau_x \Big(\nabla_{0,e}\eta_0 - \mu^\eta_{\Lambda_n, m_n(\eta)}\big(\nabla_{0,e}\eta_0\big)\Big)$$

where m is the density of the configuration η in a particular mesoscopic region centered at x. After this step, another analysis is necessary in order to have a Fick's law, i.e. in order to prove that in this new expression the terms

$$\tau_x\left[\mu^{\eta}_{\Lambda_n, m_n(\eta)}\big(\nabla_{0,e}\eta_0\big)\right] \quad \text{with } \ x \in \mathbb{T}^d_\epsilon$$

are negligible.

At this point we can describe our strategy and our results.
We set

$$\mathbb{G} = \{g \,:\, g = g(\alpha, \eta) \ \text{ local and bounded}\}$$

and we prove that the semi-inner product V_m given by (13) is well-defined and the subspace

$$\{\sum_e a_e j^{\alpha}_{0,e} + \mathcal{L}^{\alpha} g \,:\, a \in \mathbb{R}^d, \ \ g \in \mathbb{G}\} \tag{18}$$

is dense in $\mathcal{G}$ with semi-inner product V_m. The proof generalizes the main ideas developed for non disordered non gradient systems and in particular the theory of closed and exact forms. This generalization is not automatic and there are two main steps we would like to discuss.
1) Because of the presence of the disorder the structure of closed forms is richer and some new forms, which do not appear in non disordered systems, require a specific analysis.
2) It is well-known that in the hydrodynamic limit theory of non gradient systems a key role is played by the diffusive scaling of the spectral gap. As already mentioned, although for the system under consideration the spectral gap of the dynamics restricted to a cube of size L can be inferiorly bounded by CL^{-2} uniformly in the disorder configuration, our analysis of the structure of closed forms is developed in such a way that we need only the weaker estimates on the spectral gap derivable by the recursive method developed in [6], which are valid in a large class of disordered systems, also having deviations of the spectral gap from the diffusive scaling as the dilute Ising lattice gas with Kawasaki dynamics under the percolation threshold studied in [7].

As in the standard theory of non disordered non gradient systems, we consider the orthogonal decomposition of $\mathcal{G}$, endowed of the semi-inner product V_m, along $\mathcal{L}\mathbb{G}$ and its orthogonal subspace $\mathcal{L}\mathbb{G}^{\perp}$ where

$$\mathcal{L}\mathbb{G} = \{\mathcal{L}^{\alpha} g : \ g \in \mathbb{G}\}.$$

Following [21], given an integer s and $e \in \mathcal{E}$, we introduce the functions

$$W^e_s = \mathrm{Av}_{x \in \Lambda^e_{1,s}} \mathrm{Av}_{y \in \Lambda^e_{2,s}} w^{\alpha}_{x,y} \tag{19}$$

where

$$w^{\alpha}_{x,y} = \big(1 + e^{-(\alpha_x - \alpha_y)(\eta_x - \eta_y)}\big)(\eta_x - \eta_y).$$

We remember that $\Lambda^e_{1,s}, \Lambda^e_{2,s}$ is a couple of adjacent cubes of size s centered at the origin with $\Lambda^e_{2,s}$ obtained by moving $\Lambda^e_{1,s}$ along the direction determined by the vector e of the canonical basis of $\mathbb{Z}^d$.

Our main results are the following.
A) For any $e \in \mathcal{E}$

$$\lim_{n\uparrow\infty} 2m(1-m)\lambda_0'(m)\frac{\psi^e_{n,n}}{n} - \frac{W^e_n}{n} = 0$$

where W^e_n and $\psi^e_{n,n}$ are defined respectively in (19) and (15).
B) For any $e \in \mathcal{E}$ and any integer n the sequence $\{\frac{\psi^e_{n,s^4}}{n}\}_{s\geq n}$ is Cauchy. We write $\bar{\psi}^e_n$ for its limit point in the completion $\bar{\mathcal{G}}$ of $\mathcal{G}$. Then $\bar{\psi}^e_n \in \mathcal{L}\mathbb{G}^\perp$ and

$$\lim_{n\uparrow\infty} \frac{\psi^e_{n,n}}{n} - \bar{\psi}^e_n = 0.$$

Results A) and B) allow to prove that for any $e \in \mathcal{E}$ the sequence $\left\{\frac{\psi^e_{n,n}}{n}\right\}_{n\geq 1}$ is Cauchy and that its limit points ξ_e, as e varies in $\mathcal{E}$, form a basis of the subspace $\mathcal{L}\mathbb{G}^\perp$.
In fact, let P be the orthogonal projection on $\mathcal{L}\mathbb{G}^\perp$. Then B) implies that $(1-P)\frac{\psi^e_{n,n}}{n}$ goes to 0 as $n \uparrow \infty$. Since the currents and the fluctuations $\mathcal{L}^\alpha g$ form a subspace dense in $\mathcal{G}$, the set $\{Pj_e\}_{e\in\mathcal{E}}$ generates $\mathcal{L}\mathbb{G}^\perp$ and therefore A), together with the identity

$$V_m\Big(j_e, \frac{W^{e'}_n}{n}\Big) = 2m(1-m)\delta_{e,e'}, \tag{20}$$

implies that $P\frac{\psi^e_{n,n}}{n}$ converges in $\mathcal{L}\mathbb{G}^\perp$.
We note that (20) derives from an integration by parts formula valid for the function $w^\alpha_{x,y}$: for any Gibbs measure μ with disorder α and any function f

$$\mu\big(w^\alpha_{x,y} f\big) = \mu\big((\eta_x - \eta_y)\nabla_{x,y} f\big). \tag{21}$$

The proof of A) and B) is rather technical and it is explained in full details in chapter 3 and 5.

Remark. The idea to introduce $\frac{W^e_n}{n}$ is taken from [21] but referred to a completely different scale. We introduce the semi-norm $V_m^{1/2}$ after taking the limit $\epsilon \downarrow 0$ in the same spirit of the discussion preceding (12) and (13). The sequence $\{\frac{W^e_n}{n}\}_{n\geq 1}$ belongs to the space $\mathcal{G}$ which is introduced after taking the limit $\epsilon \downarrow 0$. In [21] Quastel introduces the function $\frac{W_s}{s}$ with $s = O(\epsilon^{-\frac{2}{d+2}})$, before taking the limit $\epsilon \downarrow 0$. The outlined global strategy is therefore different.
Some of the steps in the proof of A) are motivated by ideas found in [21]. However, as mentioned before, our mesoscopic context is different and a rigorous proof requires the development of a new technology.
Remark. It can be proved that for any $f \in \mathcal{G}$ and any $g \in \mathbb{G}$

$$V_m(f, \mathcal{L}^\alpha g) = -\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\tau_x f \cdot g\big)\Big]. \tag{22}$$

By (22) and the integration by parts formula (21), it can be proved that for any $g \in \mathbb{G}$

$$\lim_{n\uparrow\infty} V_m\Big(\frac{W_n^e}{n}, \mathcal{L}^\alpha g\Big) = 0$$

and therefore, by A), the same is valid for $\frac{\psi^e_{n,n}}{n}$. Since the subspace $\mathcal{L}\mathbb{G}$ has infinite dimension this is not sufficient to prove that the orthogonal projection of $\frac{\psi^e_{n,n}}{n}$ on $\mathcal{L}\mathbb{G}$ goes to 0. We move our attention on the sequence $\{\frac{\psi^e_{n,s^4}}{n}\}_{s\geq n}$. Its convergence is proved via a telescopic argument. In particular, we show

$$\sum_{s\geq n}^{\infty} V_m^{\frac{1}{2}}\Big(\frac{\psi^e_{n,s^4}}{n} - \frac{\psi^e_{n,(s+1)^4}}{n}\Big) < \infty.$$

The proof is based on estimates of the fluctuations of the gradient of the density field which are connected to the fluctuations of the random field α and which work well if $d \geq 3$. We will comment subsequently this restriction on the dimension in a similar context.
Let us give some details of the proof that $\bar{\psi}^e_n \in \mathcal{L}\mathbb{G}^\perp$. Since it can be showed that the subspace $\{\mathcal{L}^\alpha g \,:\, g \in \mathcal{G}\}$ is dense in $\mathcal{L}\mathbb{G}$, it is enough to prove that $\forall g \in \mathcal{G}$

$$\lim_{s\uparrow\infty} V_m\Big(\frac{\psi^e_{n,s}}{n}, \mathcal{L}^\alpha g\Big) = 0. \tag{23}$$

We observe that if $g \in \mathcal{G}$ then the r.h.s. of (22) is a finite sum for any local function f and is zero if f is a constant function or $f = m_n^{1,e} - m_n^{2,e}$. Therefore the proof of (23) is based on the property

$$\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}(\phi^e_{n,s}) \leq C s^{-d}$$

where $\psi^e_{x,s} = m_n^{1,e} - m_n^{2,e} + \phi^e_{n,s}$ as defined in (15).

At this point, as in the preceding discussion referred to non disordered systems, we orthogonally decompose the currents along $\mathcal{L}\mathbb{G}$ and $\mathcal{L}\mathbb{G}^\perp$ and obtain a partial Fick's law in the sense that in (4), in the limit $n \uparrow \infty$ after $\epsilon \downarrow 0$, we can substitute the current $j^\alpha_{x,x+e}$ by some negligible fluctuation $\tau_x \mathcal{L}^\alpha g$ plus

$$\sum_{e'\in\mathcal{E}} D_{e,e'}(m)\tau_x\Big[\frac{m_n^{1,e} - m_n^{2,e}}{n}\Big] - \sum_{e'\in\mathcal{E}} D_{e,e'}(m)\frac{\phi^e_{n,n}}{n} \tag{24}$$

where m is the density in a particular mesoscopic region centered at x.

In order to have gradients of the density which are macroscopically relevant, corresponding in the hydrodynamic equation to the term $\nabla_\theta m$, we have to pass from the differences

between the densities in cubes of size n to the differences between the densities in cubes of size a/ϵ where $a \downarrow 0$ after $\epsilon \downarrow 0$.
To this aim we prove a Two-Blocks estimate and introduce particular spatial averages which allow to express the difference between the densities in cubes of size a/ϵ in terms of the differences between the densities in adjacent cubes of size n. Though this step is fundamental we neglect it here and give a simplified exposition of our analysis.

We have to face again the problem of the fluctuations in the gradient of the density field in order to prove that the second addendum in (24) is negligible in the limit $n \uparrow \infty$ after $\epsilon \downarrow 0$. Because of the entropy inequality, the Feynman-Kac formula and the perturbation theory, it's enough to show that for any $e \in \mathcal{E}$ and any $\gamma > 0$

$$\overline{\lim_{n\uparrow\infty}}\, \overline{\lim_{\epsilon\downarrow 0}} \sup spec_{L^2(\mu_\epsilon^\alpha)} \{\epsilon^{-1} \mathrm{Av}_x \frac{\nabla_e H(\epsilon x)}{\epsilon} \tau_x \frac{\phi^e_{n,n}}{n} + \gamma \epsilon^{d-2} \mathcal{L}^\alpha\} \leq 0. \tag{25}$$

We observe that by (16) if $s = O(\epsilon^{-\beta})$ with $\beta > \frac{2}{d+2}$ then for almost any disorder configuration α

$$\lim_{\epsilon\downarrow 0} \sup_{x \in \mathbb{T}^d_\epsilon} \epsilon^{-1} \| \frac{\phi^e_{s,s}}{s} \|_\infty = 0, \tag{26}$$

Therefore (25) would be proved if we could substitute in (25) the term $\frac{\phi^e_{n,n}}{n}$ with $\frac{\phi^e_{s,s}}{s}$ and s as above. The mesoscopic scale $\epsilon^{-\frac{2}{d+2}}$ is the critical threshold in our analysis of the fluctuations in the gradient of the density field.
The presence of the factor $\frac{\nabla_e H(\epsilon x)}{\epsilon}$ in (25) is troublesome because we prefer to sum the largest possible number of terms of the form $\tau_x \frac{\phi^e_{n,n}}{n}$ in order to assure some cancellations. This problem can be solved by a Two-Blocks estimate and therefore, in a first approximation, in (25) we may substitute $\frac{\nabla_e H(\epsilon x)}{\epsilon}$ with 1.
At this point it is convenient to introduce some notation. Let $f_{n,\epsilon}$ be a local function depending on the parameters n, ϵ. Then we write

$$f_{n,\epsilon} \approx 0$$

if $\forall \gamma > 0$

$$\overline{\lim_{n\uparrow\infty}}\, \overline{\lim_{\epsilon\downarrow 0}} \sup spec_{L^2(\mu_\epsilon^\alpha)} \{\epsilon^{-1} \mathrm{Av}_x \tau_x f_{n,\epsilon} + \gamma \epsilon^{d-2} \mathcal{L}^\alpha\} \leq 0.$$

The proof that $\frac{\phi^e_{n,n}}{n} \approx 0$ is given in chapter 4 and consists of steps $A)$, $B)$ and $C)$ given below, where a key role is played by large deviations estimates in the random field α which work well in dimension $d \geq 3$.
Let us give an heuristic argument in order to explain this restriction on the dimension. The random variable $\phi^e_{n,n} = \mu^\alpha[m_n^{1,e} - m_n^{2,e} \,|\, m_n^e]$ depends both on the disorder configuration and on the particle configuration. In order to examine the fluctuations in the random field α, in first approximation we consider here the function

$$\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\tau_x\frac{\Phi_n}{n} \tag{27}$$

where, for an a priori fixed density m,

$$\Phi_n(\alpha)=\mu^\alpha[m_n^{1,e}-m_n^{2,e}\,|\,m_n^e=m].$$

For simplicity of notation let $n|\epsilon^{-1}$ and let us subdivide $\mathbb{T}^d_\epsilon$ in disjoint cubes of size $2n$:

$$\mathbb{T}^d_\epsilon=\sqcup_{y\in Y}\big(\Lambda_{2n}+y\big)$$

where Λ_{2n} is the cube of size $2n$ centered at the origin. Then $|Y|=O(\epsilon^{-d}n^{-d})$ and (27) can be written as

$$\epsilon^{-1}\mathrm{Av}_{z\in\Lambda_{2n}}\mathrm{Av}_{w\in Y+z}\tau_w\frac{\Phi_n}{n}. \tag{28}$$

Since the random variable $\mathrm{Av}_{w\in Y+z}\tau_w\frac{\Phi_n}{n}$ is the average of functions having disjoint supports and null expectation w.r.t. the random field α, (28) can be approximated by

$$\epsilon^{-1}\mathrm{Av}_{z\in\Lambda_{2n}}\mathrm{Var}_{\mathbb{P}}\Big[\mathrm{Av}_{w\in Y+z}\tau_w\frac{\Phi_n}{n}\Big]^{\frac12}=C\epsilon^{\frac{d}{2}-1}n^{\frac{d-1}{2}}\mathrm{Var}_{\mathbb{P}}(\Phi_n)^{\frac12}.$$

Therefore, if $d\geq 3$ the fluctuations in the random field α allow to prove that $\frac{\phi^e_{n,n}}{n}\approx 0$ while $d=2$ is the critical threshold.

As mentioned before, the rigorous proof that $\frac{\phi^e_{n,n}}{n}\approx 0$ consists of the following main steps.

A) If $s=\epsilon^{-\frac{1}{d+2}}$ then

$$\frac{\phi^e_{n,n}}{n}-\frac{\phi^e_{s,s}}{s}\approx 0.$$

B) If

$$s\equiv\epsilon^{-\frac{a}{d+2}}\qquad s'\equiv\epsilon^{-\frac{b}{d+2}}\qquad\text{where}\quad 1\leq a<b<d+2\ \text{ and }\ b<1+\frac{a}{2}$$

then

$$\frac{\phi^e_{s,s}}{s}-\frac{\phi^e_{s',s'}}{s'}\approx 0.$$

We observe that by applying recursively B) it can be proved that for any small $\gamma>0$

$$\frac{\phi^e_{s,s}}{s}-\frac{\phi^e_{s',s'}}{s'}\approx 0\qquad\text{where}\ \ s=\epsilon^{-\frac{1}{d+2}},\ \ s'=\epsilon^{-\frac{2}{d+2}+\gamma},$$

but we cannot overpass the critical threshold $\epsilon^{-\frac{2}{d+2}}$ and benefit of the estimate (26). The analysis can be concluded by the following result.

C) If $\gamma > 0$ is small enough we can show

$$\frac{\phi^e_{s,s}}{s} \approx 0 \quad \text{with } s = \epsilon^{-\frac{2}{d+2}+\gamma}$$

by proving a sort of strong Two-Blocks estimate which differs from standard Two-Blocks estimates by the presence of a factor $\epsilon^{-\beta}$ with $\beta > 0$ small enough in front of the multiplicative operator.

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Chapter 1

Hydrodynamic limit of a disordered system

In sections 1.2 and 1.3 we describe the model, its dynamics and state the main results. We concentrate on theorem 1.3 which allows the identification of the hydrodynamic equation.

Then we generalize standard methods for non gradient systems, in particular by means of stochastic calculus, the entropy inequality, the Feynman-Kac formula and perturbation theory we reduce the proof of theorem 1.3 to some eigenvalue estimates corresponding to proposition 1.10.

At the end we sketch the proof of proposition 1.10, which is based also on the results of the next chapters.

1.1 Notation

We collect here, in a schematic way, some notation useful for the following sections.

For any $x \in \mathbb{R}^d$ we set $|x| = \max\{|x_1|, \dots, |x_d|\}$. Given $l \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ we set

$$\Lambda_{x,l} \equiv \{y \in \mathbb{Z}^d \,:\, |y - x| \le l\}.$$

If $x = 0$ we write simply Λ_l. Given an integer l we set $l_1 \equiv l - \sqrt{l}$. Moreover, given a vector e we write $e > 0$ if it belongs to the canonical base of $\mathbb{Z}^d$.
We define

$$\Omega \equiv \{0,1\}^{\mathbb{Z}^d} \qquad \Omega_\epsilon \equiv \{0,1\}^{\mathbb{T}^d_\epsilon} \qquad \Omega_D \equiv [-B, B]^{\mathbb{Z}^d} \qquad \tilde{\Omega} \equiv \Omega_D \times \Omega$$

where B is a fixed positive number, ϵ is the inverse of a positive integer and $\mathbb{T}^d_\epsilon$ is the discrete torus $\mathbb{Z}^d/\epsilon^{-1}\mathbb{Z}^d$. We denote by η $[\alpha]$ a generic element of Ω $[\Omega_D]$.
The above spaces are endowed of the product topology. Measurable sets are given by Borel sets. In particular, given $\Lambda \subset \mathbb{Z}^d$, we introduce the following $\sigma-$algebras

$$K_\Lambda = \sigma\big(\eta_x \,:\, x \in \Lambda\big) \qquad \mathbb{K}_\Lambda = \sigma\big(\alpha_x, \, \eta_x \,:\, x \in \Lambda\big).$$

We call $\mathbb{F}$ the family of finite subsets of $\mathbb{Z}^d$.

A function f on $\tilde{\Omega}$ is called local if it is measurable and with finite support. We denote Δ_f its support.

We introduce some spaces of functions (we refer to section 1.2 for the definition of the measure $\mu^{\alpha}_{\Lambda_k,m}$) where $\Lambda\subset\mathbb{Z}^d$ and $k\in\mathbb{N}$:

$$\begin{aligned}
\mathbb{G} &\equiv \{g:\tilde{\Omega}\to\mathbb{R} \text{ local and bounded } \}\\
\mathbb{G}_\Lambda &\equiv \{g\in\mathbb{G} \;:\; \Delta_g\subset\Lambda\}\\
G_\Lambda &\equiv \{g:\{0,1\}^\Lambda\to\mathbb{R} \}\\
\mathcal{G}_k &\equiv \{g\in\mathbb{G} \;:\; \Delta_g\subset\Lambda_k \text{ and } \forall\alpha\in\Omega_D, \forall m\in[0,1]\ \mu^{\alpha}_{\Lambda_k,m}(g)=0\}.
\end{aligned}$$

If $\Lambda=\Lambda_k$ we write $\mathbb{G}_k$, G_k in place of $\mathbb{G}_{\Lambda_k}$, G_{Λ_k}. Moreover, we set $\mathcal{G}\equiv\cup_k\mathcal{G}_k$.

Given $\eta\in\Omega$ and $x,y\in\mathbb{Z}^d$ we set

$$\eta^{x,y}_z \equiv \begin{cases} \eta_y \text{ if } \ z=x\\ \eta_x \text{ if } \ z=y\\ \eta_z \ \text{ otherwise.}\end{cases}$$

Given $x,y\in\mathbb{Z}^d$ and a function f on $\tilde{\Omega}$ we define

$$(S_{x,y}f)(\alpha,\eta)\equiv f(\alpha,\eta^{x,y}) \qquad \nabla_{x,y}f\equiv S_{x,y}f-f. \tag{1.1}$$

Given $x\in\mathbb{Z}^d$ and $\eta\in\Omega$ we write η^x for the configuration

$$\eta^x(z)\equiv\begin{cases}1-\eta_x & \text{if } z=x\\ \eta_z & \text{otherwise.}\end{cases}$$

Given a function f on $\tilde{\Omega}$ we define $\nabla_x f$ by $\nabla_x f(\alpha,\eta)\equiv f(\alpha,\eta^x)-f(\alpha,\eta)$.
Given $z\in\mathbb{Z}^d$ we define the traslazions along z in the following way

$$(\tau_z\eta)(x)=\eta(x+z) \quad (\tau_z\alpha)(x)=\alpha(x+z) \quad (\tau_z f)(\alpha,\eta)=f(\tau_z\alpha,\tau_z\eta).$$

Given a probability measure $\mathbb{P}$ on Ω_D, we will write $\tilde{\forall}\alpha$ in order to denote "$\mathbb{P}$ almost all α".
We will denote $\mathbb{T}^d$ the d–dimensional torus with volume 1: $\mathbb{R}^d/\mathbb{Z}^d$. We define $\mathcal{M}_1$ the space of non-negative measures on $\mathbb{T}^d$ with total mass bounded by 1, endowed of the weak topology, while $\mathcal{M}_2\subset\mathcal{M}_1$ will denote the set of measures in $\mathcal{M}_1$ which are absolutely continuous w.r.t. the Lebesgue measure with density ρ satisfying $\rho\le 1$.
Moreover, we set $C^*(\mathbb{T}^d)\equiv\{J\in C(\mathbb{T}^d)\;:\;\|J\|_\infty\le 1\}$
We will call "bond" an non ordered couple of sites x,y such that $|x-y|=1$. We will denote by b a generic bond and write $b\in\Lambda$ if $b=\{x,y\}$ with $x,y\in\Lambda\subset\mathbb{Z}^d$.

Given $l\in\mathbb{N}$, $x\in\mathbb{Z}^d$ and $\eta\in\Omega$, we set $m_{x,l}(\eta):=(2l+1)^{-d}\sum_{x\in\Lambda_{x,l}}\eta_x$ while, if l is odd, $m^l_x(\eta):=l^{-d}\sum_{x\in\Lambda}\eta_x$ where Λ is the cube of side length l centered at x.

Given n parameters $\ell_1, \dots \ell_n$ we use the compact notation $\lim_{\ell_n \to \ell'_n, \dots, \ell_1 \to \ell'_1}$ for the ordered limits $\lim_{\ell_n \to \ell'_n} \dots \lim_{\ell_1 \to \ell'_1}$. The same convention is valid when "lim" is replaced by "lim sup" or "lim inf".

1.2 The model

For any odd integer ϵ^{-1} we consider the Ising lattice gas, at infinite temperature and with random chemical potential, on the discrete torus $\mathbb{T}^d_\epsilon \equiv \mathbb{Z}^d/\epsilon^{-1}\mathbb{Z}^d$. Let us describe the model.

For a fixed $B > 0$, $\Omega_D \equiv [-B, B]^{\mathbb{Z}^d}$ and $\Omega_{D,\epsilon} \equiv [-B, B]^{\mathbb{T}^d_\epsilon}$ are the sets of disorder configurations respectively on $\mathbb{Z}^d$ and on $\mathbb{T}^d_\epsilon$. On Ω_D it is defined a product and translation invariant probability measure $\mathbb{P}$ (we denote by $\mathbb{E}$ the expectation w.r.t. $\mathbb{P}$).
By identifying the cube of side ϵ^{-1} centered at the origin with the torus $\mathbb{T}^d_\epsilon$ by means of the canonical projection, $\alpha \in \Omega_D$ induces in a natural way a disorder configuration α^ϵ on $\mathbb{T}^d_\epsilon$.

Given $\lambda \in \mathbb{R}$ and $\alpha \in \Omega_{D,\epsilon}$, $\mu^{\alpha,\lambda}_\epsilon$ is the product measure on $\Omega_\epsilon \equiv \{0,1\}^{\mathbb{T}^d_\epsilon}$ such that

$$\mu^{\alpha,\lambda}_\epsilon(\eta_x) = \frac{e^{(\alpha_x+\lambda)\eta_x}}{1 + e^{\alpha_x+\lambda}}.$$

In other words, $\mu^{\alpha,\lambda}_\epsilon$ is the grand canonical Gibbs measure for the system with Hamiltonian $H_\alpha = -\sum_x \alpha_x \eta_x$.
If $\lambda = 0$ we write simply μ^α_ϵ. If $\lambda = +\infty$ [$\lambda = -\infty$] the measure $\mu^{\alpha,\lambda}_\epsilon$ is concentrated on the configuration uniformly equal to 1 [0].
Similarly, given $\alpha \in \Omega_D$ and $\lambda \in \bar{\mathbb{R}}$ we can define the grand canonical Gibbs measure $\mu^{\alpha,\lambda}$ on the set $\Omega \equiv \{0,1\}^{\mathbb{Z}^d}$.

In order to define the **dynamics** of the lattice gas on $\mathbb{T}^d_\epsilon$ let us define the transition rates of the process as

$$c_{x,x+e}(\alpha, \eta) = f_e(\alpha_x, \eta_x, \alpha_{x+e}, \eta_{x+e}) \quad \forall x \in \mathbb{Z}^d,$$

where e varies in the canonical base of $\mathbb{Z}^d$, f_e is a function on $\big([-B, B] \times \{0,1\}\big)^2$ such that $f_e(a, s, a', s') = f_e(a', s', a, s)$ and $0 < c \le f_e < c^{-1}$ for a suitable constant c. The above expressions are meaningful if referred both to $\mathbb{Z}^d$ and to $\mathbb{T}^d_\epsilon$.
The main properties of the transition rates are the following (note that a) and c) follow from the definition, while b) is assumed):
a)*Positivity and boundness*: there exists $c > 1$ such that $c^{-1} \le c_{x,y} \le c$ for any x, y such that $|x - y| = 1$.

b) *Detailed balance*: for any x, y, η, α with $|x - y| = 1$

$$c_{x,y}(\alpha, \eta) = c_{x,y}(\alpha, \eta^{x,y}) e^{-\nabla_{x,y} H_\alpha}$$

where $H_\alpha \equiv -\sum_x \alpha_x \eta_x$.
c) *Translation invariance*: $\tau_z c_{x,y} = c_{x+z,y+z}$ for any $x, y, z \in \mathbb{Z}^d$ such that $|x - y| = 1$.

For example we can take

$$c_{x,y}(\alpha,\eta) = 1 + e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}.$$

The stochastic dynamics of the lattice gas on $\mathbb{T}_\epsilon^d$ with disorder configuration $\alpha \in \Omega_{D,\epsilon}$ is determined by the diffusively rescaled Markov generator $\epsilon^{-2}\mathcal{L}_\epsilon^\alpha$ defined by

$$(\mathcal{L}_\epsilon^\alpha f)(\eta) \equiv \sum_{e>0}\sum_{x\in\mathbb{T}_\epsilon^d} c_{x,x+e}(\alpha,\eta)(\nabla_{x,x+e}f)(\eta) \qquad \eta\in\Omega_\epsilon, \quad f:\Omega_\epsilon\to\mathbb{R},$$

where $e>0$ is a generic element of the canonical base of $\mathbb{Z}^d$. We observe that by the detailed balance equation the measure $\mu_\epsilon^{\alpha,\lambda}$ is reversible w.r.t. $\mathcal{L}_\epsilon^\alpha$.

When $\alpha\in\Omega_D$ we write $\mathcal{L}_\epsilon^\alpha$ for $\mathcal{L}_\epsilon^{\alpha^\epsilon}$ (of course, when there no ambiguity we will omit some index).

For any $x\in\mathbb{T}_\epsilon^d$ and $e>0$ the current w.r.t. the bond $\{x,x+e\}$ is given by

$$j_{x,x+e}(\alpha,\eta) = c_{x,x+e}(\alpha,\eta)(\eta_x-\eta_{x+e}).$$

By the same expression we define the current when referred to $\mathbb{Z}^d$. We observe that given $\alpha\in\Omega_D$, unless $\alpha_x=\alpha_y$ for any $x,y\in\mathbb{Z}^d$, the system is *not gradient*, i.e. there is not a local function $f=f(\alpha,\eta)$ such that

$$j_{x,x+e}(\alpha,\cdot) = \tau_{x+e}f(\alpha,\cdot) - \tau_x f(\alpha,\cdot) \qquad \forall x\in\mathbb{Z}^d \ \ \forall e>0.$$

For, if the above expression is valid then we can write $f=\sum_{A\in\mathbb{F}} d_A(\alpha)\eta_A$ where $\mathbb{F}$ is the family of finite subsets of $\mathbb{Z}^d$, $\eta_A\equiv\prod_{x\in A}\eta_x$, α is a generic disorder in Ω_D and $d_A\equiv 0$ except for finite set of cases. Then $\forall x\in\mathbb{Z}^d, e>0$

$$\tau_{x+e}f-\tau_x f = \sum_{A\in\mathbb{F}}\big[\tau_{x+e}d_{A-x-e}-\tau_x d_{A-x}\big]\eta_A$$

and

$$j_{x,x+e} = \mathfrak{g}_e(\alpha_x,\alpha_{x+e})\big[\eta_x - e^{\alpha_x-\alpha_{x+e}}\eta_{x+e} + (e^{\alpha_x-\alpha_{x+e}}-1)\eta_x\eta_{x+e}\big].$$

By the linear independence of $\{\eta_A\}_{A\in\mathbb{F}}$ we obtain $\forall e,\alpha$

$$\begin{aligned} &i)\quad d_{A-e}(\tau_e\alpha) = d_A(\alpha) \qquad \text{if } A\neq\{0\},\{e\},\{0,e\},\\ &ii)\quad d_{-e,0}(\tau_e\alpha)-d_{0,e}(\alpha) = \mathfrak{g}_e(\alpha_0,\alpha_e)(e^{\alpha_0-\alpha_e}-1). \end{aligned}$$

Given $e>0$ and an integer $k\geq 0$, let $A=\{(k+1)e,(k+2)e\}$. By $i)$

$$d_{ke,(k+1)e}(\tau_e\alpha) = d_{(k+1)e,(k+2)e}(\alpha) \quad \forall\alpha$$

Since for k large enough $d_{ke,(k+1)e}$ is null, we obtain that $d_{0,e}(\alpha)=0$ for any disorder α. Similarly, $d_{-e,0}(\alpha)=0$ for any α. Therefore the r.h.s. in $ii)$ is null $\forall e$, implying $\alpha_e=\alpha_0$. Similarly, we can prove that for any $x\in\mathbb{Z}^d$ and any $e>0$ it must be $\alpha_{x+e}=\alpha_x$.

Since any canonical or grand canonical measure μ on a volume Λ with disorder configuration α is reversible for $\mathcal{L}^{\alpha}_{xy} \equiv c^{\alpha}_{xy}\nabla_{xy}$ with $x, y \in \Lambda$, we have the integration by parts formula

$$\mu(c_{xy}\nabla_{xy}f \cdot g) = \mu(c_{xy}f\nabla_{xy}g) = -\frac{1}{2}\mu(c_{xy}\nabla_{xy}f \cdot \nabla_{xy}g) \quad \forall x, y. \tag{1.2}$$

If the generalized transition rate c_{xy} is equal to a_{xy} where

$$a_{xy} \equiv 1 + \exp\big(-(\alpha_x - \alpha_y)(\eta_x - \eta_y)\big) = 1 + e^{-\nabla_{xy}H}$$

then (1.2) gives

$$\mu\big(a_{x,y}\nabla_{xy}f \cdot g\big) = -\mu(\nabla_{xy}f \cdot \nabla_{xy}g). \tag{1.3}$$

We conclude this section with some notation. Given $\alpha \in \Omega_D$ and $\Lambda \in \mathbb{F}$, we denote by $\mu^{\alpha}_{\Lambda,m}$ the Gibbs canonical measure on Λ with density m and disorder α and by $\mathcal{L}^{\alpha}_{\Lambda}$ the infinitesimal generator of the dynamics with disorder α restricted to the volume Λ, that is

$$\mathcal{L}^{\alpha}_{\Lambda} \equiv \sum_{b \in \Lambda} \mathcal{L}^{\alpha}_{b}$$

where b varies among the bonds of Λ.

1.3 Hydrodynamic limit

The macroscopic space is given by $\mathbb{T}^d$, the $d-$dimensional torus, and each macroscopic state is described by a Lebesgue absolutely continuous measure in $\mathcal{M}_1$, the space of non-negative measures on $\mathbb{T}^d$ with total mass bounded by 1. $\mathcal{M}_1$ is endowed of the weak topology.

The discrete torus $\mathbb{T}^d_{\epsilon}$ corresponds to the microscopic space at scale ϵ. The microscopic state is given by a probability measure on Ω_{ϵ} and the microscopic dynamics is determined by the Markov generator $\epsilon^{-2}\mathcal{L}^{\alpha}_{\epsilon}$ (we speed time up by the diffusive factor ϵ^{-2}).
Given a initial probability measure ν_{ϵ} we write $\nu_{\epsilon,t}$ for its evolution at time t.

The passage from the family of stochastically described microscopic worlds to the deterministic world is realized in the following way.
To each configuration $\eta \in \Omega_{\epsilon}$ we can associate the "empirical measure" $\pi_{\epsilon}(\eta) \in \mathcal{M}_1$ defined as

$$\pi_{\epsilon}(\eta) \equiv \epsilon^d \sum_{x \in \mathbb{T}^d_{\epsilon}} \eta_x \delta_{\epsilon x}.$$

Then a sequence of microscopic states $\{\nu_\epsilon\}_{\epsilon>0}$ corresponds to the macroscopic state $m_0(\theta)d\theta$ in $\mathcal{M}_2$ if, considering π_ϵ as a random variable on $(\Omega_\epsilon, \nu_\epsilon)$ with values in $\mathcal{M}_1$, we have

$$\pi_\epsilon \Rightarrow m_0(\theta)d\theta \quad \text{in probability,}$$

that is

$$\lim_{\epsilon\downarrow 0} \nu_\epsilon\Big[\Big|\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(\epsilon x)\eta_x - \int_{\mathbb{T}^d} H(\theta)m_0(\theta)d\theta\Big| > \delta\Big] = 0 \quad \forall\delta>0,\ \forall H\in C(\mathbb{T}^d).$$

In order to state the main result we introduce the diffusion matrix $D(m)$. Given a density $m\in(0,1)$ the annealed chemical potential $\lambda_0(m)\in\mathbb{R}$ is defined by

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_0)\Big] = m$$

while the empirical chemical potential $\lambda = \lambda(\alpha,\Lambda,m)$ is characterized by

$$\mu^{\alpha,\lambda}(m_\Lambda) = m$$

where m_Λ indicates the density on the volume Λ (often we use also the notation $\lambda_\Lambda(\alpha,m)$). We observe that $\lambda_0'(m) = \chi^{-1}(m)$ where $\chi(m)$ is the static compressibility given by

$$\chi(m) \equiv \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)\Big].$$

For $m\in(0,1)$ the diffusion matrix is defined by

$$D(m) \equiv \chi^{-1}(m)\sigma(m) = \lambda_0'(m)\sigma(m),$$

where $\sigma(m)$ is the symmetric matrix characterized by

$$(a,\sigma(m)a) \equiv \frac{1}{2}\inf_{g\in\mathbb{G}} \mathbb{E}\Big[\,\mu^{\alpha,\lambda_0(m)}\Big(\sum_{e>0} c_{0,e}(a_e\nabla_{0,e}\eta_0 - \nabla_{0,e}\underline{g})^2\Big)\Big] \qquad \forall a\in\mathbb{R}^d \tag{1.4}$$

where $\underline{g} \equiv \sum_{x\in\mathbb{Z}^d}\tau_x g$. Though this is a formal expression, since g is local $\nabla_{0,e}\underline{g}$ is meaningful.
In chapter 5 we will prove that $\sigma(m)$ is well-defined, the diffusion matrix $D(m)$ has lower and upper bounds

$$0 < C\mathbb{I} \le D(m) \le C'\mathbb{I} \quad \forall m\in(0,1),$$

and is continuous on $(0,1)$.

• At the moment we are not able to prove that D can be continuously extended to $0,1$. We assume the continuity. We remark that usually the diffusion matrix has no pathological behavior. Recently, the C^∞ regularity of the diffusion matrix has been proved for some models by duality methods (see [2], [16] and [17]).

We denote $D([0,T],\mathbb{T}^d_\epsilon)$ the space of right continuous paths with left limits from $[0,T]$ to $\mathbb{T}^d_\epsilon$ with Skorohod topology. Then, given a microscopic state ν_ϵ, we set P^α_ϵ to be the Markov process on $D([0,T],\mathbb{T}^d_\epsilon)$ with generator $\epsilon^{-2}\mathcal{L}^\alpha_\epsilon$ and initial distribution ν_ϵ.
We indicate by Q^α_ϵ the measure $P^\alpha_\epsilon \circ \pi_\epsilon^{-1}$ on the Skorohod space $D\big([0,T],\mathcal{M}_1\big)$. Recall the definition of $\mathcal{M}_2$ given in section 1.1.

Theorem 1.1. *Let $d \geq 3$. Then almost any disorder configuration $\alpha \in \Omega_D$ has the following property.*

Let $\{\nu_\epsilon\}_{\epsilon>0}$ be a sequence of microscopic states corresponding to the macroscopic density m_0 for which the Cauchy problem

$$\begin{cases} \partial_t m(t,\theta) = \nabla_\theta\Big(D\big(m(t,\theta)\big)\nabla_\theta m(t,\theta)\Big) \\ m(0,\theta) = m_0(\theta) \end{cases} \tag{1.5}$$

has a unique weak solution $m \in C([0,T],\mathcal{M}_2)$ satisfying the energy estimate

$$\int_0^T dt \int_{\mathbb{T}^d} d\theta\big(\nabla_\theta m(t,\theta)\big)^2 < \infty. \tag{1.6}$$

We call $m(t,\theta)$ such a solution. Then

$$Q^\alpha_\epsilon \Rightarrow \delta_{m(\cdot,\theta)} \quad \textit{in probability on } D([0,T],\mathcal{M}_1).$$

In particular, for any $t>0$, the sequence of microscopic states $\{\nu_{\epsilon,t}\}_{\epsilon>0}$ corresponds to the macroscopic state $m(t,\theta)d\theta \in \mathcal{M}_1$, that is

$$\lim_{\epsilon\downarrow 0} \nu_\epsilon\Big[\Big|\operatorname{Av}_{x\in\mathbb{T}^d_\epsilon} H(\epsilon x)\eta_x(t) - \int_{\mathbb{T}^d} H(\theta)m(t,\theta)d\theta\Big| > \delta\Big] = 0 \quad \forall\delta>0,\ \forall H \in C(\mathbb{T}^d). \tag{1.7}$$

The assumption $d \geq 3$ is due to the fluctuations in the gradients of the density field since the method developed to control them works only for $d \geq 3$. The other technical results involved in the proof of theorem 1.1 are valid for any dimension.

Let us give some comments on the weak interpretation of (1.5) and (1.6). Let $A(m)$, $m \in [0,1]$, be a $d\times d$ matrix such that $A'(m) = D(m)$ so that

$$\big(D(m(t,\theta))\nabla_\theta m(t,\theta)\big)_e = \sum_{e'\in\mathcal{E}} \partial_{\theta_{e'}} A_{e,e'}(m(t,\theta))\,, \quad \forall e \in \mathcal{E}.$$

As proven in proposition A.15, given $m \in D([0,T],\mathcal{M}_2)$ there is a measurable function $m(t,\theta)$ such that $0 \leq m(t,\theta) \leq 1$ and $m_t = m(t,\theta)d\theta$ for any $t \in [0,T]$ (such a function is well defined up to sets of zero Lebesgue measure). In what follows, we will often identify m with the funtion $m(t,\theta)$.
A path $m \in D([0,T],\mathcal{M}_2)$ is called a weak solution of (1.5) if $m(0,\cdot) = m_0(\cdot)$ and

$$\Phi(m,H) = 0 \quad \forall H \in C^{1,2}([0,T]\times\mathbb{T}^d)$$

where

$$
\begin{aligned}
\Phi(m,H) :&= \int_{\mathbb{T}^d} m(T,\theta)H(T,\theta)\,d\theta - \int_{\mathbb{T}^d} m(0,\theta)H(0,\theta)\,d\theta \\
&- \int_0^T \int_{\mathbb{T}^d} m(s,\theta)\partial_s H(s,\theta)\,d\theta\,ds \\
&- \sum_{e,e'} \int_0^T \int_{\mathbb{T}^d} A_{e,e'}\big(m(s,\theta)\big)\,\partial^2_{\theta_e,\theta_{e'}} H(s,\theta)\,d\theta\,ds.
\end{aligned}
\tag{1.8}
$$

Moreover, $m \in D([0,T],\mathcal{M}_2)$ satisfies the energy estimate (1.6) if

$$
\sup_{e\in\mathcal{E}} \sup_{H\in C^{1,2}([0,T]\times\mathbb{T}^d)} \int_0^T \int_{\mathbb{T}^d} \big(2\,m\,\partial_{\theta_e} H - H^2\big)d\theta\,ds < \infty.
\tag{1.9}
$$

The basic approach for proving the hydrodynamic limit consists of three standard steps: establishing the relatively compactness of the sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$, deriving for any limit point Q of $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ the energy estimate and identifying the support of Q with the set of weak solutions of (1.5).
In order to prove that the sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ is relatevily compact for any α it is enough to to use the Garsia-Rodemich-Rumsey inequality as done in [13], chapter 7, section 6.

Warning. In what follows, we will introduce some other mesoscopic scales in addition to the microscopic scale ϵ. For example, we will introduce some positive scale parameters a, c and consider the mesoscopic scales $\left[\frac{a}{\epsilon}\right]$ and $\left[\frac{c}{\epsilon}\right]$, where $[\cdot]$ denotes the integer part. For simplicity of notation these new scales will be denoted only by $\frac{a}{\epsilon}$ and $\frac{c}{\epsilon}$.
Moreover, in what follows a "disorder configuration", without other comments, has to be thought of as an element of Ω_D.

1.4 Energy estimate and identification of the hydrodynamic equation

In the first part of this section we prove that, for almost all α, given a sequence $\{\nu_\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ, any limit point Q^α of the sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ is concentrated on paths enjoying a certain regularity property.
First we observe that, if the sequence of $\{\nu_\epsilon\}_{\epsilon>0}$ corresponds to the macroscopic profile $m_0(\theta)$, lemma A.14 implies

$$
Q^\alpha\big(\, m \in C([0,T],\mathcal{M}_2) \,:\, m(0,\theta) = m_0(\theta)\,\big) = 1 \quad \forall \alpha.
\tag{1.10}
$$

Lemma 1.2. (Energy estimate). *Let $d \geq 3$ and $T > 0$. Then almost any disorder configurations α have the following property. For any sequence $\{\nu_\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ and any $e > 0$*

$$
\sup_{c>0} \limsup_{a\downarrow 0,\epsilon\downarrow 0} \mathbb{E}_{P^\alpha_\epsilon}\Big(\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \int_0^T \Big[\frac{m^{a/\epsilon}_{x+\frac{c}{\epsilon}e}(s) - m^{a/\epsilon}_x(s)}{c}\Big]^2 ds\Big) < +\infty.
\tag{1.11}
$$

Moreover any limit point Q^α of the sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ satisfies

$$Q^\alpha\Big\{ m \in C([0,T],\mathcal{M}_2)\ :\text{l.h.s. of (1.9)} < \infty\Big\} = 1. \tag{1.12}$$

We postpone the proof of lemma (1.2) at the end of the chapter. Let us state our microscopic indentification of the hydrodynamic equation, whose proof will be given in the nexts sections.

Theorem 1.3. *(Identification of the hydrodynamic equation)*
Let $d \geq 3$. Then almost any disorder configuration $\alpha \in \Omega_D$ has the following property.

For any function $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$ and $a, c > 0$ we define

$$\begin{aligned} B^{H,\epsilon}_{c,a} &= \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(T,\epsilon x)\eta_x(T) - \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(0,\epsilon x)\eta_x(0) - \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \partial_s H(s,\epsilon x)\eta_x(s)ds + \\ &+ \sum_{e,e'}\int_0^T \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\frac{\nabla_e H}{\epsilon}(s,\epsilon x) D_{e,e'}(m^{a/\epsilon}_x(s))\frac{\big[m^{a/\epsilon}_{x+\frac{c}{\epsilon}e'}(s) - m^{a/\epsilon}_{x-\frac{c}{\epsilon}e'}(s)\big]}{2c} \end{aligned}$$

where $\nabla_e H(s,\epsilon x) \equiv H(s,\epsilon x+\epsilon e) - H(s,\epsilon x)$ and $m^{a/\epsilon}_x$ is the density on the cube of side a/ϵ centered in x .

Then, for any $\delta > 0$ and for any sequence of microscopic states $\{\nu_\epsilon\}_{\epsilon>0}$

$$\limsup_{c\downarrow 0, a\downarrow 0,\epsilon\downarrow 0} P^\alpha_\epsilon\big[\,|B^{H,\epsilon}_{c,a}| > \delta\,\big] = 0.$$

1.5 Proof of theorem 1.1

Now we have almost all tools in order to prove our main result: theorem 1.1. Let us state a simple consequence of theorem 1.3 and lemma 1.2

Lemma 1.4. *Let $d \geq 3$ and $T > 0$. Then almost all disorder configurations α have the following property. For any sequence $\{\nu_\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ let Q^α be a limit point of the sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$. Then*

$$Q^\alpha\big\{ m \in C([0,T],\mathcal{M}_2)\ :\ \Phi(m,H) = 0\,\big\} = 1 \qquad \forall H \in C^{1,2}([0,T]\times\mathbb{T}^d) \tag{1.13}$$

where $\Phi(m,H)$ has been defined as in (1.8).

Proof. Let α be in the almost sure set of disorder configuration for which the conclusion of theorem 1.3 and lemma 1.2 apply. Given a path $\eta(\cdot) \in D([0,T],\mathbb{T}^d_\epsilon)$ and $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$, let

$$\begin{aligned} \hat{H}_{c,a,\epsilon} :&= \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big[H(T,\epsilon x)\eta_x(T) - H(0,\epsilon x)\eta_x(0) - \int_0^T \partial_s H(s,\epsilon x)\eta_x(s)ds\Big] \\ &\quad - \sum_{e,e'}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T\Big[\frac{\nabla^\epsilon_e H(s,\epsilon x + ce') - \nabla^\epsilon_e H(s,\epsilon x - ce')}{2c}\Big] A_{e,e'}(m^{a/\epsilon}_x(s))\,ds \end{aligned} \tag{1.14}$$

where the matrix $A(m)$, $m \in [0,1]$, satisfies $A'(m) = D(m)$. Using the continuity assumption on $D(m)$ and Taylor expansion we obtain that

$$|A(m') - A(m) - D(m)(m'-m)| \leq |m'-m|\vartheta(|m'-m|)\,, \qquad \forall m, m' \in [0,1]$$

where ϑ is a bounded positive function on $[0,1]$ such that $\lim_{s\downarrow 0}\vartheta(s) = 0$. An integration by parts now shows that

$$|\hat{H}_{c,a,\epsilon} - \bar{H}_{c,a,\epsilon}| \leq \\ \frac{c(H)}{c}\int_0^T ds \sum_{e\in\mathcal{E}} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \big|m^{a/\epsilon}_{x+\frac{c}{\epsilon}e}(s) - m^{a/\epsilon}_x(s)\big|\vartheta\big(\,|m^{a/\epsilon}_{x+\frac{c}{\epsilon}e}(s) - m^{a/\epsilon}_x(s)|\,\big)$$

where $\hat{H}_{c,a,\epsilon}$ is equal to $B^{H,\epsilon}_{c,a}$ of theorem 1.3.
Then, thanks to theorem 1.3, lemma 1.2 and Schwarz inequality, for any $\delta > 0$

$$\limsup_{c\downarrow 0, a\downarrow 0, \epsilon\downarrow 0} P^\alpha_\epsilon\big[\,|\hat{H}_{c,a,\epsilon}| > \delta\,\big] = 0 \tag{1.15}$$

Given $\nu \in D([0,T],\mathcal{M}_1)$, let us now define

$$\Phi^{(a)}(\nu,H) := \int_{\mathbb{T}^d} H(T,\theta)\,\nu_T(d\theta) - \int_{\mathbb{T}^d} H(0,\theta)\,\nu_0(d\theta) \\ -\int_0^T ds\int_{\mathbb{T}^d}\partial_s H(s,\theta)\,\nu_s(d\theta) - \sum_{e,e'}\int_0^T\int_{\mathbb{T}^d} A_{e,e'}(\nu^{(a)}(s,\theta))\,\partial^2_{\theta_e,\theta_{e'}}H(s,\theta)\,d\theta\,ds\,,$$

where

$$\nu^{(a)}(s,\theta) := \frac{1}{(2a)^d}\nu_s\big(\{\theta' \in \mathbb{T}^d : \sup_{i=1,\dots,d}|\theta'_i - \theta_i| \leq a\}\big) \tag{1.16}$$

and $A_{e,e'}$ has been continuously extended as $A_{e,e'}(1)$ on $[1,\infty)$.
Then

$$\int dQ^\alpha(m)\,\Phi(m,H) = \lim_{a\downarrow 0}\int dQ^\alpha(m)\Phi^{(a)}(m,H) \leq \\ \limsup_{a\downarrow 0,\epsilon\downarrow 0}\int dQ^\alpha_\epsilon(\nu)\Phi^{(a)}(\nu,H) \leq \limsup_{c\downarrow 0,a\downarrow 0,\epsilon\downarrow 0}\mathbb{E}_{P^\alpha_\epsilon}\big(\hat{H}_{c,a,\epsilon}\big) = 0 \tag{1.17}$$

where first equality follows from Lebesgue density theorem and dominated convergence theorem, the second inequality follows from the regularity of H, the last equality is a consequence of (1.15). Let us comment the first inequality. Given $a > 0$ it is simple to check that the first three addenda in the definition of $\Phi^{(a)}(\nu,H)$ are continuous functions on $D([0,T],\mathcal{M}_1)$ (see below). Let us consider the last addendum in the definition of $\Phi^{(a)}(\nu,H)$. Because of the uniform continuity of $A_{e,e'}$, the regularity of H and the support of Q^α and Q^α_ϵ, given $\delta > 0$ there exists $\rho,\epsilon_0 > 0$ such that the event

$$\sup_{s\in[0,T]}\Big|\int_{\mathbb{T}^d} A_{e,e'}(\nu^{(a)}(s,\theta))\,\partial^2_{\theta_e,\theta_{e'}}H(s,\theta)\,d\theta\,ds - \\ \sum_{\theta\in\rho\mathbb{Z}^d\cap\mathbb{T}^d} A_{e,e'}(\nu^{(a)}(s,\theta))\,\partial^2_{\theta_e,\theta_{e'}}H(s,\theta)\rho^d\Big| \leq \delta$$

has measure 1 for Q^α and Q^α_ϵ with $\epsilon \le \epsilon_0$ and the same result remains true if we substitute $\nu^{(a)}(s,\theta)$ with $(2a)^{-d}\nu_s(\tau_\theta f)$ where f is a suitable continuous function approximating $\mathbb{I}_{\sup_{i=1,\dots,d}|\theta'_i|\le a}$. Therefore, with an error of order δ the last addendum in the definition of $\Phi^{(a)}(\nu,H)$ can be substituted with a continuous function on $D([0,T],\mathcal{M}_1)$, thus proving the first inequality in (1.17).

In order to conclude the proof we need to prove that the first three addenda in the definition of $\Phi^{(a)}(\nu,H)$ are continuous functions on $D([0,T],\mathcal{M}_1)$. To this aim, given $\nu \in D([0,T],\mathcal{M}_1)$ let $\{\nu^n\}_{n>0}$ be a sequence in $D([0,T],\mathcal{M}_1)$ such that ν^n converges to ν as n goes to infinity. Then, by definition of the Skohorod distance d_0 (see section A.3), we have $\nu^n_0 \to \nu_0$ as $n \uparrow \infty$, and similarly for $\{\nu^n_T\}_{n\in\mathbb{N}}$. Therefore, the first two addenda are continuous functions.
In order to study the third addendum, let us prove that for any continuous function G

$$\lim_{n\uparrow\infty}\int_0^T ds\int_{\mathbb{T}^d} G(s,\theta)\,\nu^n_s(d\theta) = \int_0^T ds\int_{\mathbb{T}^d} G(s,\theta)\,\nu_s(d\theta) \tag{1.18}$$

Since all measures have total mass non larger than 1, it is enough to prove (1.18) with G of the form

$$G(t,\theta) = \sum_{j=0}^n \mathbb{I}_{A_j}(t)g_j(\theta),$$

$0 = t_0 < t_1, \dots, < t_n < 1$, $A_j := [t_j, t_{j+1})$ for $j < n$, $A_n = [t_n, 1]$ and $g_j \in C(\mathbb{T}^d)$.
Since $\nu^n \to \nu$ given $\epsilon > 0$ there are continuous increasing parameterizations λ_n of $[0,T]$ such that for $n \ge n_0$ $\sup_t|\lambda_n(t) - t| \le \epsilon$ and $\sup_t d(\nu^n_t, \nu_{\lambda^n(t)}) \le \epsilon$ where d is the metric on $\mathcal{M}_1$ inducing the weak topology (see section A.3). In particular, we may assume that for any $j = 0,\dots,n$, any t and any $n \ge n_0$ $|\nu^n_t(g_j) - \nu_{\lambda^n(t)}(g_j)| \le \epsilon$. At this point, the thesis follows by simple arguments. □

We are finally able to complete the proof of theorem 1.1. By lemma 1.2, lemma 1.4 and (1.10) there exists a set $\mathcal{A}\subset\Omega_D$ with $\mathbb{P}(\mathcal{A}) = 1$ such that, given $\alpha \in \mathcal{A}$ and a sequence $\{\nu_\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ corresponding to the macroscopic profile $m_0(\theta)$, any limit point Q^α of the tight sequence $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ gives full measure to the set of weak solutions of (1.5) in $C([0,T],\mathcal{M}_2)$ with finite energy. By assumption, this set is a singleton $m(\cdot,\theta)d\theta$ and therefore the almost sure weak convergence of $\{Q^\alpha_\epsilon\}_{\epsilon>0}$ to $\delta_{m(\cdot,\theta)d\theta}$ follows immediately.
In order to prove (1.7) let d_0 and d be as in section A.3. As proved in [4], given $\nu \in C([0,T],\mathcal{M}_1)$ for any $\delta > 0$ there exists $\beta = \beta(\delta,\nu) > 0$ such that $d(\mu,\nu) \le \beta$ implies $\sup_{t\in[0,T]} d(\mu_t,\nu_t) \le \delta$ for any $\mu \in D([0,T],\mathcal{M}_1)$. Let now $\nu := m(\cdot,\theta)d\theta \in C([0,T],\mathcal{M}_2)$ where $m(\cdot,\theta)$ is the unique weak solution of (1.5) and, for any given $\delta > 0$, let $\beta := \beta(\delta,\nu)$. Then, from the weak convergence result above, we immediately get

$$1 = \lim_{\epsilon\downarrow 0} Q^\alpha_\epsilon(d_S(\mu,\nu) \le \beta) \le \lim_{\epsilon\downarrow 0} Q^\alpha_\epsilon(d(\mu_t,\nu_t) < \delta)$$

thus proving (1.7).

1.6 Martingale estimate

We fix a disorder configuration α (for simplicity of notation the index α is usually understood).

Given $H \in C^{1,2}([0,T] \times \mathbb{T}^d)$ we define the function F_ϵ on $[0,T] \times \{0,1\}^{\mathbb{T}^d_\epsilon}$ by

$$F_\epsilon(s,\eta) \equiv \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(s,\epsilon x)\eta_x.$$

Since $\mathcal{L}\eta_x = \sum_{e>0}(-j_{x,x+e}(\eta) + j_{x-e,x}(\eta))$, by integration by parts we can write

$$\begin{aligned}
&\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(t,\epsilon x)\eta_x(t) - \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} H(0,\epsilon x)\eta_x(0) = \\
&\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^t \partial_s H(s,\epsilon x)\eta_x(s)ds + \int_0^t U\big(s,\eta(s)\big)ds + M^{F_\epsilon}(t)
\end{aligned}$$

where

$$U(s,\cdot) = \epsilon^{-1}\sum_{e>0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\frac{\nabla_e H(s,\epsilon x)}{\epsilon} j_{x,x+e}$$

and, by proposition $A.10$, $\{M^{F_\epsilon}(t)\}_{t>0}$ is a martingale w.r.t. the filtration $\{\mathcal{F}_t\}_{t>0}$ with $\mathcal{F}_t = \sigma(\eta_s : 0 \le s \le t)$.
Then by Doob's inequality

$$\begin{aligned}
&P^\alpha_\epsilon\Big[\sup_{0\le t\le T}|M^{F_\epsilon}(t)| > \delta\Big] \le \delta^{-2}\mathbb{E}^{P^\alpha_\epsilon}\Big(\big(M^{F_\epsilon}(T)\big)^2\Big) = \\
&\delta^{-2}\epsilon^{-2}\mathbb{E}^{P^\alpha_\epsilon}\Big[\int_0^T ds\big\{\mathcal{L}{F_\epsilon}^2\big(s,\eta(s)\big) - 2F_\epsilon\mathcal{L}F_\epsilon\big(s,\eta(s)\big)\big\}\Big].
\end{aligned} \tag{1.19}$$

By straightforward computations it can be checked that

$$\mathcal{L}{F_\epsilon}^2(s,\eta) - 2F_\epsilon\mathcal{L}F_\epsilon(s,\eta) = \epsilon^{2d+2}\sum_{\substack{(x,y):\\ |y-x|=1}}\frac{c_{xy}}{2}(\eta_x-\eta_y)^2\Big(\frac{H(s,\epsilon y)-H(s,\epsilon x)}{\epsilon}\Big)^2.$$

Since H is smooth

$$\epsilon^{-2}\big\{\mathcal{L}{F_\epsilon}^2\big(s,\eta(s)\big) - 2F_\epsilon\mathcal{L}F_\epsilon\big(s,\eta(s)\big)\big\} \le \epsilon^d C(H,B)$$

and by (1.19)

$$\limsup_{\epsilon\downarrow 0} P^\alpha_\epsilon\Big[\sup_{0\le t\le T}|\,M^{F_\epsilon}(t)\,| > \delta\Big] = 0. \tag{1.20}$$

Thanks to 1.20 in order to prove theorem 1.3 it's enough to prove the following statement.

Proposition 1.5. *For almost every disorder configuration α, given a sequence of microscopic states $\{\nu_\epsilon\}_{\epsilon>0}$, $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$, $e>0$ and $\delta>0$ then*

$$\limsup_{c\downarrow 0, a\downarrow 0, \epsilon\downarrow 0} \mathbb{E}^{P_\epsilon^\alpha}\Big[|F|>\delta\Big]=0$$

where

$$F \equiv \int_0^T \epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\frac{\nabla_e H}{\epsilon}(s,\epsilon x) j_{x,x+e}(\eta(s))ds+ \\ +\sum_{e'}\int_0^T \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\frac{\nabla_e H}{\epsilon}(s,\epsilon x) D_{e,e'}(m_x^{a/\epsilon}(s))\frac{\big[m^{a/\epsilon}_{x+\frac{c}{\epsilon}e'}(s)-m^{a/\epsilon}_{x-\frac{c}{\epsilon}e'}(s)\big]}{2c}ds \tag{1.21}$$

1.7 Reduction to the microscopic case

1.7.1 Replacing currents by averages of currents

Given an integer l we define $l_1 = l-\sqrt{l}$. We can write

$$\epsilon^{-1}\mathop{\mathrm{Av}}_{x\in\mathbb{T}_\epsilon^d}\frac{\nabla_e H_s}{\epsilon}(\epsilon x)\big(j_{x,x+e}-\mathop{\mathrm{Av}}_{|y-x|\le l_1} j_{y,y+e}\big)=\epsilon^{-1}\mathop{\mathrm{Av}}_{x\in\mathbb{T}_\epsilon^d} j_{x,x+e}\big(\frac{\nabla_e H_s}{\epsilon}(\epsilon x)-\mathop{\mathrm{Av}}_{|y-x|\le l_1}\frac{\nabla_e H_s}{\epsilon}(\epsilon y)\big).$$

By Taylor expansion,

$$\frac{\nabla_e H_s}{\epsilon}(\epsilon y)=\partial_e H_s(\epsilon y)+\frac{\epsilon}{2}\partial_e^2 H_s(\xi_{\epsilon,y})$$

where $\xi_{\epsilon,y}$ belongs to the segment in $\mathbb{T}^d$ between ϵy and $\epsilon y+\epsilon e$. Therefore

$$\begin{aligned}\epsilon^{-1}\Big\{\mathrm{Av}_{|y|\le l_1}\frac{\nabla_e H_s}{\epsilon}(\epsilon y)-\frac{\nabla_e H_s}{\epsilon}(0)\Big\}=&\\ &\frac{\epsilon^{-1}}{2}\mathrm{Av}_{|y|\le l_1}\Big[\partial_e H_s(\epsilon y)-2\partial_e H_s(0)+\partial_e H_s(-\epsilon y)\Big]\\ &+\frac{1}{4}\mathrm{Av}_{|y|\le l_1}\Big[\partial_e^2 H_s(\xi_{\epsilon,y})-2\partial_e H_s^2(\xi_{\epsilon,0})+\partial_e H_s^2(\xi_{\epsilon,-y})\Big].\end{aligned} \tag{1.22}$$

Since, for any y, $\partial_e H_s(\epsilon y)-\partial_e H_s(0)=\epsilon y\cdot\nabla\big(\partial_e H_s\big)(z_{\epsilon,y})$ with $z_{\epsilon,y}$ in the segment between 0 and ϵy, the l.h.s. of (1.22) can be bounded by

$$Cl\sup_{e'}\mathrm{Osc}\big(\partial_{e,e'}H, C(d)\epsilon l\big).$$

By the above arguments it is not restrictive to prove proposition 1.5 with the first addendum in the r.h.s. of (1.21) substituted with

$$\int_0^T \epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\frac{\nabla_e H}{\epsilon}(s,\epsilon x)\Big(\mathrm{Av}_{y:|y-x|\le l_1} j_{y,y+e}(\eta(s)\Big)ds.$$

1.7.2 Introducing fluctuations

Let $\mathbb{G}$ be the set of bounded,local and measurable functions $g = g(\alpha, \eta)$.

Lemma 1.6. *Let $g \in \mathbb{G}$. Then $\forall e > 0$, $\forall \alpha \in \Omega_D$ and $\forall \{\nu_\epsilon\}_{\epsilon > 0}$*

$$\limsup_{l\uparrow\infty,\epsilon\downarrow 0} P_\epsilon^\alpha \Big[\Big| \int_0^T \epsilon^{-1} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \mathrm{Av}_{y:|y-x|\le l_1} \tau_y \mathcal{L} g \, ds \Big| > \delta \Big] = 0.$$

Proof.
By the same arguments of subsection 1.7.1, it is enough to prove

$$\limsup_{l\uparrow\infty,\epsilon\downarrow 0} P_\epsilon^\alpha \Big[\Big| \int_0^T \epsilon^{-1} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \tau_x \mathcal{L} g \, ds \Big| > \delta \Big] = 0.$$

We set $F_\epsilon(s,\eta) = \epsilon^{-1} \mathrm{Av}_x \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \tau_x g$. Since $\mathcal{L}(\tau_x g) = \tau_x(\mathcal{L} g)$, by proposition $A.10$

$$\Big| \int_0^T \epsilon^{-1} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \tau_x \mathcal{L} g ds \Big|$$
$$= \epsilon^2 ds \Big| \Big\{ F_\epsilon\big(T, \eta(T)\big) - F_\epsilon\big(0, \eta(0)\big) - \int_0^T \partial_s F_\epsilon\big(s, \eta(s)\big) ds \} - M^{F_\epsilon}(T) \Big|,$$

which is bounded by $C(H, g)\epsilon + \epsilon^2 \big| M^{F_\epsilon}(T) \big|$, and

$$\mathbb{E}^{P_\epsilon^\alpha}\big(M^{F_\epsilon}(T)^2\big) = \epsilon^{-2} \mathbb{E}^{P_\epsilon^\alpha} \Big(\int_0^T \mathcal{L} F_\epsilon^2\big(s, \eta(s)\big) - 2F_\epsilon \mathcal{L} F_\epsilon\big(s, \eta(s)\big) \, ds \Big).$$

We assert that

$$\big| \mathcal{L} F_\epsilon^2(s,\eta) - 2F_\epsilon \mathcal{L} F_\epsilon \big| \le C \epsilon^{2d-2} \sum_{b\in\mathbb{T}_\epsilon^d} \Big(\sum_{x\in\mathbb{T}_\epsilon^d} \Big| \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \nabla_b \tau_x g \Big| \Big)^2. \tag{1.23}$$

For, by the identity $\nabla_b(fh) - f\nabla_b h - (\nabla_b f) h = (\nabla_b f)(\nabla_b h)$,

$$\mathcal{L} F_\epsilon^2 - 2F_\epsilon \mathcal{L} F_\epsilon =$$
$$\epsilon^{-2} \mathop{\mathrm{Av}}_{x\in\mathbb{T}_\epsilon^d} \mathop{\mathrm{Av}}_{y\in\mathbb{T}_\epsilon^d} \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \frac{\nabla_e H}{\epsilon}(s, \epsilon y) \{ \mathcal{L}(\tau_x g \tau_y g) - \tau_x g \mathcal{L}(\tau_y g) - \tau_y g \mathcal{L}(\tau_x g) \} =$$
$$\epsilon^{-2} \mathop{\mathrm{Av}}_{x\in\mathbb{T}_\epsilon^d} \mathop{\mathrm{Av}}_{y\in\mathbb{T}_\epsilon^d} \sum_{b\in\mathbb{T}_\epsilon^d} \frac{\nabla_e H}{\epsilon}(s, \epsilon x) \frac{\nabla_e H}{\epsilon}(s, \epsilon y) c_b (\nabla_b \tau_x g) \, (\nabla_b \tau_y g)$$

which proves (1.23).
Since $g \in \mathbb{G}$ there exists $C = C(g)$ such that $\sum_b (\sum_x |\nabla_b \tau_x g|)^2 < C\epsilon^{-d}$. Therefore, by Doob's inequality and the previous estimates, $\forall \delta > 0$

$$P_\epsilon^\alpha \big[\sup_{0\le t\le T} \epsilon^2 |M^{F_\epsilon}(t)| > \delta \big] \le \delta^{-2} \epsilon^4 \mathbb{E}^{P_\epsilon^\alpha} \big((M^{F_\epsilon}(T))^2 \big) \le C(g, H) \delta^{-2} \epsilon^d$$

which allows to conclude the proof. $\square$

1.7.3 An equivalent version of theorem 1.3

From the previous subsections we obtain that proposition 1.5, and therefore theorem 1.3, is implied by the following statement.

Proposition 1.7. *For almost any disorder configuration $\alpha \in \Omega_D$, given $\{\nu_\epsilon\}_{\epsilon>0}$, $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$, $e>0$ and a positive number δ then*

$$\inf_{g\in\mathbb{G}} \limsup_{c\downarrow 0, a\downarrow 0, l\uparrow\infty, \epsilon\downarrow 0} P_\epsilon^\alpha\Big[\Big|\int_0^T \epsilon^{-1} G_{g,\frac{\nabla_e H_s}{\epsilon}}\big(\eta(s)\big)ds\Big| > \delta\Big] = 0 \tag{1.24}$$

where, for any function J on $\mathbb{T}^d$, $G_{g,J}$ is defined by

$$G_{g,J} = \operatorname*{Av}_{x\in\mathbb{T}^d_\epsilon} J(\epsilon x)\cdot\Big\{\operatorname*{Av}_{y:|y-x|\le l_1}(j_{y,y+e}+\tau_y\mathcal{L}g) + \sum_{e'} D_{e,e'}(m_x^{a/\epsilon})\frac{m^{a/\epsilon}_{x+\frac{c}{\epsilon}e'} - m^{a/\epsilon}_{x-\frac{c}{\epsilon}e'}}{2c/\epsilon}\Big\}.$$

1.8 Entropy inequality and Feynman-Kac formula

In this section we reduce proposition 1.7 to a statement about the spectrum of some operators. To this aim we use the entropy inequality (1.25) and proposition $A.12$, which is consequence of the Feynman-Kac formula.
Let us remember that, given two measures μ,π on the same space, for any $\beta>0$ and any bounded and measurable function f,

$$\mu(f) \le \beta^{-1}\Big\{\ln\big(\pi(e^{\beta f})\big) + H(\mu|\pi)\Big\}. \tag{1.25}$$

We denote $\bar P^\alpha_\epsilon$ the probability measure on $D([0,T],\Omega_\epsilon)$ induced by the infinitesimal generator $\epsilon^{-2}\mathcal{L}^\alpha_\epsilon$ with initial distribution μ^α_ϵ.

It's simple to verify that $\mu^\alpha_\epsilon(\eta) \ge \kappa\epsilon^{-d}$ for some constant $\kappa = \kappa(B)$. By lemma $A.11$, this implies that $H[P^\alpha_\epsilon|\bar P^\alpha_\epsilon] \le \kappa\epsilon^{-d}$. By the entropy inequality, given $g\in\mathbb{G}$, a function $J\in C([0,T]\times\mathbb{T}^d)$ and $\gamma>0$,

$$\mathbb{E}^{P^\alpha_\epsilon}\Big[\Big|\int_0^T \epsilon^{-1}G_{g,J_s}ds\Big|\Big] \le \gamma^{-1}\kappa + \gamma^{-1}\epsilon^d \ln \mathbb{E}^{\bar P^\alpha_\epsilon}\Big(\exp\big\{\gamma\epsilon^{-d}\big|\int_0^T \epsilon^{-1}G_{g,J_s}ds\big|\big\}\Big). \tag{1.26}$$

By proposition $A.12$ and the reversibility of μ^α_ϵ

$$\gamma^{-1}\epsilon^d \ln \mathbb{E}^{\bar P^\alpha_\epsilon}\Big(\exp\big\{\gamma\epsilon^{-d}\big(\pm\int_0^T \epsilon^{-1}G_{g,J_s}ds\big)\big\}\Big) \le \int_0^T \sup_{L^2(\mu^\alpha_\epsilon)} spec\Big\{\epsilon^{-1}G_{g,\pm J_s} + \gamma\epsilon^{d-2}\mathcal{L}^\alpha_\epsilon\Big\}ds. \tag{1.27}$$

Since for generic positive sequences

$$\overline{\lim_{\epsilon\downarrow 0}}\,\epsilon^d\ln(a_\epsilon+b_\epsilon) \le \overline{\lim_{\epsilon\downarrow 0}}\,\epsilon^d \ln a_\epsilon \vee \overline{\lim_{\epsilon\downarrow 0}}\,\epsilon^d\ln b_\epsilon,$$

proposition 1.7 follows from the next statement.

Proposition 1.8. *Given positive numbers γ and β there exists $g \in \mathbb{G}$ such that for almost any disorder configuration α*

$$\limsup_{c\downarrow 0, a\downarrow 0, l\uparrow\infty, \epsilon\downarrow 0} \sup_{J\in C^*(\mathbb{T}^d)} \sup spec_{L^2(\mu_\epsilon^\alpha)}\left\{\epsilon^{-1}G_{g,J} + \gamma\epsilon^{d-2}\mathcal{L}\right\} \leq \beta \tag{1.28}$$

where $C^*(\mathbb{T}^d) = \{J \in C(\mathbb{T}^d) \,:\, \|J\|_\infty \leq 1\}$.

1.9 Telescopic averages

In this section we introduce some spatial averages motivated by the necessity to express the difference of density between two disjoint cubes of width a/ϵ by means of differences of density between adjacent cubes of width n.

These averages have some factorization properties (described in lemma 1.9) which are particularly useful in the recursive procedure, explained in chapter 4, which allows to estimate the fluctuations in the gradient of the density field.

We introduce some notation useful also in the sequel. Given an odd number $r = 2r' + 1$ and $e > 0$, we write

$$\Lambda^e_{1,r} \equiv \Lambda_{-(r'+1)e,r'} \qquad \Lambda^e_{2,r} \equiv \Lambda_{r'e,r'} \qquad \Lambda^e_r \equiv \Lambda^e_{1,r} \cup \Lambda^e_{2,r} \qquad \Lambda^e_{x,r} \equiv \Lambda^e_r + x.$$

For example, if $d = 1$, $\Lambda_{1,r} = \{-r, \ldots, -1\}$ and $\Lambda_{2,r} = \{0, \ldots, r-1\}$.

We set, with $j = 1, 2$,

$$m^e_{x,r} \equiv m_{\Lambda^e_{x,r}} \qquad N^e_{x,r} \equiv N_{\Lambda^e_{x,r}} \qquad m^{j,e}_r \equiv m_{\Lambda^e_{j,r}}.$$

Often we omit the index e and we write $m^{(1)}_r, m^{(2)}_r$ in place of $m^{1,e}_r, m^{2,e}_r$.

For $s \geq n$ we set

$$\phi^e_{n,s} \equiv \mu^\alpha[m^{1,e}_n - m^{2,e}_n | m^e_s] \qquad \psi^e_{n,s} \equiv m^{1,e}_n - m^{2,e}_n - \phi_{n,s}. \tag{1.29}$$

In what follows, we fix $e > 0$.

Let n, s be odd numbers such that $n|s$. We write $n = 2n' + 1$ and $s = 2s' + 1$.

We define $A \equiv n\mathbb{Z}^d \cap \Lambda_{0,s'}$. By dividing the cube $\Lambda_{1,s}$ in cubes of side n, $A - (s'+1)$ is the set of centers of such cubes.
Since $\Lambda_{2,n} = ne + \Lambda_{1,n}$, $\forall y \in \mathbb{Z}^d$

$$\sum_{v=0}^{\frac{s}{n}-1} \tau_{y+vne}(N_{\Lambda_{1,n}} - N_{\Lambda_{2,n}}) = N_{y+\Lambda_{1,n}} - N_{y+se+\Lambda_{1,n}}.$$

Therefore,

$$\sum_{a\in A}\sum_{v=0}^{\frac{s}{n}-1}\tau_{a+(-s'+n'+vn)e}(N_{\Lambda_{1,n}}-N_{\Lambda_{2,n}})=N_{\Lambda_{1,s}}-N_{\Lambda_{2,s}}. \tag{1.30}$$

We define

$$\mathrm{Av}^{n,s}_{z,x}b_z\equiv \mathrm{Av}_{v=0}^{\frac{s}{n}-1}\mathrm{Av}_{a\in A}b_{x+a+(-s'+n'+vn)e}. \tag{1.31}$$

Then, (1.30) is equivalent to

$$\mathrm{Av}^{n,s}_{z,x}\tau_z\Big(\frac{m_n^{(1)}-m_n^{(2)}}{n}\Big)=\tau_x\Big(\frac{m_s^{(1)}-m_s^{(2)}}{s}\Big). \tag{1.32}$$

We remark that $\mathrm{Av}^{n,s}_{z,x}\tau_z g_z$ has support in Λ^e_s+x if g_z is a function having support in Λ^e_n+z for any z.

We observe that given a set of numbers $\{b_z\}_{z\in\mathbb{T}^d_\epsilon}$ then

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}b_x=\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}^{n,s}_{z,x}b_z. \tag{1.33}$$

In order to prove (1.33) we note that, for a particular subset W and numbers $\{c_w\}_{w\in W}$ with $\sum_{w\in W}c_w=1$, for any x we can write $\mathrm{Av}^{n,s}_{z,x}b_z=\sum_{w\in W}c_wb_{x+w}$. Therefore,

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}^{n,s}_{z,x}b_z=\epsilon^d\sum_{x\in\mathbb{T}^d_\epsilon}\sum_{w\in W}c_wb_{w+x}=\epsilon^d\sum_{z\in\mathbb{T}^d_\epsilon}b_z\Big[\sum_{x\in\mathbb{T}^d_\epsilon}\sum_{w\in W}c_w1_{x+w=z}\Big]=\epsilon^d\sum_{z\in\mathbb{T}^d_\epsilon}b_z. \tag{1.34}$$

Lemma 1.9. *Let n,s,R be odd numbers such that $n|s$ and $s|R$. We write $n=2n'+1$ and $R=2R'+1$.*
Let

$$W\equiv\{(-R'+n')e+x\,:\,x\in n\mathbb{Z}^d,\ x\in\Lambda_{0,R'}\}.$$

Then for any $x\in\mathbb{Z}^d$ and for any set of numbers $\{b_w\}_{w\in W+x}$

$$\mathrm{Av}^{s,R}_{z,x}\big(\mathrm{Av}^{n,s}_{w,z}b_w\big)=\mathrm{Av}^{n,R}_{w,x}b_w. \tag{1.35}$$

In particular, the above expression is meaningful, that is $w\in W+x$ both in the r.h.s. and in the l.h.s. of (1.35).

Proof. For simplicity of notation we take $x=0$. We write $s=2s'+1$ and we set

$$A_0\equiv\Lambda_{0,R'}\cap n\mathbb{Z}^d\quad A_1\equiv\Lambda_{0,R'}\cap s\mathbb{Z}^d\quad A_2\equiv\Lambda_{0,s'}\cap n\mathbb{Z}^d.$$

Since $A_0 = \sqcup_{x \in A_1}(x + A_2)$,

$$\begin{aligned} \mathrm{Av}^{n,R}_{w,0} &= \mathrm{Av}^{\frac{R}{n}-1}_{v=0} \mathrm{Av}_{a \in A_0} b_{a+(-R'+n'+vn)e} \\ &= \mathrm{Av}^{\frac{R}{n}-1}_{v=0} \mathrm{Av}_{a_1 \in A_1} \mathrm{Av}_{a_2 \in A_2} b_{a_1+a_2+(-R'+n'+vn)e} \end{aligned} \tag{1.36}$$

and

$$\begin{aligned} \mathrm{Av}^{s,R}_{z,0}\big(\mathrm{Av}^{n,s}_{w,z} b_w\big) &= \mathrm{Av}^{s,R}_{z,0}\big[\mathrm{Av}^{\frac{s}{n}-1}_{u=0} \mathrm{Av}_{a_2 \in A_2} b_{z+a_2+(-s'+n'+un)e}\big] \\ &= \mathrm{Av}^{\frac{R}{s}-1}_{t=0} \mathrm{Av}^{\frac{s}{n}-1}_{u=0} \mathrm{Av}_{a_1 \in A_1} \mathrm{Av}_{a_2 \in A_2} b_{a_1+a_2+(-R'+n'+ts+un)e}. \end{aligned} \tag{1.37}$$

Since s/n divides R/n the thesis follows by writing any integer $v : 0 \le v < R/n$ as

$$v = t\frac{s}{n} + u \qquad \text{where} \quad t, u \in \mathbb{N}, \;\; 0 \le t < \frac{R}{s}, \;\; 0 \le u < \frac{s}{n}.$$

□

By lemma 1.9 or simply by applying twice (1.32) it can be proved

$$\mathrm{Av}^{s,R}_{z,x} \mathrm{Av}^{n,s}_{w,z} \tau_w \Big(\frac{m^{(1)}_n - m^{(2)}_n}{n}\Big) = \tau_x \Big(\frac{m^{(1)}_R - m^{(2)}_R}{R}\Big). \tag{1.38}$$

Let n, R be odd numbers and Q a positive integer. We assume $n|R$ and $R|2Q$. We introduce the average $\mathrm{Av}^{n,R,Q}_{z,x}$.

We write $R = 2R' + 1$. Given $x \in \mathbb{Z}^d$, we write Λ^R_x, N^R_x and m^R_x to indicate respectively the cube $\Lambda_{x,R'}$ of width R centered at x, its particles number and its density. Then

$$\begin{aligned} N^R_{-Qe} - N^R_{Qe} &= \sum_{u=0}^{\frac{2Q}{R}-1} \big(N^R_{-Qe+uRe} - N^R_{-Qe+(u+1)Re}\big) \\ &= \sum_{u=0}^{\frac{2Q}{R}-1} \big(N_{(-Q+uR+R'+1)e+\Lambda^e_{1,R}} - N_{(-Q+uR+R'+1)e+\Lambda^e_{2,R}}\big) \end{aligned}$$

which is equivalent to

$$\frac{m^R_{-Qe} - m^R_{Qe}}{2Q} = \mathrm{Av}^{\frac{2Q}{R}-1}_{u=0} \tau_{(-Q+uR+R'+1)e} \frac{m^{(1)}_R - m^{(2)}_R}{R}.$$

By the above expression and (1.32) for any $x \in \mathbb{Z}^d$

$$\tau_x \frac{m^R_{-Qe} - m^R_{Qe}}{2Q} = \mathrm{Av}^{\frac{2Q}{R}-1}_{u=0} \mathrm{Av}^{n,R}_{z,x+(-Q+uR+R'+1)e} \tau_z \frac{m^{(1)}_n - m^{(2)}_n}{n} = \mathrm{Av}^{n,R,Q}_{z,x} \tau_z \frac{m^{(1)}_n - m^{(2)}_n}{n}. \tag{1.39}$$

where in general we set

$$\mathrm{Av}^{n,R,Q}_{z,x} b_z \equiv \mathrm{Av}_{u=0}^{\frac{2Q}{R}-1} \mathrm{Av}^{n,R}_{z,x+(-Q+uR+R'+1)e} b_z. \tag{1.40}$$

It's easy to verify that

$$\mathrm{Av}^{n,R,Q}_{z,x} b_z = \mathrm{Av}_{t=0}^{\frac{2Q}{n}-1} \mathrm{Av}_{a\in A} b_{x+(-Q+n'+1+tn)e+a}$$

where $A \equiv n\mathbb{Z}^d \cap \Lambda_{0,R'}$.
In particular we have the following useful identity

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big(c_x \cdot \mathrm{Av}^{n,R,Q}_{z,x} b_z\Big) = \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big(b_x \cdot \mathrm{Av}_{t=0}^{\frac{2Q}{n}-1} \mathrm{Av}_{a\in A} c_{x+(Q-n'-1-tn)e+a}\Big). \tag{1.41}$$

1.10 Eigenvalue estimate

Let $n, \frac{a}{\epsilon}$ be odd numbers and $n|\frac{a}{\epsilon}$, $\frac{a}{\epsilon}|\frac{c}{\epsilon}$. Following the notation of section 1.9, we set

$$\mathrm{Av}^{*}_{z,x} b_z \equiv \mathrm{Av}^{n,\frac{a}{\epsilon},\frac{c}{\epsilon}}_{z,x} b_z. \tag{1.42}$$

For simplicity of notation, in what follows we assume that $n, \frac{a}{\epsilon}$ to be odd numbers such that $n|\frac{a}{\epsilon}$, $\frac{a}{\epsilon}|\frac{c}{\epsilon}$. We will comment in remark 1.16 how to treat the general case.

We define

$$\begin{aligned}
\psi^e_{n,n} &\equiv m^{1,e}_n - m^{2,e}_n - \mu^\alpha[m^{1,e}_n - m^{2,e}_n \mid m^e_n] \\
\phi_{x,y} &\equiv j_{y,y+e} + \tau_y \mathcal{L} g - \sum_{e'>0} D_{e,e'}(m_{x,l}) \tau_y \frac{\psi^{e'}_{n,n}}{n} \\
\psi^{(0)}_x &\equiv \mathrm{Av}_{y:|y-x|\le l_1} \phi_{x,y} \\
\psi^{(1)}_x &\equiv \sum_{e'>0} D_{e,e'}(m_{x,l}) \Big[\mathrm{Av}_{y:|y-x|\le l_1} \tau_y \frac{\psi^{e'}_{n,n}}{n} - \tau_x \frac{\psi^{e'}_{n,n}}{n}\Big] \\
\psi^{(2)}_x &\equiv \sum_{e'>0} \big[D_{e,e'}(m_{x,l}) - D_{e,e'}(m^{a/\epsilon}_x)\big] \tau_x \frac{\psi^{e'}_{n,n}}{n} \\
\psi^{(3)}_x &\equiv \sum_{e'>0} D_{e,e'}(m^{a/\epsilon}_x) \Big[\tau_x \frac{\psi^{e'}_{n,n}}{n} - \mathrm{Av}^*_{z,x} \tau_z \frac{\psi^{e'}_{n,n}}{n}\Big] \\
\psi^{(4)}_x &\equiv \sum_{e'>0} D_{e,e'}(m^{a/\epsilon}_x) \Big[\mathrm{Av}^*_{z,x} \tau_z \Big(\frac{m^{1,e'}_n - m^{2,e'}_n}{n}\Big) - \frac{m^{a/\epsilon}_{x+ce'/\epsilon} - m^{a/\epsilon}_{x-ce'/\epsilon}}{2c/\epsilon}\Big] \\
\psi^{(5)}_x &\equiv -\sum_{e'>0} D_{e,e'}(m^{a/\epsilon}_x) \mathrm{Av}^*_{z,x} \tau_z \frac{\mu^\alpha[m^{1,e'}_n - m^{2,e'}_n \mid m^{e'}_n]}{n}.
\end{aligned}$$

Given a function J on $\mathbb{T}^d$ we define

$$\Omega_j \equiv \sup spec_{L^2(\mu_\epsilon^\alpha)}\{\epsilon^{-1}\mathrm{Av}_x J(\epsilon x)\psi_x^{(j)} + \gamma\epsilon^{d-2}\mathcal{L}\} \quad j = 0, \ldots, 5.$$

Then proposition 1.8 is equivalent to the following statement.

Proposition 1.10. *Given positive numbers γ and β, $\exists g \in \mathbb{G}$ such that for almost any disorder configuration α*

$$\limsup_{n\uparrow\infty, l\uparrow\infty, \epsilon\downarrow 0} \sup_{J\in C^*(\mathbb{T}^d)} \Omega_0 \le \beta. \tag{1.43}$$

Given $\gamma > 0$, for almost any disorder configuration α

$$\limsup_{n\uparrow\infty, c\downarrow 0, a\downarrow 0, l\uparrow\infty, \epsilon\downarrow 0} \sup_{C^*(\mathbb{T}^d)} \Omega_j \le 0 \quad j = 1, \ldots, 5.$$

1.11 Integration by parts formula

Lemma 1.11. *Let $f \in \mathcal{G}_s$. We fix a disorder configuration α and a function $g \in \mathbb{G}_s$ such that $f = \mathcal{L}_s g$.*
Then, for a particular constant $C(s,d)$, the constant $C(f)$ defined as

$$C(f) \equiv C(s,d) \sup_{x\in\mathbb{Z}^d} \sup_{m\in[0,1]} \Big(\mu^\alpha_{\Lambda_{x,s},m}(g^2)\Big)^{\frac{1}{2}}$$

has the following property.

Let Λ be a box and $\Delta \subset \Lambda$ be such that $\Lambda_{x,s} \subset \Lambda$ $\forall x \in \Delta$. Let μ be a canonical or grand canonical measure on Λ with disorder α. Then for any function h on $\{0,1\}^\Lambda$ and for any $A > 0$

$$\Big|\mu\Big(\sum_{x\in\Delta}\tau_x f, h\Big)\Big| \le C(f)|\Delta|^{\frac{1}{2}}\mathcal{D}_\Lambda(h;\mu)^{\frac{1}{2}} \le AC(f)|\Delta| + A^{-1}C(f)\mathcal{D}_\Lambda(h;\mu). \tag{1.44}$$

Proof. Since $\forall x\ \tau_x f = \mathcal{L}_{x,s}\tau_x g$,

$$\Big|\mu\Big(\sum_{x\in\Delta}\tau_x f, h\Big)\Big| = \Big|\sum_{x\in\Delta}\sum_{b\in\Lambda_{x,s}} \mu\Big(c_b(\tau_x g)\nabla_b h\Big)\Big| \le$$
$$d(2s+1)^d\Big(\sum_{x\in\Delta}\sum_{b\in\Lambda_{x,s}} \mu\big(c_b(\tau_x g)^2\big)\Big)^{\frac{1}{2}}\mathcal{D}_\Lambda(h;\mu)^{\frac{1}{2}}.$$

∎

1.12 Central limit theorem variance

In what follows we fix a disorder configuration α.
Given $f, g \in \mathcal{G}_s$, for any $l \geq s^2$ and for any canonical or grand canonical Gibbs measure μ on $\Lambda_{z,l}$ with disorder configuration α we define

$$V_{\Lambda_{z,l}}(f, g; \mu) = (2l)^{-d} \mu\Big(\sum_{|x-z| \leq l_1} \tau_x f, -(\mathcal{L}^\alpha_{\Lambda_{z,l}})^{-1} \sum_{|x-z| \leq l_1} \tau_x g \Big).$$

If μ is concentrated on the configuration $\eta \equiv 0$ or $\eta \equiv 1$ on $\Lambda_{z,l}$ then we set $V_{\Lambda_{z,l}}(f, g; \mu) = 0$. If $z = 0$ we write simply $V_l(f, g; \mu)$ and if $f = g$ we write $V_{\Lambda_{z,l}}(f; \mu)$ and $V_l(f; \mu)$.

For simplicity of notation in what follows we assume $z = 0$. Let G_l be the set of functions $g = g(\eta)$ with support in Λ_l. Then $V_l(f; \mu)$ is characterized by

$$V_l(f; \mu) = (2l)^{-d} \sup_{h \in G_l} \Big\{ 2\mu\Big(\sum_{|x| \leq l_1} \tau_x f, h \Big) - \mathcal{D}_{\Lambda_l}(h; \mu) \Big\}. \tag{1.45}$$

Since the r.h.s. in (1.45) is equal to

$$(2l)^{-d} \sup_{h \in G_l} \sup_{\gamma \in \mathbb{R}} \Big\{ 2\gamma\mu\Big(\sum_{|x| \leq l_1} \tau_x f, h \Big) - \gamma^2 \mathcal{D}_{\Lambda_l}(h; \mu) \Big\},$$

we obtain the variational characterization

$$V_l(f; \mu) = (2l)^{-d} \sup_{\substack{h \in G_l \\ \mathcal{D}_{\Lambda_l}(h;\mu) \neq 0}} \frac{[\mu(\sum_{|x| \leq l_1} \tau_x f, h)]^2}{\mathcal{D}_{\Lambda_l}(h; \mu)}. \tag{1.46}$$

We observe that all the functions appearing in the definition of $V_l(f)$ and in its variational characterizations are referred to the cube Λ_l. Since $\Delta_f \subset \Lambda_s$ it's possible to estimate $V_l(f)$ in terms of functions with support of size much smaller than l. In what follows we give two examples of such estimates.
In the former the size of the supports is of order s and in the latter the size of the supports is of order k with $l \gg k \gg s$.

Let Δ be a box such that $\Delta_f \subset \Delta \subset \Lambda_s$ and $\mu^\alpha_{\Delta,m}(f) = 0$ for any $m \in [0, 1]$.
For any $x \in \mathbb{Z}^d$ we define

$$\Delta_x \equiv \Delta + x, \quad m_x \equiv m_{\Delta_x}, \quad \mathcal{F}_x \equiv \sigma(m_x, \{\eta_y\}_{y \notin \Delta_x}), \quad g_\alpha(\Delta_x) \equiv \min_{N \in [0, |\Delta|]} \operatorname{gap} \mathcal{L}^\alpha_{\Delta_x, N},$$

where "gap $\mathcal{L}^\alpha_{\Delta_x,N}$" is the spectral gap of the infinitesimal generator $\mathcal{L}^\alpha_{\Delta_x}$ w.r.t. the canonical measure on Δ_x with N particles and disorder α.
Then, for any Gibbs measure μ on Λ_l and any function $h \in G_l$,

$$\mu(\tau_x f, h) = \mu\big(\, \mu(\tau_x f; h \,|\, \mathcal{F}_x) \,\big) + \mu\big(\, \mu(\tau_x f \,|\, \mathcal{F}_x); h \,\big) =$$
$$\mu\big(\, \mu(\tau_x f; h \,|\, \mathcal{F}_x)\big) \le \mu\Big(\mathrm{Var}_\mu(\tau_x f \,|\, m_x)^{1/2} \mathrm{Var}_\mu(h \,|\, \mathcal{F}_x)^{1/2} \Big).$$

Then by Schwarz inequality

$$\mu(\sum_{|x|\le l_1} \tau_x f, h)^2 \le \Big(\sum_{|x|\le l_1} g_\alpha^{-1}(\Delta_x) \mu\big(Var_\mu(\tau_x f \,|\, m_x) \big) \Big) \cdot \sum_{|x|\le l_1} \mathcal{D}_{\Delta_x}(h;\mu)$$
$$\le C(d) s^d \Big(\sum_{|x|\le l_1} g_\alpha^{-1}(\Delta_x) \mu\big(Var_\mu(\tau_x f \,|\, m_x) \big) \Big) \mathcal{D}_{\Lambda_l}(h;\mu).$$

The above estimate and (1.46) imply

$$V_l(f;\mu) \le C(d) s^d \mathrm{Av}_{|x|\le l_1} g_\alpha^{-1}(\Delta_x) \mu\big(Var_\mu(\tau_x f \,|\, m_x) \big) \qquad \forall l \ge s^2. \tag{1.47}$$

By lemma 2.7 $g_\alpha(\Delta_x) \ge C(B,s,d)$, therefore

$$\sup_l V_l(f,\mu) \le C(\|f\|_\infty, B, s, d) < \infty. \tag{1.48}$$

As before, we choose a disorder configuration α and $f \in \mathcal{G}_s$. Moreover we fix $g \in \mathbb{G}_s$ such that $f = \mathcal{L}_s g$.
Let l, k be integers such that $l > k > s^2$. Since in the applications we take the limit $k \uparrow \infty$ after the limit $l \uparrow \infty$, l has to be thought much larger than k.
Let μ be a canonical or grand canonical Gibbs measure on Λ_l with disorder configuration α. In what follows, extending some standard techniques explained in [13], we give two estimates of $V_l(f;\mu)$ by expressions involving only functions with support in some of the cubes obtained by partitioning Λ_l in cubes of width $2k+1$. These estimates are given by (1.54) and (1.55) and their proof is based on the variational formula

$$V_l(f;\mu) = (2l)^{-d} \sup_{h \in G_l} \{ 2\mu(\sum_{|y|\le l_1} \tau_y f, h) - \mathcal{D}_{\Lambda_l}(h;\mu) \}. \tag{1.49}$$

We first introduce some notation.
We set $A_{k,l} \equiv \{ a \in (2k+1)\mathbb{Z}^d \mid a + \Lambda_k \subset \Lambda_l \}$. Given $a \in A_{k,l}$ we set $B_{a,k} \equiv a + \Lambda_k$, $B^0_{a,k} \equiv a + \Lambda_{k_1}$. Moreover, given $X \subset \mathbb{Z}^d$, we set $\mathcal{F}_X \equiv \sigma(\eta_x \;:\; x \in X)$.
We write $B_{k,l} = \Lambda_l \setminus \cup_{a \in A_{k,l}} B_{a,k}$, then we have the following decomposition

$$\Lambda_l = B_{k,l} \sqcup \big(\sqcup_{a \in A_{k,l}} B_{a,k} \big). \tag{1.50}$$

We observe $|\Lambda_l \setminus \cup_{a \in A_{k,l}} B^0_{a,k}| \le C(d)(l^{d-1}k + l^d k^{-\frac{1}{2}})$.

When there is no ambiguity, we will omit indexes k, l and write only A, B_a, B_a^0 and B.

In what follows, $C(f)$ and $C(k)$ are constants that can change from line to line. We require that $C(k)$ depends only on k and $C(f)$ depends only on s and

$$\sup_{x\in\mathbb{Z}^d}\sup_{m\in[0,1]}\left(\mu^{\alpha}_{\Lambda_{x,s},m}(g^2)\right)^{\frac{1}{2}}.$$

By lemma 1.11, we may assume in (1.49) that $\mathcal{D}_{\Lambda_l}(h;\mu)\le C(f)l^d$, otherwise the r.h.s. in 1.49 is null. This implies, together with lemma 1.11, that

$$\Big|\sum_{y\in\Lambda_{l_1}\setminus\cup_a B_a^0}\mu(\tau_y f,h)\Big|\le C(f)l^d\sqrt{kl^{-1}+k^{-\frac{1}{2}}}. \tag{1.51}$$

We set $h_a\equiv\mu[h|\mathcal{F}_{B_a}]$ and we indicate the density on B_a by m_a. Then by convexity

$$\sum_a\mathcal{D}_{B_a}(h_a;\mu)\le\mathcal{D}_{\Lambda_l}(h;\mu). \tag{1.52}$$

In what follows, given a generic canonical or grand canonical Gibbs measure $\bar\mu$ with disorder α, we define

$$F_a(h;\bar\mu)\equiv 2\sum_{y\in B_a^0}\bar\mu(\tau_y f,h)-\mathcal{D}_{B_a}(h;\bar\mu).$$

Then (1.51) and (1.52) imply

$$F(h;\mu)\le\sum_{a\in A}F_a(h_a;\mu)+C(f)l^d\sqrt{kl^{-1}+k^{-\frac{1}{2}}}. \tag{1.53}$$

By taking conditional expectations, $F_a(h_a;\mu)=\mu\big(F_a(h_a;\mu^{\alpha}_{B_a,m_a})\big)$. We observe that by lemma 1.11, $\exists C(f)$ such that $\forall a$ $F_a(h_a;\mu)\le F_a(\tilde h_a;\mu)$ where

$$\tilde h_a(\eta)\equiv\begin{cases}h_a(\eta) & \text{if } \mathcal{D}_{B_a}(h_a;\mu^{\alpha}_{B_a,m_a(\eta)})\le C(f)k^d\\ 0 & otherwise\end{cases}.$$

We set

$$Z_{k,l}\equiv\left\{\{h_a\}_{a\in A_{k,l}}:\forall a\in A\begin{cases}h_a \text{ is } \mathcal{F}_{B_a}-\text{measurable}\\ \mathcal{D}_{B_a}(h_a;\mu^{\alpha}_{B_a,m})\le C(f)k^d\ \forall m\in[0,1]\end{cases}\right\},$$

then by the previous estimates

$$V_l(f;\mu)\le(2l)^{-d}\sup_{\{h_a\}_{a\in A}\in Z_{k,l}}\Big(\sum_{a\in A}F_a(h_a;\mu)\Big)+C(f)\sqrt{kl^{-1}+k^{-\frac{1}{2}}}. \tag{1.54}$$

We observe that if $\{h_a\}_{a\in A} \in Z_{k,l}$ and, $\forall a \in A$, g_a is a function such that $g_a(\eta)$ depends only on $m_a(\eta)$, then $\forall a \in A$ $F_a(h_a;\mu) = F_a(h_a - g_a;\mu)$.

By using the estimate on the Dirichlet form of h_a it's simple to prove that there exists $C(k)$ such that $\forall a\ \exists g_a = g_a\big(m_a(\eta)\big)$ with $\|h_a - g_a\|_\infty \le C(f)C(k)$.

In fact, $\forall a, m$ and $\forall \eta \in \{0,1\}^{B_a}$, $\mu^\alpha_{B_a,m}(\eta) \ge C(k)$. Then the estimate on the Dirichlet form of h_a gives $\|\nabla_b h_a\|_\infty \le C(f)C(k)$ for any $b \in B_a$.
For any $m \in [0,1]$ we fix $\eta_m \in \{0,1\}^{B_a}$ with density m.Then $\forall \eta \in \{0,1\}^{B_a}$ we can find a sequence of bonds $b_1,\dots,b_r$ such that $\eta = S_{b_r} \circ \cdots \circ S_{b_1}\eta_{m_{B_a}(\eta)}$. Therefore, the thesis follows by a telescopic estimate and by taking $g(\sigma) = h(\eta_{m_{B_a}(\sigma)})$.

By the above observations

$$V_l(f;\mu) \le (2l)^{-d} \sup_{\{h_a\}_{a\in A}\in Y_{k,l}} \Big(\sum_{a\in A} F_a(h_a;\mu)\Big) + C(f)\sqrt{kl^{-1} + k^{-\frac{1}{2}}} \tag{1.55}$$

where

$$Y_{k,l} \equiv \left\{ \{h_a\}_{a\in A_{k,l}} : \forall a \in A \ \begin{cases} h_a \text{ is } \mathcal{F}_{B_a}\text{measurable}, \\ \forall m \in [0,1]\ \mathcal{D}_{B_a}(h_a;\mu^\alpha_{B_a,m}) \le C(f)k^d \\ \|h_a\|_\infty \le C(f)C(k) \end{cases} \right\}.$$

In what follows we give two estimates which are particularly useful in order to pass from expressions involving $\mu^\alpha_{l,m}$ to expressions involving $\mu^{\alpha,\lambda_0(m)}$.
By theorem 2.4 (equivalence of ensembles)

$$|\sum_{a\in A} F_a(h_a;\mu^\alpha_{l,m}) - \sum_{a\in A} F_a(h_a;\mu^{\alpha,\lambda_l(\alpha,m)})| \le C(f)C(k) \tag{1.56}$$

and the lemma $A.3$

$$|\sum_{a\in A} F_a(h_a;\mu^{\alpha,\lambda_l(\alpha,m)}) - \sum_{a\in A} F_a(h_a;\mu^{\alpha,\lambda_0(m)})| \le l^d C(f)C(k)|m - \mu^{\alpha,\lambda_0(m)}(m_l)|. \tag{1.57}$$

Let us conclude this section with the definition of the central limit theorem variance, which is extensively studied in chapter 5.
Given a density $m \in (0,1)$ and $f, g \in \mathcal{G}$ we define

$$V_m(f,g) \equiv \lim_{l\uparrow\infty} \mathbb{E}\Big[V_l(f,g;\mu^{\alpha,\lambda_0(m)})\Big].$$

In chapter 5 we prove the above limit exists and is finite and that $V_m(\cdot,\cdot)$ defines a semi-inner product on $\mathcal{G}$. The semi-norm $V_m^{\frac{1}{2}}$ is called a "central limit theorem variance".
If $f = g$ we write $V_m(f)$ in place of $V_m(f,f)$.

1.13 Estimates by the central limit theorem variance

Let $f_1, f_2 \in \mathcal{G}_s$ and D be a continuous function on $[0,1]$. For any $m \in [0,1]$ we set $\psi_m \equiv f_1 + D(m) f_2$.
The following proposition is an extension to the disordered model under consideration of some results explained in [13], chapter 7. We remark that the ergodicity of the random field is fundamental for the proof.

Proposition 1.12. *For almost any disorder configuration* α

$$\limsup_{l\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{|x|\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} V_{\Lambda_{x,l}}(\psi_m, \mu^{\alpha}_{\Lambda_{x,l},m}) \le \sup_{m\in[0,1]} \limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k\Big(\psi_m, \mu^{\alpha,\lambda_0(m)}\Big)\Big].$$

Proof. We use the results of section 1.12, in particular estimates (1.55), (1.56) and (1.57) with $f \equiv \psi_m$ and $\mu \equiv \mu^{\alpha}_{l,m}$.
We observe that these estimates are valid for a generic cube $\Lambda_{x,l}$ in place of Λ_l. Moreover, the constant $C(f)$ with $f = \psi_m$ can be bounded by a constant $\bar{C}$ independent of m.
To the notation use in (1.55), (1.56), (1.57) we add the index x when we refer to the cube $\Lambda_{x,l}$ and m when we refer to ψ_m.

The above observations imply

$$\begin{aligned}
&\mathrm{Av}_{x\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} V_{\Lambda_{x,l}}(\psi_m, \mu^{\alpha}_{\Lambda_{x,l},m}) \le \bar{C}\sqrt{kl^{-1} + k^{-\frac{1}{2}}} + \bar{C}C(k)l^{-d} + \bar{C}T_1 + C_{k,l}T_2 \\
&T_1 \equiv C(k)\mathrm{Av}_{x\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} |m - \mu^{\alpha,\lambda_0(m)}(m_{x,l})| \\
&T_2 \equiv \mathrm{Av}_{x\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} \sup_{\{h_a\}\in Y^x_{k,l}} \mathrm{Av}_{a\in A^x_{k,l}} (2k)^{-d} F^m_a(h_a; \mu^{\alpha,\lambda_0(m)})
\end{aligned} \tag{1.58}$$

where $\lim_{k\uparrow\infty,l\uparrow\infty} C_{k,l} = 1$.

We analyze term T_1.

We fix $\delta > 0$ and k. We choose $0 = \bar{m}_0 < \bar{m}_1 < \cdots < \bar{m}_N = 1$ with $\bar{m}_{i+1} - \bar{m}_i \le \delta C(k)^{-1}$ for $0 \le i < N$. Then by the Ergodic theorem and lemma $A.5$ $\tilde{\forall}\alpha$

$$\lim_{\epsilon\downarrow 0} T_1 \le C(B)\delta + C(k)\mathbb{E}\Big[\sup_i |\bar{m}_i - \mu^{\alpha,\lambda_0(\bar{m}_i)}(m_l)|\Big]$$

By taking the limit $l \uparrow \infty$ and by the arbitrariness of δ we obtain $\tilde{\forall}\alpha$ $\lim_{l\uparrow\infty,\epsilon\downarrow 0} T_1 = 0$.

We analyze term T_2.

By lemma $A.5$ and the form of ψ_m it's simple to prove that $\forall x, a, h_a$ as in the definition of T_2

$$|F_a^m(h_a;\mu^{\alpha,\lambda_0(m)}) - F_a^{m'}(h_a;\mu^{\alpha,\lambda_0(m')})| \leq \bar{C}C(k)H(|m-m'|) \tag{1.59}$$

for a particular function H such that $\lim_{\lambda\downarrow 0} H(\lambda) = 0$.

In fact, by lemma $A.5$ and the form of ψ_m

$$|F_a^m(h_a;\mu^{\alpha,\lambda_0(m)}) - F_a^m(h_a;\mu^{\alpha,\lambda_0(m')})| \leq \bar{C}C(k)|m-m'|$$

and by the form of ψ_m for any $y \in B_a^0$

(1.60)
$$|\mu^{\alpha,\lambda_0(m')}(\tau_y\psi_m, h_a) - \mu^{\alpha,\lambda_0(m')}(\tau_y\psi_{m'}, h_a)| \leq \|\psi_m - \psi_{m'}\|_\infty \|h_a\|_\infty \leq \bar{C}C(k)H(|m-m'|).$$

We fix $\delta > 0$ and k. Let λ_0 be such that $H(\lambda) \leq \delta(\bar{C}C(k))^{-1}$ if $\lambda \leq \lambda_0$.
We choose $0 = \bar{m}_0 < \bar{m}_1 < \cdots < \bar{m}_N = 1$ with $\bar{m}_{i+1} - \bar{m}_i \leq \lambda_0$ for $0 \leq i < N$. Then by (1.59)

$$\begin{aligned} T_2 &\leq \delta + \mathrm{Av}_{|x|\leq\frac{1}{\epsilon}} \sup_{i=0}^{N} \sup_{\{h_a\}\in Y_{k,l}^x} \mathrm{Av}_{a\in A_{k,l}^x} (2k)^{-d} F_a^{\bar{m}_i}(h_a;\mu^{\alpha,\lambda_0(\bar{m}_i)}) \\ &\leq \delta + \mathrm{Av}_{|x|\leq\frac{1}{\epsilon}} \sup_{i=0}^{N} \mathrm{Av}_{a\in A_{k,l}^x} V_{B_{a,k}^x}(\psi_{\bar{m}_i};\mu^{\alpha,\lambda_0(\bar{m}_i)}). \end{aligned} \tag{1.61}$$

By the Ergodic theorem, $\tilde{\forall}\alpha$ and $\forall i = 0, \ldots N$,

$$\limsup_{\epsilon\downarrow 0} T_2 \leq \delta + \mathbb{E}\Big[\sup_{i=0}^{N} \mathrm{Av}_{a\in A_{k,l}} V_{B_{a,k}}(\psi_{\bar{m}_i}, \mu^{\alpha,\lambda_0(\bar{m}_i)})\Big],$$
$$\lim_{l\uparrow\infty} \mathrm{Av}_{a\in A_{k,l}} V_{B_{a,k}}(\psi_{\bar{m}_i}, \mu^{\alpha,\lambda_0(\bar{m}_i)}) = \mathbb{E}\Big[V_{\Lambda_k}(\psi_{\bar{m}_i}, \mu^{\alpha,\lambda_0(\bar{m}_i)})\Big].$$

By the arbitrariness of δ it's enough to prove

$$\limsup_{k\uparrow\infty} \sup_{m\in[0,1]} \mathbb{E}\Big[V_k(\psi_m, \mu^{\alpha,\lambda_0(m)})\Big] \leq \sup_{m\in[0,1]} \limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(\psi_m, \mu^{\alpha,\lambda_0(m)})\Big]. \tag{1.62}$$

Let $\{m_k\}_{k>0}$ be such that the l.h.s. of (1.62) is equal to $\overline{\lim}_{k\uparrow\infty} \mathbb{E}\Big[V_k(\psi_{m_k}, \mu^{\alpha,\lambda_0(m_k)})\Big]$.

For any subsequence of $\{m_k\}_{k>0}$ there exists a converging sub-subsequence. In order to simplify notation we assume that $\lim_{k\uparrow\infty} m_k = m$ for some m. We want to prove

$$\limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(\psi_{m_k}, \mu^{\alpha,\lambda_0(m_k)})\Big] \leq \limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(\psi_m, \mu^{\alpha,\lambda_0(m)})\Big].$$

By the Dominated Convergence theorem it's enough to prove that $\tilde{\forall}\alpha$

$$\limsup_{k\uparrow\infty} V_k(\psi_{m_k}, \mu^{\alpha,\lambda_0(m_k)}) \leq \limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(\psi_m, \mu^{\alpha,\lambda_0(m)})\Big].$$

To this aim we introduce the parameter r with $r \uparrow \infty, k \uparrow \infty, l \uparrow \infty, \epsilon \downarrow 0$ and we consider the estimate (1.55) where l and k have to be substituted respectively by k and r, i.e. now we partition the cube Λ_k in cubes of width $2r+1$, and $\mu = \mu^{\alpha,\lambda_0(m_k)}$.
Then by (1.55) and by a formula similar to (1.59) for any disorder configuration α

(1.63)
$$V_k(\psi_{m_k}, \mu^{\alpha,\lambda_0(m_k)}) \le C_{r,k} \sup_{\{h_a\}_{a\in A}\in Y_{r,k}} \mathrm{Av}_{a\in A}(2r)^{-d} F_a(h_a; \mu^{\alpha,\lambda_0(m)}) + \bar{C}\sqrt{rk^{-1} + r^{-\frac{1}{2}}} + \bar{C}C(r)H(|m - m_k|).$$

where $\lim_{r\uparrow\infty, k\uparrow\infty} C_{r,k} = 1$.

The second and third terms are depressed. The first term is bounded by

$$\mathrm{Av}_{a\in A} V_{B_{a,s}}(\psi_m; \mu^{\alpha,\lambda_0(m)})$$

and the thesis follows by the Ergodic theorem.

□

1.14 Moving Particles Lemma

The standard Moving Particles Lemma, see [26], can be generalized to the disordered system under consideration as proved by Quastel [23]. In particular, given $x, y \in \mathbb{Z}^d$ we define

$$u_i \equiv (y_1, y_2, \dots, y_i, x_{i+1}, \dots, x_d) \qquad i = 0, \dots, d.$$

We observe that $u_0 = x$ and $u_d = y$. Let $\gamma_{x,y}$ be the path connecting u_0 to u_1 by moving along the first direction, then connecting u_1 to u_2 by moving along the second direction and so on until arriving to u_d. We denote by $|\gamma_{x,y}|$ the length of the path.

Lemma 1.13. *(Moving Particles Lemma) There exists a constant $C = C(B)$ having the following property.*
Given any canonical measure μ with disorder α on a box containing x, y and given a measurable function f

$$\mu\big((\nabla_{x,y} f)^2 \big) \le C|\gamma_{x,y}| \sum_{b\in\gamma_{x,y}} \mu\big((\nabla_b f)^2 \big).$$

1.15 Two Blocks estimate

We observe that given $\{b_x\}_{x\in\mathbb{Z}^d}$ such that $|b_x| \le C_0$ for any x then for any positive integers k and K

$$|\mathrm{Av}_{|y|\le K} b_y - \mathrm{Av}_{|y|\le K}\mathrm{Av}_{z:|z-y|\le k} b_z| \le C_0 C(d)\frac{k}{K}. \tag{1.64}$$

Proposition 1.14. *Let $\gamma > 0$. Then for almost any disorder configuration α*

$$\limsup_{c\downarrow 0,k\uparrow\infty,\epsilon\downarrow 0}\ \sup_{w:|w|\le\frac{c}{\epsilon}}\ \sup spec_{L^2(\mu^\alpha_\epsilon)}\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x,k} - m_{x+w,k}| + \gamma\epsilon^{d-2}\mathcal{L}\} \le 0. \tag{1.65}$$

Proof. Let us observe that it is enough to prove the theorem in the case $c_{x,y} = 1 + e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}$. In particular, we can extend the defininition $\mathcal{L}_{x,y} \equiv c_{x,y}\nabla_{x,y}$ to the case $|x-y| \ne 1$. We introduce the parameter l with $l \uparrow \infty, c \downarrow 0, k \uparrow \infty, \epsilon \downarrow 0$. By (1.64) we can write

$$m_{x,k} = \mathrm{Av}_{y\in\Lambda_k} m_{x+y,l} + O\Big(\frac{l}{k}\Big)$$
$$m_{x+w,k} = \mathrm{Av}_{y'\in\Lambda_k\,:\,|w+y'-y|>2l} m_{x+w+y',l} + O\Big(\frac{l}{k}\Big) \quad \forall y \in \Lambda_k.$$

Therefore, the sequence in the l.h.s. of (1.65) can be approximated with a depressed error by

$$\sup_{w:|w|\le\frac{c}{\epsilon}} \mathrm{Av}_{y\in\Lambda_k}\mathrm{Av}_{y'\in\Lambda_k\,:\,|w+y'-y|>2l} \sup spec_{L^2(\mu^\alpha_\epsilon)}\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x,l} - m_{x+w+y'-y,l}| + \gamma\epsilon^{d-2}\mathcal{L}\}$$

and, by changing names, in order to prove (1.65) it's enough to show that, given $\gamma > 0$, for almost any disorder configuration α

$$\limsup_{l\uparrow\infty,c\downarrow 0,\epsilon\downarrow 0}\ \sup_{w:2l<|w|\le\frac{c}{\epsilon}}\ \sup_{L^2(\mu^\alpha_\epsilon)} spec\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x,l} - m_{x+w,l}| + \gamma\epsilon^{d-2}\mathcal{L}\} \le 0. \tag{1.66}$$

In what follows we fix $w : 2l < |w| \le \frac{c}{\epsilon}$.

We assert that if $l \le k \le \frac{c}{\epsilon}$ then

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}_{u\in\Lambda_{x,l}}\mathrm{Av}_{v\in\Lambda_{x+w,l}}(-\mathcal{L}_{u,v}) \le Cc^2\epsilon^{d-2}(-\mathcal{L}). \tag{1.67}$$

In order to prove (1.67) we observe that by the Moving Particles Lemma the l.h.s. of (1.67) can be bounded by

$$C\,\frac{c}{\epsilon}\epsilon^d l^{-2d}\sum_{b\in\mathbb{T}^d_\epsilon} a_b(-\mathcal{L}_b) \ \text{ where }\ a_b \equiv |\{(x,u,v)\,:\,x\in\mathbb{T}^d_\epsilon,\ u\in\Lambda_{x,l},\ v\in\Lambda_{x+w,l},\ b\in\gamma_{u,v}\}|. \tag{1.68}$$

In what follows we estimate a_b. Let (x,u,v) be as in the definition of a_b and let the bond b be equal to $\{y, y+e_j\}$ where $y = (y_1, \dots, y_d)$.
Since $b \in \gamma_{u,v}$ we have $u \in A_b$ and $v \in B_b$ where

$$\begin{aligned} A_b &\equiv \{z \, : \, z_{j+1} = y_{j+1}, \dots, z_d = y_d\} \\ B_b &\equiv \{z \, : \, z_1 = y_1, \dots, z_{j-1} = y_{j-1}\}. \end{aligned}$$

Since $|x-u| \le l$, $|x+w-v| \le l$, $u \in A_b$ and $v \in B_b$ then $|x_i - y_i| \le l$ for $i = j+1, \dots, d$ and $|x_i + w_i - y_i| \le l$ for $i = 1, \dots, j-1$. Moreover $dist(x,b) \le C\frac{c}{\epsilon}$ and therefore, given b, x can be chosen in $O(\frac{c}{\epsilon} l^{d-1})$ ways.
Given x, since $u \in A_b \cap \Lambda_{x,l}$, u can be chosen in $O(l^j)$ ways and, by a similar argument, v can be chosen in $O(l^{d-j+1})$ ways.
In conclusion, $a_b \le C\frac{c}{\epsilon} l^{2d}$ and by (1.68) we obtain (1.67).

By (1.67) the supremum of the spectrum in (1.66) is bounded, for some positive constant γ', by

$$\text{Av}_{x \in \mathbb{T}^d_\epsilon} \sup_{\nu \in \Theta} \sup spec_{L^2(\nu)} \{ |m_{x,l} - m_{x+w,l}| + \gamma' c^{-2} \text{Av}_{u \in \Lambda_{x,l}} \text{Av}_{v \in \Lambda_{x+w,l}} (-\mathcal{L}_{u,v}) \}$$

where Θ_l is the set of canonical Gibbs measures on $\Lambda_{x,l} \cup \Lambda_{x+w,l}$ with density m and disorder configuration α.
Let $\nu \in \Theta_l$. Since for some constant $\kappa = \kappa(l) > 0$

$$gap_{L^2(\nu)} \text{Av}_{u \in \Lambda_{x,l}} \text{Av}_{v \in \Lambda_{x+w,l}} (-\mathcal{L}_{u,v}) \ge \kappa(l)$$

by proposition $A.13$ if $c \le c_0(l, \gamma')$

$$\sup_{L^2(\nu)} spec \{ |m_{x,l} - m_{x+w,l}| + \gamma' c^{-2} \text{Av}_{\substack{v \in \Lambda_{x+w,l} \\ u \in \Lambda_{x,l}}} (-\mathcal{L}_{u,v}) \} \le C\nu(|m_{x,l} - m_{x+w,l}|) + C(\gamma') \frac{c^2}{\kappa(l)}.$$

By the previous estimates and lemma 2.6 it's enough to prove, for almost any configuration disorder α,

$$\limsup_{l \uparrow \infty, \epsilon \downarrow 0} \sup_{w: 2l < |w| \le \frac{c}{\epsilon}} \text{Av}_{x \in \mathbb{T}^d_\epsilon} \sup_m \mu^{\alpha, \lambda_{\Lambda_{x,l} \cup \Lambda_{x+w,l}}(\alpha, m)} (|m_{x,l} - m_{x+w,l}|). \tag{1.69}$$

To this aim we introduce another parameter s with $s \uparrow \infty, l \uparrow \infty, \epsilon \downarrow 0$. Then by (1.64)

$$|m_{x,l} - m_{x+w,l}| \le \text{Av}_{y \in \Lambda_{x,l}} |m_{y,s} - m_{y+w,s}| + O\Big(\frac{s}{l}\Big)$$

and by lemma $A.7$

$$\begin{aligned} &\mu^{\alpha, \lambda_{\Lambda_{x,l} \cup \Lambda_{x+w,l}}(\alpha, m)} (|m_{x,l} - m_{x+w,l}|) \le \\ &\qquad C\Big(\mathop{\text{Av}}_{y \in \Lambda_{x,l}} \mu^{\alpha, \lambda_0(m)} (|m_{y,s} - m_{y+w,s}|) + s^d \big| m - \mu^{\alpha, \lambda_0(m)} \big(m_{\Lambda_{x,l} \cup \Lambda_{x+w,l}} \big) \big| \Big) + O\Big(\frac{s}{l}\Big). \end{aligned}$$

Therefore, it's enough to prove

$$\limsup_{s\uparrow\infty,l\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_m \mathrm{Av}_{y\in\Lambda_l}\mu^{\alpha,\lambda_0(m)}(|m_{x+y,s}-m|)=0, \tag{1.70}$$

$$\limsup_{l\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_m |m-\mu^{\alpha,\lambda_0(m)}(m_{x,l})|=0. \tag{1.71}$$

We consider equation (1.70). By lemma $A.5$, for any x,

$$\left|\mu^{\alpha,\lambda_0(m)}(|m_{x,s}-m|)-\mu^{\alpha,\lambda_0(m')}(|m_{x,s}-m'|)\right|\le C(s)|m-m'|$$

We fix $\delta>0$ and a partition $0=\bar m_0<\bar m_1<\cdots<\bar m_N=1$ where $N=N(\delta,s)$ and $|\bar m_i-\bar m_{i+1}|\le\delta C(s)^{-1}$ for any $i=0,\dots,N-1$. Then

$$\mathop{\mathrm{Av}}_{x\in\mathbb{T}^d_\epsilon}\sup_m \mathop{\mathrm{Av}}_{y\in\Lambda_l}\mu^{\alpha,\lambda_0(m)}(|m_{x+y,s}-m|)\le\delta+\mathop{\mathrm{Av}}_{x\in\mathbb{T}^d_\epsilon}\sup_{i=0}^{N}\mathop{\mathrm{Av}}_{y\in\Lambda_l}\mu^{\alpha,\lambda_0(\bar m_i)}(|m_{x+y,s}-\bar m_i|). \tag{1.72}$$

By the Ergodic Theorem, by taking the limit $\epsilon\downarrow 0$ $\tilde\forall\alpha$ the r.h.s. in (1.72) becomes

$$\delta+\mathbb{E}\Big(\sup_{i=0}^{N}\mathrm{Av}_{y\in\Lambda_l}\mu^{\alpha,\lambda_0(\bar m_i)}(|m_{y,s}-\bar m_i|)\Big). \tag{1.73}$$

By the Ergodic Theorem, $\tilde\forall\alpha$ the limit with $l\uparrow\infty$ of (1.73) can be estimated by the supremum over $m\in[0,1]$ of $\mathbb{E}\Big(\mu^{\alpha,\lambda_0(m)}(|m_s-m|)\Big)$ which is bounded, uniformly in m, by

$$\left\{\mathbb{E}\Big(\mu^{\alpha,\lambda_0(m)}\big([m_s-\mu^{\alpha,\lambda_0(m)}(m_s)]^2\big)\Big)\right\}^{\frac12}+\left\{\mathbb{E}\Big(\big[\mu^{\alpha,\lambda_0(m)}(m_s)-m\big]^2\Big)\right\}^{\frac12}\le 2s^{-\frac{d}{2}}$$

and that proves (1.70).

We consider equation (1.71).
By lemma $A.5$, it's enough to prove (1.71) for a density m which is fixed a priori.
By the Ergodic Theorem, for almost any disorder configuration α

$$\lim_{\epsilon\downarrow 0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m-\mu^{\alpha,\lambda_0(m)}(m_{x,l})|=\mathbb{E}\big(|m-\mu^{\alpha,\lambda_0(m)}(m_l)|\big)$$

that goes to 0 when $l\uparrow\infty$.

□

1.16 Reduction to the Two Blocks estimate

The following proposition is particularly useful in order to reduce some eigenvalue estimates to a Two Blocks estimate.

Proposition 1.15. *Let $g \in \mathcal{G}$ have support included in Λ_s. Then for any disorder configuration α, any $\gamma > 0$ and any family of functions $F := \{f_x\}_{x\in\mathbb{T}^d_\epsilon}$ on Ω_ϵ,*

$$\begin{aligned}\sup spec_{L^2(\mu^\alpha_\epsilon)}\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}(\tau_x g\, f_x) + \gamma\epsilon^{d-2}\mathcal{L}\} \le \\ \epsilon^{-1}c(g, \|F\|_\infty)\|\nabla F\|_\infty + \sup spec_{L^2(\mu^\alpha_\epsilon)}\{c(g)\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} f_x^2 + \frac{1}{2}\gamma\epsilon^{d-2}\mathcal{L}\},\end{aligned} \tag{1.74}$$

where $\|F\|_\infty := \sup_{x\in\mathbb{T}^d_\epsilon}\|f_x\|_\infty$ and $\|\nabla F\|_\infty := \sup_{x\in\mathbb{T}^d_\epsilon}\sup_{b\in\Lambda_{x,s}}\|\nabla_b f_x\|_\infty$.

Proof. Let $h \in \mathbb{G}$ be such that $g = \mathcal{L}_{\Lambda_s} h$ and $\Delta_h \subset \Lambda_\ell$. Then

$$\text{l.h.s. of (1.74)} = \sup_\rho\big\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\,\mu^\alpha_\epsilon\big(\mathcal{L}_{\Lambda_{x,s}}(\rho\, f_x)\,\tau_x h\big) - \gamma\epsilon^{d-2}\mathcal{D}(\sqrt{\rho};\mu^\alpha_\epsilon)\big\}$$

where ρ varies among the non-negative measurable functions such that $\mu^\alpha_\epsilon(\rho) = 1$. Since $\nabla_b(\rho\, f_0)(\eta) = \rho(\eta^b)\nabla_b f(\eta) + (\nabla_b\rho)(\eta) f_0(\eta)$, we can write

$$\mu^\alpha_\epsilon\big(\mathcal{L}_{\Lambda_\ell}(\rho\, f_0)h\big) = \sum_{b\in\Lambda_\ell}\big[\mu^\alpha_\epsilon\big(c_b\,\rho(\eta^b)(\nabla_b f_0)h\big) + \mu^\alpha_\epsilon\big(c_b(\nabla_b\rho) f_0\, h\big)\big].$$

By Schwarz inequality and the identity $\nabla_b\rho(\eta) = \nabla_b\sqrt{\rho}(\eta)\big(\sqrt{\rho}(\eta^b)+\sqrt{\rho}(\eta)\big)$, for any $\beta > 0$

$$\big|\,\mu^\alpha_\epsilon\big(c_b(\nabla_b\rho) f_0\, h\big)\,\big| \le \beta\mu^\alpha_\epsilon\big(c_b(\nabla_b\sqrt{\rho})^2\big) + \frac{1}{\beta}\mu^\alpha_\epsilon\Big(c_b\big(\sqrt{\rho}\circ S_b + \sqrt{\rho}\big)^2 f_0^2\, h^2\Big).$$

Since μ^α_ϵ is reversible and $f_0^2(\eta^b) = f_0^2(\eta) + \nabla_b f_0(\eta)\big(f_0(\eta^b) + f_0(\eta)\big)$,

$$\mu^\alpha_\epsilon\big(c_b\rho(\eta^b) f_0^2\, h^2\big) = \mu^\alpha_\epsilon\big(c_b\rho\, f_0^2(\eta^b)\, h^2(\eta^b)\big) \le C\mu^\alpha_\epsilon(\rho\, f_0^2)\|h\|^2_\infty + C\|f_0\|_\infty\|\nabla_b f_0\|_\infty\|h\|^2_\infty.$$

Similarly for f_0 and h replaced by f_x and $\tau_x h$ respectively. In conclusion

$$\begin{aligned}\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\,\mu^\alpha_\epsilon\big(\mathcal{L}_{\Lambda_{x,\ell}}(\rho\, f_x)\tau_x h\big) \le \\ c(g)\Big(\epsilon^{-1}\big(1+\frac{1}{\beta}\big)(1+k)k' + \frac{1}{\beta}\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\,\mu^\alpha_\epsilon(\rho\, f_x^2) + \beta\epsilon^{d-1}\mathcal{D}(\sqrt{\rho};\mu^\alpha_\epsilon)\Big).\end{aligned}$$

The thesis follows by taking $\beta = 1/(2c(g)\,\epsilon)$. □

1.17 Proof of proposition 1.10

1.17.1 Ω_0

The proof of proposition 1.10 for Ω_0 is based on perturbation theory, proposition 1.12 and the results concerning semi-norm V_m of chapter 5.

Let $\beta, \gamma > 0$ be given. We have to determine a function $g \in \mathbb{G}$ in the definition of Ω_0 such that

$$\limsup_{n\uparrow\infty, l\uparrow\infty, \epsilon\downarrow 0}\ \sup_{J\in C^*(\mathbb{T}^d)}\ \Omega_0 \le \beta.$$

For any $m \in [0, 1]$ and any positive integer n we define

$$\psi_m \equiv j_e + \mathcal{L}g - \sum_{e'>0} D_{e,e'}(m)\tau_y \frac{\psi^{e'}_{n,n}}{n} \qquad \text{and} \qquad \psi^{(l)}_m \equiv Av_{y:|y|\le l_1}\tau_y\psi_m.$$

Then

$$\begin{aligned}(1.75)\qquad &\sup_{J\in C^1_\chi(\mathbb{T}^d)} \sup spec_{L^2(\mu^\alpha_\epsilon)}\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} J(\epsilon x)\tau_x\big(\psi^{(l)}_{m_l}\big) + \gamma\epsilon^{d-2}\mathcal{L}\} \le \\ &\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{C_0:|C_0|\le 1}\sup_{m\in[0,1]} \sup spec_{L^2(\mu^{\alpha^\epsilon}_{\Lambda_{x,l},m})}\{\epsilon^{-1}C_0\tau_x\psi^{(l)}_m + C(d)\gamma\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,l}}\mathcal{L}_b\}.\end{aligned}$$

Lemma 2.7 allows to apply perturbation theory, in particular proposition $A.13$. Therefore, if $\epsilon \le \epsilon_0(l,\gamma)$, then $\forall m \in [0,1]$, $\forall C_0 : |C_0| \le 1$

$$\sup spec_{L^2(\mu^{\alpha^\epsilon}_{\Lambda_{x,l},m})}\big\{\epsilon^{-1}C_0\tau_x\psi^{(l)}_m + C(d)\gamma\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,l}}\mathcal{L}_b\big\} \le 2\gamma^{-1}V_{\Lambda_{x,l}}(\psi_m, \mu^{\alpha^\epsilon}_{\Lambda_{x,l},m})$$

and the r.h.s. of (1.75) can be bounded by

$$2\gamma^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{m\in[0,1]} V_{\Lambda_{x,l}}(\psi_m, \mu^{\alpha^\epsilon}_{\Lambda_{x,l},m}).$$

By the previous estimate and (1.48) (which allows to pass from α^ϵ to α),

$$\limsup_{n\uparrow\infty,l\uparrow\infty,\epsilon\downarrow 0}\sup_{J\in C^*(\mathbb{T}^d)}\Omega_0 \le \limsup_{n\uparrow\infty,l\uparrow\infty,\epsilon\downarrow 0} 2\gamma^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{m\in[0,1]} V_{\Lambda_{x,l}}(\psi_m, \mu^{\alpha}_{\Lambda_{x,l},m}).$$

Therefore, by proposition 1.12, it's enough to find a function $g \in \mathbb{G}$ such that

$$\limsup_{n\uparrow\infty}\sup_{m\in[0,1]} V_m\Big(j_e + \mathcal{L}g - \sum_{e'>0} D_{e,e'}(m)\tau_y\frac{\psi^{e'}_{n,n}}{n}\Big) \le \beta.$$

The existence of such a function g follows from section 5.7.

1.17.2 Ω_1, Ω_2, Ω_3

In order to prove proposition 1.10 for Ω_1, Ω_2, Ω_3 we apply proposition 1.15.
For simplicity of notation we write $\frac{c}{\epsilon} = Q$, $\frac{a}{\epsilon} = R = 2R' + 1$ and $n = 2n' + 1$ and we set $A' \equiv n\mathbb{Z}^d \cap \Lambda_{0,R'}$. By (1.41) we can write

$$\epsilon^{-1}\mathrm{Av}_x J(\epsilon x)\psi^{(i)}_x = \sum_{e'>0}\epsilon^{-1}\mathrm{Av}_x\tau_x\frac{\psi^{e'}_{n,n}}{n}\cdot B^{(i)}_x \quad \forall i = 1,2,3$$

where

$$\begin{aligned}B^{(1)}_x &= \mathrm{Av}_{y:|y-x|\le l_1} J(\epsilon y)D_{e,e'}(m_{y,l}) - J(\epsilon x)D_{e,e'}(m_{x,l}),\\ B^{(2)}_x &= J(\epsilon x)[D_{e,e'}(m_{x,l}) - D_{e,e'}(m^{a/\epsilon}_x)],\\ B^{(3)}_x &= J(\epsilon x)D_{e,e'}(m^{a/\epsilon}_x) - \mathrm{Av}^{\frac{2Q}{n}-1}_{t=0}\mathrm{Av}_{a'\in A'} J(\epsilon x_{t,a'})D_{e,e'}(m^{a/\epsilon}_{x_{t,a'}})\end{aligned}$$

and $x_{t,a'} \equiv x + (Q - n' - 1 - tn)e + a'$.
We want to estimate $|\nabla_b B_x^{(i)}|$ for $i = 1, 2, 3$, $x \in \mathbb{T}_\epsilon^d$ and $b \in \Lambda_{x,n}$. We assume $n < \sqrt{l} < \frac{a}{\epsilon}$.
Then $\nabla_b B_x^{(i)} = 0$ for $i = 1, 2$.
We consider $|\nabla_b B_x^{(3)}|$. We observe that the set

$$\{x \in \mathbb{Z}^d \,:\, b \cap \Lambda_{x,\frac{a}{\epsilon}} \neq \emptyset \text{ and } b \cap \Lambda_{x,\frac{a}{\epsilon}}^c \neq \emptyset\}$$

consists of two boxes of dimension $d - 1$ and with sides of length $O\big(\frac{a}{\epsilon}\big)$.
We remark that given t, $\{x_{t,a'}\}_{a' \in A'}$ forms a finite lattice of step n. Therefore, for any given t,

$$|\{a' \in A' \,:\, \nabla_b D_{e,e'}(m^{a/\epsilon}_{x_{t,a'}}) \neq 0\}| \leq C\Big(\frac{a}{n\epsilon}\Big)^{d-1}$$

and then $|\nabla_b B_x^{(3)}| \leq C\frac{n\epsilon}{a}\mathrm{Osc}\Big(D, C(d)\frac{\epsilon^d}{a^d}\Big)$.
By proposition 1.15 it's enough to prove that, give $\gamma > 0$, for almost any disorder configuration α

$$\lim_{c\downarrow 0, a\downarrow 0, l\uparrow\infty, \epsilon\downarrow 0} \sup spec_{L^2(\mu_\epsilon^\alpha)}\Big\{\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\big(B_x^{(j)}\big)^2 + \gamma\epsilon^{d-2}\mathcal{L}\Big\} = 0 \qquad \forall j = 1, 2, 3. \tag{1.76}$$

Since Lipschitz continuous functions are dense in $C\big([0,1]\big)$ endowed of the uniform norm, we may assume that D is Lipschitz continuous.
Since J is Lipschitz continuous too, it's simple to verify

$$\begin{aligned}
&|B_x^{(1)}|^2 \leq C\epsilon^2 l^2 + C\mathrm{Av}_{y:|y-x|\leq l_1}|m_{y,l} - m_{x,l}|\\
&|B_x^{(2)}|^2 \leq C|m_{x,l} - m_x^{a/\epsilon}|\\
&|B_x^{(3)}|^2 \leq C\, c^2 + C\mathrm{Av}_{t=0}^{\frac{2Q}{n}-1}\mathrm{Av}_{a'\in A'}|m_x^{a/\epsilon} - m^{a/\epsilon}_{x_{t,a'}}|.
\end{aligned}$$

We introduce another parameter k with $c \downarrow 0, a \downarrow 0, l \uparrow \infty, k \uparrow \infty, \epsilon \downarrow 0$ in order to reduce the thesis to proposition 1.14.
We consider for example the case $j = 3$ (the other cases are similar).
By (1.64) and, for simplicity of notation, let us prove this case with $m_z^{a/\epsilon}$ substituted with $m_{z,\frac{a}{\epsilon}}$. Since

$$\begin{aligned}
\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,\frac{a}{\epsilon}} - m_{x_{t,a'},\frac{a}{\epsilon}}| &\leq \mathrm{Av}_{y\in\Lambda_{\frac{a}{\epsilon}}}\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x+y,k} - m_{x_{t,a'}+y,k}| + O(\epsilon k a^{-1})\\
&= \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,k} - m_{x_{t,a'},k}| + O(\epsilon k a^{-1}),
\end{aligned}$$

for any $\gamma > 0$

$$\begin{aligned}
&\sup spec_{L^2(\mu_\epsilon^\alpha)}\Big\{\mathrm{Av}_{t=0}^{\frac{2Q}{n}-1}\mathrm{Av}_{a'\in A'}\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,\frac{a}{\epsilon}} - m_{x_{t,a'},\frac{a}{\epsilon}}| + \gamma\epsilon^{d-2}\mathcal{L}\Big\} \leq\\
&\quad \mathrm{Av}_{t=0}^{\frac{2Q}{n}-1}\mathrm{Av}_{a'\in A'}\sup spec_{L^2(\mu_\epsilon^\alpha)}\Big\{\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,k} - m_{x_{t,a'},k}| + \gamma\epsilon^{d-2}\mathcal{L}\Big\} + O(\epsilon k a^{-1}).
\end{aligned} \tag{1.77}$$

By proposition 1.14 and the previous estimates we obtain (1.76).

1.17.3 Ω_4

Let us remember that we have assumed $n, \frac{a}{\epsilon}$ to be odd numbers such that $n|\frac{a}{\epsilon}$, $\frac{a}{\epsilon}|\frac{c}{\epsilon}$. Moreover, the average $\mathrm{Av}^*_{z,x}$ is defined by

$$\mathrm{Av}^*_{z,x} b_z \equiv \mathrm{Av}^{n,\frac{a}{\epsilon},\frac{c}{\epsilon}}_{z,x} b_z.$$

By (1.39)

$$\frac{m^{a/\epsilon}_{x-c/\epsilon} - m^{a/\epsilon}_{x+c/\epsilon}}{2c/\epsilon} = \mathrm{Av}^*_{z,x} \tau_z \frac{m^{(1)}_n - m^{(2)}_n}{n}$$

which concludes the proof.
We now consider more general cases. Suppose for example that $n, \frac{a}{\epsilon}$ are odd numbers such that $n|\frac{a}{\epsilon}$ but $\frac{a}{\epsilon}$ does not divide $\frac{c}{\epsilon}$. Let $R = \frac{a}{\epsilon}$, $Q = \frac{c}{\epsilon}$ and $Q' = R[Q/R]$, where $[\cdot]$ denotes the integer part.
Then by (1.39)

$$\tau_x \frac{m^R_{-Q'e} - m^R_{Q'e}}{2Q'} = \mathrm{Av}^{n,R,Q'}_{z,x} \tau_z \frac{m^{(1)}_n - m^{(2)}_n}{n}.$$

We assert that for any $\gamma > 0$ and for almost any disorder configuration α,

(1.78)

$$\lim_{c\downarrow 0, a\downarrow 0, \epsilon\downarrow 0} \sup spec_{L^2(\mu^\alpha_\epsilon)} \{\epsilon^{-1} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \tau_x | \frac{m^R_{-Q'e} - m^R_{Q'e}}{2Q'} - \frac{m^R_{-Qe} - m^R_{Qe}}{2Q} | + \gamma\epsilon^{d-2}\mathcal{L}\} = 0.$$

Since $|Q^{-1} - (Q')^{-1}| = O(a\epsilon c^{-2})$, it's enough to prove $\forall\gamma > 0$ and $\tilde{\forall}\alpha$

$$\lim_{a\downarrow 0, \epsilon\downarrow 0} \sup_{w:|w|\leq\frac{a}{\epsilon}} \sup spec_{L^2(\mu^\alpha_\epsilon)} \{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} |m^{a/\epsilon}_x - m^{a/\epsilon}_{x+w}| + \gamma\epsilon^{d-2}\mathcal{L}\} = 0.$$

We introduce the scale parameter l such that $a \downarrow 0, l \uparrow \infty, \epsilon \downarrow 0$. Since

$$m^{a/\epsilon}_x = \mathrm{Av}_{y\in\Lambda} m_{y,l} + O(l\epsilon a^{-1}),$$

where Λ is the cube centered at x of side a/ϵ, it's enough to prove that

$$\lim_{a\downarrow 0, l\uparrow\infty, \epsilon\downarrow 0} \sup_{|w|\leq\frac{a}{\epsilon}} \sup spec_{L^2(\mu^\alpha_\epsilon)} \{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} |m_{x,l} - m_{x+w,l}| + \gamma\epsilon^{d-2}\mathcal{L}\} = 0$$

which follows from the Two Blocks estimate in proposition 1.14.

Remark 1.16. *In conclusion, in the general case we can define an average, which reduces to* $\mathrm{Av}^*_{z,x}$ *if some divisibility criteria are satisfied, for which the analog of* Ω_4 *is depressed thanks to Two-Blocks estimate and section* 1.9.
The same average has to replace $\mathrm{Av}^*_{z,x}$ *also in* Ω_3, Ω_5. *By standard arguments based on Two-Blocks estimates this average works well.*

1.17.4 Ω_5

The thesis follows by theorem 4.1.

1.18 Proof of lemma 1.2

In this section we prove lemma 1.2.
It is simple to check that

$$\text{spatial–time average in (1.11)} = \sup_{H\in\mathcal{H}_c} H^\star_{c,a,\epsilon} \tag{1.79}$$

where $\mathcal{H}_c := \{H \text{ smooth on } [0,T]\times\mathbb{T}^d,\ \|H\|_\infty \le \frac{1}{c}\}$ and

$$H^\star_{c,a,\epsilon} := \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \Big(2H(s,\epsilon x)\Big[\frac{m^{a/\epsilon}_{x+\frac{c}{\epsilon}e}(s) - m^{a/\epsilon}_x(s)}{c}\Big] - H(s,\epsilon x)^2\Big)ds.$$

In what follows let H belong to $\mathcal{H}_c$. By the entropy inequality and the Feynman-Kac formula (see section 1.8), for any $\gamma > 0$,

(1.80)
$$\begin{aligned}\mathbb{E}_{P^\alpha_\epsilon}\big(H^\star_{2c,a,\epsilon}\big) &\le \frac{\kappa}{\gamma} - \gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T ds H(s,\epsilon x)^2 \\ &\quad + \int_0^T ds \sup_{L^2(\mu^\alpha_\epsilon)} spec\Big\{\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} 2H(s,\epsilon x)\Big[\frac{m^{a/\epsilon}_{x+\frac{2c}{\epsilon}e}(s) - m^{a/\epsilon}_x(s)}{2c/\epsilon}\Big] + \epsilon^{d-2}\mathcal{L}_\epsilon\Big\}.\end{aligned}$$

It is convenient to introduce a free scale parameter n, with $n\uparrow\infty$ after $a\downarrow 0$ and $\epsilon\downarrow 0$, and write the gradient of masses appearing in (1.80) as $-\mathrm{Av}^*_{z,x}\tau_z\big(\frac{\psi^e_{n,n}}{n} + \frac{\phi^e_{n,n}}{n}\big)$ (see chapter 4).
By the definition of $\mathrm{Av}^*_{z,x}$, the subadditivity of $\sup spec$ and theorem 4.2,

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\int_0^T ds \sup_{L^2(\mu^\alpha_\epsilon)} spec\Big\{-\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} 2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\phi^e_{n,n}}{n} + \epsilon^{d-2}\mathcal{L}_\epsilon\Big\} \le 0.$$

Let us consider, for fixed c,n,a,

$$\sup spec_{L^2(\mu^\alpha_\epsilon)}\Big\{-\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} 2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\psi^e_{n,n}}{n} + \epsilon^{d-2}\mathcal{L}_\epsilon\Big\}. \tag{1.81}$$

Let $\mathrm{Av}^\star_{z,x}$ be the average dual to $\mathrm{Av}^*_{z,x}$, then we can write

$$-\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} 2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\psi^e_{n,n}}{n} = \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} a_x\tau_x\frac{\psi^e_{n,n}}{n}$$

where $a_x := -\mathrm{Av}^\star_{z,x}2H(s,\epsilon z)$. Since $\mathrm{Av}^\star_{z,x}$ is translationally invariant w.r.t. x and H is regular, we can proceed as in section 1.7 and safely replace $\tau_x\frac{\psi^e_{n,n}}{n}$ by a local average $\mathrm{Av}_{|y-x|\le\ell_1}\tau_y\frac{\psi^e_{n,n}}{n}$, $\ell \gg n$, to get

$$(1.81) \le \sup spec_{L^2(\mu^\alpha_\epsilon)}\Big\{\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} a_x\mathrm{Av}_{|y-x|\le\ell_1}\tau_y\frac{\psi^e_{n,n}}{n} + \epsilon^{d-2}\mathcal{L}_\epsilon\Big\} + c(H)\gamma\epsilon\ell^2. \tag{1.82}$$

By the usual trick of localizing on boxes $\Lambda_{x,\ell}$ and proposition A.13, if ϵ is small enough then the first term in the r.h.s. of (1.82) is bounded from above by

$$C\,\gamma^2 \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\, a_x^2 \sup_{\nu\in\mathcal{M}(\Lambda_{x,l})} V_{x,\ell}\big(\frac{\psi^{e'}_{n,n}}{n};\nu\big)$$

which in turn is bounded from above by

$$C\,\gamma^2 \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\, a_x^2 \le C^*\,\gamma^2 \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\, H(s,\epsilon x)^2$$

for some suitable positive constant C^* (let us here for simplicity to prove the thesis assuming diffusive spectral gap: then (1.47) and equivalence of ensembles imply $V_{x,\ell}\big(\frac{\psi^{e'}_{n,n}}{n};\nu\big)\le C$). Let us now choose γ so small that $C^*\gamma^2-\gamma<0$. Then, by the previous estimates, if ϵ is small enough,

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \text{ r.h.s. of (1.80)} \le \frac{\kappa}{\gamma}+(C^*\gamma^2-\gamma)\int_0^T\int_{\mathbb{T}^d} H(s,\theta)^2 d\theta\, ds \le \frac{\kappa}{\gamma}. \tag{1.83}$$

In order to conclude the proof it is enough to observe that there exists a finite set $\mathcal{H}^*_c\subset\mathcal{H}_c$ depending on c such that

$$\sup_{H\in\mathcal{H}_c} H^\star_{c,a,\epsilon}\le 1+\sup_{H\in\mathcal{H}^*_c} H^\star_{c,a,\epsilon}$$

so that

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \mathbb{E}_{P^\alpha_\epsilon}\big(\sup_{H\in\mathcal{H}_c} H^\star_{c,a,\epsilon}\big)\le 1+\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \mathbb{E}_{P^\alpha_\epsilon}\big(\sup_{H\in\mathcal{H}^*_c} H^\star_{c,a,\epsilon}\big)\le 1+\frac{\kappa}{\gamma} \tag{1.84}$$

thus allowing to conclude the proof of (1.11).
Let us now sketch the proof of (1.12). Since $C^1([0,T]\times\mathbb{T}^d)$ has a countable base, by Beppo–Levi theorem it is enough to prove that there exists a constant c_0 such that, given $H_1,\dots,H_n$ in $C^1([0,T]\times\mathbb{T}^d)$, then

$$\int dQ^\alpha(m)\Big[\sup_{i=1,\dots,n}\int_0^T\int_{\mathbb{T}^d}\big(2\,m(s,\theta)\frac{\partial}{\partial\theta_e}H_i(s,\theta)-H_i(s,\theta)^2\big)d\theta\, ds\Big]\le c_0. \tag{1.85}$$

By the Lebesgue density theorem and the dominated convergence theorem, the l.h.s. of (1.85) is equal to $\lim_{a\downarrow 0}\mathbb{E}_Q\big(\Theta^{(a)}\big)$ where, for any $\nu\in D([0,T],\mathcal{M}_1)$,

$$\Theta^{(a)}(\nu):=\sup_{i=1,\dots,n}\int_0^T\int_{\mathbb{T}^d}\big(2\,\nu^{(a)}(s,\theta)\frac{\partial}{\partial\theta_e}H_i(s,\theta)-H_i^2(s,\theta)\big)d\theta\, ds, \tag{1.86}$$

with $\nu^{(a)}$ is as in (1.16).
Similarly to the proof of lemma 1.4, it can be proved that

$$\begin{aligned}
&\lim_{a\downarrow 0}\int dQ^\alpha(m)\big(\Theta^{(a)}(m)\big)\le\limsup_{a\downarrow 0,\epsilon\downarrow 0}\int dQ^\alpha_\epsilon(\nu)\big(\Theta^{(a)}(\nu)\big)=\\
&\limsup_{c\downarrow 0,a\downarrow 0,\epsilon\downarrow 0}\mathbb{E}_{P^\alpha_\epsilon}\Big(\sup_{i=1,\dots,n}\int_0^T \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big(2m_{x,\frac{a}{\epsilon}}(s,\epsilon x)\big[\frac{H_i(s,\epsilon x+ce)-H_i(s,\epsilon x)}{c}\big]\\
&\qquad\qquad -\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H_i^2(s,\epsilon x)\big)\Big)
\end{aligned}$$

By integrating by parts and observing that

$$\sup_{H\in\mathcal{H}_c} H^\star_{c,a,\epsilon} = \sup_{H\in C^1([0,T]\times\mathbb{T}^d)} H^\star_{c,a,\epsilon},$$

the thesis follows from (1.84).

Chapter 2

Equivalence of Ensembles and Spectral Gap

In this chapter we prove the equivalence between multi-canonical and multi-grand canonical ensembles. The proof is based on [5], with strong simplifications since here the grand canonical measures are product.

In section 2.2 we give a bound on some special covariances. In particular, propositions 2.11 and 2.13 are used in the proof of theorem 3.9.

In section 2.3 we obtain some bounds on the spectral gap by the recursive method developed in [6]. The final estimate deviates from the usually diffusive scaling by a term depending on the disorder, which can be controlled by large deviations estimates.
As observed by P.Caputo it can be proved, by a specific analysis, that the spectral gap on a cube of width L is larger than CL^{-2} uniformly in the disorder. The proof needs that the grand canonical Gibbs measure are product.
We remark that in other disordered models, as the dilute Ising model with Kawasaki dynamics under the percolation threshold, this uniform estimate is violated and only weaker estimates, as those described in section 2.3, are valid. Therefore, the method we develop for the study of the hydrodynamic behavior of the disordered system under consideration is more general.

2.1 Equivalence of ensembles

The main result of this section is theorem 2.4 which corresponds to the equivalence between multi-canonical and multi-grand canonical ensembles. Its statement and proof are based on [5] with strong simplifications since here the grand canonical measures are product.

Let $\Lambda \in \mathbb{F}$ and let $\Lambda = \sqcup_{i=1}^k \Lambda_i$ be a partition of Λ.
We set $\mathbf{N}_\Lambda = \{N_{\Lambda_i}\}_{i=1}^k$ and given $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ we define $\mu^{\alpha,\underline{\lambda}}$ as the multi-grand canonical measure on Λ with disorder configuration α and chemical potential λ_i on the region Λ_i for $i = 1, \dots, k$ (if λ_i is ∞ or $-\infty$ the measure is concentrated on configurations which are respectively equal to 1 and to 0 in Λ_i).

Remark In this section we do not use the notation Λ_l to indicate the cube centered at the origin of side $2l+1$.

Let α be a disorder configuration, $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible particle numbers in each atom Λ_i and $\rho_i = \frac{N_i}{|\Lambda_i|}$.
We define μ as the multi-grand canonical measure on Λ with disorder α such that, $\forall i = 1, \dots, k$, $\mu(N_{\Lambda_i}) = N_i$.
We write ν for the multi-canonical measure on Λ defined as $\mu^\alpha(\cdot|\mathbf{N}_\Lambda = \mathbf{N})$.

For any random variable X and any subset $W \subset \Lambda$ we set $\bar{X} \equiv X - \mu(X)$ and $\sigma_W^2 \equiv \mu(N_W; N_W)$. Given $s = 1, \dots, k$ we write

$$\sigma_s^2 \equiv \mu(N_{\Lambda_s}; N_{\Lambda s}) \qquad F(t_1, \dots, t_s) \equiv \mu\Big(e^{i\sum_{j=1}^s \frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}}\Big) \qquad T_s \equiv \pi\sigma_s.$$

The equivalence of ensembles is expressed by an estimate of $|\nu(f) - \mu(f)|$ where f is a local function with support in Λ. The basic idea is to rewrite the difference $\nu(f) - \mu(f)$ by Fourier transforms as follows.
Let $\chi_j \equiv \mathbb{I}_{N_{\Lambda_j} = N_j}$ $\forall j = 1, \dots, k$. Then

$$\nu(f) - \mu(f) = \frac{\mu\big(\prod_j \chi_j; f\big)}{\mu\big(\prod_j \chi_j\big)}. \tag{2.1}$$

By Fourier transform, $\int_{-T_j}^{T_j} dt_j\, e^{i\frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}} = 2T_j \chi_j$ and

$$\nu(f) - \mu(f) = \frac{\int_{-T_1}^{T_1} dt_1 \dots \int_{-T_k}^{T_k} dt_k \mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}}; f\big)}{\int_{-T_1}^{T_1} dt_1 \dots \int_{-T_k}^{T_k} dt_k F(t_1, \dots, t_k)}. \tag{2.2}$$

Lemma 2.1. *(Gaussian upper bound on characteristic functions).*
There exists a universal constant a such that for any disorder configuration α, for any $\Lambda = \sqcup_{i=1}^k \Lambda_i$ and $\mathbf{N} = \{N_i\}_{i=1}^k$, for any $s = 1, \dots, k$

$$|F(t_1, \dots, t_s)| \le e^{-a\sum_{j=1}^s t_j^2} \quad \textit{if } t_j \in [-T_j, T_j] \textit{ for } j = 1, \dots, s.$$

Proof. Since $x \le \exp\big(\frac{1}{2}(x^2 - 1)\big)$ with $0 \le x \le 1$, there exists $a > 0$ such that $\forall t \in [-\pi, \pi]$

$$|\mu(e^{it\eta_x})| \le e^{\frac{1}{2}\left(|\mu(e^{it\eta_x})|^2 - 1\right)} = e^{-\mu(\eta_x)(1-\mu(\eta_x))(1-\cos t)} \le e^{-a\mu(\eta_x;\eta_x)t^2}.$$

In particular, if $t_j \in [-T_j, T_j]$ $\forall j = 1, \dots, s$,

$$|F(t_1, \dots, t_s)| = \prod_{j=1}^s \prod_{x \in \Lambda_j} |\mu(e^{i\frac{t_j}{\sigma j}\eta_x})| \le \prod_{j=1}^s \prod_{x \in \Lambda_j} e^{-a\mu(\eta_x;\eta_x)\frac{t_j^2}{\sigma j^2}} = e^{-a\sum_{j=1}^s t_j^2}.$$

$\square$

Lemma 2.2. *(A lower bound on the integral of characteristic functions)*
$\exists A_0 = A_0(k)$ such that for any disorder configuration α, for any $\Lambda = \sqcup_{i=1}^k \Lambda_i$ and $\mathbf{N} = \{N_i\}_{i=1}^k$, for any $A \geq A_0$

$$\int_{-A}^{A} dt_1 \dots \int_{-A}^{A} dt_k \; Re\; F(t_1, \dots, t_k) \geq 1 \quad \text{if } \sigma_j \geq A^{k+4}.$$

Proof. Let $(t_1, \dots, t_k) \in [-A, A]^k$ and $\sigma_j \geq A^{k+4}$. We set $\bar{\eta}_x \equiv \eta_x - \mu(\eta_x)$ and $\gamma \equiv k + 4$. By Taylor expansion

$$\mu(e^{i\frac{t_j}{\sigma_j}\bar{\eta}_x}) = 1 - \frac{t_j^2}{2\sigma_j^2}\mu(\eta_x;\eta_x) + R_x \quad \text{with} \quad |R_x| \leq \frac{|t_j|^3}{\sigma_j^3}\mu(|\bar{\eta}_x|^3) \leq \frac{|t_j|^3}{\sigma_j^3}\mu(\eta_x;\eta_x).$$

Since $F(t_1, \dots, t_k) = \prod_{j=1}^k \prod_{x\in\Lambda_j} \mu(e^{i\frac{t_j}{\sigma_j}\bar{\eta}_x})$,

$$|\; F(t_1, \dots, t_k) - \prod_{j=1}^{k} \prod_{x\in\Lambda_j} \big(1 - \frac{t_j^2}{2\sigma_j^2}\mu(\eta_x;\eta_x)\big) \;| \leq \sum_{j=1}^{k} \frac{|t_j|^3}{\sigma_j} \leq kA^{3-\gamma}.$$

Since $1 - \frac{z}{2} \geq e^{-z}$ with $z \in [0, z_0]$, if A is enough large $1 - \frac{t_j^2}{2\sigma_j^2}\mu(\eta_x;\eta_x) \geq e^{-\frac{t_j^2}{\sigma_j^2}\mu(\eta_x;\eta_x)}$ and

$$\int_{-A}^{A} dt_1 \dots \int_{-A}^{A} dt_k \; Re\; F(t_1, \dots, t_k) \geq \big(\int_{-A}^{A} e^{-t^2}\big)^k - k2^k A^{k+3-\gamma} = \big(\int_{-A}^{A} e^{-t^2}\big)^k - k2^k A^{-1}.$$

The thesis follows from the above estimate. □

Proposition 2.3. *There are constants C_0 and $v_0(k)$ such that for any disorder configuration α, for any $\Lambda = \sqcup_{i=1}^k \Lambda_i$ with $|\Lambda_i| \geq v_0$ and for any $\mathbf{N} = \{N_i\}_i$*

$$\int_{-T_1}^{T_1} dt_1 \dots \int_{-T_k}^{T_k} dt_k F(t_1, \dots, t_k) \geq C_0^k.$$

Proof. Since the measure is product, it's enough to consider the case $k = 1$. In this case we write simply T, σ, N in place of T_1, σ_1 and N_1. Let a and A_0 as in lemmas 2.1 and 2.2.

We fix $A \geq A_0$ such that $\int_A^\infty dt\, e^{-at^2} \leq 1/4$.
We consider two cases:
a) $\sigma \geq A^5$,
b) $\sigma < A^5$.

We first consider case *a*). We write

$$\int_{-T}^{T} dt\; F(t) = \int_{-A}^{A} dt\; Re\; F(t) + \int_{A\leq|t|\leq T} dt\; Re\; F(t) \tag{2.3}$$

By lemmas 2.1 and 2.2 and the choice of A, (2.3) is larger than 1/2.

We consider case b).

By duality, it's not restrictive to assume $\rho \le 1/2$. By lemma $A.2$, condition b) corresponds to $N < CA^{10}$.
Since $\int_{-T}^{T} dt\ F(t) > C\mu(N_\Lambda = N)$ it's enough to inferiorly bound $\mu(N_\Lambda = N)$.
We can write

$$\mu(N_\Lambda = N) = \sum_{V \subset \Lambda, |V| = N} \prod_{x \in V} \mu(\eta_x) \prod_{x \in \Lambda \setminus V} (1 - \mu(\eta_x)). \tag{2.4}$$

By lemma $A.2$, $\exists c : 0 < c < 1$ such that $\forall x \in \Lambda\ c\rho \le \mu(\eta_x) \le c^{-1}\rho$. Since $N \le C(A)$ if $|\Lambda| \ge v_0(A, c)$ then $c^{-1}\rho \le 2^{-1}$. In this case, by (2.4),

$$\mu(N_\Lambda = N) \ge \binom{|\Lambda|}{N} (c\rho)^N \Big(1 - \frac{\rho}{c}\Big)^{|\Lambda| - N}. \tag{2.5}$$

We can write

$$\Big(1 - \frac{\rho}{c}\Big)^{|\Lambda| - N} = \Big(1 - \frac{\rho}{c}\Big)^{\frac{c}{\rho} \cdot \frac{N}{c}(1-\rho)}. \tag{2.6}$$

If $|\Lambda| \ge v_0'(A, c)$ then $\rho/c \ll 1$ and $(1 - \frac{\rho}{c}\Big)^{\frac{c}{\rho}} \ge 3^{-1}$. Therefore (2.6) is inferiorly bounded by $C(N) > 0$.
Moreover, since $\rho \le 1/2$ it's simple to check that $\binom{|\Lambda|}{N} \ge C(N)|\Lambda|^N$.
Therefore, (2.5) is inferiorly bounded by a positive constant depending on N. Since, given A, N can assume a finite set of values, the thesis is proved.

$\square$

Theorem 2.4. *(Equivalence of ensembles)*
Let $0 < \epsilon < 1$ and $0 < \delta < 1$ be given.
Let α be a disorder configuration, $\Lambda = \sqcup_{i=1}^{k} \Lambda_i$ and $\mathbf{N} = \{N_i\}_{i=1}^{k}$. We assume that $\forall i = 1, \dots, k\ \frac{|\Lambda_i|}{|\Lambda|} \ge \delta$. Let f be a local function with support $\Delta_f \subset \Lambda$ and $|\Delta_f| \le |\Lambda|^{1-\epsilon}$.

If $|\Lambda| \ge v(\epsilon, \delta, k)$ then

$$|\nu(f) - \mu(f)| \le C(\delta, k)\|f\|_\infty \frac{|\Delta_f|}{|\Lambda|}.$$

Proof. We use Fourier representation (2.2). By proposition 2.3, if $|\Lambda_i| \ge v_0(k)$ then the denominator of (2.2) is larger than C_0^k.

In order to estimate the numerator we apply lemma 2.5 (to which we refer for notation) with $\Delta' \equiv \Delta_f$. We write $f = F + \sum_{j=1}^{k} \alpha_j N_{\Delta_f \cap \Lambda_j}$.
We observe that F is a sort of projection of f orthogonal to each N_{Λ_j} since $\Delta_F \subset \Delta f$ and $\mu(F; N_{\Lambda_j}) = 0$.
If $|\Lambda|$ is enough large, by lemma 2.5 and (2.16),

$$
\begin{aligned}
|\mu(f)-\nu(f)| &\le |\mu(F)-\nu(F)| + \sum_{j=1}^{k} |\alpha_j|\, |\mu(N_{\Delta_f\cap\Lambda_j}) - \nu(N_{\Delta_f\cap\Lambda_j})| \\
&\le C(\delta,k)\|f\|_\infty \frac{|\Delta_f|}{|\Lambda|} + C\|f\|_\infty |\Delta_f| \sup_{y\in\Lambda} |\mu(\eta_y)-\nu(\eta_y)|.
\end{aligned}
\tag{2.7}
$$

We set $a_{xy} \equiv \frac{\mu(\eta_x;\eta_x)}{\mu(\eta_y;\eta_y)}$ and we fix $i : 1 \le i \le k$.

By the lemma 2.5, $\forall x, y \in \Lambda_i$,

$$\mu(\eta_x) - \nu(\eta_x) - a_{xy}(\mu(\eta_y) - \nu(\eta_y)) = \epsilon_{xy}$$

with $|\epsilon_{xy}| \le C|\Lambda_i|^{-1}$. Since

$$\sum_{x\in\Lambda_i} \big(\mu(\eta_x) - \nu(\eta_x)\big) = 0$$

we can conclude that for any $y \in \Lambda_i$

$$|\mu(\eta_y) - \nu(\eta_y)| \le \frac{\sum_{x\in\Lambda_i} |\epsilon_{xy}|}{\sum_{x\in\Lambda_i} a_{xy}} \le \frac{C}{|\Lambda_i|} \le \frac{C(\delta)}{|\Lambda|}$$

The above estimates and (2.7) imply the thesis.

□

Lemma 2.5. *In the same setting of theorem 2.4, given $\Delta' \subset \Lambda$ we define $A_j \equiv \Delta_f \cap \Lambda_j$, $A'_j \equiv \Delta' \cap \Lambda_j$ and $\Delta_j \equiv A_j \cup A'_j$.*
We assume that $|A_j| = |A'_j|$ for any $j = 1, \dots, k$.
If $|\Lambda| \ge v(\epsilon, \delta, k)$ then

$$\int_{-T_1}^{T_1} dt_1 \dots \int_{-T_k}^{T_k} dt_k |\mu\big(e^{i\sum_{j=1}^{k} \frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}}; F\big)| \le C(\delta,k)\|f\|_\infty \frac{|\Delta_f|}{|\Lambda|} \tag{2.8}$$

where

$$F \equiv f - \sum_{j=1}^{k} \alpha_j N_{A'_j} \qquad \textit{and} \qquad \alpha_j \equiv \begin{cases} \frac{\mu(f;N_{\Delta_j})}{\mu(N_{A'_j};N_{\Delta_j})} & \textit{if } A'_j \ne \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

In particular,

$$|\mu(F) - \nu(F)| \le C(\delta,k)\|f\|_\infty \frac{|\Delta_f|}{|\Lambda|}. \tag{2.9}$$

Proof. We observe that, by the choice of α_j, F has support in $\Delta' \cup \Delta_f$ and $\mu(F; N_{\Delta_j}) = 0$ for each j.
By the Fourier representation (2.2) and proposition 2.3, (2.9) follows from (2.8).

We set $a_m \equiv \min(m, 1-m)$ for any density m and $a_j \equiv a_{\rho_j}$.

In order to prove (2.8) we set $V_j \equiv \Lambda_j \setminus \Delta_j$ and $t_j \in [-T_j, T_j]$. Then

$$\mu\big(e^{i\sum_j \frac{t_j}{\sigma_j}\bar N_{\Lambda_j}}; F\big) = \Big(\prod_j \mu\big(e^{i\frac{t_j}{\sigma_j}\bar N_{V_j}}\big)\Big)\, \mu\big(e^{i\sum_j \frac{t_j}{\sigma_j}\bar N_{\Delta_j}}; F\big). \tag{2.10}$$

We observe that

$$\Big|1 - \frac{|V_j|}{|\Lambda_j|}\Big| \le \frac{|\Delta_f|}{|\Lambda_j|} \le \frac{|\Lambda|^{-\epsilon}}{\delta}.$$

Therefore by lemmas 2.1 and $A.2$ if $|\Lambda| \ge v(\epsilon, \delta)$ then

$$|\prod_j \mu(e^{i\frac{t_j}{\sigma_j}\bar N_{V_j}})| \le \prod_j e^{-at_j^2\frac{\sigma^2_{V_j}}{\sigma^2_{\Lambda_j}}} \le e^{-\frac{a}{2}\sum_j t_j^2}. \tag{2.11}$$

By Taylor expansion,

$$\mu(e^{i\sum_j \frac{t_j}{\sigma_j}\bar N_{\Delta_j}}; F) = \mu\Big(\prod_j (1 + i\frac{t_j}{\sigma_j}\bar N_{\Delta_j} - \frac{t_j^2}{2\sigma_j^2}\bar N^2_{\Delta_j} e^{iG_j}); F\Big) \tag{2.12}$$

where G_j is a real local function with support in Δ_j.
We observe that $\mu(N_{\Delta_j}; F) = 0$ by the choice of α_j.
We expand the product in the second member of (2.12) and we integrate on $t_1, \dots, t_k$. By (2.11) the first member in (2.8) is bounded by a finite sum of terms of the following form:

$$\begin{aligned} &a) \quad |\mu\Big(\frac{\bar N^2_{\Delta_j}}{\sigma_j^2} e^{iG_j}; \bar F\Big)| \\ &b) \quad |\mu\Big(\frac{\bar N_{\Delta_j}}{\sigma_j}\frac{\bar N_{\Delta_s}}{\sigma_s}; F\Big)| \qquad j \neq s \\ &c) \quad \mu\Big(\frac{|\bar N^{n_1}_{\Delta_{j_1}}|}{\sigma^{n_1}_{j_1}} \cdots \frac{|\bar N^{n_s}_{\Delta_{j_s}}|}{\sigma^{n_s}_{j_s}} |\bar F|\Big) \quad n_1 + \dots + n_s \ge 3, \;\; j_a \neq j_b \text{ if } a \neq b, \quad n_a \in \{1, 2\}. \end{aligned} \tag{2.13}$$

We observe that if $j = 1$ then we have to estimate only terms of the form $|\mu\Big(\frac{\bar N^2_{\Delta_j}}{\sigma_j^2} e^{iG_j}; \bar F\Big)|$ and the proof is much shorter.
We assert that each of the above terms can be bounded by the second member of (2.8).
First we observe that by lemma $A.2$

$$Ca_j|\Lambda_j| \le \sigma_j^2 \le C'a_j|\Lambda_j| \qquad Ca_j \le \mu(N_{\Delta_j})\big(1 - \mu(N_{\Delta_j})\big) \le C'a_j \tag{2.14}$$

Case a).

It's enough to estimate $\mu\Big(\frac{\bar N^2_{\Delta_j}}{\sigma_j^2}|\bar f|\Big)|$ and $\mu\Big(\frac{\bar N^2_{\Delta_j}}{\sigma_j^2}|\bar N_{A'_j}|\Big)|\cdot|\alpha_j|$.
For the former it's enough to apply lemma $A.2$ in order to bound it by $C\|f\|_\infty|\Delta_j||\Lambda_j|^{-1}$ which is bounded by $C(\delta)\|f\|_\infty|\Delta_f||\Lambda|^{-1}$.
In order to estimate the latter we first estimate α_j in the case that $A_j\neq\emptyset$. We set $\mathcal{F}_j\equiv\sigma(\eta_x\,:\,x\notin\Delta_j)$.
By lemma $A.2$ and property (2.14),

$$|\mu(f;N_{\Delta_j})|=|\mu\big(\,\mu(f;N_{\Delta_j}\,|\,\mathcal{F}_j)\,\big)|\le C\|f\|_\infty\min\Big(|\Delta_j|a_j,\sqrt{|\Delta_j|a_j}\Big). \tag{2.15}$$

Since $\mu(N_{A'_j};N_{\Delta_j})\ge Ca_j|\Delta_j|$,

$$|\alpha_j|\le C\|f\|_\infty\min(1,\frac{1}{\sqrt{a_j|\Delta_j|}}). \tag{2.16}$$

Since $\mu(\bar N^4_{\Delta_j})\le Ca_j|\Delta_j|+Ca_j^2|\Delta_j|^2$,

$$\mu(\bar N^2_{\Delta_j}|\bar N_{A'_j}|)\le\mu\big(\bar N^4_{\Delta_j}\big)^{\frac12}\mu\big(\bar N^2_{A'_j}\big)^{\frac12}\le C\max(a_j|\Delta_j|,\sqrt{a_j|\Delta_j|})\sqrt{a_j|\Delta_j|}. \tag{2.17}$$

By (2.14), (2.16) and (2.17)

$$\mu\Big(\frac{\bar N^2_{\Delta_j}}{\sigma_j^2}|\bar N_{A'_j}|\Big)|\cdot|\alpha_j|\le C\|f\|_\infty\frac{|\Delta_j|}{|\Lambda_j|}\le C(\delta)\|f\|_\infty\frac{|\Delta_f|}{|\Lambda|}.$$

Case b).

By lemma $(A.2)$, (2.14) and Schwarz inequality,

$$|\mu\Big(\frac{\bar N_{\Delta_j}}{\sigma_j}\frac{\bar N_{\Delta_s}}{\sigma_s};f\Big)|\le C\|f\|_\infty\frac{\sigma_{\Delta_j}}{\sigma_j}\frac{\sigma_{\Delta_s}}{\sigma_s}\le C(k,\delta)\|f\|_\infty\frac{|\Delta_f|}{|\Lambda|}.$$

Moreover, since $j\neq s$, for any $r=1,\dots k$, $\mu(\bar N_{\Delta_j}\bar N_{\Delta_s};N_{\Delta_r})=0$.

Case c).

By (2.16),

$$\|\bar F\|_\infty\le\|f\|_\infty+C\|f\|_\infty\sum_j|\bar N_{A'_j}|\le C'\|f\|_\infty|\Delta_f|.$$

Since the measure is product, the term of case c) in (2.13) can be bounded by

$$\|F\|_\infty\prod_{r=1}^s\mu\Big(\frac{|\bar N^{n_r}_{\Delta_{j_r}}|}{\sigma^{n_r}_{j_r}}\Big)\le\|F\|_\infty\prod_{r=1}^s\frac{|\Delta_j|^{n_r/2}}{|\Lambda_j|^{n_r/2}}\le C(k,\delta)\|f\|_\infty|\Delta_f|\Big(\frac{|\Delta_f|}{|\Lambda|}\Big)^{\frac32}. \tag{2.18}$$

As $|\Delta_f|\le|\Lambda|^{1-\epsilon}$, if $|\Lambda|\ge v(\epsilon)$ then $|\Delta_f|^{3/2}|\Lambda|^{-3/2}\le|\Lambda|^{-1}$ and by (2.18) we have the thesis.

□

We conclude the section with a technical tool.

Lemma 2.6. *Let $\delta \in (0,1)$, $\Lambda = \sqcup_{i=1}^{k} \Lambda_i$ and f be a function with support $\Delta_f \subset \Lambda$ such that $|\Lambda_i \setminus \Delta_f| \geq \delta |\Lambda_i|$ for any $i = 1, \cdots, k$.*
Then, there are constants $C = C(k, \delta)$ and $v_0(k)$ such that if $|\Lambda_i| \geq v_0$ $\forall i = 1, \ldots, k$ then

$$\nu(|f|) \leq C\mu(|f|). \tag{2.19}$$

In particular,

$$\mathrm{Var}_{\nu}(f) \leq C \mathrm{Var}_{\mu}(f). \tag{2.20}$$

Proof. (2.20) is an easy consequence of (2.19). In order to prove (2.19), let $f \geq 0$ and $V_j \equiv \Lambda_j \setminus \Delta_f$. By Fourier transform

$$\nu(f) = \frac{\int_{-T_1}^{T_1} dt_1 \ldots \int_{-T_k}^{T_k} dt_k \mu\big(e^{i \sum_{j=1}^{k} \frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}} f\big)}{\int_{-T_1}^{T_1} dt_1 \ldots \int_{-T_k}^{T_k} dt_k F(t_1, \ldots, t_k)}. \tag{2.21}$$

By proposition 2.3, if $\Lambda \geq v_0(k)$ then the denominator in (2.21) is greater than C_0^k.
In order to bound the numerator, we observe that by lemma 2.1, lemma $A.2$ and the hypothesis $|\Lambda_i \setminus \Delta_f| \geq \delta |\Lambda_i|$,

$$|\mu\big(e^{i \sum_{j=1}^{k} \frac{t_j}{\sigma_j} \bar{N}_{\Lambda_j}} f\big)| \leq \mu(f) \prod_{j=1}^{k} |\mu\big(e^{i \frac{t_j}{\sigma_j} \bar{N}_{V_j}}\big)| \leq \mu(f) \prod_{j=1}^{k} e^{-a t_j^2 \frac{\sigma_{V_j}^2}{\sigma_j^2}} \leq \mu(f) \prod_{j=1}^{k} e^{-a t_j^2 C \delta}.$$

$\square$

2.2 A key bound on special covariances

In this section we give some bounds of special covariances by applying the main ideas of [6], [7]. These special covariances will play an important role in chapter 3 and in derivation of estimates on the spectral gap by means of the recursive procedure of [6], [7].
We fix two functions $h = h(\alpha, \eta)$ and $h' = h'(\alpha, \eta)$ depending only on α_0, η_0 and for any $z \in \mathbb{Z}^d$ we set $h_z \equiv \tau_z h$ and $h'_z = \tau_z h'$.

Given disjoint boxes V and W and a canonical Gibbs measure ν on $V \cup W$, with disorder configuration α and conditioned to have N_V particles in V and N_W particles in W, by proposition 2.11 we give an estimate of the covariance

$$\nu(Av_{v \in V} h_v Av_{w \in W} h'_w; g)$$

for a generic function g.

The estimate of the above covariance contains a term depending on α of which we give in proposition 2.13 some large deviations estimates thanks to the ergodicity of the disorder.

A fundamental tool is the following simple observation. Given a generic probability measure μ, a subalgebra $\mathcal{F}$ of measurable set and two functions $f, g \in L^2(\mu)$, then

$$\mu(f;g) \equiv \mu\big(\mu(f;g\,|\,\mathcal{F})\big) + \mu\big(\mu(f\,|\,\mathcal{F});\mu(g\,|\,\mathcal{F})\big). \tag{2.22}$$

In particular

$$\mathrm{Var}(f) \equiv \mu\big(\mathrm{Var}(f\,|\,\mathcal{F})\big) + \mathrm{Var}\big(\mu(f\,|\,\mathcal{F})\big). \tag{2.23}$$

Given a box let $l_{\min}$ and $l_{\max}$ be respectively the length of its shortest side and the length of its longest side. Then for any integer L we define

$$R_L \equiv \{A \subset \mathbb{Z}^d \;:\; A \text{ is a box}, \quad L \le l_{\min} \le l_{\max} \le 1000L\}. \tag{2.24}$$

In the following lemma we use the notation of section 2.3. The proof can be found in [8].

Lemma 2.7. $\exists \kappa(B), C(B) > 0$ *such that for any integer* L*, any* $\Lambda \in R_L$ *and any disorder configuration* α

$$\min_{N\in[0,|\Lambda|]} gap\ \mathcal{L}^{\alpha}_{\Lambda,N} \ge Ce^{-\kappa L^{d-1}}. \tag{2.25}$$

Lemma 2.8. *There exist constant* v_0 *and* C_0 *having the following property.*

Let ν *be a generic canonical measure on* $V \in \mathbb{F}$ *with particles number* N *and disorder* α. *We write* $\rho \equiv \frac{N}{|V|}$.
If $|V| \ge v_0$ *then, for any function* g *with support in* V*,*

$$\nu(Av_{v\in V}h_v;g)^2 \le C\|h\|^2_\infty \min(\rho, 1-\rho)|V|^{-1}Var_\nu(g).$$

Proof. We set $\mu \equiv \mu^{\alpha,\lambda_V(\alpha,N)}$. By Schwarz inequality and lemma 2.6,

$$\nu(Av_{v\in V}h_v;g)^2 \le |V|^{-2}Var_\mu(\sum_{v\in V}h_v)Var_\nu(g).$$

It's enough to observe that for $s \in \{0,1\}$

$$Var_\mu(h_x) \le \mu([h_x - h_x(s)]^2) \le \|h\|^2_\infty\mu\big((\delta_{0,s}\eta_x + \delta_{1,s}(1-\eta_x)\big).$$

$\square$

Let $L \geq l$ be integers and $V \in R_L$. We fix an integer $N \in [0, |V|]$. We set $m_V \equiv N/|V|$ and

$$\mu \equiv \mu^{\alpha, \lambda_V(\alpha, N_V)} \qquad \nu \equiv \mu^{\alpha}_{V, N_V}.$$

We subdivide V as follows

$$V = \sqcup_{i \in I} Q_i \quad \text{with} \quad Q_i \in R_l \quad \forall i \in I \tag{2.26}$$

and we define $h_i \equiv \sum_{v \in Q_i} h_v$ and $N_i \equiv N_{Q_i}$. Then we set

$$G^{V,\alpha}_{l,L,N} \equiv Av_{i \in I} \Delta_i^2, \tag{2.27}$$

where $\Delta_i \equiv \frac{a_i}{b_i} - \frac{A}{B}$ with

$$\begin{aligned} a_i &= \mu^{\alpha, \lambda_V(\alpha, N)}(h_i; N_i) \qquad & A &= \mathbb{E}\big[\mu^{\alpha, \lambda_0(m_V)}(h_0; \eta_0)\big] \\ b_i &= \mu^{\alpha, \lambda_V(\alpha, N)}(N_i; N_i) \qquad & B &= \mathbb{E}\big[\mu^{\alpha, \lambda_0(m_V)}(\eta_0; \eta_0)\big]. \end{aligned}$$

When there is no ambiguity, we often write $G^{V,\alpha}$ in place of $G^{V,\alpha}_{l,L,N}$.

We fix a canonical measure ν on V with density m_V and disorder configuration α.

The following proposition corresponds to the Lu-Yau's Two Blocks Estimate, see [20], [6], [7].

Proposition 2.9. *Let $0 < \delta < \frac{1}{2}$. If $L \geq l \geq l_0(\delta)$ then for α, ν, V and $\{Q_i\}_{i \in I}$ as above, given a local function g with support in V,*

$$\nu(Av_{v \in V} h_v; g)^2 \leq \frac{C(l)}{|V|} \mathcal{D}_\nu(g) + \frac{C\delta}{|V|} Var_\nu(g) + \frac{C}{|V|} F(m_V) Var_\nu(g) G^{V,\alpha}_{l,L,N}, \tag{2.28}$$

where $F(m) \equiv \mathbb{I}_{\{m \in [\delta, 1-\delta]\}}$.

Proof. For simplicity of notation we assume V to be a cube with side L. We assume that l_0 is large enough to apply the results of section 2.1 when necessary.
We fix a disorder α and when there is no ambiguity we omit it from standard notation.

We write $H \equiv Av_{v \in V} h_v$ and given a function f we set

$$\tilde{\nabla}_x f(\eta) \equiv (1 - \eta_x) \nabla_x f(\eta) - \eta_x \nabla_x f(\eta).$$

Extreme density

If $m_V \in [0, \delta) \cup (1 - \delta, 1]$ then the thesis follows from lemma 2.8.

Non extreme density

In what follows we assume $m_V \in [\delta, 1-\delta]$.
Let $\mathcal{F} \equiv \sigma\big(N_{Q_i} : i \in I\big)$ and $\lambda_i(\alpha, S) \equiv \lambda_{Q_i}(\alpha, S)$ for any $S \in [0, |Q_i|]$.
The proof is based on the fundamental estimate

$$\nu(H;g)^2 \le 2\nu\big(\,\nu(H;g|\mathcal{F})\,\big)^2 + 2\nu\big(\,\nu(H|\mathcal{F});g\,\big)^2. \tag{2.29}$$

• We consider $\nu\big(\nu(H;g|\mathcal{F})\big)^2$.

By observing that $\nu(\cdot|\mathcal{F})(\eta)$ is the product measure $\prod_{i\in I}\mu^{\alpha}_{Q_i,N_i(\eta)}$ and applying lemma 2.7,

$$\nu\big(Var_\nu(g|\mathcal{F})\big) \le C(l)\nu\big(\,\mathcal{D}_\nu(g|\mathcal{F})\,\big) = C(l)\mathcal{D}_\nu(g).$$

Therefore, by Schwarz inequality and by lemma 2.6

$$\nu\big(\nu(H;g|\mathcal{F})\big)^2 \le Var_\mu(H)\nu\big(Var_\nu(g|\mathcal{F})\big) \le C(l)L^{-d}\mathcal{D}_\nu(g). \tag{2.30}$$

• The rest of the proof is devoted to the estimate of $\nu\big(\nu(H|\mathcal{F});g\big)^2$.

For any $\gamma \in \mathbb{R}$

$$\nu\big(\nu(H|\mathcal{F});g\big)^2 \le Var_\nu(\nu(H|\mathcal{F}))Var_\nu(g) = L^{-2d}Var_\nu\big(\sum_{i\in I}\nu(h_i-\gamma N_i|N_i)\big)Var_\nu(g).$$

We define

$$G_i^\gamma(\eta) \equiv \nu(h_i-\gamma N_i|N_i) = \mu^{\alpha}_{Q_i,N_i(\eta)}(h_i-\gamma N_i) \;\text{ and }\; \xi_i^\gamma(\eta) \equiv \mu^{\alpha,\lambda_i(\alpha,N_i(\eta))}(h_i-\gamma N_i).$$

By lemma 2.6,

$$Var_\nu\big(\sum_{i\in I}G_i^\gamma\big) \le C\sum_{i\in I}Var_\mu(G_i^\gamma).$$

By the equivalence of ensembles, $\|G_i^\gamma - \xi_i^\gamma\|_\infty \le C$.
By collecting the previous estimates we obtain

$$\nu\big(\nu(H|\mathcal{F});g\big)^2 \le CL^{-d}l^{-d}\mathrm{Var}_\nu(g) + CL^{-2d}\mathrm{Var}_\nu(g)\sum_{i\in I}\mathrm{Var}_\mu(\xi_i^\gamma) \tag{2.31}$$

and if l is enough large $L^{-d}l^{-d}\mathrm{Var}_\nu(g)$ is bounded by $C|\Lambda|^{-1}\delta\mathrm{Var}_\nu(g)$.

By Poincaré inequality for Glauber dynamics,

$$\sum_{i\in I} Var_\mu(\xi_i^\gamma) \le C \sum_{i\in I}\sum_{y\in Q_i} \mu((\tilde\nabla_y \xi_i^\gamma)^2). \tag{2.32}$$

Since $\partial_S \lambda_i(S) = [\,\mu^{\lambda_i(S)}(N_i;N_i)\,]^{-1}$, we can write

$$\tilde\nabla_y \xi_i^\gamma(\eta) = \int_{N_i(\eta)+\eta_y-1}^{N_i(\eta)+\eta_y} \Big(\frac{\mu^{\lambda_i(S)}(h_i;N_i)}{\mu^{\lambda_i(S)}(N_i;N_i)} - \gamma\Big) dS. \tag{2.33}$$

By choosing $\gamma = A/B$, the integrand in (2.33) can be written as $R_i(S) - \Delta_i$ where

$$R_i(S) \equiv \frac{\mu^{\lambda_i(S)}(h_i;N_i)}{\mu^{\lambda_i(S)}(N_i;N_i)} - \frac{\mu(h_i;N_i)}{\mu(N_i;N_i)}, \tag{2.34}$$

and then $|\tilde\nabla_y \xi_i^\gamma(\eta)| \le \Delta_i + \int_{N_i(\eta)-1}^{N_i(\eta)+1} |R_i(S)| dS$.

Let $n_i = \mu(N_i)$. We observe that $\lambda_i(n_i) = \lambda_V(\alpha, N)$. By writing the denominator and the numerator of the first addendum in (2.34) as integrals of their derivatives in S, we obtain the following representation

$$R_i(S) = \\ \frac{1}{\mu(N_i;N_i)}\Big(\int_{n_i}^S \frac{\mu^{\lambda_i(S')}(h_i;N_i;N_i)}{\mu^{\lambda_i(S')}(N_i;N_i)} dS' + \frac{\mu^{\lambda_i(S)}(h_i;N_i)}{\mu^{\lambda_i(S)}(N_i;N_i)} \int_{n_i}^S \frac{\mu^{\lambda_i(S')}(N_i;N_i;N_i)}{\mu^{\lambda_i(S')}(N_i;N_i)} dS'\Big).$$

Since

$$|\mu^{\lambda_i(S')}(h_i;N_i;N_i)| \le \sum_{x\in Q_i} |\mu^{\lambda_i(S')}(\bar h_x \bar\eta_x^2)| \le \|h\|_\infty \mu^{\lambda_i(S')}(N_i;N_i)$$

and by lemma $A.2$, (2.33) implies

$$|R_i(S)| \le C\frac{|S-n_i|}{\mu(N_i;N_i)} \qquad |\tilde\nabla_y \xi_i^\gamma| \le |\Delta_i| + C\frac{2+|N-n_i|}{\mu(N_i;N_i)}.$$

Therefore, thanks to (2.32),

$$\sum_{i\in I} Var_\mu(\xi_i^\gamma) \le Cl^d \sum_{i\in I} \Delta_i^2 + Cl^d \sum_{i\in I} \frac{1}{\mu(N_i;N_i)^2} + Cl^d \sum_{i\in I} \frac{1}{\mu(N_i;N_i)}. \tag{2.35}$$

By lemma $A.2$ and the hypothesis $m_a \in [\delta, 1-\delta]$, $\mu(N_i;N_i) \ge C\delta l^d$. Therefore, the second and third addendum in (2.35) are bounded respectively by $C(\delta)L^d l^{-2d}$ and $C(\delta)L^d l^{-d}$. These estimates and (2.31) permit to conclude the proof.

□

Lemma 2.10. *In the same setting of* (2.26) *and* (2.27) *, if $m_V \neq 0, 1$, then for any $i \in I$*

$$
\begin{aligned}
m_V(1-m_V)|\Delta_i| \leq\ & |m_V - \mu^{\alpha,\lambda_0(m_V)}(m_V)| \\
& + |\mathrm{Av}_{x\in Q_i}\ (\ \mu^{\alpha,\lambda_0(m_V)}(h_x;\eta_x) - \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m_V)}(h_x;\eta_x)\ )\Big]| \\
& + |\mathrm{Av}_{x\in Q_i}\ (\ \mu^{\alpha,\lambda_0(m_V)}(\eta_x;\eta_x) - \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m_V)}(\eta_x;\eta_x)\ )\Big]\ |.
\end{aligned}
$$

Proof. Let for example $m_V \leq \frac{1}{2}$, $A_i \equiv |Q_i|\, A$ and $B_i \equiv |Q_i|\, B$. Then, by lemma $A.2$,

$$
b_i \geq C m_V l^d, \quad B_i \geq C m_V l^d \quad |a_i| \leq C\|h\|_\infty m_V l^d \quad |A_i| \leq C\|h\|_\infty m_V l^d.
$$

Since $\Delta_i = \frac{a_i}{b_i B_i}(B_i - b_i) + \frac{a_i - A_i}{B_i}$, we obtain

$$
|\Delta_i| \leq \frac{C}{m_V l^d}(|a_i - A_i| + |b_i - B_i|).
$$

We can estimate $|a_i - A_i| \leq D + E$ with

$$
\begin{aligned}
D &\equiv |\mu^{\alpha,\lambda_V(\alpha,m_V)}(h_i;N_i) - \mu^{\alpha,\lambda_0(m_V)}(h_i;N_i)| \\
E &\equiv |\mu^{\alpha,\lambda_0(m_V)}(h_i;N_i) - \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m_V)}(h_i;N_i)\Big]|.
\end{aligned}
$$

By lemma $A.4$, $D \leq C l^d |m_V - \mu^{\alpha,\lambda_0(m_V)}(m_V)|$. It's simple to conclude the proof with a similar estimate of $|b_i - B_i|$. ☐

Given a pair of boxes V and W we write

$$
V \sim W \tag{2.36}
$$

if $V \cap W = \emptyset$ and there is a face F_V of V, a face F_W of W and a unit vector z such that $F_V = F_W + z$.

We consider two disjoint boxes $V, W \in R_L$, $V \sim W$, which are subdivided in boxes belonging to R_l:

$$
V = \sqcup_{i\in I} Q_i \qquad W = \sqcup_{j\in J} Q_j.
$$

We fix integers $N_V \in [0, |V|]$ and $N_W \in [0, |W|]$. Let α be a disordered configuration and ν be a multi-canonical measure on $V \cup W$ with N_V $[N_W]$ particles in V $[W]$ and disorder configuration α.
Let $G^{V,\alpha}$ be as in (2.27) and let $G^{W,\alpha}$ be its analog for the box W with respect to the subdivision $W = \sqcup_{j\in J} Q_j$ and the function h'.

Proposition 2.11. *(Bound on special covariances)*
Let $0 < \delta < \frac{1}{2}$. If $L \geq l \geq l_0(\delta)$ then for α, V, W and ν as above and for any local function g with support in $\Lambda \equiv V \cup W$,

$$\nu(Av_{v\in V}h_v Av_{w\in W}h'_w;g)^2 \leq \\ \frac{C(l)}{|\Lambda|}\mathcal{D}_\nu(g)+\frac{C\delta}{|\Lambda|}Var_\nu(g)+\frac{C}{|\Lambda|}Var_\nu(g)\big[F(m_V)G^{V,\alpha}+F(m_W)G^{W,\alpha}\big]$$

where $F(m)\equiv \mathbb{I}_{m\in[\delta,1-\delta]}$.

Proof. Let $H\equiv Av_{v\in V}h_v$, $H'\equiv Av_{w\in W}h'_w$, $\mathcal{F}_V\equiv\sigma(\eta_V)$ and $\mathcal{F}_W\equiv\sigma(\eta_W)$. Then

$$\nu(HH';g)=\nu\big(\nu(HH';g|\mathcal{F}_W)\big)+\nu\big(\nu(HH'|\mathcal{F}_W);g\big)=\nu(H'\nu(H;g|\mathcal{F}_W))+\nu(H)\nu(H';g).$$

Since $\nu(H';g)=\nu\big(\nu(H';g|\mathcal{F}_V)\big)$

$$\nu(HH';g)^2\leq 2\|h'\|_\infty^2\nu\big(\nu(H;g|\mathcal{F}_W)^2\big)+2\|h\|_\infty^2\nu\big(\nu(H';g|\mathcal{F}_V)^2\big). \tag{2.37}$$

The thesis follows by (2.37) and proposition 2.9.

□

Lemma 2.12. *Let* $f=f(\alpha_0)$ *and* $f_x\equiv\tau_x f$. *Let* $V\in R_L$ *be subdivided as* $V=\sqcup_{i\in I}Q_i$ *with* $Q_i\in R_l$.
For any $\gamma>0$ *if* $L\geq l\geq l(\gamma,\|f\|_\infty)$ *then*

$$\mathbb{P}(\,\alpha\,:\,\mathrm{Av}_{i\in I}(\mathrm{Av}_{x\in Q_i}f_x-\mathbb{E}(f))^2\geq\gamma\,)\leq 2e^{-C(\|f\|_\infty)\gamma^2\frac{L^d}{l^d}}. \tag{2.38}$$

Proof. Let $F_i\equiv(\mathrm{Av}_{x\in Q_i}f_x-\mathbb{E}(f))^2$ and $\bar{F}_i\equiv F_i-\mathbb{E}(F_i)$. By proposition $A.1$, for any $0<\delta<1$,

$$\mathbb{E}(F_i)\leq\|f\|_\infty^2\mathbb{P}\big(\,|\mathrm{Av}_{x\in Q_i}f_x-\mathbb{E}(f)|\geq\delta\,\big)+\|f\|_\infty^2\delta^2\leq\|f\|_\infty^2(2e^{-C(\|f\|_\infty)\delta^2l^d}+\delta^2).$$

Therefore, by choosing δ small enough, if $l\geq l(\gamma,\|f\|_\infty)$ then $\mathbb{E}(F_i)\leq\gamma/2$ for any i. In this case,

$$\mathbb{P}(\,\alpha\,:\,\mathrm{Av}_{i\in I}F_i\geq\gamma\,)\leq\mathbb{P}(\,\alpha\,:\,\mathrm{Av}_{i\in I}\bar{F}_i\geq\frac{\gamma}{2}\,). \tag{2.39}$$

As in the proof of proposition $A.1$, for any $t\geq 0$ and $\forall i\in I$, $\mathbb{E}(e^{t\bar{F}_i})\leq e^{Kt^2}$ where $K=K(\|f\|_\infty)$. Since the boxes Q_i are disjoint, $\mathbb{E}(e^{t\mathrm{Av}_{i\in I}\bar{F}_i})\leq e^{Kt^2|I|^{-1}}$. Therefore, by the same arguments of proposition $A.1$ and by (2.39) we obtain the thesis.

□

Proposition 2.13. *We assume the same setting of proposition* 2.11 *and we define* $h_x^{(1)}\equiv\eta_x$, $h_x^{(2)}\equiv h_x$ *and* $h_x^{(3)}\equiv h'_x$.

Then $G^{V,\alpha}+G^{W,\alpha}\leq CY^\alpha_{V,W}$ *where*

$$
\begin{aligned}
&Y^\alpha_{V,W} \equiv Y^\alpha_V + Y^\alpha_W + \sum_{r=1}^2 Y^\alpha_{V,r} + \sum_{r=1}^2 Y^\alpha_{W,r},\\
&Y^\alpha_V \equiv \sup_{m\in M_L} Y^\alpha_V(m) \quad Y^\alpha_{V,r} \equiv \sup_{m\in M_L} Y^\alpha_{V,r}(m),\\
&Y^\alpha_V(m) \equiv m^{-2}(1-m)^{-2}(m-\mu^{\alpha,\lambda_0(m)}(m_V))^2,\\
&Y^\alpha_{V,r} \equiv m^{-2}(1-m)^{-2}\mathrm{Av}_{i\in I}\Big(\mathrm{Av}_{x\in Q_i}\mu^{\alpha,\lambda_0(m)}(h^{(r)}_x;\eta_x) - \mathbb{E}\big[\mu^{\alpha,\lambda_0(m)}(h^{(r)}_0;\eta_0)\big]\Big)^2,\\
&M_L \equiv \big\{\frac{1}{(1000!L)^d}, \frac{2}{(1000!L)^d}, \dots, 1-\frac{1}{(1000!L)^d}\big\}
\end{aligned}
$$

and Y^α_W, $Y^\alpha_{W,r}$, $Y^\alpha_W(m)$, $Y^\alpha_{W,r}(m)$ are defined in the same way with W in place of V and J in place of I.
Moreover, for any $\gamma > 0$ if $L \ge l \ge l(\gamma)$ then

$$\mathbb{P}(\ \alpha\ :\ Y^\alpha_{V,W} \ge \gamma\ ) \le C(\gamma,l)e^{-C'(\gamma,l)L^d} \tag{2.40}$$

Proof. The estimate $G^{V,\alpha} + G^{W,\alpha} \le CY^\alpha_{V,W}$ derives from lemma 2.10. In what follows, we prove (2.40).
Let $Y(m)$ be one of the functions $Y^\alpha_V(m)$, $Y^\alpha_{V,r}(m)$, $Y^\alpha_W(m)$ and $Y^\alpha_{W,r}(m)$. Then

$$\mathbb{P}(\alpha : \sup_{m\in M_L} Y(m) \ge \gamma) \le \sum_{m\in M_L} \mathbb{P}(\alpha : Y(m) \ge \frac{\gamma}{2}). \tag{2.41}$$

The r.h.s. of (2.41) can be estimated by proposition $A.1$ if $Y(m) = Y^\alpha_V(m)$, $Y^\alpha_W(m)$ and by lemma 2.12 if $Y(m) = Y^\alpha_{V,r}(m)$, $Y^\alpha_W(m)$.

□

2.3 Spectral Gap

Given a disorder configuration α, a box Λ and an integer $N \in [0, |\Lambda|]$ we write ρ for the density $N/|\Lambda|$, ν for the canonical measure $\nu^\alpha_{\Lambda,N}$ and μ for the grand canonical measure $\mu^{\alpha,\lambda_\Lambda(\alpha,N)}$.
Let $\mathcal{L}^\alpha_{\Lambda,N}$ be the infinitesimal generator $\sum_{b\in\Lambda}\mathcal{L}_b$, understood as operator in $L^2(\nu)$.
Moreover, we define

$$g_\alpha(\Lambda) \equiv \min_{N\in[0,|\Lambda|]} \text{gap}\ \mathcal{L}^\alpha_{\Lambda,N} \qquad \gamma_\alpha(\Lambda) \equiv g_\alpha^{-1}(\Lambda)L^{-2}\ \text{ with } \Lambda \in R_L. \tag{2.42}$$

We observe that $\gamma_\alpha(\Lambda)$ is well-defined except for a multiplicative factor in $[10^{-2}, 10^2]$.

In this section, we inferiorly estimate the spectral gap $g_\alpha(\Lambda)$ by following the method introduced in [6].
In particular, we concentrate on boxes not too slim, that is in R_L for some L.

In theorem 2.14 we give an estimate of $g_\alpha(\Lambda)^{-1}$ in terms of $g_\alpha(V)^{-1}$ and $g_\alpha(W)^{-1}$ where V, W are two disjoint adjacent boxes in R_L such that $\Lambda = V \cup W$.
The fundamental difference from [6] is that now the above estimate is not close and present a term which depend on the disorder α, of which we give some large deviations estimates.

2.3.1 A theorem for the recursive procedure

We fix a disorder configuration α. Let $V, W \in R_L$ with $V \sim W$ and $\Lambda = V \cup W$, where R_L is defined as in (2.24) and the relation $\sim$ is defined as in (2.36).

Theorem 2.14. *Let $0 < \epsilon < 1$. If $l \geq l_0(\epsilon)$, $l' = (\epsilon)^{-\frac{1}{4d}}$ and $L \geq L(l, \epsilon) \geq l \vee l'$ then*

$$g_\alpha^{-1}(\Lambda) \leq (1 - A)^{-1}\big(g_\alpha^{-1}(V) \vee g_\alpha^{-1}(W) + CL^2\big) \tag{2.43}$$

where

$$A \equiv C_0\epsilon + C_0\epsilon^{\frac{1}{4}} + C_0(\epsilon)Y^\alpha_{V,W} + C_0 Z^\alpha_{V,W} \tag{2.44}$$

and $Y^\alpha_{V,W}$ and $Z^\alpha_{V,W}$ are defined as follows.
We fix some subdivisions of V and W in boxes belonging to R_l and then in boxes belonging to $R_{l'}$

$$V = \sqcup_{i\in I} Q_i \qquad W = \sqcup_{j\in J} Q_j \qquad Q_i \in R_l,\ Q_j \in R_l, \tag{2.45}$$

$$V = \sqcup_{i'\in I'} Q_{i'} \qquad W = \sqcup_{j'\in J'} Q_{j'} \qquad Q_{i'} \in R_{l'},\ Q_{j'} \in R_{l'}. \tag{2.46}$$

Then $Y^\alpha_{V,W}$ is defined as in proposition 2.13 with respect to the subdivisions in (2.45) and $Z^\alpha_{V,W}$ is defined as

$$\sum_{s=\pm 1} Av_{i'\in I'}\big(Av_{x\in Q_{i'}} e^{s\alpha_x} - \mathbb{E}(e^{s\alpha_0})\big)^2 + \sum_{s=\pm 1} Av_{j'\in J'}\big(Av_{x\in Q_{j'}} e^{s\alpha_x} - \mathbb{E}(e^{s\alpha_0})\big)^2.$$

Remark. For any integer l and any box A with sides of length not smaller than l we fix once for all a subdivision $A = \sqcup_{s\in S} Q_s$ with $Q_s \in R_l$.
Therefore, if $l \vee l' \leq L$ and $V, W \in R_L$ as in theorem 2.14 then $Y^\alpha_{V,W}$ and $Z^\alpha_{V,W}$ are univocally determined.

The rest of this subsection is devoted to the proof of theorem 2.14.
We follow the method introduced in [6]. Almost all the results in [6] are valid also for the disordered model under consideration and can be proved with slight modifications of the arguments given in [6]. Often, the proofs are shorter as the measure μ^α is product. We will only briefly describe this common part.
The true difference appears in the estimate of the term (that we will explain)

$$\sum_n (\gamma(n) \wedge \gamma(n-1)) B(n)^2$$

which is based on bounds of special covariances, some of which have been developed in section 2.2.

Let ν be a canonical measure on Λ with N particles. We write $\rho = N/|\Lambda|$.
We assume $0 < \rho \le 1/2$. The cases $\rho = 0, 1$ are trivial, while the case $1/2 \le \rho < 1$ can be treated in a similar way by duality.
Let $n_{\min}$ and $n_{\max}$ be the smallest and largest possible value of N_V under the constraint that $N_\Lambda = N$. Moreover, we set $I \equiv \{n \in [n_{\min}, n_{\max}] : n \text{ is an integer}\}$.

We fix a function f with support in Λ.

• Let $\mathcal{F} \equiv \sigma(\, N_V, N_W\,)$. By rule (2.23),

$$\mathrm{Var}_\nu(f) = \nu\big(\,\mathrm{Var}_\nu(f \,|\, \mathcal{F})\,\big) + \nu\big(\,\mathrm{Var}(f_1)\,\big) \tag{2.47}$$

where $f_1 \equiv \nu(f \,|\, \mathcal{F})$.

We observe that the measure $\nu(\cdot \,|\, \mathcal{F})(\eta)$ is the product the canonical measures on V and W. Therefore

$$\nu\big(\,\mathrm{Var}_\nu(f \,|\, \mathcal{F})\,\big) \le \big(g_\alpha(V)^{-1} \vee g_\alpha(W)^{-1}\big)\mathcal{D}_\nu(f). \tag{2.48}$$

In what follows we concentrate on $\nu\big(\,\mathrm{Var}(f_1)\,\big)$. Since we can think f_1 as a function of N_V, given $n \in I$ we write $f_1(n)$ for $\nu(f \,|\, N_V = n)$.

• A fundamental result is the following estimate of the spectral gap of one-dimensional discrete random walks, [6] section E.

Proposition 2.15. *Let γ be a positive probability measure on the integers $\Omega \equiv \{N_1, N_1 + 1, \ldots, N_2\}$ and let N^* be the largest integer such that $\sum_{n \le N^*} \gamma(n) \le 1/2$. Then, for all functions f on Ω we have*

$$\mathrm{Var}_\gamma(f) \le C_\gamma \sum_{n=N_1+1}^{N_2} (\,\gamma(n) \wedge \gamma(n-1)\,)[f(n) - f(n-1)]^2$$

where

$$C_\gamma \equiv 4 \max\Big[\Big(\sup_{n \le N^*+1} \sum_{j \le n} \frac{\gamma(j)}{\gamma(n)}\Big)^2, \Big(\sup_{n \ge N^*} \sum_{j \ge n} \frac{\gamma(j)}{\gamma(n)}\Big)^2\Big].$$

If for some $\bar{N} \in [N_1, N_2]$ and $\alpha > 0$

$$\frac{\gamma(j-1)}{\gamma(j)} \le e^{-\alpha(\bar{N}-j)} \quad \text{if } j \le \bar{N} \qquad\qquad \frac{\gamma(j+1)}{\gamma(j)} \le e^{-\alpha(j-\bar{N})} \quad \text{if } j \ge \bar{N}$$

then $C_\gamma \le C \max\big\{\frac{1}{\alpha}, 1\big\}$.

• We will apply proposition 2.15 in the case that γ is the probability measure on I such that

$$\gamma(n) \equiv \nu(N_V = n).$$

Let $N_V^* \equiv \mu(N_V)$, then by lemma $A.2$ $N_V^* \leq C\rho|V|$. By proposition 2.15 it can be proved the following theorem.

Theorem 2.16. *For any function f depending on N_V*

$$\mathrm{Var}_\nu(f) \leq C N_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n) \wedge \gamma(n-1))[f(n) - f(n-1)]^2.$$

In particular, theorem 2.16 can be applied to the function f_1.

• We set

$$\begin{aligned} E_{xz} &\equiv \{\eta \in \Omega_\Lambda \,:\, \eta(x) = 1\, \eta(z) = 0\} \quad \forall x, z \in \Lambda \\ c_n &\equiv n(|W| - N + n) \quad n \in I \\ c_n' &\equiv n(|V| - N + n) \quad n \in I. \end{aligned}$$

We indicate by $[x]$ the integer part of x. Then we set $u \equiv [\rho|V|]$ and $H \equiv -\sum_{x\in\Lambda} \alpha_x \eta_x$. Given n such that $n_{\min} < n \leq n_{\max}$, we define

$$A(n) \equiv \begin{cases} \frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{x\in V, z\in W} \nu[(\nabla_{xz} f)\mathbb{I}_{E_{zx}} e^{-\nabla_{xz}H} \,|\, N_V = n-1] & \text{if } n \leq u \\ \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)} \sum_{x\in V, z\in W} \nu[(\nabla_{xz} f)\mathbb{I}_{E_{xz}} e^{-\nabla_{xz}H} \,|\, N_V = n] & \text{otherwise} \end{cases}$$

$$B(n) \equiv \begin{cases} \frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{x\in V, z\in W} \nu[f ; e^{-\nabla_{xz}H}\mathbb{I}_{E_{zx}} \,|\, N_V = n-1] & \text{if } n \leq u \\ \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)} \sum_{x\in V, z\in W} \nu[f ; e^{-\nabla_{xz}H}\mathbb{I}_{E_{xz}} \,|\, N_V = n] & \text{otherwise.} \end{cases} \tag{2.49}$$

We observe that the functions

$$\sum_{x\in V, z\in W} \nu(\mathbb{I}_{E_{z,x}} \,|\, N_V = n-1) \;\text{ and }\; \sum_{x\in V, z\in W} \nu(\mathbb{I}_{E_{x,z}} \,|\, N_V = n)$$

are constant, and therefore the above definition of $B(n)$ doesn't differ to that in [6].

As in section G of [6], it can be proved that

$$\sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n) \wedge \gamma(n-1))[f_1(n) - f_1(n-1)]^2 \leq \\ 2 \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n) \wedge \gamma(n-1))[A(n)^2 + B(n)^2].$$

By the above equation and theorem 2.16,

$$\text{(2.50)}\quad \mathrm{Var}_\nu(f_1) \le \\ CN_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))A(n)^2 + CN_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))B(n)^2.$$

• Estimate of the first addendum in the r.h.s. of (2.50).

Given $x, z \in \mathbb{Z}^d$ let $\gamma_{x,z}$ be defined as in section 1.14. We set

$$\mathcal{G} \equiv \max_{b\in\Lambda} |\{(x,z) \in V \times W \;:\; b \in \gamma_{x,z}\}|$$
$$\mathcal{D} \equiv \max_{(x,z)\in V\times W} |\gamma_{x,z}|.$$

Similarly to section H of [6], by the Moving Particles lemma it can be proved that

$$CN_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))A(n)^2 \le CN_V^* \frac{\mathcal{G}\mathcal{D}}{\rho(1-\rho)|V||W|} \sum_{b\in\Lambda} \nu\big(\,(\nabla_b f)^2\,\big).$$

Given $b \in \Lambda$ with $b = \{y, y+e_i\}$ and $b \in \gamma_{x,z}$ then $y_j = z_j$ if $j = 1, \ldots, i-1$ and $y_j = x_j$ if $j = i+1, \ldots, n$. Therefore, $|\mathcal{G}| \le CL^{d+1}$.
Since $|\mathcal{D}| \le CL$,

$$\text{(2.51)}\qquad N_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))A(n)^2 \le CL^2\mathcal{D}_\nu(f)$$

• Estimate of the second addendum in the r.h.s. of (2.50).

We assume, for example, $n \le u$.
In what follows we write $\bar\nu$ for the multi-canonical measure $\nu(\cdot\,|\, N_V = n-1)$ and $\bar\mu$ for the grand multi-canonical measure on Λ with chemical potentials $\lambda_V(\alpha, n-1)$ on V and $\lambda_W(\alpha, N-n+1)$ on W.
As in [6] it can be proved that

$$\frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \le \frac{1}{\rho|V||W|}.$$

Therefore

$$\text{(2.52)}\qquad N_V^* B(n)^2 \le C\frac{|\Lambda|}{\rho}\bar\nu\Big(f; \big(\mathrm{Av}_{x\in V} e^{\alpha_x}(1-\eta_x)\big)\,\big(\mathrm{Av}_{z\in W} e^{-\alpha_z}\eta_z\big)\Big)^2.$$

Let $0 < \epsilon < 1$. In what follows we distinguish two cases.

Non extreme density: $\rho \geq \epsilon$.

By (2.52) and by proposition 2.11 with $\delta \equiv \epsilon^2$, given $L \geq l \geq l_0(\delta)$ and subdivisions

$$V = \sqcup_{i\in I} Q_i \qquad W = \sqcup_{j\in J} Q_j \quad \text{where } Q_i, Q_j \in R_l,$$

if $\rho \geq \epsilon$ then

(2.53)
$$N_V^* B(n)^2 \leq C(l,\epsilon)\mathcal{D}_{\bar\nu}(f) + C\epsilon Var_{\bar\nu}(f) + C(\epsilon) Var_{\bar\nu}(f)\bar\nu\big(F(m_V)G^{V,\alpha} + F(m_W)G^{W,\alpha}\big)$$

where $m_V \equiv N_V/|V|$, $m_W \equiv N_W/W$ and $F(m) \equiv \mathbb{I}_{m\in[\epsilon^2,1-\epsilon^2]}$.

By (2.53) and proposition 2.13,

$$\begin{aligned} &N_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))B(n)^2 \leq \\ &C(l,\epsilon)\mathcal{D}_\nu(f) + C\epsilon Var_\nu(f) + C(\epsilon) Var_\nu(f) Y_{V,W}^\alpha. \end{aligned} \tag{2.54}$$

Extreme density $\rho \leq \epsilon$.

By proposition 2.17, since $l' = \epsilon^{-\frac{1}{4}}$ if $\rho \leq \epsilon$,

$$\begin{aligned} &N_V^* \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))B(n)^2 \leq \\ &C(l')\mathcal{D}_\nu(f) + C\epsilon^{\frac{1}{4}}\mathrm{Var}_\nu(f) + C\mathrm{Var}_\nu(f)\mathrm{Av}_{j'\in J'}\big(\mathrm{Av}_{z\in Q_{j'}} e^{-\alpha_z} - \mathbb{E}(e^{-\alpha_0})\big)^2. \end{aligned} \tag{2.55}$$

• By (2.47), (2.48), (2.50), (2.51), (2.54) and (2.55) we obtain (2.43).

Proposition 2.17. *Let $\Lambda = V \cup W$ with $V, W \in R_L$ and $V \sim W$. Let ν be a canonical measure on Λ with density ρ.*
Let $0 < \rho < \frac{1}{2}$, $L \geq l$ and W be subdivided as $W = \sqcup_{j\in J} Q_j$ with $Q_j \in R_l$.
Let n be a possible values in N_V for the canonical measure ν. We assume that $n \leq [\rho|V|]$ and write $\bar\nu$ for the multi-canonical measure $\nu(\cdot \,|\, N_V = n)$.
Then for any function f with support in Λ,

$$\begin{aligned} &\bar\nu\Big(f; \big(\mathrm{Av}_{x\in V} e^{\alpha_x}(1-\eta_x)\big)\big(\mathrm{Av}_{z\in W} e^{-\alpha_z}\eta_z\big)\Big)^2 \leq \\ &C(l)\frac{\rho}{|\Lambda|}\mathcal{D}_{\bar\nu}(f) + Cl^{3d}\frac{\rho^2}{|\Lambda|}\mathrm{Var}_{\bar\nu}(f) + C\frac{\rho}{|\Lambda|}\mathrm{Var}_{\bar\nu}(f)\mathrm{Av}_{j\in J}\big(\mathrm{Av}_{z\in Q_j} e^{-\alpha_z} - \mathbb{E}(e^{-\alpha_0})\big)^2. \end{aligned} \tag{2.56}$$

Proof. We write $\bar\mu$ for the grand multi-canonical measure on Λ with chemical potentials $\lambda_V(\alpha,n)$ on V and $\lambda_W(\alpha,N-n)$ on W.
We observe that $n/|V|\le C\rho$ and $(N-n)/|W|\le C\rho$.
We first prove that in (2.56) the term $e^{\alpha_x}(1-\eta_x)$ can be substituted by e^{α_x}. In particular, we assert that

$$\mathrm{Var}_{\bar\nu}\Big(\big(\mathrm{Av}_{x\in V}e^{\alpha_x}\eta_x\big)\,\big(\mathrm{Av}_{z\in W}e^{-\alpha_z}\eta_z\big)\Big)\le C\frac{\rho^2}{|\Lambda|}. \tag{2.57}$$

In fact, by lemma 2.6 the first member of (2.57) is bounded by

$$\frac{C}{|\Lambda|^4}\sum_{x,x'\in V}\sum_{z,z'\in W}|\bar\mu(\eta_x\eta_z;\eta_{x'}\eta_{z'})|$$

which is bounded by $C\big(\frac{\rho^3}{|\Lambda|^3}+\frac{\rho^2}{|\Lambda|^2}\big)$ and that proves (2.57).

By (2.57) and Schwarz inequality, it remains to estimate $C|\Lambda|^{-2}\bar\nu\big(f;\sum_{z\in W}e^{-\alpha_z}\eta_z\big)^2$.
Let $N_j\equiv N_{Q_j}$, $\mathcal{F}\equiv\sigma(\eta_V,\ \{N_j\}_{j\in J})$ and $F\equiv\sum_{z\in W}e^{-\alpha_z}\eta_z$. By rule (2.23),

$$\bar\nu(f;F)^2\le 2\bar\nu\big(\bar\nu(f;F\,|\,\mathcal{F})\big)^2+2\bar\nu\big(f;\bar\nu(F\,|\,\mathcal{F})\big)^2. \tag{2.58}$$

Since $\bar\nu(\cdot\,|\,\mathcal{F})(\eta)$ on W is the product of the canonical measures $\mu^\alpha_{Q_j,N_{Q_j}(\eta)}$,

$$|\Lambda|^{-2}\bar\nu\big(\bar\nu(f;F\,|\,\mathcal{F})\big)^2\le|\Lambda|^{-2}C(l)\mathcal{D}_{\bar\nu}(f)\mathrm{Var}_{\bar\nu}(F)\le C(l)\frac{\rho}{|\Lambda|}\mathcal{D}_{\bar\nu}(f). \tag{2.59}$$

By Schwarz inequality the second member in (2.58) is bounded by $2\mathrm{Var}_{\bar\nu}(f)\mathrm{Var}_{\bar\nu}\big(\bar\nu(F\,|\,\mathcal{F})\big)$.
By lemma 2.6, for any $\gamma\in\mathbb{R}$

$$\mathrm{Var}_{\bar\nu}\big(\bar\nu(F\,|\,\mathcal{F})\big)=\mathrm{Var}_{\bar\nu}\big(\bar\nu(F\,|\,\mathcal{F})-\gamma N_W\big)\le C\sum_{j\in J}\mathrm{Var}_{\bar\mu}(\xi_j)$$

where

$$\xi_j(\eta)\equiv\sum_{z\in Q_j}\mu^\alpha_{Q_j,N_j(\eta)}([e^{-\alpha_z}-\gamma]\eta_z)$$

We write $\xi_j=\xi_j\mathbb{I}_{N_j=1}+\xi_j\mathbb{I}_{N_j\ge2}$. We observe that

$$\xi_j\mathbb{I}_{N_j=1}=\Big(\big(\mathrm{Av}_{z\in Q_j}e^{\alpha_z}\big)^{-1}-\gamma\Big)\mathbb{I}_{N_j=1}.$$

By choosing $\gamma\equiv\big(\mathbb{E}(e^{\alpha_0})\big)^{-1}$, then

$$|\xi_j|\mathbb{I}_{N_j=1}\le C|\mathrm{Av}_{z\in Q_j}e^{\alpha_z}-\mathbb{E}(e^{\alpha_0})|\mathbb{I}_{N_j=1}.$$

Since, $\bar\mu(\mathbb{I}_{N_j=1})\le Cl^d\rho$

$$|\Lambda|^{-2}\sum_{j\in J}\mathrm{Var}_{\bar\mu}(\xi_j\mathbb{I}_{N_j=1})\le C\frac{\rho}{|\Lambda|}\mathrm{Av}_{j\in J}\Big(\mathrm{Av}_{z\in Q_j}e^{\alpha_z}-\mathbb{E}e^{\alpha_0}\Big)^2. \tag{2.60}$$

We observe that $\|\xi_j\le Cl^d$ and $\bar\mu(\mathbb{I}_{N_j\ge 2})\le C\rho^2 l^{2d}$. Therefore

$$|\Lambda|^{-2}\sum_{j\in J}\mathrm{Var}_{\bar\mu}(\xi_j\mathbb{I}_{N_j\ge 2})\le C|\Lambda|^{-2}l^{2d}\sum_{j\in J}\bar\mu(N_j\ge 2)\le C|\Lambda|^{-1}\rho^2 l^{3d}. \tag{2.61}$$

Thanks to (2.59), (2.60) and (2.61), we estimate $|\Lambda|^{-2}\bar\nu\big(f;\sum_{z\in W}e^{-\alpha_z}\eta_z\big)^2$ obtaining the thesis.

$\square$

2.3.2 Recursive Estimate

Given $\Lambda\in\mathbb{F}$ and an integer S we define

$$\mathcal{A}_{S,\Lambda}\equiv\{\ \{V,W\}\ :\ V,W\subset\Lambda,\ V,W\in R_S \text{ and } V\sim W\}.$$

We fix $\gamma\equiv(1-2^{-\frac{1}{d}})$. In what follows we explain how to choose the constants ϵ, l', l.

Let C_0 be as in (2.44).
By lemma 2.12, there exists ϵ_0 such that if $\epsilon\le\epsilon_0$ then $\forall L\ge l'$ with $l'\equiv\epsilon^{-\frac{1}{4d}}$ and for any $V,W\in R_L$ with $V\sim W$

$$\mathbb{P}(\alpha\ :\ C_0 Z^{\alpha}_{V,W}\ge\frac{\gamma}{4})\le 2e^{-C(\epsilon)L^d}. \tag{2.62}$$

Let ϵ_1 such that $C_0(\epsilon_1+\epsilon_1^{\frac{1}{4}})\le\gamma/4$ and we choice $\epsilon\equiv\epsilon_0\wedge\epsilon_1$.
We choose l with the property that $\forall L\ge l$ and for any $V,W\in R_L$ with $V\sim W$

$$\mathbb{P}(\alpha\ :\ C_0(\epsilon)Y^{\alpha}_{V,W}\ge\frac{\gamma}{4})\le Ce^{-C'L^d} \tag{2.63}$$

C,C' depending on γ,l,ϵ. By proposition 2.13 this choice is legitim.

We observe that if

$$C_0(\epsilon)Y^{\alpha}_{V,W}\le\frac{\gamma}{4}\quad\text{and}\quad C_0Z^{\alpha}_{V,W}\le\frac{\gamma}{4} \tag{2.64}$$

then $A\le\gamma$, where A is as in theorem 2.14.

Definition 2.18. *Let $\Lambda \in R_L$. Given a positive number L' we define*

$$\bar{\Theta}(\Lambda, L')$$

as the set of disorder configurations α such that, for any integer $S : L' \le S \le L$ and for any $(V, W) \in \mathcal{A}_{S,\Lambda}$, (2.64) is satisfied.

Given $M > 0$ we set

$$\Theta(\Lambda, M) \equiv \bar{\Theta}(\Lambda, (M \ln L)^{\frac{1}{d}}).$$

In what follows L is a positive integer, L' a positive number and given a positive number x let $[x]$ be its integer part and $[x]_+$ as

$$[x]_+ \equiv \begin{cases} x & \text{if } x \in \mathbb{N} \\ [x]+1 & \text{if } x \notin \mathbb{N} \end{cases}.$$

Lemma 2.19. *There exist positive constants $\bar{C}$ and L_0 with $L_0 \ge l' \vee l$ such that if $L_0 \le L' \le L$ then*

$$\mathbb{P}(\ \bar{\Theta}(\Lambda, L')\ ) \ge 1 - L^{3d+1} e^{-\bar{C}(L')^d} \tag{2.65}$$

and if $M > 0$ and $L_0 \le (M \ln L)^{\frac{1}{d}}$ then

$$\mathbb{P}(\ \Theta(\Lambda, M)\ ) \ge 1 - L^{3d+1-\bar{C}M}. \tag{2.66}$$

Proof. (2.66) follows by (2.65) with $L' \equiv (M \ln L)^{\frac{1}{d}}$.
In order to prove (2.65) we observe that if $S \le L$ then $|\mathcal{A}_{S,\Lambda}| \le CL^{3d}$. By (2.62) and (2.63), if L' is large enough

$$\begin{aligned} 1 - \mathbb{P}(\ \bar{\Theta}(\Lambda, L')\ ) \le & \sum_{S=[L']_+}^{L} \sum_{(V,W)\in\mathcal{A}_{S,\Lambda}} \Big(\mathbb{P}(C_0(\epsilon) Y^{\alpha}_{V,W} > \frac{\gamma}{4}) + \mathbb{P}(C_0 Z^{\alpha}_{V,W} > \frac{\gamma}{4}) \Big) \\ \le & C L^{3d+1} e^{-C(L')^d}. \end{aligned}$$

That proves (2.65) by an opportune choice of L_0 and $\bar{C}$.

□

We define $\bar{R}_L \subset R_L$ as

$$\bar{R}_L \equiv \{ A \in \mathbb{F} \,:\, A \text{ is a box }, \ L \le l_{\min} \le l_{\max} \le 10L \}.$$

Theorem 2.20. *There exist positive numbers $k, C, \bar{C}, L_0$ such that, given $L_0 \le L' < L$ and $\Lambda \in \bar{R}_L$,*

$$\sup_{N \in [0, |\Lambda|]} \big(gap\ \mathcal{L}^{\alpha}_{\Lambda,N} \big)^{-1} \le (L^{-1} e^{k(L')^{d-1}} + C) L^2 \qquad \text{if } \alpha \in \bar{\Theta}(\Lambda, L') \tag{2.67}$$

and

$$\mathbb{P}(\ \bar{\Theta}(\Lambda, L') \geq 1 - L^{3d+1} e^{-\bar{C}(L')^d}. \tag{2.68}$$

In particular, for any $\beta > 0$ let $M(\beta) \equiv (\bar{C})^{-1}(3d+1+\beta)$. Then there exist C_0 and $L(\beta)$ such that $\forall \beta > 0$, $\forall L \geq L(\beta)$ and for any cube $\Lambda \in R_L$

$$\sup_{N \in [0,|\Lambda|]} \left(gap\ \mathcal{L}^{\alpha}_{\Lambda,N} \right)^{-1} \leq C_0 L^2 \quad \forall \alpha \in \Theta(\Lambda, M(\beta)) \tag{2.69}$$

and

$$\mathbb{P}(\ \Theta(\Lambda, M(\beta))\ ) \geq 1 - L^{-\beta}. \tag{2.70}$$

Proof. Let $\Lambda \in \bar{R}_L$ and $\alpha \in \bar{\Theta}(\Lambda, L')$. We set $R \equiv 10L$ and we define by induction $L_0 \equiv L$ and $L_{n+1} \equiv [2^{-1}L_n]$ for any $n \geq 0$, $R_0 \equiv R$ and $R_{n+1} \equiv [2^{-1}L_n]_+$ for any $n \geq 0$.
It's simple to check that $|2^{-n}L - L_n| \leq 1$ and $|2^{-n}R - R_n| \leq 1$.

We describe some subdivisions of Λ. For simplicity of notation we consider the case $d = 2$ and $\Lambda = \{1, l_1\} \times \{1, l_2\}$. Let $a_i \equiv [l_i/2]$ for $i = 1, 2$.
We first subdivide Λ as $\Lambda = V \sqcup W$ with

$$V \equiv \{1, a_1\} \times \{1, l_2\} \quad W \equiv \{a_1 + 1, a_2\} \times \{1, l_2\}$$

.
Afterwards we subdivide Λ as $\Lambda = \sqcup_{i=1}^{4} Q_i$ with

$$\begin{array}{ll} Q_1 \equiv \{1, a_1\} \times \{1, a_2\} & Q_2 \equiv \{1, a_1\} \times \{a_2 + 1, l_1\} \\ Q_3 \equiv \{a_1 + 1, l_1\} \times \{1, a_2\} & Q_4 \equiv \{a_1 + 1, l_1\} \times \{a_2 + 1, l_2\} \end{array}$$

We can continue with other subdivisions. The criterion is that at each subdivision we divide each side of length l in two sides of length respectively $[l/2]$ and $[l/2]_+$. Therefore the boxes of the $(2k-1)^{th}$ subdivision have sides of length in $[L_k, R_{k-1}]$ and the boxes of the $2k^{th}$ subdivision have sides of length in $[L_k, R_k]$. By our choice of $\bar{R}_L$ we obtain that the boxes of the $(2k-1)^{th}$ subdivision or $2k^{th}$ subdivision are in R_{L_k}.
By theorem 2.14, if $L' \leq L_1$, for some constant $k_0 = k_0(\gamma)$

$$g_\alpha^{-1}(\Lambda) \leq (1-\gamma)^{-2} \big(\max_{i=1,\ldots,4} g_\alpha^{-1}(Q_i) + k_0 L^2 \big).$$

It's simple to generalize the above step to a generic dimension d and obtain for $L' \leq L_1$

$$g_\alpha^{-1}(\Lambda) \leq (1-\gamma)^{-d} \big(\max_{V \in R_{L_1},\, V \subset \Lambda} g_\alpha^{-1}(V) + k_0 L^2 \big) \qquad (1-\gamma)^{-d} = 2.$$

Let n be the minimal integer such that $L' \leq L_n$. By considering the subdivisions of Λ described at the beginning it's simple to check that

$$g_\alpha^{-1}(\Lambda) \le 2^n\big(\max_{V\in R_{L_n},\, V\subset\Lambda} g_\alpha^{-1}(V)\big) + k_0 \sum_{j=1}^{n} 2^j L_j^2. \tag{2.71}$$

Since $L_j \le 2^{-j+1}L$ the sum in the r.h.s. is bounded by CL^2.
For the choice of n, $2^n \le C\frac{L}{L'}$ and by lemma 2.7 if $V \in R_{L_n}$ then

$$g_\alpha^{-1}(V) \le Ce^{\kappa L_n^{d-1}} \le Ce^{\kappa'(L')^{d-1}}.$$

It's simple to obtain (2.67).

In order to prove (2.69) it's enough to apply (2.67) with $L' \equiv [(M(\beta)\ln L)^{\frac{1}{d}}]$. Then the condition $L_0 \le L' < L$ corresponds to an inequality of the form $L \ge L(\beta)$. By (2.69), if L is enough large

$$g_\alpha^{-1}(V) \le \big(L^{-1+kM(M\ln L)^{-1/d}} + C\big)L^2 \le 2CL^2.$$

The estimates (2.68) and (2.70) derive from lemma 2.19.

□

Proposition 2.21. *For any $r > 0$, there exists a constant C_r such that*

$$\sup_L \sup_{\Lambda\in\bar R_L} \mathbb{E}[(\gamma_\alpha(\Lambda))^r] \le C_r$$

Proof. We apply theorem 2.20. In what follows the constant C_r can change from line to line.

We fix $\beta, L^* > 0$ such that $3d+2-\frac{\bar C}{2}M(\beta) = 0$ and $e^{rk(L')^{d-1}-\bar C(L'-1)^d} \le e^{-\frac{\bar C}{2}\bar L^d}$.
Let $\bar L \equiv (M(\beta)\ln L)^{\frac{1}{d}}$ and $\Lambda \in \bar R_L$.

If $L \le L^*$, by lemma 2.7, for any α $(\gamma_\alpha(\Lambda))^r \le e^{rk(L^*)^{d-1}}$.

If $L \ge L^*$, by lemma 2.19 and theorem 2.20,

$$\begin{aligned}
\mathbb{E}[(\gamma_\alpha(\Lambda))^r] &\le C_r\mathbb{P}(\ \Theta(\Lambda, M(\beta))\ ) \\
&\quad + \sum_{L'=[\bar L]_+ +1}^{L} (L^{-1}e^{k(L')^{d-1}} + C)^r \mathbb{P}(\ \bar\Theta(\Lambda, L') \setminus \bar\Theta(\Lambda, L'-1)\ ) \\
&\le C_r + \sum_{L'=[\bar L]_+ +1}^{L} (L^{-1}e^{k(L')^{d-1}} + C)^r L^{3d+1} e^{-\bar C(L'-1)^d} \\
&\le C_r + C_r L^{3d+1} \sum_{L'=[\bar L]_+ +1}^{L} e^{rk(L')^{d-1}-\bar C(L'-1)^d} \\
&\le C_r + C_r L^{3d+2} e^{-\frac{\bar C}{2}\bar L^d} = C_r + C_r L^{3d+2-\frac{\bar C}{2}M}
\end{aligned}$$

□

Chapter 3

The method of long jumps revisited

Let us remember that, given an odd integer n and $e > 0$, $\Lambda^e_{1,n}, \Lambda^e_{2,n}$ form a couple of adjacent cubes of size n centered at the origin with $\Lambda^e_{2,n}$ obtained by moving $\Lambda^e_{1,n}$ along the direction determined by the vector e.
Following [21] we introduce the function

$$W^e_n = \mathrm{Av}_{x\in\Lambda^e_{1,n}} \mathrm{Av}_{y\in\Lambda^e_{2,n}} w^\alpha_{x,y} \tag{3.1}$$

where

$$w^\alpha_{x,y} = \left(1 + e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}\right)(\eta_x - \eta_y).$$

Let us remember that $\psi^e_{n,n}$ is defined as

$$\psi^e_{n,n} \equiv m^{1,e}_n - m^{2,e}_n - \mu^\alpha[m^{1,e}_n - m^{2,e}_n \,|\, m^e_n].$$

The main result of this chapter is given by theorem 3.9. In particular, we prove that for any $m \in (0,1)$

$$\lim_{n\uparrow\infty} 2m(1-m)\lambda_0'(m)\frac{\psi^e_{n,n}}{n} - \frac{W^e_n}{n} = 0$$

in the space $\mathcal{G}$ endowed of the semi-inner product V_m.

The functions W^e_n are taken from [21], but in our analysis the mesoscopic parameter n is such that $n \uparrow \infty$ after $\epsilon \downarrow 0$ while in [21] the parameter n is a function of ϵ, more precisely $n = O(\epsilon^{-\frac{2}{d+2}})$. This different choice strongly changes the strategy of the proof of the hydrodynamic limit and the role of the functions W^e_n.

Some steps in the proof of theorem 3.9 are motivated by ideas given in [21], but a rigorous proof requires the development of a proper technology since the mesoscopic context is different.

3.1 Criterion

Let f_n be a function with support in Λ_n^e such that $\mu^\alpha[f_n|m_n^e] = 0$ for any disorder configuration α. By (1.47) for any $k \geq n^2$ and for any Gibbs measure μ with disorder α

$$V_k\Big(\frac{f_n}{n};\mu\Big) \leq C(d)n^d \mathrm{Av}_{|x|\leq k_1}\gamma_\alpha(\Lambda_{x,n}^e)\mu\Big(\mathrm{Var}_\mu[\tau_x f_n|m_{x,n}^e]\Big). \tag{3.2}$$

Since in the disordered model under consideration $\gamma_\alpha(\Lambda_{x,n}^e)$ is bounded uniformly in α, x, n we have the following useful criterion.

Proposition 3.1. *For any n, let f_n and h_n be functions on $\tilde{\Omega}$ such that f_n has support in Λ_n^e and $\mu^\alpha[f_n|m_n^e] = 0$ for any disorder α, $|f_n| \leq |h_n|$ $\forall n$ and $\sup_n \|h_n\|_\infty < \infty$.*
If

$$\lim_{n\uparrow\infty} n^d \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(h_n^2\big)\Big] = 0 \tag{3.3}$$

then

$$\lim_{n\uparrow\infty, k\uparrow\infty} \mathbb{E}\Big[V_k\Big(\frac{f_n}{n};\mu^{\alpha,\lambda_0(m)}\Big)\Big] = 0. \tag{3.4}$$

Remark. For the proof of theorem 3.9 it is not necessary to have a uniform bound on the spectral gap. The estimates given in chapter 2, valid also for the dilute Ising model under the percolative threshold [7], are sufficient. We explain the reason.
Let h_n be a bounded function on $\tilde{\Omega}$ such that $|f_n| \leq |h_n|$. Let $p, q > 1$ with $p^{-1} + q^{-1} = 1$. Then by (3.2), the Hoelder inequality and proposition 2.21

$$\begin{aligned}\mathbb{E}\Big[V_k\Big(\frac{f_n}{n};\mu^{\alpha,\lambda_0(m)}\Big)\Big] &\leq C(d)n^d \|h_n^2\|_\infty^{\frac{1}{p}} \big(\mathbb{E}\gamma_\alpha^p(\Lambda_n^e)\big)^{\frac{1}{p}} \big(\mathbb{E}\mu^{\alpha,\lambda_0(m)}(h_n^2)\big)^{\frac{1}{q}} \\ &\leq C(d)n^d \|h_n^2\|_\infty^{\frac{1}{p}} C(p) \big(\mathbb{E}\mu^{\alpha,\lambda_0(m)}(h_n^2)\big)^{\frac{1}{q}}.\end{aligned}$$

Therefore in the general case proposition 3.1 remains valid if condition (3.3) is substituted by the request that $\exists q > 1$ such that

$$\lim_{n\uparrow\infty} n^{dq} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(h_n^2\big)\Big] = 0. \tag{3.5}$$

It can be verified that when we apply proposition 3.1 condition (3.5) is always satisfied.

The following estimates give some candidates for h_n.

Proposition 3.2. *For any $n \in \mathbb{N}$ let $\Delta_n \in \mathbb{F}$ with $|\Delta_n| = Cn^d$.*
Then $\forall c > 0$ and for $i = 1, 2$ there exists $C > 0$ such that $\forall m \in [0,1]$

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}([m - m_{\Delta_n}]^4)\Big] \le Cn^{-2d} \tag{3.6}$$

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\mathbb{I}_{\{|m-m_{\Delta_n}|\ge c\}})\Big] \le Cn^{-2d} \tag{3.7}$$

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big([m_n^e - \mu^{\alpha,\lambda_0(m_n^e)}(m_n^{i,e})]^4\big)\Big] \le Cn^{-2d} \tag{3.8}$$

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big([m - \mu^{\alpha,\lambda_0(m)}(m_n^{i,e})]^4\big)\Big] \le Cn^{-2d} \tag{3.9}$$

Proof. We prove (3.6). We consider the estimate

$$\mu^{\alpha,\lambda_0(m)}([m - m_{\Delta_n}]^4) \le 4[m - \mu^{\alpha,\lambda_0(m)}(m_{\Delta_n})]^4 + 4\mu^{\alpha,\lambda_0(m)}([\mu^{\alpha,\lambda_0(m)}(m_{\Delta_n}) - m_{\Delta_n}]^4).$$

Since $m = \mathbb{E}\mu^{\alpha,\lambda_0(m)}(m_{\Delta_n})$, setting $g_x(\alpha) = \mu^{\alpha,\lambda_0(m)}(\eta_x) - \mathbb{E}\mu^{\alpha,\lambda_0(m)}(\eta_x)$,

$$\mathbb{E}([m - \mu^{\alpha,\lambda_0(m)}(m_{\Delta_n})]^4) = \mathbb{E}([\mathrm{Av}_{x\in\Delta_n} g_x]^4) \le Cn^{-2d}.$$

Similarly

$$\mu^{\alpha,\lambda_0(m)}([\mu^{\alpha,\lambda_0(m)}(m_{\Delta_n}) - m_{\Delta_n}]^4) \le Cn^{-2d}. \tag{3.10}$$

(3.6) follows from the above estimates.

(3.7) follows from Chebyshev inequality and (3.6).

We prove (3.8). By lemma $A.5$

$$\big[\mu^{\alpha,\lambda_0(m_n^e)}(m_n^{i,e}) - \mu^{\alpha,\lambda_0(m)}(m_n^{i,e})\big]^4 \le C[m_n^e - m]^4.$$

(3.6) implies

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big([m_n^e - m]^4\big)\Big] \le Cn^{-2d} \quad \text{and} \quad \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big([m_n^{i,e} - m]^4\big)\Big] \le Cn^{-2d}.$$

Therefore it's enough to prove that

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big([m_n^{i,e} - \mu^{\alpha,\lambda_0(m)}(m_n^{i,e})]^4\big)\Big] \le Cn^{-2d},$$

which is consequence of (3.10).

The proof of (3.9) is similar.

∎

3.2 Main statement

In what follows Q is the unitary square $[0,1]\times[0,1]$. If $f : D(f)\subset Q\to\mathbb{R}$ is a function well defined on a subset $D(f)$ of Q and $g : Q\to\mathbb{R}$ is such that $g\equiv 0$ on $Q\setminus D(f)$ then we convey to define the product fg as null on $Q\setminus D(f)$.

We fix a versor $e>0$. In what follows, when there is no ambiguity, the index e is omitted.

We define

$$F_n^\alpha(m_1,m_2)\equiv \mu^{\alpha,\underline{\lambda}_n(\alpha,m_1,m_2)}(W_n)$$

where $\underline{\lambda}_n(\alpha,m_1,m_2)$ is the multi-chemical potential equal to $\lambda(\alpha,\Lambda^e_{i,n},m_i)$ on $\Lambda^e_{i,n}$ for $i=1,2$.
We observe F_n^α is continuous in Q, it's regular on $(0,1)\times(0,1)$ and $\|F_n^\alpha\|_\infty\le C(B)$ $\forall n$ and $\forall\alpha$.

We define $\lambda_{i,n}(m)\equiv\lambda(\alpha,\Lambda^e_{i,n},m)$. When there is no ambiguity we omit the index n writing simply λ_1 and λ_2.
Since

$$W_n = m_n^{(1)}-m_n^{(2)}+\mathrm{Av}_{\substack{x\in\Lambda^e_{1,n}\\ y\in\Lambda^e_{2,n}}}\eta_x e^{-\alpha_x}(1-\eta_y)e^{\alpha_y}-\mathrm{Av}_{\substack{x\in\Lambda^e_{1,n}\\ y\in\Lambda^e_{2,n}}}(1-\eta_x)e^{\alpha_x}\eta_y e^{-\alpha_y} \tag{3.11}$$

and thanks to the computational rules

$$\mu^{\alpha,\lambda}(\eta_x e^{-\alpha_x})=e^{\lambda}\mu^{\alpha,\lambda}(1-\eta_x)\qquad \mu^{\alpha,\lambda}((1-\eta_x)e^{\alpha_x})=e^{-\lambda}\mu^{\alpha,\lambda}(\eta_x),$$

F_n^α and its derivatives can be computed $\forall(m_1,m_2)\in(0,1)\times(0,1)$:

$$F_n^\alpha(m_1,m_2)=m_1-m_2+e^{\lambda_1(m_1)-\lambda_2(m_2)}(1-m_1)m_2-e^{\lambda_2(m_2)-\lambda_1(m_1)}m_1(1-m_2) \tag{3.12}$$

and

$$\begin{aligned}\partial_{m_1}F(m_1,m_2)&=1+e^{\lambda_1(m_1)-\lambda_2(m_2)}m_2[\lambda_1'(m_1)(1-m_1)-1]\\&\quad+e^{\lambda_2(m_2)-\lambda_1(m_1)}(1-m_2)[\lambda_1'(m_1)m_1-1],\\ \partial_{m_2}F(m_1,m_2)&=-1+e^{\lambda_1(m_1)-\lambda_2(m_2)}(1-m_1)[-\lambda_2'(m_2)m_2+1]\\&\quad+e^{\lambda_2(m_2)-\lambda_1(m_1)}m_1[-\lambda_2'(m_2)(1-m_2)+1].\end{aligned} \tag{3.13}$$

In what follows, given functions f_n and g_n on $\tilde{\Omega}$ with support in Λ_n^e and which can depend on the parameter $m\in(0,1)$, we write

$$f_n\approx g_n$$

if $\forall m\in(0,1)$

$$\lim_{n\uparrow\infty,k\uparrow\infty}\mathbb{E}\Big[V_k\Big(\frac{f_n-\mu^\alpha[f_n\mid m_n^e]}{n}-\frac{g_n-\mu^\alpha[g_n\mid m_n^e]}{n},\mu^{\alpha,\lambda_0(m)}\Big)\Big]=0.$$

3.2.1 Step 1

Proposition 3.3.

$$W_n \approx \mu^\alpha[W_n | m_n^{(1)}, m_n^{(2)}] \tag{3.14}$$

Proof. For simplicity of notation, we set $\mu \equiv \mu^{\alpha,\lambda_0(m)}$, $f_n \equiv W_n - \mu^\alpha[W_n|m_n^{(1)}, m_n^{(2)}]$ and we write $\mathcal{F}_{x,n}$ for the $\sigma-$algebra generated by the density in $\Lambda_{1,n}^e + x$, the density in $\Lambda_{2,n}^e + x$ and the particles configuration outside $\Lambda_n^e + x$.
By definition of $\mathcal{F}_{x,n}$ $\mu[\tau_x f_n | \mathcal{F}_{x,n}] = 0$ for any x.
Since for any site x and any function g

$$\mu(\tau_x f_n; g) = \mu\big(\mu[\tau_x f_n|\mathcal{F}_{x,n}]; g\big) + \mu\big(\mu[\tau_x f_n; g|\mathcal{F}_{x,n}]\big) = \mu\big(\mu[\tau_x W_n; g|\mathcal{F}_{x,n}]\big)$$

the variational characterization (1.46) corresponds to

$$V_k\Big(\frac{f_n}{n};\mu\Big) = n^{-2}(2k)^{-d} \sup_{\substack{g\in G_k \\ \mathcal{D}_{\Lambda_k}(g;\mu)\neq 0}} \frac{\Big(\sum_{|x|\le k_1} \mu\big(\,\mu[\tau_x W_n; g|\mathcal{F}_{x,n}]\,\big)\Big)^2}{\mathcal{D}_{\Lambda_k}(g;\mu)}. \tag{3.15}$$

By (3.11) W_n can be written as sum of expressions of the following form

$$\mathrm{Av}_{a\in\Lambda_{1,n}^e, b\in\Lambda_{2,n}^e} h_a h'_b,$$

where $h_a = \tau_a h_0$, $h'_b = \tau_b h'_0$ and h_0, h'_0 are functions only of α_0, η_0.

Therefore it's enough to prove for such functions h_a, h'_b that

$$\lim_{n\uparrow\infty}\lim_{k\uparrow\infty} n^{-2}(2k)^{-d}\mathbb{E}\Big[\sup_{\substack{g\in G_k \\ \mathcal{D}_{\Lambda_k}(g;\mu)\neq 0}} \frac{\Big(\sum_{|x|\le k_1} \mu\big(\,\mu[\mathrm{Av}_{a,b} h_a h'_b; g|\mathcal{F}_{x,n}]\,\big)\Big)^2}{\mathcal{D}_{\Lambda_k}(g;\mu)}\Big] = 0 \tag{3.16}$$

where we write $\mathrm{Av}_{a,b}$ in place $\mathrm{Av}_{a\in\Lambda_{1,n}^e, b\in\Lambda_{2,n}^e}$.

We fix $\delta > 0$. For simplicity of notation we write $\nu_{x,n} \equiv \mu[\cdot\,|\,\mathcal{F}_{x,n}]$. By proposition 2.11, with $L \equiv n$ and $n \ge l \ge l_0(\delta)$, and proposition 2.13

$$\nu_{x,n}(\mathrm{Av}_{a,b} h_a h'_b; g)^2 \le \frac{C(l)}{n^d}\mathcal{D}(g;\nu_{x,n}) + \frac{\delta}{n^d}\mathrm{Var}(g;\nu_{x,n}) + \frac{C}{n^d}\mathrm{Var}(g;\nu_{x,n})Y_{x,n}^\alpha \tag{3.17}$$

where $Y_{x,n}^\alpha \equiv Y_{\Lambda_{1,n}^e+x, \Lambda_{2,n}^e+x}^\alpha$, according to the notation of proposition 2.13.

We set

$$\rho_\alpha(\Lambda_{x,n}^e) \equiv \gamma_\alpha(\Lambda_{1,n}^e + x) + \gamma_\alpha(\Lambda_{2,n}^e + x).$$

Since $\nu_{x,n}(\eta)$ is the product between a canonical measure on $\Lambda_{1,n}^e + x$ and a canonical measure on $\Lambda_{2,n}^e + x$, (3.17) implies

$$\nu_{x,n}(\mathrm{Av}_{a,b}h_a h'_b; g)^2 \le n^{-d}\mathcal{D}(g;\nu_{x,n})\Big(C(l)+\delta\rho_\alpha(\Lambda^e_{x,n})n^2 + C\rho_\alpha(\Lambda^e_{x,n})Y^{\alpha,l}_{x,n}n^2\Big). \tag{3.18}$$

We indicate how to conclude the proof using the bound on the spectral gap uniformly in the disorder.
By Schwarz inequality and (3.18)

$$\Big[\sum_{|x|\le k_1}\mu\big(\nu_{x,n}(\mathrm{Av}_{a,b}h_a h'_b; g)\big)\Big]^2 \le$$
$$\Big[n^{-d}\sum_{|x|\le k_1}\mathcal{D}_{\Lambda^e_{x,n}}(g;\mu)\Big]\cdot\Big[k^d C(l)+C\delta k^d n^2 + C\sum_{|x|\le k_1}Y^{\alpha,l}_{x,n}n^2\Big].$$

By the arbitrariness of δ in order to prove (3.16) it's enough to show

$$\lim_{l\uparrow\infty}\lim_{n\uparrow\infty}\mathbb{E}\Big(Y^{\alpha,l}_{0,n}\Big)=0$$

and this follows from the estimate (2.40) of proposition 2.13.

We observe that the proof can be concluded with less information on the spectral gap. In particular, we can use only the estimates of section 2.3.
By Schwarz inequality, (3.18) and the arbitariness of δ in order to prove (3.16) it's enough to show

$$\sup_n \mathbb{E}\Big(\rho_\alpha(\Lambda^e_{0,n})\Big)<\infty \qquad \lim_{l\uparrow\infty}\lim_{n\uparrow\infty}\mathbb{E}\Big(\rho_\alpha(\Lambda^e_{0,n})Y^{\alpha,l}_{0,n}\Big).$$

The first estimate follows from proposition 2.21 while for the second one it's enough to observe

$$\mathbb{E}\Big(\rho_\alpha(\Lambda^e_{0,n})Y^{\alpha,l}_{0,n}\Big)\le\mathbb{E}\Big(\big[\rho_\alpha(\Lambda^e_{0,n})\big]^2\Big)+\mathbb{E}\Big(\big[Y^{\alpha,l}_{0,n}\big]^2\Big)$$

and the thesis follows by proposition 2.21 and estimate (2.40) of proposition 2.13.

□

3.2.2 Step 2

Proposition 3.4.

$$\mu^\alpha[W_n|m^{(1)}_n, m^{(2)}_n]\approx F^\alpha_n(m^{(1)}_n, m^{(2)}_n) \tag{3.19}$$

Proof. By theorem 2.4,

$$\|\mu^\alpha[w_{xy}\mid m^{(1)}_n, m^{(2)}_n]-\mu^{\alpha,\underline{\lambda}_n(\alpha,m_1,m_2)}(w_{xy})\|_\infty\le Cn^{-d}.$$

Since $\mu^\alpha[W_n \,|\, m_n^e] = 0$ the above estimate implies

$$\|\mu^\alpha[F_n^\alpha(m_n^{(1)}, m_n^{(2)}) \mid m_n^e]\|_\infty \leq C n^{-d}$$

and

$$\|\mu^\alpha[W_n | m_n^{(1)}, m_n^{(2)}] - F_n^\alpha(m_n^{(1)}, m_n^{(2)})\|_\infty \leq C n^{-d}.$$

Therefore it's enough to apply proposition 3.1 with $h_n \equiv C n^{-d}$.

□

3.2.3 Step 3

Proposition 3.5.
(3.20)

$$F_n^\alpha(m_n^{(1)}, m_n^{(2)}) \approx [\partial_{m_1} F_n^\alpha(m_n^e, m_n^e)(m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)] H_m(m_n^e)$$

where $H_m(m_n^e) \equiv \mathbb{I}_{\{|m_n^e - m| < a_m\}}$ *with* $a_m \equiv \frac{1}{2}(m \wedge (1-m))$.

Proof. Given $m \in (0,1)$ we set

$$X_m \equiv \{(m_1, m_2) \in Q \;:\; |m_1 - m| < a_m, |m_2 - m| < a_m\}$$

and $G_m(m_1, m_2) \equiv \mathbb{I}_{X_m}$.
We divide the proof in three steps.

Step A:

$$F_n^\alpha(m_n^{(1)}, m_n^{(2)}) \approx F_n^\alpha(m_n^{(1)}, m_n^{(2)}) G_m(m_n^{(1)}, m_n^{(2)})$$

Proof. As

$$|F_n^\alpha(m_n^{(1)}, m_n^{(2)})[1 - G_m(m_n^{(1)}, m_n^{(2)})]| \leq C\mathbb{I}_{\{|m_n^{(1)} - m| \geq a_m\}} + C\mathbb{I}_{\{|m_n^{(2)} - m| \geq a_m\}}$$

the thesis follows from propositions 3.1, 3.2.

Step B:

$$\begin{aligned} &F_n^\alpha(m_n^{(1)}, m_n^{(2)}) G_m(m_n^{(1)}, m_n^{(2)}) \approx \\ &[F_n^\alpha(m_n^e, m_n^e) + \partial_{m_1} F_n^\alpha(m_n^e, m_n^e)(m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)] \cdot \\ &G_m(m_n^{(1)}, m_n^{(2)}) \end{aligned}$$

Proof.
Given m_1, m_2 we define $\bar{m} = \frac{m_1+m_2}{2}$. We observe that if $(m_1, m_2) \in X_m$ then $(\bar{m}, \bar{m}) \in X_m$ and that if $m_1 = m_n^{(1)}$ and $m_2 = m_n^{(2)}$ then $\bar{m} = m_n^e$.
Since $F_n^\alpha \in C^\infty(\ (0,1) \times (0,1)\ )$, by Taylor expansion for any $(m_1, m_2) \in X_m$

$$
\begin{aligned}
&F_n^\alpha(m_1, m_2) = \\
&F_n^\alpha(\bar{m}, \bar{m}) + \partial_{m_1} F_n^\alpha(\bar{m}, \bar{m})(m_1 - \bar{m}) + \partial_{m_2} F_n^\alpha(\bar{m}, \bar{m})(m_2 - \bar{m}) + \frac{1}{2} R_n^\alpha(m_1, m_2)
\end{aligned} \tag{3.21}
$$

where

$$
\begin{aligned}
R_n^\alpha \equiv\ & \partial^2_{m_1} F_n^\alpha(\tilde{m}, \tilde{m})(m_1 - \bar{m})^2 + 2\partial_{m_1}\partial_{m_2} F_n^\alpha(\tilde{m}, \tilde{m})(m_1 - \bar{m})(m_2 - \bar{m}) + \\
& + \partial^2_{m_2} F_n^\alpha(\tilde{m}, \tilde{m})(m_2 - \bar{m})^2
\end{aligned} \tag{3.22}
$$

and $(\tilde{m}, \tilde{m})$ is a point in the segment between (m_1, m_2) and $(\bar{m}, \bar{m})$, and therefore in X_m.
Since

$$|R_n^\alpha(m_n^{(1)}, m_n^{(2)}) G_m(m_n^{(1)}, m_n^{(2)})| \le C(m)(m_n^{(1)} - m_n^e)^2 + C(m)(m_n^{(2)} - m_n^e)^2$$

by propositions 3.1 and 3.2

$$R_n^\alpha(m_n^{(1)}, m_n^{(2)}) G_m(m_n^{(1)}, m_n^{(2)}) \approx 0.$$

Step C:

$$
\begin{aligned}
&[F_n^\alpha(m_n^e, m_n^e) + \partial_{m_1} F_n^\alpha(m_n^e, m_n^e)(m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)] \cdot \\
&G_m(m_n^{(1)}, m_n^{(2)}) \approx [\partial_{m_1} F_n^\alpha(m_n^e, m_n^e)(m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)] H_m(m_n^e)
\end{aligned} \tag{3.23}
$$

Proof. Since

$$|H_m(m_n^e) - G_m(m_n^{(1)}, m_n^{(2)})| \le H_m(m_n^e),$$

by propositions 3.1 and 3.2 the l.h.s. of (3.23) is equivalent to

$$[F_n^\alpha(m_n^e, m_n^e) + \partial_{m_1} F_n^\alpha(m_n^e, m_n^e)](m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)] H_m(m_n^e)$$

Since $F_n^\alpha(m_n^e, m_n^e) H_m(m_n^e)$ is a function of m_n^e, by definition $F_n^\alpha(m_n^e, m_n^e) H_m(m_n^e) \approx 0$.
$\square$

3.2.4 Step 4

Proposition 3.6.

$$[\partial_{m_1} F_n^\alpha(m_n^e, m_n^e)(m_n^{(1)} - m_n^e) + \partial_{m_2} F_n^\alpha(m_n^e, m_n^e)(m_n^{(2)} - m_n^e)]H_m(m_n^e) \approx \tag{3.24}$$
$$\big[2m_n^e(1-m_n^e)\lambda'_{1,n}(m_n^e)[m_n^{(1)} - m_n^e] - 2m_n^e(1-m_n^e)\lambda'_{2,n}(m_n^e)[m_n^{(2)} - m_n^e]\big]H_m(m_n^e)$$

Proof. By (3.13)

$$\partial_{m_1} F_n^\alpha(m_n^e, m_n^e) - 2m_n^e(1-m_n^e)\lambda'_1(m_n^e) =$$
$$[e^{\lambda_1(m_n^e)-\lambda_2(m_n^e)} - 1]m_n^e[\lambda'_1(m_n^e)(1-m_n^e) - 1] + [e^{\lambda_2(m_n^e)-\lambda_1(m_n^e)} - 1](1-m_n^e)\cdot$$
$$[\lambda'_1(m_n^e)m_n^e - 1],$$

$$\partial_{m_2} F_n^\alpha(m_n^e, m_n^e) + 2m_n^e(1-m_n^e)\lambda'_2(m_n^e) =$$
$$[e^{\lambda_1(m_n^e)-\lambda_2(m_n^e)} - 1](1-m_n^e)[-\lambda'_2(m_n^e)m_n^e + 1] + [e^{\lambda_2(m_n^e)-\lambda_1(m_n^e)} - 1]m_n^e\cdot$$
$$[-\lambda'_2(m_n^e)(1-m_n^e) + 1].$$

For $i = 1,2$ and $\forall s \in [0,1]$, $s = \mu^{\alpha,\lambda_i(s)}(m_n^{(i)})$ and therefore $\lambda'_i(s) = \frac{|\Lambda_n^{i,e}|}{\mu^{\alpha,\lambda_i(s)}(N_{\Lambda_n^{i,e}}, N_{\Lambda_n^{i,e}})}$. By lemma $A.2$ if $|s-m| < a_m$ then

$$|\lambda'_i(s)| \le \frac{C(B)}{s \wedge (1-s)} \le A(m) - 1 \quad \text{where} \quad A(m) \equiv 1 + \frac{C(B)}{(m-a_m) \wedge (1-m-a_m)},$$

and

$$|(1-s)[\lambda'_i(s)s - 1]| \le A(m) \qquad |s[\lambda'_i(s)(1-s) - 1]| \le A(m).$$

Therefore it's enough to prove that, given $i \in \{1,2\}$ and given a function p_n on $(0,1)$ such that $\|p_n H_m\|_\infty \le A(m)$, then

$$[e^{\lambda_1(m_n^e)-\lambda_2(m_n^e)} - 1]p_n(m_n^e)H_m(m_n^e)(m_n^{(i)} - m_n^e) \approx 0,$$
$$[e^{\lambda_2(m_n^e)-\lambda_1(m_n^e)} - 1]p_n(m_n^e)H_m(m_n^e)(m_n^{(i)} - m_n^e) \approx 0.$$

As example we prove $g_n \approx 0$ where

$$g_n \equiv [e^{\lambda_1(m_n^e)-\lambda_2(m_n^e)} - 1]p(m_n^e)H_m(m_n^e)(m_n^{(i)} - m_n^e).$$

Since, $\forall a \in \mathbb{R}$, $|e^a - 1| = |\int_0^a e^s ds| \le \max(1, e^a)|a| \le e^{|a|}|a|$,

$$\begin{aligned}|g_n| &\le C(m)H_m(m_n^e)|\lambda_1(m_n^e) - \lambda_2(m_n^e)|\,|m_n^{(i)} - m_n^e| \\ &\le 2C(m)H_m(m_n^e)(\lambda_1(m_n^e) - \lambda_2(m_n^e))^2 + 2C(m)(m_n^{(i)} - m_n^e)^2 \\ &\le \sum_{j=1}^{2} 4C(m)H_m(m_n^e)(\lambda_j(m_n^e) - \lambda_0(m_n^e))^2 + 2C(m)(m_n^{(i)} - m_n^e)^2\end{aligned}$$

By lemma $A.6$, for $j = 1, 2$,

$$|\lambda_j(m_n^e) - \lambda_0(m_n^e)| H_m(m_n^e) \leq C(m) |m_n^e - \mu^{\alpha,\lambda_0(m_n^e)}(m_n^{(j)})|.$$

Therefore

$$|g_n| \leq \sum_{j=1}^{2} C(m)[m_n^e - \mu^{\alpha,\lambda_0(m_n^e)}(m_n^{(j)})]^2 + C(m)[m_n^{(i)} - m_n^e]^2$$

and the thesis follows by proposition 3.1 and 3.2.

□

3.2.5 Step 5

Proposition 3.7.

(3.25)
$$\left[2m_n^e(1-m_n^e)\lambda'_{1,n}(m_n^e)[m_n^{(1)} - m_n^e] - 2m_n^e(1-m_n^e)\lambda'_{2,n}(m_n^e)[m_n^{(2)} - m_n^e]\right] H_m(m_n^e) \approx 2m_n^e(1-m_n^e)\lambda'_0(m)[m_n^{(1)} - m_n^{(2)}] H_m(m_n^e)$$

Proof. In equation (3.25) the r.h.s. is obtained by the l.h.s. by substituting $\lambda'_{1,n}(m_n^e)$ and $\lambda'_{2,n}(m_n^e)$ by $\lambda'_0(m)$.
Therefore it's enough to prove that given $i \in \{1, 2\}$ and given a bounded function p on $(0, 1)$ then

$$p(m_n^e) H_m(m_n^e)[\lambda'_i(m_n^e) - \lambda'_0(m)](m_n^{(i)} - m_n^e) \approx 0.$$

By proposition 3.1, it's enough to prove

$$\lim_{n\uparrow\infty} n^{\frac{3}{2}d} \mathbb{E}\Big[\, \mu^{\alpha,\lambda_0(m)}\big(H_m(m_n^e)[\lambda'_i(m_n^e) - \lambda'_0(m)]^2[m_n^{(i)} - m_n^e]^2\big)\,\Big] = 0.$$

Since

$$H_m(m_n^e)[\lambda'_i(m_n^e) - \lambda'_0(m)]^2[m_n^{(i)} - m_n^e]^2 \leq 2H_m(m_n^e)[\lambda'_i(m_n^e) - \lambda'_0(m)]^4 + 2[m_n^{(i)} - m_n^e]^4$$

by proposition 3.2 it's enough to prove

$$\lim_{n\uparrow\infty} n^{\frac{3}{2}d} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(H_m(m_n^e)[\lambda'_i(m_n^e) - \lambda'_0(m)]^4\big)\Big] = 0. \tag{3.26}$$

Since $s = \mu^{\alpha,\lambda_i(s)}(m_n^{(i)})$ and $s = \mathbb{E}\Big[\mu^{\alpha,\lambda_0(s)}(\eta_0)\Big]$,

$$\lambda'_i(s) = \frac{|\Lambda_n^{i,e}|}{\mu^{\alpha,\lambda_i(s)}(N_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}})} \qquad \lambda'_0(s) = \frac{1}{\mathbb{E}\Big[\mu^{\alpha,\lambda_0(s)}(\eta_0;\eta_0)\Big]}. \tag{3.27}$$

By (3.27) and lemma $A.2$, $|m_n^e - m| < a_m$ implies $\lambda_i'(m_n^e) \le C(m)$ and then

$$|\lambda_i'(m_n^e) - \lambda_0'(m)| \le A(m)\Big|\frac{1}{\lambda_i'(m_n^e)} - \frac{1}{\lambda_0'(m)}\Big|.$$

In particular, we obtain that (3.26) is implied by

$$\lim_{n\uparrow\infty} n^{\frac{3}{2}d}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(H_m(m_n^e)[\mu^{\alpha,\lambda_i(m_n^e)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}}) - \mathbb{E}\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)]^4\ \big)\Big] = 0. \tag{3.28}$$

For a particular constant C

$$H_m(m_n^e)[\mu^{\alpha,\lambda_i(m_n^e)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}}) - \mathbb{E}\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)]^4 \le C(A_n^{(1)} + A_n^{(2)} + A_n^{(3)})$$

where

$$\begin{aligned} A_n^{(1)} &= H_m(m_n^e)\Big[\mu^{\alpha,\lambda_i(m_n^e)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}}) - \mu^{\alpha,\lambda_i(m)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}})\Big]^4 \\ A_n^{(2)} &= \Big[\mu^{\alpha,\lambda_i(m)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}}) - \mu^{\alpha,\lambda_0(m)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}})\Big]^4 \\ A_n^{(3)} &= \Big[\mu^{\alpha,\lambda_0(m)}(m_{\Lambda_n^{i,e}}; N_{\Lambda_n^{i,e}}) - \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)\Big]\ \Big]^4. \end{aligned} \tag{3.29}$$

We prove that $\lim_{n\uparrow\infty} n^{\frac{3}{2}d}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(A_n^{(j)}\big)\Big] = 0$ for $j = 1,2,3$. This result permits to conclude the proof.

By lemma $A.4$

$$A_n^{(1)} \le C[m_n^e - m]^4 \quad \text{and} \quad A_n^{(2)} \le C[m - \mu^{\alpha,\lambda_0(m)}(m_n^{(i)})]^4$$

and by proposition 3.2 $\lim_{n\uparrow\infty} n^{\frac{3}{2}d}\mathbb{E}\mu^{\alpha,\lambda_0(m)}\big(A_n^{(i)}\big) = 0$ for $i = 1,2$.

Let $g_x(\alpha) = \mu^{\alpha,\lambda_0(m)}(\eta_x;\eta_x) - \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_x;\eta_x)\Big]$. We observe that g_x depends only on α_x and $\mathbb{E}(g_x) = 0$.

Then

$$A_n^{(3)} = [\mathrm{Av}_{x\in\Lambda_{i,n}} g_x]^4 \le Cn^{-2d}$$

and therefore $\lim_{n\uparrow\infty} n^{\frac{3}{2}d}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(A_n^{(3)}\big)\Big] = 0$.

□

3.2.6 Step 6

Proposition 3.8.

$$2m_n^e(1-m_n^e)\lambda_0'(m)[m_n^{(1)}-m_n^{(2)}]H_m(m_n^e) \approx 2m(1-m)\lambda_0'(m)[m_n^{(1)}-m_n^{(2)}] \tag{3.30}$$

Proof. We set

$$g_n \equiv 2[m_n^e(1-m_n^e)-m(1-m)]\lambda_0'(m)[m_n^{(1)}-m_n^{(2)}]H_m(m_n^e).$$

Then

$$\begin{aligned} &g_n^2 \le C(m)[m_n^e(1-m_n^e)-m(1-m)]^2[m_n^{(1)}-m_n^{(2)}]^2 \le \\ &C'(m)[m-m_n^e]^4 + C'(m)[m-m_n^{(1)}]^4 + C'(m)[m-m_n^{(2)}]^4. \end{aligned}$$

Therefore, by propositions 3.1 and 3.2, $g_n \approx 0$.
It remains to prove

$$2m(1-m)\lambda_0'(m)[m_n^{(1)}-m_n^{(2)}]H_m(m_n^e) \approx 2m(1-m)\lambda_0'(m)[m_n^{(1)}-m_n^{(2)}]$$

and this follows from propositions 3.1, 3.2.

□

Steps $1,\dots,6$ imply the following theorem.

Theorem 3.9. *For any $e>0$*

$$\lim_{n\uparrow\infty}\lim_{k\uparrow\infty}\mathbb{E}\Big[V_k\Big(\frac{W_n^e}{n}-2m(1-m)\lambda_0'(m)\frac{\psi_{n,n}^e}{n};\mu^{\alpha,\lambda_0(m)}\Big)\Big]=0. \tag{3.31}$$

Lemma 3.10. *The function $(0,1)\ni m \to m(1-m)\lambda_0'(m)\in\mathbb{R}$ can be extended with continuity in $0,1$. In particular,*

$$\lim_{m\downarrow 0} m(1-m)\lambda_0'(m)=1 \qquad \textit{and} \qquad \lim_{m\uparrow 1} m(1-m)\lambda_0'(m)=1.$$

Proof. Since $\lambda_0'(m)=\frac{1}{\mathbb{E}\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)}$ and $m=\mathbb{E}\mu^{\alpha,\lambda_0(m)}(\eta_0)$ we have to prove

$$\lim_{\lambda\downarrow-\infty}\frac{\mathbb{E}\mu^{\alpha,\lambda}(\eta_0)}{\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)}=1 \qquad \lim_{\lambda\uparrow\infty}\frac{\mathbb{E}\mu^{\alpha,\lambda}(1-\eta_0)}{\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)}=1.$$

These limits are consequence of the following estimates.

$$\frac{|\mathbb{E}\mu^{\alpha,\lambda}(\eta_0)-\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)|}{\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)}=\frac{\int\mathbb{P}(d\alpha_0)\big(\frac{e^{\alpha_0+\lambda}}{1+e^{\alpha_0+\lambda}}\big)^2}{\int\mathbb{P}(d\alpha_0)\frac{e^{\alpha_0+\lambda}}{(1+e^{\alpha_0+\lambda})^2}}\le e^{B+\lambda},$$

$$\frac{|\mathbb{E}\mu^{\alpha,\lambda}(1-\eta_0)-\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)|}{\mathbb{E}\mu^{\alpha,\lambda}(\eta_0;\eta_0)}=\frac{\int\mathbb{P}(d\alpha_0)\ \frac{1}{(1+e^{\alpha_0+\lambda})^2}}{\int\mathbb{P}(d\alpha_0)\frac{e^{\alpha_0+\lambda}}{(1+e^{\alpha_0+\lambda})^2}}\le e^{B-\lambda}.$$

□

Chapter 4

Fluctuations in the gradient of the density field

In this chapter we follow the notation introduced in section 1.9. In particular, let us remember that given an odd number $r = 2r' + 1$ and $e > 0$, we write

$$\Lambda^e_{1,r} \equiv \Lambda_{-(r'+1)e,r'} \quad \Lambda^e_{2,r} \equiv \Lambda_{r'e,r'} \quad \Lambda^e_r \equiv \Lambda^e_{1,r} \cup \Lambda^e_{2,r} \quad \Lambda^e_{x,r} \equiv \Lambda^e_r + x.$$

For example, if $d = 1$, $\Lambda_{1,r} = \{-r, \ldots, -1\}$ and $\Lambda_{2,r} = \{0, \ldots, r-1\}$.

We set, with $j = 1, 2$,

$$m^e_{x,r} \equiv m_{\Lambda^e_{x,r}} \quad N^e_{x,r} \equiv N_{\Lambda^e_{x,r}} \quad m^{j,e}_r \equiv m_{\Lambda^e_{j,r}}.$$

Moreover, we will often use the properties of the telescopic averages $\mathrm{Av}^{n,s}_{z,x} b_z$, $\mathrm{Av}^{n,R,Q}_{z,x} b_z$ and $\mathrm{Av}^*_{z,x} b_z$ defined respectively in (1.31), (1.40) and (1.42). We remember that $\mathrm{Av}^{n,s}_{z,x} b_z$ requires n, s to be odd numbers with $n|s$. The other averages needs similar properties. We always assume these conditions to be satisfied. The general case can be solved in the same spirit of remark 1.16.

In what follows we fix $e > 0$ which will be often understood. We remember that, if $0 < n \leq s$,

$$\phi_{n,s} \equiv \mu^\alpha[m^{1,e}_n - m^{2,e}_n | m^e_s]$$
$$\psi_{n,s} \equiv m^{1,e}_n - m^{2,e}_n - \phi_{n,s}.$$

We introduce, for $n \leq s$, the function $\hat{\phi}_{n,s}$:

$$\hat{\phi}_{n,s}(\eta) = \mu^{\alpha, \lambda_{\Lambda^e_s}(\alpha, m^e_s(\eta))}(m^{1,e}_n - m^{2,e}_n).$$

In this chapter, given an integer s, we set

$$I_s \equiv \{0, \frac{1}{2s^d}, \frac{2}{2s^d}, \ldots, 1\}. \tag{4.1}$$

Moreover, given a disorder configuration α and a set $\Lambda \in \mathbb{F}$, we denote $\mathcal{M}_\alpha(\Lambda)$ the set of canonical measures on Λ with disorder α.

4.1 The main theorem

This chapter is devoted to the proof of the following statement.

Theorem 4.1. *Let G be a real continuous function on $[0,1]$. Then, given $e>0$ and a positive number γ_0, for almost any configuration disorder α*

$$(4.2)\quad \limsup_{n\uparrow\infty,c\downarrow 0,a\downarrow 0,\epsilon\downarrow 0}\ \sup_{J\in C^*(\mathbb{T}^d)}$$
$$\sup spec_{L^2(\mu_\epsilon^\alpha)}\big\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}J(\epsilon x)G(m_x^{a/\epsilon})\mathrm{Av}^*_{z,x}\tau_z\frac{\phi_{n,n}}{n}+\gamma_0\epsilon^{d-2}\mathcal{L}\big\}\leq 0.$$

In this section we reduce theorem 4.1 to a simpler form. To this aim we introduce some notation. Given functions $\{f_x\}_{x\in\mathbb{T}^d_\epsilon}$ and $\{g_x\}_{x\in\mathbb{T}^d_\epsilon}$ depending on the parameter a,n,ϵ we write

$$f_x\approx g_x$$

if, given $\gamma_0>0$, for almost any disorder configuration α

$$(4.3)\quad \limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{|C_0|\leq 1}\ \sup_\nu\ \sup spec_{L^2(\nu)}\big\{\epsilon^{-1}C_0[f_x-g_x]+\gamma_0\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\big\}\leq 0$$

where ν varies among $\mathcal{M}_\alpha(\Lambda_{x,2\frac{a}{\epsilon}})$.

Theorem 4.2.
$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{n,n}}{n}\approx 0.$$

We first prove that theorem 4.1 follows from theorem 4.2. At the end of the section we sketch the proof of theorem 4.2, which is based on the results of next sections.

For simplicity of notation we set $R\equiv\frac{a}{\epsilon}$ and $Q\equiv\frac{c}{\epsilon}$. Moreover, given an integer u and $x\in\mathbb{T}^d_\epsilon$, we define $x_u\equiv x+(-Q+uR+R'+1)e$.
Since in general

$$\mathrm{Av}^{n,R,Q}_{z,x}b_z\equiv\mathrm{Av}_{u=0}^{\frac{2Q}{R}-1}\mathrm{Av}^{n,R}_{z,x_u}b_z,$$

the supremum of the spectrum in (4.2) is bounded by $\mathrm{Av}_{u=0}^{\frac{2Q}{R}-1}F_u$ where

$$F_u\equiv\sup spec_{L^2(\mu_\epsilon^\alpha)}\big\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}J(\epsilon x)G(m_x^R)\mathrm{Av}^{n,R}_{z,x_u}\tau_z\frac{\phi_{n,n}}{n}+\gamma_0\epsilon^{d-2}\mathcal{L}\big\}.$$

We assume $R>n$. Then $\mathrm{Av}^{n,R}_{z,x_u}\tau_z\frac{\phi_{n,n}}{n}$ has support in $\Lambda^e_{R+n}+x_u\subset\Lambda_{x_u,2R}$.
Let

$$U\equiv\{u\,:\,|-Q+uR+R'+1|\leq 3R\}.$$

Then $|U| \leq 6$ and if $u \notin U$ then $\Lambda_{x_u,2R}$ and Λ_x^R are disjoint.
We set

$$R_{x,u} \equiv \begin{cases} 20R & \text{if } \Lambda_{x_u,2R} \cap \Lambda_x^R \neq \emptyset \\ R & \text{otherwise.} \end{cases}$$

Then

$$F_u \leq F_u^{(1)} + F_u^{(2)} + F_u^{(3)}$$

where

$$F_u^{(i)} \equiv \sup spec_{L^2(\mu_\epsilon^\alpha)} \{ \epsilon^{-1} \mathrm{Av}_{x \in \mathbb{T}_\epsilon^d} J(\epsilon x) H_{x,u}^{(i)} + \frac{\gamma_0}{3} \epsilon^{d-2} \mathcal{L} \}$$

and

$$H_{x,u}^{(1)} \equiv G(m_x^{R_u}) \mathrm{Av}_{z,x_u}^{n,R} \tau_z \frac{\phi_{n,n}}{n}$$
$$H_{x,u}^{(2)} \equiv \big(G(m_x^R) - G(m_x^{R_u}) \big) \mathrm{Av}_{z,x_u}^{n,R} \tau_z \frac{-\psi_{n,n}}{n}$$
$$H_{x,u}^{(3)} \equiv \big(G(m_x^R) - G(m_x^{R_u}) \big) \mathrm{Av}_{z,x_u}^{n,R} \tau_z \frac{m_n^{1,e} - m_n^{2,e}}{n}.$$

By means of lemma 4.3 and 4.4, it's enough to prove theorem 4.2 in order to prove theorem 4.1.

Lemma 4.3. *For almost any disorder configuration α, for $i = 2, 3$,*

$$\limsup_{c\downarrow 0, a\downarrow 0, \epsilon\downarrow 0} \sup_{J \in C^*(\mathbb{T}^d)} \mathrm{Av}_{u=0}^{\frac{2Q}{R}-1} F_u^{(i)} \leq 0. \tag{4.4}$$

Proof. We first consider the case $i = 2$.
We set $Y \equiv \{ y \in n\mathbb{Z}^d \ : \ y \in \Lambda_{0,R'} \}$. By (1.31), for a particular integer $w = w(n, R, Q, u)$,

$$\mathrm{Av}_{x \in \mathbb{T}_\epsilon^d} \Big(a_x \cdot \mathrm{Av}_{z,x_u}^{n,R} b_z \Big) = \mathrm{Av}_{x \in \mathbb{T}_\epsilon^d} \Big(b_x \cdot \big[\mathrm{Av}_{v=0}^{\frac{R}{n}-1} \mathrm{Av}_{y \in Y} a_{x+y+(w-vn)e} \big] \Big).$$

We set

$$B_{x,u} \equiv \mathrm{Av}_{v=0}^{\frac{R}{n}-1} \mathrm{Av}_{y \in Y} a_{x+y+(w-vn)e} \qquad \text{where} \quad a_z \equiv J(\epsilon z)[G(m_z^R) - G(m_z^{R_{z,u}})] \ \ \forall z \in \mathbb{T}_\epsilon^d,$$

then

$$F_u^{(2)} = \sup spec_{L^2(\mu_\epsilon^\alpha)} \{ \epsilon^{-1} \mathrm{Av}_{x \in \mathbb{T}_\epsilon^d} \big(B_{x,u} \cdot \tau_x \frac{-\psi_{n,n}}{n} \big) + \frac{\gamma_0}{3} \epsilon^{d-2} \mathcal{L} \}.$$

By the same arguments used to estimate $\nabla_b B_x^{(3)}$ in section 1.17 we obtain that for any bond b in Λ_n

$$\|\nabla_b B_{x,u}\|_\infty \le C\frac{n\epsilon}{a}\mathrm{Osc}\Big(G, C(d)\frac{\epsilon^d}{a^d}\Big).$$

Since $\|B_{x,u}\|_\infty \le C(G,J)$ and $\psi_{n,n} \in \mathcal{G}_n$, by proposition 1.15 and the property that $F_u^{(2)} \ne 0$ for $O(1)$ possible values of u,

$$\mathrm{Av}_{u=0}^{\frac{2Q}{R}-1} F_u^{(2)} \le C(n,a,c,\gamma_0,G,J)\mathrm{Osc}\Big(G, \frac{\epsilon^d}{a^d}\Big) + C(n,G,J,\gamma_0)\frac{a}{c}$$

which implies the thesis.

We consider the case $i = 3$. We write $R = 2R' + 1$, then by (1.32),

$$\mathrm{Av}_{u=0}^{\frac{2Q}{R}-1} F_u^{(3)} \le \frac{1}{c}\sup spec_{L^2(\mu_\epsilon^\alpha)}\big\{C(J,G)\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,R'} - m_{x+Re,R'}| + a\frac{\gamma_0}{3}\epsilon^{d-2}\mathcal{L}\big\}.$$

We introduce the parameter s with $c \downarrow 0, s \uparrow \infty, a \downarrow 0, \epsilon \downarrow 0$. Since

$$m_{x,R'} = \mathrm{Av}_{y\in\Lambda_{x,R'}} m_{y,s} + O(s/R'),$$

it's enough to prove that, for almost any disorder configuration α and for any $\gamma > 0$,

$$\limsup_{s\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \sup spec_{L^2(\mu_\epsilon^\alpha)}\big\{\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}|m_{x,s} - m_{x+Re,s}| + \gamma a\epsilon^{d-2}\mathcal{L}\big\} \le 0. \tag{4.5}$$

We use arguments standard for Two-Blocks estimates. In particular, by the Moving Particle Lemma

$$\epsilon^{d-2}\mathcal{L} \le Ca^{-2}\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\mathrm{Av}_{y\in\Lambda_{x,s}}\mathcal{L}_{y,y+Re}$$

and the supremum of the spectrum in (4.5) can be bounded by

$$\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_{\nu\in\mathcal{M}_\alpha(\Lambda_{x,s}\cup\Lambda_{x+Re,s})} a^{-1}\sup spec_{L^2(\nu)}\big\{a|m_{x,s} - m_{x+Re,s}| + \gamma C\mathrm{Av}_{y\in\Lambda_{x,s}}\mathcal{L}_{y,y+Re}\big\}.$$

By perturbation theory, in particular by proposition A.13, it's enough to prove that, for almost any disorder configuration α,

$$\lim_{s\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_{\nu\in\mathcal{M}_\alpha(\Lambda_{x,s}\cup\Lambda_{x+Re,s})} \nu\big(|m_{x,s} - m_{x+Re,s}|\big) = 0.$$

By lemma $A.8$ the thesis follows from the limit

$$\lim_{s\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d}\big(m - \mu^{\alpha,\lambda_0(m)}(m_{x,s})\big)^2 = 0.$$

□

Lemma 4.4. *Let A, B be two boxes in $\mathbb{T}^d_\epsilon$ and g, h be a couple of bounded measurable functions depending respectively on m_A and η_B.*

If $A \cap B = \emptyset$ or $B \subset A$ then

(4.6)
$$\sup spec_{L^2(\mu^\alpha_\epsilon)}\big\{g(m_A)h(\eta_B) + C\mathcal{L}_B\big\} \le \sup_{|t|\le\|g\|_\infty} \sup_{m\in[0,1]} \sup spec_{L^2(\mu^\alpha_{B,m})}\{th(\eta_B) + C\mathcal{L}_B\big\}.$$

Proof. We set $\mu \equiv \mu^\alpha_\epsilon$ and for a generic $X \in \mathbb{F}$

$$\mathcal{F}_X \equiv \sigma(m_X; \{\eta_y\}_{y\notin X}).$$

The l.h.s. of (4.6) corresponds to

$$\sup_f \Big(\mu(g(m_A)h(\eta_B)f) - C\mathcal{D}_B(\sqrt{f};\mu)\Big)$$

where the supremum is carried over the density functions f for μ.

We consider the case $A \cap B = \emptyset$. We set $\nu_\tau = \mu(\cdot\,|\,\mathcal{F}_B)(\tau)$. Then the thesis follows by the identity

$$\mu(ghf) - C\mathcal{D}_B(\sqrt{f};\mu) = \int \mu(d\tau)\nu_\tau(f)\Big[g\big(m_A(\tau)\big)\nu_\tau\Big(h\frac{f}{\nu_\tau(f)}\Big) - \mathcal{D}_B\Big(\sqrt{\frac{f}{\nu_\tau(f)}};\nu_\tau\Big)\Big].$$

We consider the case $B \subset A$.
Since $\mu(F) = \mu\big(\mu(F\,|\,\mathcal{F}_A)\big)$ for any function F, by the above argument we obtain that the l.h.s. of (4.6) is bounded by

$$\sup_m \sup spec_{L^2(\mu_{A,m})}\{g(m)h(\eta_B) + C\mathcal{L}_B\big\}$$

Since $\mu_{A,m}(F) = \mu_{A,m}\big(\mu_{A,m}(F\,|\,\mathcal{F}_B)\big)$ for any function F, by the usual argument, we obtain the thesis.

□

4.1.1 Roadmap of the proof of theorem 4.2.

We sketch the proof of theorem 4.2, which is based on the results of next sections and is valid for $d \ge 3$.

Let $s_0 = O\big(\epsilon^{-\frac{1}{d+2}}\big)$. By proposition 4.16

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\Big[\frac{\phi_{n,n}}{n} - \frac{\phi_{n,s_0}}{n}\Big] \approx 0.$$

Moreover, by proposition 4.19,

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{n,s_0}}{n} \approx \mathrm{Av}^{s_0,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{s_0,s_0}}{s_0}.$$

By lemma 4.20, if

$$s \equiv \epsilon^{-\frac{\mathfrak{a}}{d+2}} \qquad s' \equiv \epsilon^{-\frac{\mathfrak{b}}{d+2}} \qquad 1 \le \mathfrak{a} < \mathfrak{b} < d+2 \qquad \mathfrak{b} < 1 + \frac{\mathfrak{a}}{2},$$

then

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} \approx \mathrm{Av}_{z,x}^{s',\frac{a}{\epsilon}} \tau_z \frac{\phi_{s',s'}}{s'}.$$

Therefore, as stated by proposition 4.21, for any γ small enough we can find

$$1 < \mathfrak{a}_1 < \mathfrak{a}_2 < \cdots < \mathfrak{a}_r < 2$$

with $\mathfrak{a}_r > 2-\gamma$ such that, setting $s_i = O(\epsilon^{-\frac{\mathfrak{a}_i}{d+2}})$ for $i = 1, \dots r$,

$$\mathrm{Av}_{z,x}^{s_i,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_i,s_i}}{s_i} \approx \mathrm{Av}_{z,x}^{s_{i+1},\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_{i+1},s_{i+1}}}{s_{i+1}} \qquad \forall i = 0, \dots, r.$$

It remains to prove that, for γ small enough,

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} \approx 0 \qquad s = O(\epsilon^{-\frac{2}{d+2}+\gamma}).$$

This last statement follows from proposition 4.25 which is an "exotic" Two Block estimate.

4.2 An estimate of $\nabla_y\left[\tau_x \hat{\phi}_{n,s}\right]$

In what follows n, s are positive integers with $n \le s$. Let $0 < \gamma < d$ and $m \in (0,1)$ such that

$$a_m \ge 4 s^{-d+\gamma} \quad \text{where } a_m \equiv \min(m, 1-m). \tag{4.7}$$

Given $m' \in [0,1]$ we define

$$\begin{aligned}
\xi(\alpha, m') &\equiv \mu^{\alpha,\lambda_{\Lambda_s^e}(\alpha,m')}(\, m_n^{1,e} - m_n^{2,e}\,; N_n^e\,)\\
\xi_0(\alpha, m') &\equiv \mu^{\alpha,\lambda_0(m')}(\, m_n^{1,e} - m_n^{2,e}\,; N_n^e\,)\\
\sigma^2(\alpha, m') &\equiv \mu^{\alpha,\lambda_{\Lambda_s^e}(\alpha,m')}(\, m_s^e; N_s^e\,)\\
\sigma_0^2(\alpha, m') &\equiv \mu^{\alpha,\lambda_0(m')}(\, m_s^e; N_s^e\,)\\
\sigma_0^2(m') &\equiv \mathbb{E}\left(\mu^{\alpha,\lambda_0(m)}(\,\eta_0;\eta_0\,)\right).
\end{aligned} \tag{4.8}$$

Let us remember the definition

$$\hat{\phi}_{n,s}(\eta) = \mu^{\alpha,\lambda_{\Lambda_s^e}(\alpha,m_s^e(\eta))}(m_n^{1,e} - m_n^{2,e}).$$

We observe that by the equivalence of ensembles $\|\phi_{n,s} - \hat{\phi}_{n,s}\|_\infty \le C s^{-d}$.

In this section we concentrate on the quantity $\nabla_y \tau_x \hat{\phi}_{n,s}(\eta)$ (which is zero if $y \notin \Lambda_{x,s}^e$) under the assumption $\eta \notin \mathcal{M}_x(m)$ and $\alpha \notin \mathcal{A}_x^{(2)}(m) \cup \mathcal{A}_x^{(3)}(m)$ where

$$
\begin{aligned}
&\mathcal{M}_x(m)\equiv\{|m^e_{x,s}(\eta)-m|\geq\sqrt{a_m}s^{-\frac{d}{2}+\frac{\gamma}{2}}\}\\
&\mathcal{A}^{(2)}_x(m)\equiv\{\frac{1}{a_m}|m-\tau_x\,\mu^{\alpha,\lambda_0(m)}(m^e_s)|\geq s^{-\frac{d}{2}+\frac{\gamma}{2}}\}\\
&\mathcal{A}^{(3)}_x(m)\equiv\{|\tau_x\frac{\sigma_0^2(\alpha,m)}{\sigma_0^2(m)}-1|\geq s^{-\frac{d}{2}+\frac{\gamma}{2}}\}.
\end{aligned}
\tag{4.9}
$$

Let I be the interval with extremes $m^e_{x,s}(\eta)$ and $m^e_{x,s}(\eta^y)$. Since $\eta\notin\mathcal{M}_x(m)$, it's simple to check that if $s\geq s_0(d,\gamma)$ then $\forall m'\in I$

$$
a_{m'}\geq\frac{a_m}{4}.
\tag{4.10}
$$

In what follows we assume $s\geq s_0(d,\gamma)$. Then we can write

$$
\nabla_y\tau_x\hat\phi_{n,s}(\eta)=\int_{m^e_{x,s}(\eta)}^{m^e_{x,s}(\eta^y)}\tau_x\frac{\xi(\alpha,m')}{\sigma^2(\alpha,m')}dm'.
\tag{4.11}
$$

By lemma $A.2$, for some $C>0$,

$$
\xi(\alpha,m'),\ \xi_0(\alpha,m')\leq Ca_{m'}\quad\sigma^2(\alpha,m'),\ \sigma_0^2(\alpha,m'),\ \sigma_0^2(m')\geq C^{-1}a_{m'}
$$

By lemma $A.4$ and $A.5$,

$$
\begin{aligned}
|\tau_x\xi(\alpha,m')-\tau_x\xi_0(\alpha,m')|&\leq C|m'-\mu^{\alpha,\lambda_0(m')}(m^e_{x,s})|\\
|\tau_x\sigma^2(\alpha,m')-\tau_x\sigma_0^2(\alpha,m')|&\leq C|m'-\mu^{\alpha,\lambda_0(m')}(m^e_{x,s})|\\
|\tau_x\xi_0(\alpha,m')-\tau_x\xi_0(\alpha,m)|&\leq C|m'-m|\\
|\tau_x\sigma_0^2(\alpha,m')-\tau_x\sigma_0^2(\alpha,m)|&\leq C|m'-m|.
\end{aligned}
\tag{4.12}
$$

In what follows we often use the obvious estimate $|\frac{a}{b}-\frac{A}{B}|\leq\frac{a}{bB}|b-B|+\frac{1}{B}|a-A|$.
By (4.12),

$$
\Big|\tau_x\frac{\xi(\alpha,m')}{\sigma^2(\alpha,m')}-\tau_x\frac{\xi_0(\alpha,m')}{\sigma_0^2(\alpha,m')}\Big|\leq\frac{C}{a_{m'}}|m'-\mu^{\alpha,\lambda_0(m')}(m^e_{x,s})|
\tag{4.13}
$$

$$
\Big|\tau_x\frac{\xi_0(\alpha,m')}{\sigma_0^2(\alpha,m')}-\tau_x\frac{\xi_0(\alpha,m)}{\sigma_0^2(\alpha,m)}\Big|\leq\frac{C}{a_m}|m'-m|\leq\frac{C}{a_m}|m^e_{x,s}(\eta)-m|+\frac{C}{a_m}s^{-d}
\tag{4.14}
$$

$$
\Big|\tau_x\frac{\xi_0(\alpha,m)}{\sigma_0^2(\alpha,m)}-\tau_x\frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\Big|\leq C|\tau_x\frac{\sigma_0^2(\alpha,m)}{\sigma_0^2(m)}-1|.
\tag{4.15}
$$

Let $f_{m'}\equiv|m'-\mu^{\alpha,\lambda_0(m')}(m^e_{x,s})|$. By lemma $A.2$ $f_{m'}\leq Ca_{m'}$. Let I the interval with extremes $m^e_{x,s}(\eta)$ and $m^e_{x,s}(\eta^y)$. By (4.10) and lemma $A.5$, if $m',m''\in I$,

$$|\frac{f_{m'}}{a_{m'}} - \frac{f_{m''}}{a_{m''}}| \leq \frac{C}{a_{m''}}|a_{m'} - a_{m''}| + \frac{C}{a_{m''}}|m' - m''| \leq \frac{C'}{a_m}s^{-d}. \tag{4.16}$$

Therefore, the r.h.s. of (4.13) can be estimated by

$$\frac{C}{a_m}s^{-d} + \frac{C}{a_m}|m^e_{x,s}(\eta) - \mu^{\alpha,\lambda_0(m^e_{x,s}(\eta))}(m^e_{x,s})|. \tag{4.17}$$

By (4.11) and the previous estimates,

(4.18)

$$\Big|\nabla_y \tau_x \hat{\phi}_{n,s}(\eta) - (1-2\eta_y)(2s^d)^{-1}\tau_x \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\Big| \leq$$

$$Cs^{-d}\Big\{\frac{s^{-d}}{a_m} + \frac{1}{a_m}|m^e_{x,s}(\eta) - \mu^{\alpha,\lambda_0(m^e_{x,s}(\eta))}(m^e_{x,s})| + \frac{1}{a_m}|m^e_{x,s}(\eta) - m| + |\tau_x \frac{\sigma_0^2(\alpha,m)}{\sigma_0^2(m)} - 1|\Big\}.$$

By lemma $A.5$,

$$|m^e_{x,s}(\eta) - \mu^{\alpha,\lambda_0(m^e_{x,s}(\eta))}(m^e_{x,s})| \leq C|m^e_{x,s}(\eta) - m| + C|m - \mu^{\alpha,\lambda_0(m)}(m^e_{x,s})|. \tag{4.19}$$

The following lemma follows easily from (4.9), (4.18) and (4.19).

Lemma 4.5. *Let n, s be positive integers with $n \leq s$. Let $0 < \gamma < d$ and $m \in (0,1)$ such that $a_m \geq 4s^{-d+\gamma}$ where $a_m \equiv \min(m, 1-m)$.*
Let $x, y \in \mathbb{Z}^d$ and $\mathcal{M}_x(m), \mathcal{A}^{(2)}_x(m), \mathcal{A}^{(3)}_x(m)$ be as in (4.9).
Then, if $s \geq s_0(\gamma)$, $\forall \eta \notin \mathcal{M}_x(m)$ and $\forall \alpha \notin \mathcal{A}^{(2)}_x(m) \cup \mathcal{A}^{(3)}_x(m)$,

$$|\nabla_y \tau_x \hat{\phi}_{n,s}(\eta) - (1-2\eta_y)(2s^d)^{-1}\tau_x \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}| \leq Cs^{-d}\Big\{\frac{s^{-d}}{a_m} + \frac{1}{\sqrt{a_m}}s^{-\frac{d}{2}+\frac{\gamma}{2}}\Big\}. \tag{4.20}$$

In what follows we give some estimates that will be useful in the applications of lemma 4.5.

Lemma 4.6. *Let n, s, γ, m be as in the previous lemma. Let $\gamma' : 0 < \gamma' < \gamma$ and $\lambda = \lambda(\alpha)$ be a bounded measurable function such that for any disorder configuration α*

$$|\lambda(\alpha) - \lambda_0(m)| \leq s^{-\frac{d}{2}+\frac{\gamma}{4}}.$$

If $s \geq s_0(\gamma, \gamma')$ then $\forall \Delta \in \mathbb{F}$

$$\mathbb{P}\Big(\mu^{\alpha,\lambda}\big(\cup_{x\in\Delta}\mathcal{M}_x(m)\big) \geq |\Delta| e^{-s^{\gamma'}}\Big) \leq e^{-s^{\frac{\gamma}{2}}}. \tag{4.21}$$

Proof. By Chebyshev inequality and the translation invariance of $\mathbb{P}$, the l.h.s. of (4.21) can be bounded by

$$e^{s^{\gamma'}}\,\mathbb{E}\big(\,\mu^{\alpha,\lambda}(\mathcal{M}_0(m))\,\big).$$

We consider for example the case $m \le \frac{1}{2}$ (the proof is similar if $m > \frac{1}{2}$).

We write $\mathcal{M}_0(m) = \mathcal{M}_-(m) \cup \mathcal{M}_+(m)$ with

$$\mathcal{M}_\pm(m) \equiv \{\,\pm(m^e_{0,s}(\eta) - m) \ge \sqrt{m}s^{-\frac{d}{2}+\frac{\gamma}{2}}\,\}.$$

As example we estimate

$$e^{s^{\gamma'}}\,\mathbb{E}\big(\,\mu^{\alpha,\lambda}(\mathcal{M}_+(m))\,\big), \tag{4.22}$$

the estimate for $\mathcal{M}_-(m)$ is similar.
Let $0 < t < 1$. By Chebyshev inequality, (4.22) can be bounded by

$$e^{s^{\gamma'}-2t\sqrt{m}s^{\frac{d}{2}+\frac{\gamma}{2}}}\,\mathbb{E}\Big(\mu^{\alpha,\lambda}\big(e^{t(N^e_{0,s}-2s^dm)}\big)\Big) = e^{s^{\gamma'}-2t\sqrt{m}s^{\frac{d}{2}+\frac{\gamma}{2}}}\,\mathbb{E}\Big(\prod_{x\in\Lambda^e_s}\mu^{\alpha,\lambda}\big(e^{t(\eta_x-m)}\big)\Big). \tag{4.23}$$

We fix a disorder configuration α and denote I the interval with extremes $\lambda = \lambda(\alpha)$ and $\lambda_0(m)$. Then, for any $x \in \mathbb{Z}^d$,

$$\mu^{\alpha,\lambda}\big(e^{t(\eta_x-m)}\big) \le \sup_{\tilde\lambda\in I}\Big(1 + \frac{|\mu^{\alpha,\tilde\lambda}\big(\eta_x; e^{t(\eta_x-m)}\big)|}{\mu^{\alpha,\lambda_0(m)}\big(e^{t(\eta_x-m)}\big)}|\lambda - \lambda_0(m)|\Big)\mu^{\alpha,\lambda_0(m)}\big(e^{t(\eta_x-m)}\big). \tag{4.24}$$

Since $\mathbb{E}\big(\mu^{\alpha,\lambda_0(m)}(\eta_0)\big) = m$, $|\lambda_0(m) - \ln\frac{m}{1-m}| \le B$. Therefore $|\tilde\lambda - \ln\frac{m}{1-m}| \le 2B$ if s is large enough. In particular,

$$\mu^{\alpha,\tilde\lambda}\big(\eta_x\big) \le C(B)m.$$

We observe that $\mu^{\alpha,\lambda_0(m)}\big(e^{t(\eta_x-m)}\big) \ge e^{-1}$ while

$$|\mu^{\alpha,\tilde\lambda}\big(\eta_x; e^{t(\eta_x-m)}\big)| = \mu^{\alpha,\tilde\lambda}\big(\eta_x;\eta_x\big)\big|e^{t(1-m)} - e^{-tm}\big| \le Ctm.$$

By the previous estimates and (4.24),

$$\mu^{\alpha,\lambda}\big(e^{t(\eta_x-m)}\big) \le (1 + Ctms^{-\frac{d}{2}+\frac{\gamma}{4}})\mu^{\alpha,\lambda_0(m)}\big(e^{t(\eta_x-m)}\big)$$

and therefore (4.23) is bounded by

$$e^{s^{\gamma'}-2t\sqrt{m}s^{\frac{d}{2}+\frac{\gamma}{2}}+Ctms^{\frac{d}{2}+\frac{\gamma}{4}}}\,\mathbb{E}\Big(\mu^{\alpha,\lambda_0(m)}\big(e^{t(\eta_0-m)}\big)\Big)^{2s^d}. \tag{4.25}$$

Since by Taylor expansion

$$\mathbb{E}\Big(\mu^{\alpha,\lambda_0(m)}(e^{t(\eta_0-m)})\Big) \le 1 + Ct^2m \le e^{Ct^2m},$$

(4.25) can be bounded by

$$\exp\big(s^{\gamma'} - 2t\sqrt{m}s^{\frac{d}{2}+\frac{\gamma}{2}} + Ctms^{\frac{d}{2}+\frac{\gamma}{4}} + Ct^2ms^d\big). \tag{4.26}$$

We choose t such that $t^2m = s^{-d+\gamma'}$. Then (4.26), which bounds (4.22), corresponds to

$$\exp\big((1+C)s^{\gamma'} - 2s^{\frac{\gamma'}{2}+\frac{\gamma}{2}} + C\sqrt{m}s^{\frac{\gamma'}{2}+\frac{\gamma}{4}}\big).$$

It's simple to derive the thesis.

□

The following lemma will be useful in order to apply lemma 4.6.

Lemma 4.7. *Let s be positive integer. Let $0<\gamma<d$ and $m\in(0,1)$ with $a_m \geq 4s^{-d+\gamma}$. We define*

$$\mathcal{B}_x(m) \equiv \{\,|\lambda_{\Lambda^e_{x,s}}(\alpha,m) - \lambda_0(m)| \geq s^{-\frac{d}{2}+\frac{\gamma}{4}}\,\}. \tag{4.27}$$

Then there exists a constant $C = C(B) > 0$ such that

$$\mathbb{P}\big(\mathcal{B}_x\big) \leq 2e^{-Cs^{\frac{\gamma}{2}}}.$$

Proof. We consider the case $m \leq 1/2$ (otherwise the proof is similar). By lemma $A.6$,

$$|\lambda_{\Lambda^e_{x,s}}(\alpha,m) - \lambda_0(m)| \leq C|1 - \frac{1}{m}\mu^{\alpha,\lambda_0(m)}(m_{\Lambda^e_{x,s}})|.$$

The thesis follows from proposition $A.1$ and lemma $A.2$.

□

Remark 4.8. *In the applications of the previous results we use also other parameters in place of n,s. When n,s are substituted by n',s' we use the following notation:*

$$\mathcal{M}_x(m,s'),\quad \mathcal{A}^{(2)}_x(m,s'),\quad \mathcal{A}^{(3)}_x(m,s'),\quad \mathcal{B}_x(m,s'),\quad \xi(\alpha,m,n',s'),\quad \xi_0(\alpha,m,n')...$$

4.3 An example

In this section we describe a method for estimating the expression

$$\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}\right) \tag{4.28}$$

where n,s,k are positive integers such that $n\leq s\leq k$ and $m\in[0,1]$.

We fix $0<\gamma'<\gamma<d$ and we distinguish two cases:

a) $a_m \leq 4s^{-d+\gamma}$,
b) $a_m \geq 4s^{-d+\gamma}$,

where $a_m \equiv \min(m, 1-m)$.

Case a).

For example let $m \leq 2^{-1}$. Since $\forall y, z \in \Lambda_s^e$

$$\mu^\alpha[\eta_y \,|\, m_s^e] \leq C\mu^\alpha[\eta_z \,|\, m_s^e],$$

it's simple to show that $\mu^\alpha[\eta_y \,|\, m_s^e] \leq C m_s^e$ for any $y \in \Lambda_s^e$. Therefore $|\tau_x \phi_{n,s}| \leq C m_{x,s}^e$ for any x and then $|\mathrm{Av}_{x\in\Lambda_k} \tau_x \phi_{n,s}| \leq C' m_{2k}$. We observe that

$$\mu^{\alpha,\lambda_0(m)}(m_{2k}^2) \leq C k^{-2d}\big(\sum_{x\in\Lambda_{2k}} \mu^{\alpha,\lambda_0(m)}(\eta_x) + \sum_{x\neq y \text{ in } \Lambda_{2k}} \mu^{\alpha,\lambda_0(m)}(\eta_x\eta_y)\big) \leq C(k^{-d}m + m^2).$$

Therefore (4.28) can be bounded by

$$C\mu^{\alpha,\lambda_0(m)}(m_{2k}^2) \leq C' s^{-2d+2\gamma}. \tag{4.29}$$

Case b).

By the equivalence of ensembles, in (4.28) we can substitute $\phi_{n,s}$ by $\hat{\phi}_{n,s}$ with a global error of order s^{-2d}.

In order to estimate $\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\Big(\mathrm{Av}_{x\in\Lambda_k} \tau_x \hat{\phi}_{n,s}\Big)$ we use Poincaré inequality in Λ_{k+s} w.r.t. Glauber dynamics. More precisely, by proposition $A.9$ it's enough to estimate

$$a_m \mu^{\alpha,\lambda_0(m)}\Big(\sum_{y\in\Lambda_{k+s}} \big[\,\mathrm{Av}_{x\in\Lambda_k} \nabla_y \tau_x \hat{\phi}_{n,s}\,\big]^2\Big) \tag{4.30}$$

or equivalently

$$a_m \mu^{\alpha,\lambda_0(m)}\Big(\sum_{y\in\Lambda_{k+s}} \big[\,k^{-d} \sum_{x\in\Lambda_k\cap\Lambda_{y,s}} \nabla_y \tau_x \hat{\phi}_{n,s}\,\big]^2\Big). \tag{4.31}$$

We refer to definition (4.9) and set

$$\mathcal{M}(m) \equiv \cup_{x\in\Lambda_k} \mathcal{M}_x(m)$$

and

$$\begin{aligned}
&\mathcal{A}_1 \equiv \big\{\mu^{\alpha,\lambda_0(m)}\big(\mathcal{M}(m)\big) \geq k^d \exp(-s^{\gamma'})\,\big\}\\
&\mathcal{A}_2 \equiv \cup_{x\in\Lambda_k} \mathcal{A}_x^{(2)}(m)\\
&\mathcal{A}_3 \equiv \cup_{x\in\Lambda_k} \mathcal{A}_x^{(3)}(m)\\
&\mathcal{A}_4 \equiv \cup_{y\in\Lambda_{k+s}} \big\{\,|\mathrm{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}} \big[\tau_x \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\big]\,| \geq |\Lambda_k \cap \Lambda_{y,s}|^{-\frac{1}{2}+\frac{\gamma}{2d}}\,\big\}
\end{aligned}$$

$$\mathcal{A} \equiv \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4.$$

We first estimate

$$\mathbb{I}_{\mathcal{A}^c}(\alpha) a_m \mu^{\alpha,\lambda_0(m)} \Big(\mathbb{I}_{\mathcal{M}^c(m)}(\eta) \sum_{y \in \Lambda_{k+s}} \big[k^{-d} \sum_{x \in \Lambda_k \cap \Lambda_{y,s}} \nabla_y \tau_x \hat{\phi}_{n,s} \big]^2 \Big). \tag{4.32}$$

By definition of $\mathcal{M}(m)$, since $\alpha \notin \mathcal{A}_2 \cup \mathcal{A}_3$ and by lemma 4.5, if $s \geq s_0(\gamma)$ then (4.32) can be bounded by a constant C multiplied by the following expression

$$k^{-d} \mathbb{I}_{\mathcal{A}^c}(\alpha) \mathrm{Av}_{y \in \Lambda_{k+s}} \big[(2s^d)^{-1} \sum_{x \in \Lambda_k \cap \Lambda_{y,s}} \tau_x \frac{\xi_0(\alpha, m)}{\sigma_0^2(m)} \big]^2 + k^{-d} s^{-d+\gamma} \tag{4.33}$$

We analyze the first addendum in (4.33) with $\alpha \notin \mathcal{A}$. It can be bounded by $C(A+B)$ where C is a positive constant and A, B are defined as follows.

$$\begin{aligned} A &\equiv k^{-2d} \sum_{|y| \leq k} \big[(2s^d)^{-1} \sum_{x \in \Lambda_k \cap \Lambda_{y,s}} \tau_x \frac{\xi_0(\alpha, m)}{\sigma_0^2(m)} \big]^2, \\ B &\equiv k^{-2d} \sum_{k < |y| \leq k+s} \big[(2s^d)^{-1} \sum_{x \in \Lambda_k \cap \Lambda_{y,s}} \tau_x \frac{\xi_0(\alpha, m)}{\sigma_0^2(m)} \big]^2. \end{aligned}$$

If $|y| \leq k$ then $|\Lambda_k \cap \Lambda_{y,s}| = O(s^d)$. By definition of $\mathcal{A}_4$, $A \leq C k^{-d} s^{-d+\gamma}$.
Given $l : 0 \leq l < s$, if $|y| = k + s - l$ then $|\Lambda_k \cap \Lambda_{y,s}| = O(s^{d-1}(l+1))$ and by definition of $\mathcal{A}_4$

$$\Big| \sum_{x \in \Lambda_k \cap \Lambda_{y,s}} \tau_x \frac{\xi_0(\alpha, m)}{\sigma_0^2(m)} \Big| \leq C (s^{d-1}(l+1))^{\frac{1}{2} + \frac{\gamma}{2d}}$$

and therefore

$$B \leq C k^{-d-1} \sum_{l=1}^{s+1} \Big[s^{-d} (s^{d-1} l)^{\frac{1}{2} + \frac{\gamma}{2d}} \Big]^2 \leq C' k^{-d-1} s^{-d+1+\gamma}.$$

In conclusion, (4.32) can be bounded by $k^{-d} s^{-d+\gamma} + C k^{-d-1} s^{-d+1+\gamma}$.
Moreover, if we consider expression (4.32) with $\mathbb{I}_{\mathcal{M}^c(m)}(\eta)$ substituted by $\mathbb{I}_{\mathcal{M}(m)}(\eta)$ then, by definition of $\mathcal{A}_1$, it can be bounded by $s^{2d} e^{-s^{\gamma'}}$.
Taking into account the estimate by means of the equivalence of ensembles, we obtain that in case $b)$ expression (4.28) can be bounded by

$$C \mathbb{I}_{\mathcal{A}^c}(\alpha) \Big(s^{-2d} + k^{-d} s^{-d+\gamma} + k^{-d-1} s^{-d+1+\gamma} + s^{2d} e^{-s^{\gamma'}}) + \mathbb{I}_{\mathcal{A}}(\alpha).$$

By the above estimate, (4.29) and lemma 4.10 we have the following proposition.

Proposition 4.9. *Let $d \geq 2$, $m \in [0,1]$ and n, s, k be positive integers with $n \leq s \leq k$. Let $0 < \gamma' < \gamma < d$.*
There exist constants C, C' and a measurable set $\mathcal{A}$ with $\mathbb{P}(\mathcal{A}) \leq Ck^{2d}e^{-C's^{\frac{\gamma}{2}}}$ such that, if $s \geq s_0(\gamma, \gamma')$, then

$$\begin{aligned}&\operatorname{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\operatorname{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}\right) \leq \\ &C\mathbb{I}_{\mathcal{A}^c}(\alpha)\left(s^{-2d+2\gamma} + k^{-d}s^{-d+\gamma} + k^{-d-1}s^{-d+1+\gamma} + s^{2d}e^{-s^{\gamma'}}\right) + \mathbb{I}_{\mathcal{A}}(\alpha).\end{aligned} \tag{4.34}$$

We remark that the restriction $d \geq 2$ is needed in order to prove that $\mathbb{P}(\mathcal{A}_4)$ is negligible as stated in lemma 4.10.

Lemma 4.10. *Let $d \geq 2$. If $s \geq s_0(\gamma, \gamma')$ then*

$$\mathbb{P}(\mathcal{A}) \leq Ck^{2d}e^{-C's^{\frac{\gamma}{2}}}. \tag{4.35}$$

Proof. By lemma 4.6, $\mathbb{P}(\mathcal{A}_1) \leq Ce^{-s^{\frac{\gamma}{2}}}$ if $s \geq s_0(\gamma, \gamma')$.

By proposition $A.1$, $\mathbb{P}(\mathcal{A}_i) \leq Ck^de^{-C's^\gamma}$ where $i = 2, 3$.

Let us consider $\mathbb{P}(\mathcal{A}_4)$. We define

$$g_x \equiv (\sigma_0^2(m))^{-1}[\mu^{\alpha,\lambda_0(m)}(\eta_x;\eta_x) - \mu^{\alpha,\lambda_0(m)}(\eta_{x+ne};\eta_{x+ne})], \tag{4.36}$$

then

$$\operatorname{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}\tau_x\frac{\xi_0(\alpha,m)}{\sigma_0^2(m)} = \operatorname{Av}_{z\in\Lambda_{1,n}^e}\operatorname{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}g_{x+z}.$$

Therefore

$$\mathcal{A}_4 \subset \cup_{y\in\Lambda_{k+s}}\cup_{z\in\Lambda_{1,n}^e}\{|\operatorname{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}g_{x+z}| \geq |\Lambda_k\cap\Lambda_{y,s}|^{-\frac{1}{2}+\frac{\gamma}{2d}}\}.$$

Since $\|g_0\|_\infty \leq C$ and $\mathbb{E}(g_0) = 0$, by arguments similar to those of proposition $A.1$

$$\mathbb{P}(\mathcal{A}_4) \leq Ck^dn^de^{-C's^{\frac{d-1}{d}\gamma}}.$$

$\square$

For the proof of theorem 4.2 it's useful to give an estimate of the expression

$$\operatorname{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\operatorname{Av}^*_{x\in\Lambda_k}\tau_x\phi_{n,s}\right) \tag{4.37}$$

where n, s, k are positive integers such that $n \leq s \leq k \leq 100s$, $m \in [0,1]$ and

$$\operatorname{Av}^*_{x\in\Lambda_k} \equiv \operatorname{Av}_{x\in\Lambda_k\cap n\mathbb{Z}^d}.$$

Let $0 < \gamma < d$ and $\gamma' \equiv \frac{\gamma}{2}$.

We indicate the main differences w.r.t. the estimate of (4.28).
It's simple to check that the contribution due to extreme densities, that is $a_m \le 4s^{-d+\gamma}$, can be bounded by $Cs^{-2d+2\gamma}$. For example if $m \le 1/2$ then it's enough to observe that $|\tau_x \phi_{n,s}| \le C m_{x,s}$ for any x and therefore (4.37) can be bounded by

$$C\mathrm{Av}^*_{x\in\Lambda_k}\mu^{\alpha,\lambda_0(m)}(m^2_{x,s}) \le C\, s^{-2d+2\gamma}.$$

We assume $a_m \ge 4s^{-d+\gamma}$. The contribution due to the equivalence of ensembles is of order s^{-2d}. After applying Poincaré inequality, in order to estimate

$$\sum_{y\in\Lambda_{k+s}} a_m \mu^{\alpha,\lambda_0(m)}\Big(\big[\mathrm{Av}^*_{x\in\Lambda_k}\nabla_y \tau_x \hat\phi_{n,s}\big]^2\Big), \tag{4.38}$$

we define $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ as done for the estimate of (4.28) and we set

$$\mathcal{A}_4 \equiv \Big\{ \Big|\mathrm{Av}^*_{x\in\Lambda_k}\tau_x\, \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\Big| \ge k^{-\frac{d}{2}+\frac{\gamma}{2}}\Big\}.$$

By lemma 4.5 if $s \ge s_0(\gamma)$ then

$$\sum_{y\in\Lambda_{k+s}} \mathbb{I}_{\mathcal{A}^c}(\alpha) a_m \mu^{\alpha,\lambda_0(m)}\Big(\mathbb{I}_{\mathcal{M}^c(m)}(\eta)\big[\mathrm{Av}^*_{x\in\Lambda_k}\nabla_y \tau_x \hat\phi_{n,s}\big]^2\Big) \tag{4.39}$$

can be bounded by

$$Ck^d \mathbb{I}_{\mathcal{A}^c}(\alpha)\big[s^{-d}\mathrm{Av}^*_{x\in\Lambda_k}\tau_x\, \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\big]^2 + Ck^{-2d+\gamma} \le Ck^{-2d+\gamma}.$$

Moreover, (4.39) with $\mathbb{I}_{\mathcal{M}^c(m)}$ replaced by $\mathbb{I}_{\mathcal{M}(m)}$ is bounded by $c\,k^{2d}e^{-s^{\gamma/2}}$.
$\mathbb{P}(\mathcal{A}_i)$, with $i = 1, 2, 3$, can be bounded as in the proof of lemma 4.10, while we claim that, for a suitable positive constant C,

$$\mathbb{P}(\mathcal{A}_4) \le e^{-Ck^\gamma}.$$

Let us consider the case $d = 1$ for simplicity of notation. We can write

$$n \sum_{x\in\Lambda_k\cap n\mathbb{Z}} \tau_x\, \frac{\xi_0(\alpha,m)}{\sigma_0^2(m)} = \Big(\sum_{x\in\Lambda_k\cap 2n\mathbb{Z}}\ \sum_{z\in\Lambda^e_{1,n}} g_{x+z}\Big) + \Big(\sum_{x\in\Lambda_k\cap(2n+n)\mathbb{Z}}\ \sum_{z\in\Lambda^e_{1,n}} g_{x+z}\Big) \tag{4.40}$$

where g_x has been defined in (4.36). We remark that both in the first addendum and in the second addendum of the r.h.s. of (4.40) the appearing functions have disjoint support and form a set of cardinality $O(k^d)$. By the same argument used to prove proposition A.1, we obtain the thesis. By collecting the previous results we obtain the following proposition.

Proposition 4.11. *Let $m \in [0, 1]$ and n, s, k be positive integers with $n \le s \le k \le 100s$. Given $0 < \gamma < d$, if $s \ge s_0(\gamma)$ then there exists a measurable set $\mathcal{A}$ such that for a suitable positive constants C*

$$\mathbb{P}(\mathcal{A}) \le e^{-k^{C\gamma}}$$

and such that

$$\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\big(\mathrm{Av}^*_{x\in\Lambda_k}\tau_x\phi_{n,s}\big) \le \mathbb{I}_{\mathcal{A}}(\alpha) + C\mathbb{I}_{\mathcal{A}^c}k^{-2d+2\gamma}. \tag{4.41}$$

In a similar way we can prove the following statement.

Proposition 4.12. *Let $n \leq s$ be positive integers and let $0 < \delta \ll 1$. Let also $\gamma > 0$ and set $J_s = \{1/s^\gamma, 2/s^\gamma, \dots 1 - 1/s^\gamma\}$. Then there exists a set $\mathcal{A}$ of disorder configurations α in Λ_{2s} satisfying*

$$\mathbb{P}(\mathcal{A}) \leq s^\gamma \, e^{-c\, s^\delta}$$

and such that, for s large enough depending on δ,

$$\sup_{m \in J_s} \operatorname{Var}_{\mu_{\Lambda_{2s}}^{\lambda(\alpha,\Lambda_{2s},m)}} \left(\operatorname{Av}^*_{x\in\Lambda_s} \tau_x \phi_{n,s}\right) \leq c\mathbb{I}_{\mathcal{A}^c}(\alpha) s^{-2d+2\delta} + \mathbb{I}_{\mathcal{A}}(\alpha). \tag{4.42}$$

4.4 Perturbation theory

In the next sections we develop the proof of theorem 4.2. A basic tool in the proof is given by perturbation theory and in particular by proposition $A.13$.
We remark that in our proof of theorem 4.2 the estimates on the spectral gap needed in order to apply proposition $A.13$ can be derived by the results of chapter 2, which are valid also in disordered models without diffusive scaling in the spectral gap. In particular, it can be proved that if $s = O(\epsilon^{-\delta})$, with $0 < \delta < 1$, then there exists $C_0 > 0$ such that almost any disorder configuration α has the following property: $\forall \epsilon \leq \epsilon_0(\alpha)$ and $\forall x \in \mathbb{T}^d_\epsilon$

$$\sup_{\nu\in\mathcal{M}} \operatorname{gap}\left(\mathcal{L}^\alpha_{\Lambda_{x,s}}, \nu\right)^{-1} \leq C_0 s^2 \tag{4.43}$$

where $\mathcal{M} \equiv \mathcal{M}_\alpha(\Lambda_{x,s})$.
In fact, let β be such that $\epsilon^{-d} s^{-\beta} \leq \epsilon^2$. Then by theorem 2.20, in particular by (2.69) and (2.70), there exists $C_0 > 0$ such that if ϵ is small enough

$$\mathbb{P}\Big(\exists x \in \mathbb{T}^d_\epsilon \ : \ \sup_{\nu\in\mathcal{M}} \operatorname{gap}\left(\mathcal{L}^\alpha_{\Lambda_{x,s}}, \nu\right)^{-1} > C_0 s^2 \Big) \leq \epsilon^{-d} s^{-\beta}.$$

Therefore (4.43) follows by Borel-Cantelli lemma.

In many applications of proposition $A.13$ it's useful the following estimate of $\|\phi_{s,s'}\|_\infty$.

Lemma 4.13. *Let $0 < \beta < 1$ and $s = O\big(\epsilon^{-\beta}\big)$. If $0 < \gamma < \frac{d}{2}$ then for almost any configuration disorder α, $\forall \epsilon \leq \epsilon_0(\alpha)$, $\forall s' \geq s$ and $\forall x \in \mathbb{T}^d_\epsilon$*

$$|\tau_x \phi_{s,s'}| \leq C s^{-\frac{d}{2}+\gamma}. \tag{4.44}$$

Proof. By the equivalence of ensembles and lemma $A.3$,

$$\begin{gathered} |\tau_x\phi_{s,s'} - \tau_x\hat{\phi}_{s,s'}| \leq C(s')^{-d}, \\ |\tau_x\hat{\phi}_{s,s'}(\eta) - \mu^{\alpha,\lambda_0(\, m^e_{x,s'}(\eta)\,)}\big(\tau_x(m^{(1)}_s - m^{(2)}_s)\big)| \leq C|m^e_{x,s'}(\eta) - \mu^{\alpha,\lambda_0(\, m^e_{x,s'}(\eta)\,)}(m^e_{x,s'})|. \end{gathered} \tag{4.45}$$

Let $\mathcal{B} \equiv \cup_{m\in I_{s'}} \cup_{x\in\mathbb{T}^d_\epsilon} \Big(\mathcal{B}'{}_x(m) \cup \mathcal{B}''{}_x(m)\Big)$ where

$$\mathcal{B}'_x(m) \equiv \{\, |m - \mu^{\alpha,\lambda_0(m)}(m^e_{x,s'})| \geq (s')^{-\frac{d}{2}+\gamma}\,\},$$
$$\mathcal{B}''_x(m) \equiv \{\, |\mu^{\alpha,\lambda_0(m)}\big(\tau_x(m^{(1)}_s - m^{(2)}_s)\big)| \geq s^{-\frac{d}{2}+\gamma}\,\}.$$

By proposition $A.1$ $\mathbb{P}(\mathcal{B}) \leq 2\epsilon^{-2d}e^{-Cs^{2\gamma}}$ and since $\epsilon^{-\beta} = s \leq s'$, by Borel-Cantelli lemma, $\tilde{\forall}\alpha$ if $\epsilon \leq \epsilon_0(\alpha)$ then $\alpha \notin \mathcal{B}$ and (4.45) allows to conclude the proof.

□

Remark 4.14. *Let s be an odd integer such that $s|\frac{a}{\epsilon}$. Let $s \leq s' < \frac{a}{\epsilon}$. Let $x \in \mathbb{T}^d_\epsilon$, $\nu \in \mathcal{M}_\alpha(\Lambda_{x,2\frac{a}{\epsilon}})$ and F be a function with support in $\Lambda^e_{0,s'}$. Then*

$$\sup spec_{L^2(\nu)}\{\epsilon^{-1}\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z F + \epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\} \leq \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\sup_{\nu'}\sup spec_{L^2(\nu')}\{\epsilon^{-1}\tau_z F + C(d)\epsilon^{-2}\mathrm{Av}_{b\in\Lambda^e_{z,s'}}\mathcal{L}_b\} \tag{4.46}$$

where ν' varies in $\mathcal{M}_\alpha(\Lambda^e_{z,s'})$.

In fact, let $a/\epsilon = R = 2R' + 1$, $s = 2s' + 1$ and $A \equiv s\mathbb{Z}^d \cap \Lambda_{0,R'}$. By the definition of the average $\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}$ the l.h.s. of (4.46) is bounded by

$$Av^{\frac{R}{s}-1}_{v=0}\sup spec_{L^2(\nu)}\{\epsilon^{-1}\mathrm{Av}_{z\in x+A_v}\tau_z F + \epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2R}}\mathcal{L}_b\}$$

where $A_v \equiv A + (-R' + s' + vs)e$. The thesis follows easily by observing that, for any $v : 0 \leq v \leq R/s$,

$$\mathrm{Av}_{b\in\Lambda_{x,2R}}\mathcal{L}_b \leq C(d)\mathrm{Av}_{z\in x+A_v}\mathrm{Av}_{b\in\Lambda^e_{z,s'}}\mathcal{L}_b.$$

4.5 From $n^{-1}\phi_{n,n}$ to $n^{-1}\phi_{n,s}$ with $s = O(\epsilon^{-\frac{1}{d+2}})$ and $n \uparrow \infty, \epsilon \downarrow 0$

A fundamental tool of this section is given by lemma 4.15.

Lemma 4.15. *Let A be a box in $\mathbb{Z}^d$ and $\{A_i\}_{i\in I}$ be a family of cubes $A_i \subset A$ with side R such that*

$$|\{i \in I \,:\, x \in A_i\}| \leq 10^{6d} \qquad \forall x \in A.$$

For any $i \in I$ we define $\mathcal{F}_i \equiv \sigma(m_{A_i}; \eta_x$ with $x \notin A_i)$.

Let $F = \mathrm{Av}_{i\in I}F_i$ where F_i is a local function with support in A_i and such that

$$\mu^\alpha(F_i \,|\, \mathcal{F}_i) = 0 \qquad \forall i \in I.$$

Then, given a canonical Gibbs measure μ on A with disorder configuration α,

$$\mu\big(F, \frac{1}{-\mathcal{L}^\alpha_A}F\big) \leq C(d)R^2|I|^{-1}\mathrm{Av}_{i\in I}\gamma_\alpha(A_i)\mu\big(\,\mathrm{Var}_\mu(F_i\,|\,\mathcal{F}_i)\,\big)$$

where $\gamma_\alpha(A_i)$ is defined by (2.42).

Proof. Let g be a local function with support in A. Then

$$\begin{aligned}|\mu(F,g)| &\le \mathrm{Av}_{i\in I}\big|\mu\big(\,\mu(F_i;g\,|\,\mathcal{F}_i)\,\big)\big|\\ &\le R\ \mathrm{Av}_{i\in I}\mu\Big(\,\big[\mathrm{Var}_\mu(F_i\,|\,\mathcal{F}_i)\gamma_\alpha(A_i)\mathcal{D}_{A_i}\big(g;\mu(\cdot,\mathcal{F}_i)\big)\big]^{\frac12}\,\Big)\le\\ &C(d)R\Big(\mathrm{Av}_{i\in I}\gamma_\alpha(A_i)\mu\big(\,\mathrm{Var}_\mu(F_i\,|\,\mathcal{F}_i)\big)\Big)^{\frac12}|I|^{-\frac12}\mathcal{D}_A(g;\mu)^{\frac12}.\end{aligned}$$

It's enough to take g such that $f=\mathcal{L}^\alpha_A g$. $\square$

Proposition 4.16. *Let $d\ge 3$, $s=O\big(\epsilon^{-\frac{1}{d+2}}\big)$ such that $s=N^4$ for some $N\in\mathbb{N}$. Then*

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\Big[\frac{\phi_{n,n}}{n}-\frac{\phi_{n,s}}{n}\Big]\approx 0.$$

Proof. Let us first prove that

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\Big[\frac{\phi_{n,n^4}}{n}-\frac{\phi_{n,s}}{n}\Big]\approx 0. \tag{4.47}$$

We set $F_{n,s}\equiv n^{-1}(\phi_{n,n^4}-\phi_{n,s})$. We have to prove that, given $\gamma_0>0$, for almost any disorder configuration α

(4.48)
$$\limsup_{n\uparrow\infty,a\downarrow0,\epsilon\downarrow0}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{|C_0|\le1}\ \sup_\nu\sup spec_{L^2(\nu)}\{\epsilon^{-1}C_0\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_zF_{n,s}+\gamma_0\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\}\le 0$$

where ν varies in $\mathcal{M}_\alpha\big(\Lambda_{x,2\frac{a}{\epsilon}}\big)$.
For simplicity of notation we take $\gamma_0=1$. The proof does not change in the general case.
Let $\frac{a}{\epsilon}=R=2R'+1$ and $s=2s'+1$ (we assume $n|s|R$) and set $A\equiv n\mathbb{Z}^d\cap\Lambda_{0,R'}$.
Let us remember that in general

$$\mathrm{Av}^{n,R}_{z,x}b_z=\mathrm{Av}^{\frac{R}{n}-1}_{v=0}\mathrm{Av}_{a\in A}b_{x+a+(-R'+n'+vn)e}$$

and that for $0\le v\le\frac{R}{n}-1$

$$A+x+(-R'+n'+vn)e\subset\Lambda^e_{x,R}.$$

By taking out the average $\mathrm{Av}^{\frac{R}{n}-1}_{v=0}$ from the spectrum in (4.48) we reduce to prove that for almost any disorder configuration α

$$\limsup_{n\uparrow\infty,a\downarrow0,\epsilon\downarrow0}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{|C_0|\le1}\ \sup_\nu\sup spec_{L^2(\nu)}\{\epsilon^{-1}C_0\mathrm{Av}_{z\in A+x}\tau_zF_{n,s}+C\epsilon^{-2}\mathrm{Av}_{b\in\mathcal{L}_{x,\frac32R}}\}\le 0$$

where ν varies in $\mathcal{M}_\alpha(\Lambda_{x,\frac32R})$.
Dividing each cube $\Lambda_{x,R}$ in cubes of side s, by standard arguments it's simple to check that it's enough prove that $\tilde\forall\alpha$

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|C_0|\le 1} \sup_\nu \sup spec_{L^2(\nu)}\{\epsilon^{-1}C_0\mathrm{Av}^*_{y\in\Lambda_{s'}}\tau_{x+y}F_{n,s}+C\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2s}}\mathcal{L}_b\}\le 0$$

where ν varies among $\mathcal{M}_\alpha(\Lambda_{x,2s})$ and for a generic set $\Lambda\in\mathbb{F}$ we write

$$\mathrm{Av}^*_{y\in\Lambda}\equiv\mathrm{Av}_{y\in\Lambda\cap n\mathbb{Z}^d}.$$

Since $\epsilon s^{d+2}\|F_{n,s}\|_\infty$ goes to 0 when n is large enough we can apply perturbation theory (see section 4.4), and in particular proposition $A.13$. Therefore it's enough to prove that $\tilde{\forall}\alpha$

$$\limsup_{n\uparrow\infty,\epsilon\downarrow 0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{\nu\in\mathcal{M}_\alpha(\Lambda_{x,2s})}F_{x,\nu}=0$$

where

$$F_{x,\nu}=n^{-2}s^d\nu\Big(\mathrm{Av}^*_{y\in\Lambda_{s'}}\tau_{x+y}\big[\phi_{n,n^4}-\phi_{n,s}\big],\frac{1}{-\mathcal{L}_{\Lambda_{x,2s}}}\mathrm{Av}^*_{y\in\Lambda_{s'}}\tau_{x+y}\big[\phi_{n,n^4}-\phi_{n,s}\big]\Big).$$

We fix $\nu\in\mathcal{M}_\alpha(\Lambda_{2s})$ and we indicate how to estimate $F_{0,\nu}$. The method is completely general.
For a given function f with support in Λ_s we set

$$\Psi(f)\equiv s^d\nu\Big(\mathrm{Av}^*_{y\in\Lambda_{s'}}\tau_y f,\frac{1}{-\mathcal{L}_{\Lambda_{2s}}}\mathrm{Av}^*_{y\in\Lambda_{s'}}\tau_y f\Big). \tag{4.49}$$

In what follows for any $k\in\mathbb{N}$ we set $s_k\equiv k^4$. In particular, $s=s_N$.
By Schwarz inequality it's simple to check that, given $\delta>0$ and a semi-norm $\|\ \|$ on a generic space X, then there exists C_δ such that for any $x_1,\cdots,x_s\in X$

$$\|x_1+\cdots+x_s\|^2\le C_\delta\sum_{k=1}^s k^{1+\delta}\|x_k\|^2. \tag{4.50}$$

Let $\delta>0$ be small enough to satisfy the conditions which will follow.
By (4.50)

$$\Psi(\phi_{n,n^4}-\phi_{n,s})\le C_\delta\sum_{k=n}^{N-1}k^{1+\delta}\Psi(\phi_{n,s_k}-\phi_{n,s_{k+1}}).$$

For any k we divide $\Lambda_{s'}$ in boxes having sides of length in $[s_k,2s_k]$. For simplicity of notation (but the proof works also in the general case) we assume that $n|s_k|s$. In particular $\Lambda_{s'}$ can be divided in cubes of side s_k:

$$\Lambda_{s'}=\sqcup_{i\in I_k}\Delta_{i,k}$$

where $\Delta_{i,k}$ is a cube of side s_k. Let $\Delta'_{i,k}$ the cube of side $10s_k$ concentric to $\Delta_{i,k}$.
In order to estimate $\Psi(\phi_{n,s_k} - \phi_{n,s_{k+1}})$ we apply lemma 4.15 with

$$I = I_{k+1}, \qquad A = \Lambda_{2s}, \qquad A_i = \Delta'_{i,k+1}, \qquad F_i = \mathrm{Av}^*_{x\in\Delta_{i,k+1}}\tau_x[\phi_{n,s_k} - \phi_{n,s_{k+1}}].$$

Lemma 4.15 and lemma 2.6 give the estimate

$$\Psi(\phi_{n,s_k} - \phi_{n,s_{k+1}}) \le C s_{k+1}^{d+2} \mathrm{Av}_{i\in I_{k+1}} \gamma_{i,k+1} \mathrm{Var}_{\mu^{\alpha,\lambda}}\Big(\mathrm{Av}^*_{x\in\Delta_{i,k+1}}\tau_x[\phi_{n,s_k} - \phi_{n,s_{k+1}}]\Big)\Big] \tag{4.51}$$

where $\gamma_{i,k+1} = \gamma_\alpha(\Delta'_{i,k+1})$ and $\lambda = \lambda(\alpha, \Lambda_{2s}, m)$ with m such that ν is the canonical measure on Λ_{2s} with density m.

In what follows we use the diffusive scaling of the spectral gap, although it can be easily checked that the results of chapter 2 are enough.

Let J be the set of possible densities on Λ_{2s} and $\lambda_x(m) \equiv \lambda(\alpha, \Lambda_{x,2s}, m)$.
Then by the previous results it's enough to prove , for δ small enough, that $\tilde{\forall}\alpha$

$$\lim_{n\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{m\in J} n^{-2} \sum_{k=n+1}^{N} k^{1+\delta} s_k^{d+2} \mathrm{Av}_{i\in I_k} \mathrm{Var}_{\mu^{\alpha,\lambda_x(m)}}\Big(\mathrm{Av}^*_{y\in\Delta_{i,k}}\tau_{x+y}\phi_{n,s_k}\Big) = 0 \tag{4.52}$$

and to prove a similar limit with ϕ_{n,s_k} substituted by $\phi_{n,s_{k-1}}$. We prove (4.52), the other limit can be proved in a similar way.

Given $\gamma > 0$ we set $J_k = \{s_k^{-\gamma}, 2s_k^{-\gamma}, \ldots, 1 - s_k^{-\gamma}\}$. Then, using (A.2), the variance in (4.52) can be bounded from above by

$$\mathrm{Var}_{\mu^{\alpha,\lambda(\bar m)}_{x+\Delta'_{i,k}}}\Big(\mathrm{Av}^*_{y\in\Delta_{i,k}}\tau_{x+y}\phi_{n,s_k}\Big) + C s_k^{d-\gamma}$$

provided that $\bar m \in J_k$ satisfies $|\bar m - m| \le s_k^{-\gamma}$. Therefore, by choosing γ large enough, it's sufficient to prove that the limit with $n \uparrow \infty, \epsilon \downarrow 0$ of

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} n^{-2} \sum_{k=n+1}^{N} k^{1+\delta} s_k^{d+2} \mathrm{Av}_{i\in I_k} \sup_{m\in J_k} \mathrm{Var}_{\mu^{\alpha,\lambda(\bar m)}_{x+\Delta'_{i,k}}}\Big(\mathrm{Av}^*_{y\in\Delta_{i,k}}\tau_{x+y}\phi_{n,s_k}\Big) \tag{4.53}$$

goes to 0.
By applying proposition 4.12 to get that

$$\sup_{m\in J_k} \mathrm{Var}_{\mu^{\alpha,\lambda(\bar m)}_{x+\Delta'_{i,k}}}\Big(\mathrm{Av}^*_{y\in\Delta_{i,k}}\tau_{x+y}\phi_{n,s_k}\Big) \le C\mathbb{I}_{\mathcal{A}^c_{x,i,k}}(\alpha) {s_k}^{-2d+2\delta} + \mathbb{I}_{\mathcal{A}_{x,i,k}}(\alpha)\,, \tag{4.54}$$

where $\mathcal{A}_{x,i,k}$ is a set of disorder configurations in $x + \Delta'_{i,k}$ with $\mathbb{P}(\mathcal{A}_{x,i,k}) \le e^{-C\, s_k^\delta}$, $\delta > 0$.
Therefore

$$\begin{aligned} \text{l.h.s. of (4.53)} \ \le\ & \lim_{n\uparrow\infty,\epsilon\downarrow 0} n^{-2} \sum_{k=n+1}^{N} k^{1+\rho} s_k^{d+2} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \\ & + \lim_{n\uparrow\infty,\epsilon\downarrow 0} n^{-2} \sum_{k=n+1}^{N} k^{1+\rho} s_k^{2-d+2\delta}. \end{aligned} \tag{4.55}$$

The second addendum in the r.h.s. of (4.55) is zero because of the definition of s_k and the condition $d \geq 3$.
Let us consider the first addendum in the r.h.s. of (4.55). By Chebyschev inequality, for any $q > 0$ and any x, k

$$\begin{aligned} \mathbb{P}\big(\mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq s_k^{-q} \big) &\leq \mathbb{P}\big(\exists i \in I_k \,:\, \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq s_k^{-q} \big) \\ &\leq s_0^d \, s_k^{q-d} \, e^{-c\, s_k^\delta}. \end{aligned} \tag{4.56}$$

Moreover, by setting $\bar{\mathbb{I}}_{\mathcal{A}_{x,i,k}} = \mathbb{I}_{\mathcal{A}_{x,i,k}} - \mathbb{P}\big(\mathcal{A}_{x,i,k}\big)$, we have for any $r \in \mathbb{N}$ and any x, k

$$\mathbb{P}\big(\mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq s_k^{-q} \big) \leq c_r s_k^{2rq} \mathbb{E}\big[\big(\mathrm{Av}_{i\in I_k} \bar{\mathbb{I}}_{\mathcal{A}_{x,i,k}} \big)^{2r} \big] \leq c_r' s_k^{2rq+dr} s_0^{-dr} \tag{4.57}$$

By taking the geometric average of the two estimates (4.56) and (4.57) we finally obtain

$$\mathbb{P}\big(\mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq l_k^{-q} \big) \leq c(q,r) s_0^{-d(r-1)/2}.$$

It is enough at this point to choose q and r large enough, define

$$\Theta_\epsilon := \{\, \exists x \in \mathbb{T}_\epsilon^d : \mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq s_k^{-q} \quad \text{for some } k \leq N \,\},$$

and apply Borel-Cantelli lemma to get that also the first addendum in the r.h.s. of (4.55) is negligible.
In order to conclude the proof we show that

$$\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}} \tau_z \Big[\frac{\phi_{n,n}}{n} - \frac{\phi_{n,n^4}}{n} \Big] \approx 0.$$

Fix $\beta > 0$ such that $\beta^{-1} \in \mathbb{N}$. β as to be thought of as very small and only at the end we will say how much small. By considering the finite sequence of mesoscopic scales $n, n^{1+\beta}, n^{1+2\beta}, \ldots, n^4 = n^{1+k\beta}$, with $k := 3\beta^{-1}$, it is enough to prove that

$$\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}} \tau_z \Big[\frac{\phi_{n,n^{j(1-\beta)}}}{n} - \frac{\phi_{n,n^j}}{n} \Big] \approx 0$$

where $j \geq 1 + \beta$. To this aim, by reasoning as in the beginning, we need to prove that for almost any disorder configuration α

$$\limsup_{n\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_{\nu\in\mathcal{M}_\alpha(\Lambda_{x,2s})} \tau_x F_{x,\nu} = 0, \tag{4.58}$$

where now

$$F_{x,\nu} := n^{-2} s^d \nu \Big(\mathrm{Av}^*_{y\in\Lambda_{s'}} \tau_{x+y} \big[\phi_{n,n^{j(1-\beta)}} - \phi_{n,n^j} \big], \frac{1}{-\mathcal{L}_{\Lambda_{x,2s}}} \mathrm{Av}^*_{y\in\Lambda_{s'}} \tau_{x+y} \big[\phi_{n,n^{j(1-\beta)}} - \phi_{n,n^j} \big] \Big).$$

Let us indicate how to estimate $F_{0,\nu}$ (the same arguments work also for $F_{x,\nu}$). To this aim we write

$$\begin{aligned} &\mathrm{Av}^*_{y\in\Lambda_{s'}} \tau_y \big[\phi_{n,n} - \phi_{n,n^j} \big] = \mathrm{Av}_{y\in\Lambda_{s'}\cap(2n^j+1)\mathbb{Z}^d} \tau_y H, \\ &H := \mathrm{Av}_{z\in\Lambda_{n^j}\cap n\mathbb{Z}^d} \tau_z \big[\phi_{n,n^{j(1-\beta)}} - \phi_{n,n^j} \big] = \mathrm{Av}^*_{z\in\Lambda_{n^j}} \tau_z \big[\phi_{n,n^{j(1-\beta)}} - \phi_{n,n^j} \big] \end{aligned}$$

Thanks to lemma 4.15 and lemma 2.6

$$F_{0,\nu} \le C\, n^{-2+2j+dj} \mathrm{Av}_{y\in\Lambda_{s'}\cap(2n^j+1)\mathbb{Z}^d} \sup_{\mu_y} \mathrm{Var}_{\mu_y}(\tau_y H) \tag{4.59}$$

where μ_y varies among the measures $\mu_\Delta^{\lambda_\Delta(\alpha,m)}$, $\Delta=\Lambda_{y,n^j}$ and m varies among the possible densities in Λ_{y,n^j}. Let us estimate the r.h.s. of (4.59) with H replaced by:

$$H \to \mathrm{Av}^*_{z\in\Lambda_{n^j}} \tau_z \phi_{n,n^j}$$

Thanks to proposition 4.12, given $0<\delta\ll 1$ and n large enough, we obtain the bound

$$C\, n^{-2+2j-dj+2\delta} + C\, n^{-2+2j+dj} \mathrm{Av}_{y\in\Lambda_{s'}\cap(2n^j+1)\mathbb{Z}^d} \mathbb{I}_{\mathcal{A}_y} \tag{4.60}$$

where $\mathcal{A}_y$ is a set of disorder configurations in Λ_{y,n^j} such that $\mathbb{P}(\mathcal{A}_y)\le Ce^{-C'n^{j\delta}}$. Since $d\ge 3$, the first addendum in (4.60) is negligible. By reasoning as in the first part of the proof, it is simple to prove that the global contribution of the latter to (4.58) is negligible. Let us now estimate the r.h.s. of (4.59) with H replaced by:

$$H \to H' := \mathrm{Av}^*_{z\in\Lambda_{n^j}} \tau_z \phi_{n,n^{(1-\beta)j}}$$

The idea is to estimate variances on cubes of side $O(n^{(1-\beta)j})$. To this aim let $A := \Lambda_{n^j}\cap(2n^{(1-\beta)j}+1)\mathbb{Z}^d$ and $B:=\Lambda_{n^{(1-\beta)j}}$. Then

$$H' = \mathrm{Av}_{a\in A}\mathrm{Av}^*_{b\in B}\tau_{a+b}\phi_{n,n^{(1-\beta)j}}$$

and since the functions in the average defining H' have disjoint supports

$$\mathrm{Var}_{\mu_y}(\tau_y H') = \mathrm{Av}_{a\in y+A}\mathrm{Var}_{\mu_y}(\mathrm{Av}^*_{b\in B}\tau_{a+b}\phi_{n,n^{(1-\beta)j}}). \tag{4.61}$$

By means of proposition 4.12, given $0<\delta\ll 1$ and n large enough,

$$\mathrm{Var}_{\mu_y}(\tau_y H') \le C\, n^{(j-\beta)(-2d+2\delta)} + \mathrm{Av}_{a\in y+A}\mathbb{I}_{\mathcal{A}_a}$$

where $\mathcal{A}_a$ is an n–exponentially small set of disorder configurations on $\Lambda_{a,2n^{(1-\beta)j}}$. Let us remark that, thanks to our choise of β and d, we have

$$n^{-2+2j+dj+(j-\beta)(-2d+2\delta)} = n^{-2+2j-dj+C\beta+C'\delta} \downarrow 0$$

if $0<\beta, d\ll 1$. At this point it's simple to conclude the proof. □

4.6 From $s^{-1}\phi_{s,s}$ to $s^{-1}\phi_{s,s'}$ with $s=O(\epsilon^{-\frac{\mathfrak{a}}{d+2}})$ and $s'=O(\epsilon^{-\frac{\mathfrak{b}}{d+2}})$

Proposition 4.17. *Let $2\le d$, $1\le \mathfrak{a}\le\mathfrak{b}<d+2$, $s=O(\epsilon^{-\frac{\mathfrak{a}}{d+2}})$, $s'=O(\epsilon^{-\frac{\mathfrak{b}}{d+2}})$ and $s\le s'$.*
If $\mathfrak{b}<1+\frac{\mathfrak{a}}{2}$ then

$$\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z\Big[\frac{\phi_{s,s}}{s}-\frac{\phi_{s,s'}}{s}\Big]\approx 0$$

Proof. By remark 4.14 and the identity $\mathrm{Av}_x b_x = \mathrm{Av}_x \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x} b_z$, it's enough to prove that

$$\limsup_{a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|C_0|\le 1} \sup\nolimits_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,s'})} \bar{F}_{x,C_0,\nu} \le 0$$

where

$$\bar{F}_{x,C_0,\nu} \equiv \sup spec_{L^2(\nu)} \Big\{ \epsilon^{-1} C_0 \tau_x \Big[\frac{\phi_{s,s}}{s} - \frac{\phi_{s,s'}}{s} \Big] + C(d)\gamma_0 \epsilon^{-2} \mathrm{Av}_{b\in\Lambda^e_{x,s'}} \mathcal{L}_b \Big\}.$$

We define

$$f_\epsilon \equiv \frac{\phi_{s,s}}{s} - \frac{\phi_{s,s'}}{s}.$$

We choose γ such that $0 < \gamma < \frac{d}{2}$ and γ small enough to satisfy all the requests which will follow.
By lemma 4.13, $\tilde{\forall}\alpha$ and $\forall \epsilon \le \epsilon_0(\alpha)$,

$$\|f_\epsilon\|_\infty \le C s^{-\frac{d+2}{2}+\gamma}.$$

Therefore

$$\epsilon (s')^{d+2} \|f_\epsilon\|_\infty \le C \epsilon^{1-\mathfrak{b}+\frac{\mathfrak{a}}{2}-\frac{\mathfrak{a}\gamma}{d+2}}$$

which goes to 0 by taking the limit $\epsilon \downarrow 0$ if γ is small enough. This allows to apply proposition $A.13$ in order to estimate $\bar{F}_{x,C_0,\nu}$ and by lemma 2.6 it's enough to prove that for almost any α

(4.62)
$$\lim_{\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{m\in I_{s'}} F(x,m) = 0 \qquad \text{where } F(x,m) \equiv s^{-2}(s')^{d+2} \mathrm{Var}_{\mu^{\alpha,\lambda(\alpha,\Lambda^e_{x,s'},m)}}(\tau_x \phi_{s,s}).$$

In order to estimate $F(x,m)$ we use the main ideas of sections 4.2 and 4.3 to which we refer for notation and details. In particular we follows the notation given in (4.8), (4.9), (4.27) and remark 4.8.

We first consider the case of extreme densities $a_m \le 4 s^{-d+\gamma}$. For example let $m \le 1/2$. Since $|\phi_{s,s}| \le C m^e_s$,

$$F(x,m) \le C s^{-2}(s')^{d+2}(s^{-d} m + m^2) \le 2C(s')^{d+2} s^{-2d-2+2\gamma} = O(\epsilon^{-\mathfrak{b}+\mathfrak{a}+\mathfrak{a}\frac{d-2\gamma}{d+2}}).$$

By the hypothesis on $\mathfrak{a}, \mathfrak{b}$ and d, for γ small enough $\epsilon^{-\mathfrak{b}+\mathfrak{a}+\mathfrak{a}\frac{d-2\gamma}{d+2}} \downarrow 0$ as $\epsilon \downarrow 0$.

In what follows let $a_m \ge 4 s^{-d+\gamma}$. By the equivalence of ensembles and Poincaré inequality for Glauber dynamics, with an error of order $s^{-2-2d}(s')^{d+2}$, $F(x,m)$ can be bounded by

$$C s^{d-2} (s')^{d+2} a_m\, \mu^{\alpha,\lambda_{\Lambda^e_{x,s'}}(\alpha,m)} \big([\nabla_x \tau_x \hat{\phi}_{s,s}]^2 \big). \tag{4.63}$$

We use the results of section 4.2. In particular, by lemma 4.5, if $\alpha \notin \mathcal{A}^{(2)}_x(m,s)\cup\mathcal{A}^{(3)}_x(m,s)$ then for ϵ small enough

$$
\begin{aligned}
&a_m\, \mu^{\alpha,\lambda_{\Lambda^e_{x,s'}},(\alpha,m)}\big([\nabla_x\tau_x\hat{\phi}_{s,s}]^2\big) \le \\
&C\Big(s^{-2d}[\tau_x\frac{\xi_0(\alpha,m,s,s)}{\sigma_0^2(m,s)}]^2 + s^{-3d+\gamma} + \mu^{\alpha,\lambda_{\Lambda^e_{x,s'}},(\alpha,m)}(\mathcal{M}_x(m,s)\,)\,\Big).
\end{aligned}
\tag{4.64}
$$

Let $I' \equiv I_{s'} \cap \{a_m \ge 4s^{-d+\gamma}\}$ and $\gamma' \equiv \gamma/2$.
We define

$$
\begin{aligned}
\mathcal{A} &\equiv \cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I'}\Big(\mathcal{B}_x(m,s')\cup\mathcal{A}^{(2)}_x(m,s)\cup\mathcal{A}^{(3)}_x(m,s)\Big)\\
\mathcal{A}' &\equiv \Big(\cap_{x\in\mathbb{T}^d_\epsilon}\cap_{m\in I'}\mathcal{B}^c_x(m,s')\Big)\cap\Big(\cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I'}\{\ \mu^{\alpha,\lambda_{\Lambda^e_{x,s'}},(\alpha,m)}\big(\mathcal{M}_x(m,s)\big)\ge e^{-(2s)^{\gamma'}}\ \}\Big)\\
\mathcal{A}'' &\equiv \cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I'}\{\ |\tau_x\frac{\xi_0(\alpha,m,s)}{\sigma_0^2(m,s)}|\ge(2s)^{-\frac{d}{2}+\frac{\gamma}{2}}\ \}.
\end{aligned}
$$

By lemmas 4.6 and 4.7, proposition $A.1$ and some arguments as in the proof of lemma 4.10, it can be proved that there are positive constants C, r, r', depending on γ and γ', such that

$$
\mathbb{P}\big(\mathcal{A}\cup\mathcal{A}'\cup\mathcal{A}''\big)\le C\epsilon^{-r}e^{-\epsilon^{-r'}}
$$

By Borel-Cantelli lemma, $\tilde{\forall}\alpha$ if $\epsilon\le\epsilon_0(\alpha)$ then $\alpha\notin\mathcal{A}\cup\mathcal{A}'\cup\mathcal{A}''$. In this case, by (4.64), if ϵ is small enough,

$$
\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{m\in I_{s'}}F(x,m)\le Cs^{-2d-2+\gamma}(s')^{d+2} = C'\epsilon^{\mathfrak{a}+\frac{\mathfrak{a}d}{d+2}-\frac{\mathfrak{a}\gamma}{d+2}-\mathfrak{b}}
$$

which goes to 0 as $\epsilon\downarrow 0$ because of the conditions on $\mathfrak{a},\mathfrak{b},\gamma,d$.

$\square$

4.7 Estimates to pass from $n^{-1}\phi_{n,s}$ to $s^{-1}\phi_{s,2s}$

Proposition 4.18. *There exist $k_0, C_0 > 0$ having the following property.
Let $a, \gamma_0, T > 0$, $s = O(\epsilon^{-\delta})$ with $0<\delta<1$. Let γ be such that $0<\gamma<\frac{d}{2}$.*

Let n be a positive parameter satisfying one of the following cases.

Case A. $n\uparrow\infty$ after $\epsilon\downarrow 0$ and

$$
\gamma_0^{-1}\epsilon s^{d+2}n^{-1}\le k_0 \quad \textit{with } \epsilon\le\epsilon_0, n\ge n_0. \tag{4.65}
$$

Case B. $n = O(\epsilon^{-\delta'})$ with $0<\delta'<\delta$ and

$$
\gamma_0^{-1}\epsilon s^{d+2}n^{-\frac{d+2}{2}+\gamma}\le k_0 \quad \textit{with } \epsilon\le\epsilon_0. \tag{4.66}
$$

Then, in both cases, there exists a measurable set $\mathcal{A}$ with $\mathbb{P}(\mathcal{A}) \le C\epsilon^{-r}e^{-\epsilon^{-r'}}$ for positive r, r', C such that $\forall\alpha \notin \mathcal{A}$, if ϵ is small enough and n large enough,

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|C_0|\le 1} \sup_{\nu} F_{x,C_0,\nu} \le \gamma_0^{-1} n^{-2} s^{2-d+2\gamma} \tag{4.67}$$

where ν varies in $\mathcal{M}_\alpha(\Lambda_{x,2\frac{a}{\epsilon}})$ and

$$F_{x,C_0,\nu} \equiv \sup spec_{L^2(\nu)} \Big\{ \epsilon^{-1} C_0 \Big[\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{n,s}}{n} - \mathrm{Av}^{s,\frac{a}{\epsilon}}_{w,x} \tau_w \frac{\phi_{s,2s}}{s} \Big] + \gamma_0 \epsilon^{-2} \mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}} \mathcal{L}_b \Big\}$$

Proof. By lemma 1.9 and (1.32)

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{n,s}}{n} - \mathrm{Av}^{s,\frac{a}{\epsilon}}_{w,x} \tau_w \frac{\phi_{s,2s}}{s} = \mathrm{Av}^{s,\frac{a}{\epsilon}}_{w,x} \Big[\mathrm{Av}^{n,s}_{z,w} \tau_z \frac{\phi_{n,s}}{n} - \tau_w \frac{\phi_{s,2s}}{s} \Big].$$

By remark 4.14 the l.h.s. of (4.67) is bounded by

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|C_0|\le T} \sup\nolimits_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,2s})} F'_{x,C_0,\nu}$$

with

$$F'_{x,C_0,\nu} \equiv \sup spec_{L^2(\nu)} \Big\{ \epsilon^{-1} C_0 \Big[\mathrm{Av}^{n,s}_{z,x} \tau_z \frac{\phi_{n,s}}{n} - \frac{\phi_{s,2s}}{s} \Big] + C\gamma_0 \epsilon^{-2} \mathrm{Av}_{b\in\Lambda^e_{x,2s}} \mathcal{L}_b \Big\}$$

We consider $\|\frac{\phi_{n,s}}{n}\|_\infty$. In case A we bound it by n^{-1} while in case B, by lemma 4.13, for almost any α if ϵ is enough small we can bound it by $n^{-\frac{d+2}{2}+\gamma}$.
Therefore, by an appropriate choice of k_0, (4.65) in case A, or (4.66) in case B, allows to apply proposition $A.13$. By applying also lemma 2.6, we obtain that in order to prove (4.67) it's enough to prove that $\tilde{\forall}\alpha$, if ϵ is small enough, then

(4.68)
$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{m\in I_{2s}} F(x,m) \le C s^{-2d+2\gamma} \quad \text{where } F(x,m) \equiv \mathrm{Var}_{\mu^{\alpha,\lambda_{\Lambda^e_{x,2s}}(\alpha,m)}} \Big(\mathrm{Av}^{n,s}_{z,x} \tau_z \phi_{n,s} \Big).$$

For simplicity of notation we set $\lambda_x \equiv \lambda_{\Lambda^e_{x,2s}}(\alpha,m)$.
In order to estimate $F(x,m)$ we use the main ideas of sections 4.2 and 4.3 to which we refer for notation and details. In particular we follows the notation given in (4.8), (4.9), (4.27) and remark 4.8.

We consider first the case $m \le 4s^{-d+\gamma}$. Let, for example, $m \le 1/2$. Since $|\phi_{n,s}(\eta)| \le m^e_s(\eta)$ and $\mu^{\alpha,\lambda_x}(\eta_y) \le C(B)m$ for any $y \in \Lambda^e_{x,2s}$, $F(x,m)$ can be bounded by

$$C\mu^{\alpha,\lambda_x}\Big(\big[\mathrm{Av}^{n,s}_{z,x} m^e_{z,s}\big]^2 \Big) \le \mathrm{Av}^{n,s}_{z,x} \mu^{\alpha,\lambda_x}\Big([m^e_{z,s}]^2 \Big) \le C(s^{-d}m + m^2) \le C' s^{-2d+2\gamma}$$

Therefore in (4.68) we can restrict to the densities $m \in I'$ where $I' \equiv I_{2s} \cap \{a_m \ge 4s^{-d+\gamma}\}$.
By the equivalence of ensembles and Poincaré inequality for Glauber dynamics, with an error of order s^{-2d}, $F(x,m)$ can be bounded by

$$Ca_m\mu^{\alpha,\lambda_x}\Big(\sum_{y\in\Lambda^e_{x,2s}}[\mathrm{Av}^{n,s}_{z,x}\nabla_y\tau_z\hat{\phi}_{n,s}]^2\Big) \tag{4.69}$$

By lemma 4.5, if

$$\alpha \notin \cup_{z\in\Lambda^e_{x,2s}}\Big(\mathcal{A}^{(2)}_z(m)\cup\mathcal{A}^{(3)}_z(m)\Big),$$

(4.69) is bounded by

$$Cs^{-d}\big[\mathrm{Av}^{n,s}_{z,x}\tau_z\frac{\xi_0(\alpha,m)}{\sigma_0^2(m)}\big]^2 + Cs^{-2d+\gamma} + Cs^d\mu^{\alpha,\lambda_x}\big(\cup_{z\in\Lambda^e_{x,2s}}\mathcal{M}_z(m)\big). \tag{4.70}$$

By (1.32)

$$\mathrm{Av}^{n,s}_{z,x}\tau_z\xi_0(\alpha,m) = \mu^{\alpha,\lambda_0(m)}\Big(\mathrm{Av}^{n,s}_{z,x}\tau_z[m^{(1)}_n - m^{(2)}_n]; N^e_s\Big) = ns^{-1}\xi_0(\alpha,m,s).$$

We fix $0<\gamma'<\gamma$ and we define

$$\begin{aligned}
&\mathcal{B} \equiv \cap_{x\in\mathbb{T}^d_\epsilon}\cap_{m\in I'}\mathcal{B}^c_x(m,2s)\\
&\mathcal{A} \equiv \cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I'}\cup_{z\in\Lambda^e_{x,2s}}\Big(\mathcal{B}_x(m,2s)\cup\mathcal{A}^{(2)}_z(m)\cup\mathcal{A}^{(3)}_z(m)\Big)\\
&\mathcal{A}' \equiv \mathcal{B}\cap\Big(\cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I'}\{\ \mu^{\alpha,\lambda_{\Lambda^e_{x,2s}}(\alpha,m)}\big(\cup_{z\in\Lambda^e_{x,2s}}\mathcal{M}_z(m)\big) \ge 2(2s)^d e^{-(2s)^{\gamma'}}\ \}\Big)\\
&\mathcal{A}'' \equiv \cup_{x\in\mathbb{T}^d_\epsilon}\cup_{m\in I}\{\ |\tau_x\frac{\xi_0(\alpha,m,s)}{\sigma_0^2(m)}| \ge s^{-\frac{d}{2}+\frac{\gamma}{2}}\ \}.
\end{aligned}$$

By lemmas 4.6 and 4.7, proposition $A.1$ and some arguments as in the proof of lemma 4.10, it can be proved that there are positive constants C, r, r', depending on γ and γ', such that

$$\mathbb{P}\big(\mathcal{A}\cup\mathcal{A}'\cup\mathcal{A}''\big) \le C\epsilon^{-r}e^{-\epsilon^{-r'}}$$

By the Borel-Cantelli lemma, $\tilde{\forall}\alpha$ if $\epsilon \le \epsilon_0(\alpha)$ then $\alpha \notin \mathcal{A}\cup\mathcal{A}'\cup\mathcal{A}''$. In this case, we obtain the thesis by the estimate (4.70) and by the previous estimates related to extreme densities and the equivalence of ensembles.

□

4.8 From $n^{-1}\phi_{n,s}$ to $s^{-1}\phi_{s,s}$ with $s = O(\epsilon^{-\frac{1}{d+2}})$ and $n \uparrow \infty, \epsilon \downarrow 0$

Proposition 4.19. *Let $d \ge 3$, $s = O(\epsilon^{-\frac{1}{d+2}})$. Then $F^{(i)}_x \approx 0$ where $i = 1,2$ and*

$$F^{(1)}_x \equiv \mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{n,s}}{n} - \mathrm{Av}^{s,\frac{a}{\epsilon}}_{w,x}\tau_w\frac{\phi_{s,2s}}{s} \quad \textit{and} \quad F^{(2)}_x \equiv \mathrm{Av}^{s,\frac{a}{\epsilon}}_{w,x}\tau_w\Big[\frac{\phi_{s,2s}}{s} - \frac{\phi_{s,s}}{s}\Big].$$

Proof. The thesis with $i=1$ is a consequence of proposition 4.18 and $d \ge 3$.
In order to prove the thesis with $i=2$ it's enough to apply proposition 4.17 with $\mathfrak{a} = \mathfrak{b} = 1$.

□

4.9 From $s^{-1}\phi_{s,s}$ to $(s')^{-1}\phi_{s',s'}$ with $s = O(\epsilon^{-\frac{1}{d+2}})$ and $s' = O(\epsilon^{-\frac{2}{d+2}+\gamma})$

The main result of this section is proposition 4.21, whose proof is based on a recursive procedure having two fundamental steps given by lemma 4.20.

Lemma 4.20. *Let $d \geq 3$ and*

$$s \equiv \epsilon^{-\frac{\mathfrak{a}}{d+2}} \qquad s' \equiv \epsilon^{-\frac{\mathfrak{b}}{d+2}} \qquad \textit{where} \;\; 1 \leq \mathfrak{a} < \mathfrak{b} < d+2.$$

If $\mathfrak{b} < 1 + \frac{\mathfrak{a}}{2}$ then:
step 1): $\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \big[\frac{\phi_{s,s}}{s} - \frac{\phi_{s,s'}}{s}\big] \approx 0$,
step 2): $\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s'}}{s} \approx \mathrm{Av}_{z,x}^{s',\frac{a}{\epsilon}} \tau_z \frac{\phi_{s',s'}}{s'}$

Proof. Proposition 4.17 implies the identity of the first step and also $\mathrm{Av}_{z,x}^{s',\frac{a}{\epsilon}} \tau_z \big[\frac{\phi_{s',2s'}}{s'} - \frac{\phi_{s',s'}}{s'}\big] \approx 0$.
Therefore it's enough to prove

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s'}}{s} \approx \mathrm{Av}_{z,x}^{s',\frac{a}{\epsilon}} \tau_z \frac{\phi_{s',2s'}}{s'}.$$

For this purpose it's enough to apply proposition 4.18 where n, s have to be substituted with the current s, s' and γ is chosen such that $\mathfrak{b} < 1 + \frac{\mathfrak{a}}{2} - \frac{\mathfrak{a}\gamma}{d+2}$ and $2\gamma < d-2$.

□

Proposition 4.21. *Let $d \geq 3$.*
Given $0 < \gamma < \frac{2}{d+2}$ there exists γ' with $0 < \gamma' \leq \gamma$ such that

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} \approx \mathrm{Av}_{z,x}^{s',\frac{a}{\epsilon}} \tau_z \frac{\phi_{s',s'}}{s'}$$

where

$$s \equiv \epsilon^{-\frac{1}{d+2}} \quad \textit{and} \quad s' \equiv \epsilon^{-\frac{2}{d+2}+\gamma'}.$$

Proof. Let $a_1 \equiv 1$ and $a_{n+1} \equiv 1 + 2^{-1}a_n - 2^{-n}a_n$. We define $s_n \equiv \epsilon^{-\frac{a_n}{d+2}}$ and observe that $s_1 = s$. It's simple to check that the sequence $\{a_n\}_{n\geq 1}$ is increasing and converges to 2
By lemma 4.20, for any n,

$$\mathrm{Av}_{z,x}^{s_n,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_n,s_n}}{s_n} \approx \tau_z \mathrm{Av}_{z,x}^{s_{n+1},\frac{a}{\epsilon}} \frac{\phi_{s_{n+1},s_{n+1}}}{s_{n+1}}.$$

Since $\{a_n\}_{n\geq 1}$ converges to 2, the exists $\bar{n}$ such that $a_{\bar{n}} \geq 2 - \gamma(d+2)$. Then it's enough to take $s' \equiv s_{\bar{n}}$. □

4.10 Strong Two Blocks estimate

In what follows

$$s = O(\epsilon^{-\frac{2}{d+2}+\gamma})$$

where $0 < \gamma < \frac{2}{d+2}$ and γ is small enough to satisfy the conditions which will follow.
We observe that if $\beta > \frac{d+2}{2}$ then, for γ small enough, $\epsilon^{-1}s^{-\beta}$ goes to zero by taking the limit $\epsilon \downarrow 0$.

We write Δm_s for $m_s^{1,e} - m_s^{2,e}$. Let us remember that $I_s \equiv \{0, \frac{1}{2s^d}, \frac{2}{2s^d}, \dots, 1\}$.

Proposition 4.22. *If $d \geq 3$ and γ is small enough, then*

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} \approx 0.$$

That is, given $\gamma_0 > 0$, for almost any disorder configuration α

(4.71)
$$\limsup_{a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_{|C_0|\leq 1} \sup_\nu \sup spec_{L^2(\nu)}\big\{\epsilon^{-1}C_0 \mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} + \gamma_0\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\big\} \leq 0$$

where ν varies in $\mathcal{M}_\alpha(\Lambda_{x,2\frac{a}{\epsilon}})$.

The rest of the section is devoted to the proof of proposition 4.22.
By the equivalence of ensembles we can substitute in (4.71) $\phi_{s,s}$ by $\hat{\phi}_{s,s}$ where

$$\hat{\phi}_{s,s}(\eta) \equiv \mu^{\alpha,\lambda_{\Lambda_s^e}(\alpha,m_s^e(\eta))}\big(\Delta m_s\big)$$

with a global error of order not more than $\epsilon^{-1}s^{-(1+d)}$ which goes to zero if γ is small enough.

Lemma 4.23. *For almost any disorder configuration α, $\forall\epsilon$, $\forall x \in \mathbb{T}_\epsilon^d$ and $\forall m \in I_s$*

$$|\mu^{\alpha,\lambda_{\Lambda_{x,s}^e}(\alpha,m)}(\tau_x\Delta m_s) - \mu^{\alpha,\lambda_0(m)}(\tau_x\Delta m_s)| \leq C(\alpha)s^{-d+2\gamma}. \tag{4.72}$$

Proof. We prove the above lemma with I_s substituted by $I_s\cap[0,\frac{1}{2}]$. The case $m \geq \frac{1}{2}$ can be analyzed in a similar way. Since the thesis is obvious for $m = 0$, we assume $m \in I_s\cap(0,\frac{1}{2}]$.
We set $\lambda \equiv \lambda_{\Lambda_{x,s}^e}(\alpha, m)$ and $\lambda_0 \equiv \lambda_0(m)$.
Then, there are $\tilde{\lambda}, \lambda'$ between λ and λ_0 such that

$$\begin{aligned}
&\mu^{\alpha,\lambda}(\tau_x\Delta m_s) - \mu^{\alpha,\lambda_0}(\tau_x\Delta m_s) = \mu^{\alpha,\tilde{\lambda}}(\tau_x\Delta m_s; N_{x,s}^e)(\lambda-\lambda_0),\\
&\mu^{\alpha,\tilde{\lambda}}(\tau_x\Delta m_s; N_{x,s}^e) - \mu^{\alpha,\lambda_0}(\tau_x\Delta m_s; N_{x,s}^e) = \mu^{\alpha,\lambda'}(\tau_x\Delta m_s; N_{x,s}^e; N_{x,s}^e)(\lambda'-\lambda_0).
\end{aligned} \tag{4.73}$$

By lemma $A.6$,

$$|\lambda - \lambda_0| \leq Cm^{-1}|m - \mu^{\alpha,\lambda_0}(m_{x,s}^e)|.$$

Let $\mathcal{B} \equiv \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 \equiv \{\exists x \in \mathbb{T}^d_\epsilon,\ \exists m \in I_s \setminus \{0,1\}\ :\ m^{-1}|m - \mu^{\alpha,\lambda_0(m)}(m^e_{x,s})| \geq s^{-\frac{d}{2}+\gamma}\}$$
$$\mathcal{B}_2 \equiv \{\exists x \in \mathbb{T}^d_\epsilon,\ \exists m \in I_s \setminus \{0,1\}\ :\ |\mu^{\alpha,\lambda_0(m)}(\tau_x \Delta m_s; N^e_{x,s})| \geq s^{-\frac{d}{2}+\gamma}\}.$$

By proposition $A.1$ there exists a polynomial p such that $\mathbb{P}(\mathcal{B}) \leq p(\epsilon^{-1})e^{-Cs^{2\gamma}}$.
By Borel-Cantelli lemma, $\tilde{\forall}\alpha$ if $\epsilon \leq \epsilon(\alpha)$ then $\alpha \notin \mathcal{B}$.
Since $\mu^{\alpha,\lambda'}(\tau_x \Delta m_s; N^e_{x,s}; N^e_{x,s}) \leq C$, the thesis follows by (4.73) and the definition of $\mathcal{B}$. □

By the previous lemma in (4.71) we can substitute $\hat{\phi}_{s,s}$ by $\mu^{\alpha,\lambda_0(m^e_s)}(\Delta m_s)$.

Lemma 4.24. *If $0 < \beta$ is small enough, then for almost any disorder configuration α, for any ϵ small enough, $\forall m, m' \in [0,1]$ and $\forall x \in \mathbb{T}^d_\epsilon$*

$$\left|\mu^{\alpha,\lambda_0(m)}\big(\tau_x \Delta m_s\big) - \mu^{\alpha,\lambda_0(m')}\big(\tau_x \Delta m_s\big)\right| \leq C\, s^{-\frac{d}{2}+\beta}|m - m'| + C\, s^{-\frac{d}{2}-\beta}. \tag{4.74}$$

Proof. By the continuity in m, m' of both members of (4.74), it is not restrictive to assume $0 < m < m' < 1$. Moreover, by lemma $A.5$, we can assume $m, m' \in I_s$ with an error of order s^{-d}.
We set

$$\mathcal{A} \equiv \{\exists x \in \mathbb{T}^d_\epsilon, \exists r \in I_s \cap (0,1)\ :\ \frac{1}{r}|\mu^{\alpha,\lambda_0(r)}\Big(\tau_x \Delta m_s\Big)| \geq s^{-\frac{d}{2}+\beta}\},$$
$$\mathcal{B} \equiv \{\exists x \in \mathbb{T}^d_\epsilon,\ \exists r \in I_s \cap (0,1)\ :\ \frac{1}{r}|\mu^{\alpha,\lambda_0(r)}(\tau_x \Delta m_s; \tau_x N^e_s)| \geq s^{-\frac{d}{2}+\beta}\}.$$

By proposition $A.1$, $\mathbb{P}(\mathcal{A} \cup \mathcal{B}) \leq C\epsilon^{-2d}e^{-Cs^{2\beta}}$ and therefore, by Borel-Cantelli lemma, we have that $\alpha \notin \mathcal{A} \cup \mathcal{B}$ if ϵ is small enough (as we assume in what follows).
If $m' - m \geq \frac{1}{2}$ then the thesis is valid because $\alpha \notin \mathcal{A}$. Let us assume $m, m' \leq \frac{3}{4}$ (the case $m, m' \geq \frac{1}{4}$ can be analyzed similarly) and, setting $\delta \equiv s^{-\frac{d}{2}+\beta}$, let us distinguish the cases $m \leq \delta$ and $m > \delta$.

Let $m \leq \delta$. Since $\alpha \notin \mathcal{A}$ the l.h.s. of (4.74) is bounded by

$$Cs^{-\frac{d}{2}+\beta}\delta + Cs^{-\frac{d}{2}+\beta}m' \leq C'\, s^{-\frac{d}{2}+\beta}(m' - m) + C' s^{-d+2\beta}.$$

Let us consider the case $\delta < m < m' \leq \frac{3}{4}$. For some $\tilde{m} \in (m, m')$, the l.h.s. of (4.74) is equal to

$$|\mu^{\alpha,\lambda_0(\tilde{m})}(\tau_x \Delta m_s; \tau_x N^e_s)|\lambda_0'(\tilde{m})(m' - m). \tag{4.75}$$

We remark that

$$\lambda_0'(\tilde{m}) = \frac{1}{\mathbb{E}\Big[\mu^{\alpha,\lambda_0(\tilde{m})}(\eta_0; \eta_0)\Big]} \leq \frac{C}{\tilde{m}}.$$

Let $r \in I_s$ be such that $|r - \tilde{m}| \le s^{-d}$. By lemma $A.5$,

$$|\mu^{\alpha,\lambda_0(\tilde{m})}(\tau_x \Delta m_s; \tau_x N_s^e) - \mu^{\alpha,\lambda_0(r)}(\tau_x \Delta m_s; \tau_x N_s^e)| \le C s^{-d}.$$

Therefore

$$\frac{1}{\tilde{m}}|\mu^{\alpha,\lambda_0(\tilde{m})}(\tau_x \Delta m_s; \tau_x N_s^e)| \le C\frac{s^{-d}}{\tilde{m}} + C\frac{1}{r}|\mu^{\alpha,\lambda_0(r)}(\tau_x \Delta m_s; \tau_x N_s^e)|\frac{r}{\tilde{m}}. \tag{4.76}$$

Since $\tilde{m} \ge \delta$ and $\alpha \notin \mathcal{B}$ the r.h.s. of (4.76) is bounded by $Cs^{-\frac{d}{2}+\beta}$ and the thesis follows from (4.75).

□

Let us introduce a new mesoscopic scale $L = O(\epsilon^{\delta-1})$ with $1 - \delta > \frac{2}{d+2} - \gamma$.
By lemma 1.9

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \mu^{\alpha,\lambda_0(m_{z,s}^e)}\Big(\tau_z \frac{\Delta m_s}{s}\Big) = \mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} Av_{z,w}^{s,L} \mu^{\alpha,\lambda_0(m_{z,s}^e)}\Big(\tau_z \frac{\Delta m_s}{s}\Big).$$

By (1.32) and lemma 4.13, if δ is small enough, for almost any disorder configuration α,

$$\mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} Av_{z,w}^{s,L} \mu^{\alpha,\lambda_0(m_{w,L}^e)}\Big(\tau_z \frac{\Delta m_s}{s}\Big) = \mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} \mu^{\alpha,\lambda_0(m_{w,L}^e)}\Big(\tau_w \frac{\Delta m_L}{L}\Big) \approx 0.$$

Therefore, in order to prove proposition 4.22, it's enough to prove

$$\mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} Av_{z,w}^{s,L}\Big[\mu^{\alpha,\lambda_0(m_{z,s}^e)}\Big(\tau_z \frac{\Delta m_s}{s}\Big) - \mu^{\alpha,\lambda_0(m_{w,L}^e)}\Big(\tau_z \frac{\Delta m_s}{s}\Big)\Big] \approx 0. \tag{4.77}$$

Given $0 < \beta \ll 1$, by lemma 4.24 the l.h.s. of (4.77) is bounded (for almost any α and for ϵ small enough) by

$$s^{-\frac{d+2}{2}+\beta} \mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} Av_{z,w}^{s,L}|m_{z,s}^e - m_{w,L}^e| + s^{-\frac{d+2}{2}-\beta}.$$

Therefore, in order to conclude the proof of proposition 4.22 it is enough to apply the following proposition (which is a sort of strong Two Blocks Estimate) after choosing suitable $0 < \beta \ll \gamma \ll \rho \ll 1$.

Proposition 4.25. *(Strong Two Blocks estimate)*
Let $s = O(\epsilon^{-t})$ and $L = O(\epsilon^{\delta-1})$ with $t < 1 - \delta$.
If $0 < r < \min\big(\frac{2\delta}{d+4}, \frac{t}{2}\big)$ then for any $\gamma_0 > 0$

$$\mathrm{Av}_{x\in\mathbb{T}_\epsilon^d} \sup_\nu \sup spec_{L^2(\nu)}\big\{\epsilon^{-r} \mathrm{Av}_{w,x}^{L,\frac{a}{\epsilon}} Av_{z,w}^{s,L}|m_{z,s}^e - m_{w,L}^e| + \gamma_0 \epsilon^{-2} \mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}} \mathcal{L}_b\big\} \le 0 \tag{4.78}$$

where ν varies among $\mathcal{M}_\alpha(\Lambda_{x,2\frac{a}{\epsilon}})$.

Proof. In what follows C is a generic positive constant which can depend only on d, B. It can change from line to line.
We write $\frac{a}{\epsilon} = R = 2R' + 1$, $L = 2L' + 1$ and $s = 2s' + 1$. Moreover, given $x \in \mathbb{Z}^d$ and $u, w \in \mathbb{Z}$ we define $A_1 = A_1(x, u)$ and $A_2 = A_2(v)$ in the following way:

$$\begin{aligned} A_1 &\equiv \{x + (-R' + L' + uL)e + y \,:\, y \in L\mathbb{Z}^d \cap \Lambda_{0,R'}\} \\ A_2 &\equiv \{(-L' + s' + vs) + y \,:\, y \in s\mathbb{Z}^d \cap \Lambda_{0,L'}\}. \end{aligned}$$

Then, thanks to the definition (1.31) and the subadditivity of sup *spec*, the supremum of the spectrum in (4.78) is bounded by

$$\begin{aligned} (4.79)\quad \mathrm{Av}_{u=0}^{\frac{R}{L}-1}\mathrm{Av}_{v=0}^{\frac{L}{s}-1} \sup spec_{L^2(\nu)}\{\epsilon^{-r}\mathrm{Av}_{a_1\in A_1}\mathrm{Av}_{a_2\in A_2}|m^e_{a_1+a_2,s} - m^e_{a_1,L}| \\ + \gamma_0\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\}. \end{aligned}$$

Let $l \equiv \epsilon^{-r}$. Uniformly in z

$$(4.80)\qquad m^e_{z,s} = \mathrm{Av}_{y\in\Lambda^e_{z,s}} m_{y,l} + O\Big(\frac{l}{s}\Big) \qquad m^e_{z,L} = \mathrm{Av}_{y\in\Lambda^e_{z,L}} m_{y,l} + O\Big(\frac{l}{L}\Big),$$

therefore, if r is small enough that $l^2/s \downarrow 0$, we can perform substitution (4.80) by neglecting the error terms. Moreover, since $l^{d+1}L^{-d} < l^2/s \downarrow 0$, the spectrum in (4.79) can be estimated (with an error depressed in the limit $\epsilon \downarrow 0$) by

$$\begin{aligned} (4.81)\quad \sup spec_{L^2(\nu)}\{\epsilon^{-r}\mathrm{Av}_{a_1}\mathrm{Av}_{a_2}\mathrm{Av}_{y\in\Lambda^e_{a_1+a_2,s}}\mathrm{Av}_{y'\in\Lambda^e_{a_1,L}}\mathbb{I}_{|y-y'|>2l}|m_{y,l} - m_{y',l}| + \\ \gamma_0\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}}\mathcal{L}_b\}. \end{aligned}$$

Let $z \in \Lambda_{0,l}$ with $z_e = 0$. We assert that

$$(4.82)\qquad \mathrm{Av}_{a_1}\mathrm{Av}_{a_2}\mathrm{Av}_{y\in\Lambda^e_{a_1+a_2,s}}\mathrm{Av}_{y'\in\Lambda^e_{a_1,L}}(-\mathcal{L}_{y+le+z,y'-le+z}) \le CL^2\mathrm{Av}_{b\in\Lambda_{x,\frac{a}{\epsilon}}}(-\mathcal{L}_b).$$

In fact, by the Moving Particle lemma, $-\mathcal{L}_{u,v} \le CL\sum_{b\in\gamma_{u,v}}(-\mathcal{L}_b)$ for any u, v at distance $O(L)$ and therefore the l.h.s. of (4.82) is bounded by

$$C\epsilon^d a^{-d} L^{1-d} \sum_{b\in\Lambda_{x,2\frac{a}{\epsilon}}} (-a_b\mathcal{L}_b)$$

where a_b is the numbers quadruples (a_1, a_2, y, y') with $a_1 \in A_1$, $a_2 \in A_2$, $y \in \Lambda^e_{a_1+a_2,s}$, $y' \in \Lambda^e_{a_1,L}$ and $b \in \gamma_{y+le+z,y'-le+z}$.

Let (a_1, a_2, y, y') as above. We observe that, given b, y, y' are at distance less than CL from b, therefore there are $O(L^{d+1})$ possible couples (y, y'). Given (y, y') there are $O(1)$ possible couples (a_1, a_2). Therefore $a_b \le L^{d+1}$ and (4.82) is proven.

By some trivial computation we obtain

(4.83)
$$\mathrm{Av}_{a_1}\mathrm{Av}_{a_2}\mathrm{Av}_{y\in\Lambda^e_{a_1+a_2,s}}\mathrm{Av}_{y'\in\Lambda^e_{a_1,L}}\Big(\mathrm{Av}_{b\in\Lambda_{y,l}}(-\mathcal{L}_b)+\mathrm{Av}_{b\in\Lambda_{y',l}}(-\mathcal{L}_b)\Big)\le C\mathrm{Av}_{b\in\Lambda_{x,\frac{a}{\epsilon}}}(-\mathcal{L}_b).$$

Given y, y' with $|y-y'|>2l$, we define

$$T^\alpha_{y,y'}\equiv \mathrm{Av}_{b\in\Lambda_{y,l}}\mathcal{L}_b+\mathrm{Av}_{b\in\Lambda_{y',l}}\mathcal{L}_b+(2l+1)^{-d}\sum_{z\in\Lambda_{0,l}\,:\,z_e=0}\mathcal{L}_{y+z+le,y'+z-le}$$

and we write $\Theta^\alpha_{y,y'}$ for the set of canonical measures on $\Lambda_{y,l}\cup\Lambda_{y',l}$ with disorder configuration α.
By (4.82) and (4.83), (4.81) is bounded by

$$(4.84)\quad \mathop{\mathrm{Av}_1}_{a_1}\mathop{\mathrm{Av}_2}_{a_2}\mathop{\mathrm{Av}}_{y\in\Lambda^e_{a_1+a_2,s}}\mathop{\mathrm{Av}}_{y'\in\Lambda^e_{a_1,L}}\sup_{\nu\in\Theta^\alpha_{y,y'}}\mathbb{I}_{|y-y'|>2l}\sup_{L^2(\nu)} spec\{l|m_{y,l}-m_{y',l}|+C\gamma_0\epsilon^{-2\delta}T^\alpha_{y,y'}\}.$$

By large deviations arguments as in chapter 2 or by the diffusive scaling limit for Kawasaki dynamics we obtain that $\exists C$ such that $\tilde\forall\alpha$, if $\epsilon\le\epsilon_0(\alpha)$, $\forall y,y'\in\mathbb{T}^d_\epsilon$ with $|y-y'|>2l$

$$\sup_{\nu\in\Theta^\alpha_{y,y'}} gap_{L^2(\nu)}T^\alpha_{y,y'}\ge Cl^{-(d+2)}.$$

Therefore, if $\epsilon^{2\delta}l^{d+3}\downarrow 0$ we can apply proposition $A.13$ which gives

$$\sup spec_{L^2(\nu)}\{l|m_{y,l}-m_{y',l}|+C\gamma_0\epsilon^{-2\delta}T^\alpha_{y,y'}\}\le l\nu(|m_{y,l}-m_{y',l}|)+C'\epsilon^{2\delta}l^{d+4}\nu(|m_{y,l}-m_{y',l}|^2).$$

If $\epsilon^{2\delta}l^{d+4}\downarrow 0$ then, by collecting the previous estimates, the l.h.s. of (4.78) can be bounded, with an error depressed in the limit $\epsilon\downarrow 0$, by

$$(4.85)\qquad \sup_{y,y'\in\mathbb{T}^d_\epsilon:|y-y'|>2l}\ \sup_{\nu\in\Theta^\alpha_{y,y'}} l\nu(|m_{y,l}-m_{y',l}|).$$

Let ν have density m on $\Lambda_{y,l}\cup\Lambda_{y',l}$. By lemma $A.8$

$$(4.86)\quad \nu\big((m_{y,l}-m_{y',l})^2\big)\le C\Big(l^{-d}+\big(m-\mu^{\alpha,\lambda_0(m)}(m_{y,l})\big)^2+\big(m-\mu^{\alpha,\lambda_0(m)}(m_{y',l})\big)^2\Big).$$

Given $0<\gamma'<d$ we define

$$\mathcal{B}\equiv\{\,\exists m\in I_{2l},\,\exists y\in\mathbb{T}^d_\epsilon\ :\ |m-\mu^{\alpha,\lambda_0(m)}(m_{y,l})|\ge l^{-\frac{d}{2}+\frac{\gamma'}{2}}\}$$

By proposition $A.1$ and Borel-Cantelli lemma, $\tilde\forall\alpha$ if $\epsilon\le\epsilon_0(\alpha)$ then $\alpha\notin\mathcal{B}$ and in this case, by (4.86), (4.85) is bounded by $Cl^{1-\frac{d}{2}+\frac{\gamma'}{2}}$. As $d\ge 3$, if γ' is small enough then (4.85) goes to zero by taking the limit $\epsilon\downarrow 0$ and that completes the proof.

□

Chapter 5

Central Limit Theorem Semi-norm

In this chapter we analyze the structure of the space $\mathcal{G}$ endowed of the semi-inner product $V_m(\cdot,\cdot)$. The final aim is to prove that for any $e>0$

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty} \sup_{m\in[0,1]} V_m\Big(j_e + \mathcal{L}g - \sum_{e'>0} D_{e,e'}(m)\frac{\psi^{e'}_{n,n}}{n}\Big) = 0. \tag{5.1}$$

This result, together with theorem 4.1, allows to prove the generalized Fick's law for the disordered system under consideration.
(5.1) is related to the orthogonal decomposition of the currents along $\mathcal{L}\mathbb{G} = \{\mathcal{L}g : g\in\mathbb{G}\}$ and its orthogonal subspace $\mathcal{L}\mathbb{G}^{\perp}$, which we prove to be generated by the limit points of the sequences $\left\{\frac{\psi^{e'}_{n,n}}{n}\right\}_{n\geq 1}$ with e' varying in the canonical basis of $\mathbb{Z}^d$.

The diffusion matrix $D(m)$ has lower and upper bounds

$$0 < C\mathbb{I} \leq D(m) \leq C'\mathbb{I} \quad \forall m\in(0,1),$$

and it is continuous on $(0,1)$. At the moment we are not able to prove that D can be continuously extended to $0,1$. We assume that. We remark that generally the diffusion matrix has no pathological behavior. Recently, the C^∞ regularity of the diffusion matrix has been proved for some models by duality methods, we refer to [2], [16] and [17].

5.1 Table of computations

In this section,for particular functions $f,g\in\mathcal{G}$ and $\forall m\in(0,1)$ we compute the limit

$$V_m(f,g) \equiv \lim_{k\uparrow\infty} V_k(f,g;\mu^{\alpha,\lambda_0(m)}).$$

Given $f\in\mathbb{G}$, $g\in\mathcal{G}$ and $e>0$ we set

$$
\begin{aligned}
(g,f)_0 &\equiv \sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(g,\tau_x f)\Big],\\
t_e(g) &\equiv \sum_{x\in\mathbb{Z}^d}(x,e)\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(\eta_x,g)\Big].
\end{aligned}
$$

We remark that, since $g\in\mathcal{G}$, the above sums are finite. Let us remember that $\underline{g}\equiv\sum_{x\in\mathbb{Z}^d}\tau_x g$ for any $g\in\mathbb{G}$.

We define $\mathbb{P}^*$ as the probability measure on $\tilde{\Omega}$ such that for any $f\in\mathbb{G}$

$$\int \mathbb{P}^*(d\alpha,d\eta)f(\alpha,\eta) = \int \mathbb{P}(d\alpha)\int \mu^{\alpha,\lambda_0(m)} f(\alpha,\eta).$$

A fundamental property of $\mathbb{P}^*$ is its translation invariance. We write $\mathbb{E}^*$ for the expectation w.r.t. $\mathbb{P}^*$.

Proposition 5.1. *For any $g\in\mathcal{G}$, $\forall u\in\mathbb{G}$, $\forall n$ and $e,e'>0$*

$$V_m(g,\mathcal{L}u) = -(g,u)_0 \tag{5.2}$$

$$V_m(\mathcal{L}u,\mathcal{L}u) = \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\sum_{e>0}c_e(\nabla_{0,e}\underline{u})^2\ \big)\Big] \tag{5.3}$$

$$V_m(j_e,\mathcal{L}u) = -\frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\ c_e\nabla_{0,e}\underline{u}\cdot\nabla_{0,e}\eta_0\ \big)\Big] \tag{5.4}$$

$$V_m(j_e,g) = -t_e(g) \tag{5.5}$$

$$V_m(j_e,j_{e'}) = \frac{\delta_{e,e'}}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\ c_e(\nabla_{0,e}\eta_0)^2\ \big)\Big] \tag{5.6}$$

$$V_m\Big(j_e,\frac{W_n^{e'}}{n}\Big) = \delta_{e,e'}2m(1-m). \tag{5.7}$$

Proof. The proof consists in standard computations in which the translation invariance of $\mathbb{P}^*$ is crucial.
It's simple to verify (5.3) and (5.4) follow from (5.2), while (5.6) follows from (5.5).

We prove (5.2).

Let $g,u\in\mathcal{G}_s$ and $\sqrt{k}>s$, then $\mathcal{L}\tau_x u=\mathcal{L}_k\tau_x u$ if $|x|\le k_1$. Therefore

$$\mathbb{E}\Big[V_k(g,\mathcal{L}u;\mu)\Big] = -(2k)^{-d}\sum_{\substack{|x|\le k_1\\|y|\le k_1}}\mathbb{E}^*(\tau_x g,\tau_y u) = -\sum_{|z|\le 2s}c_{z,k}\mathbb{E}^*(g,\tau_z u) \tag{5.8}$$

where

$$c_{z,k}\equiv(2k)^{-d}|\{(x,y)\in\Lambda_{k_1}\times\Lambda_{k_1}\ :\ y-x=z\}|.$$

Since $\lim_{k\uparrow\infty} c_{z,k} = 1$, we gain the thesis.

We prove (5.5).
By (1.3) and Schwarz inequality, for any finite set B,

$$[\mu\big((\sum_{b\in B} j_b)v\big)]^2 \leq \frac{1}{4}\mu\big(\sum_{b\in B} c_b(\nabla_b\eta)^2\big)\cdot\mu\big(\sum_{b\in B} c_b(\nabla_b v)^2\big).$$

In particular, by taking $B \subset \Lambda_k$ and $v \equiv -\mathcal{L}_k^{-1}(\sum_{b\in B} j_b)$,

$$\mu(\sum_{b\in B} j_b, -\mathcal{L}_k^{-1}\sum_{b\in B} j_b) \leq C|B|. \tag{5.9}$$

Thanks to (5.9), Schwarz inequality and the estimate (proved in section 1.12)

$$\mathbb{E}\Big[V_k(g;\mu)\Big] \leq C(g) \quad \forall k,$$

$V(j_e, g)$ is equal to

$$\lim_{l\uparrow\infty} (2k)^{-d}\mathbb{E}\Big[\mu\big(\sum_{x\in\Lambda_{k_1}} \tau_x g, -\mathcal{L}_k^{-1} \sum_{\substack{x\in\Lambda_k:\\ x+e\in\Lambda_k}} j_{x,x+e}\big)\Big].$$

The thesis follows from

$$\mathcal{L}_k \sum_{x\in\Lambda_k}(x,e)\eta_x = \sum_{\substack{x\in\Lambda_k:\\ x+e\in\Lambda_k}} j_{x,x+e}$$

and from the identity, valid with $s \leq \sqrt{k}$,

$$\sum_{y\in\Lambda_{k_1}}\sum_{x\in\Lambda_k}(x,e)\mathbb{E}^*(\tau_y g\cdot\eta_x) = \sum_{|a|\leq s}\Big(\mathbb{E}^*(g\eta_a)\sum_{x\in\Lambda_k\cap(\Lambda_{k_1}+a)}(x,e)\Big) =$$
$$\sum_{|a|\leq s}\Big(\mathbb{E}^*(g\eta_a)\sum_{x\in(\Lambda_{k_1}+a)}(x,e)\Big) = (2k_1+1)^d\sum_{|a|\leq s}(a,e)\mathbb{E}^*(g\eta_a).$$

In order to prove (5.7) in what follows we assume that x, y and z vary respectively in $\mathbb{Z}^d$, $\Lambda_{1,n}^{e'}$ and $\Lambda_{2,n}^{e'}$. Then, by (5.5) and (1.3),

$$V_m(j_e, \frac{W_n^{e'}}{n}) = -n^{-(2d+1)}\sum_{x,y,z}(x,e)\mathbb{E}^*((\eta_z-\eta_y)\nabla_{yz}\eta_x)$$
$$= -n^{-(2d+1)}2m(1-m)\sum_{y,z}(y-z,e) = 2m(1-m).$$

□

5.2 Semi-norm $V_m^{1/2}$

By (1.48) for any $f \in \mathcal{G}$ the limits

$$V^*(f) \equiv \limsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(f;\mu^{\alpha,\lambda_0(m)})\Big] \tag{5.10}$$

$$V_*(f) \;\equiv \liminf_{k\uparrow\infty} \mathbb{E}\Big[V_k(f;\mu^{\alpha,\lambda_0(m)})\Big] \tag{5.11}$$

are finite. By the following theorem they are equal.

Theorem 5.2. *$\forall m \in (0,1)$ and $\forall f, g \in \mathcal{G}$ the limit*

$$V_m(f,g) \equiv \lim_{k\uparrow\infty} \mathbb{E}\Big[V_k(f,g;\mu^{\alpha,\lambda_0(m)})\Big] \tag{5.12}$$

exists, is finite and defines a semi-inner product on $\mathcal{G}$.
In particular, it is well-defined the semi-norm $V_m^{1/2}$:

$$V_m(f) \equiv \lim_{k\uparrow\infty} \mathbb{E}\Big[V_k(f;\mu^{\alpha,\lambda_0(m)})\Big]. \tag{5.13}$$

Moreover,

$$\begin{aligned} V_m(f) &= \sup_{\substack{a\in\mathbb{R}^d\\ u\in\mathbb{G}}} \big\{2V_m(f,\sum_{e>0} a_e j_e + \mathcal{L}u) - V_m(\sum_{e>0} a_e j_e + \mathcal{L}u)\big\} \\ &= \sup_{\substack{a\in\mathbb{R}^d\\ u\in\mathbb{G}}} \Big\{\sum_{e>0} 2a_e t_e(f) + 2(f,u)_0 - \sum_{e>0} \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(c_e(a_e\nabla_e\eta_0 - \nabla_e\underline{u})^2\big)\Big]\Big\} \end{aligned} \tag{5.14}$$

and the subspace

$$\{\sum_{e>0} a_e j_e + \mathcal{L}g \;:\; a \in \mathbb{R}^d,\; g \in \mathbb{G}\} \tag{5.15}$$

is dense in $\mathcal{G}$ endowed of the semi-inner product V_m.

Since the proof of theorem 5.2 requires several preliminary results we first sketch the proof and we develop all the necessary technical tools in next sections.

Proof. We observe that, by proposition 5.1, the terms appearing in equation (5.14) are well defined and the second member is equal to the third one.
By proposition 5.9 and theorem 5.6 for any $f \in \mathcal{G}$, $V_*(f) = V^*(f)$ and therefore the limit (5.13) exists and is finite. Moreover, (5.35) implies (5.14).
We observe that the first equality in (5.14) corresponds to

$$\inf_{\substack{a\in\mathbb{R}^d \\ u\in\mathbb{G}}} V_m(f - \sum_{e>0} a_e j_e - \mathcal{L}u) = 0$$

and therefore the subspace (5.15) is dense in $\mathcal{G}$.
Given $f, g \in \mathcal{G}$, by the polarization formula,

$$V_k(f, g; \mu^{\alpha,\lambda_0(m)}) = \frac{1}{4} V_k(f + g; \mu^{\alpha,\lambda_0(m)}) - \frac{1}{4} V_k(f - g; \mu^{\alpha,\lambda_0(m)})$$

Therefore, as limit (5.13) exists and is finite, limit (5.12) exists and is finite too. It's simple to verify $V_m(\cdot,\cdot)$ defines a semi-inner product.

□

We observe that (5.14) corresponds to the property that, in the semi-norm $V_m^{1/2}$, any $f \in \mathcal{G}$ can be approximated by functions of the form $\sum_{e>0} a_e j_e + \mathcal{L}g$, with $a \in \mathbb{R}^d$ and $g \in \mathbb{G}$. By the following lemma we can restrict to functions g in $\mathcal{G}$.

Lemma 5.3.

$$\overline{\mathcal{L}\mathbb{G}} = \overline{\mathcal{L}\mathcal{G}}$$

Proof. For any integer s we define $g_s \equiv \mu^\alpha[g \mid m_s]$, then $g - g_s \in \mathcal{L}\mathcal{G}$ with s large enough. We assert that $\mathcal{L}g$ can be approximated by $\mathcal{L}(g - g_s)$, that is $\lim_{s\uparrow\infty} V_m(\mathcal{L}g_s) = 0$.

To this aim we define $X_s \equiv \{x : s - 1 \le |x| \le s + 1\}$. Then proposition 5.1 implies

$$V_m(\mathcal{L}g_s) = \sum_{e>0} \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(c_e(\nabla_e \underline{g_s})^2\big)\Big] = \sum_{e>0} \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(c_e\big(\sum_{x\in X_s} \nabla_e \tau_x g_s\big)^2\big)\Big]. \tag{5.16}$$

Let $\hat{g}_s(\alpha, \eta) \equiv \mu^{\alpha,\lambda(\alpha,\Lambda_s,m_s(\eta))}(g)$. By the equivalence of ensembles $\|\hat{g}_s - g_s\|_\infty \le C(g)s^{-d}$ and therefore in (5.16) we can replace g_s by $\hat{g}_s$ with an error of the order $O(s^{-2})$.
By $A.3$

$$|\nabla_e \tau_x \hat{g}_s| \le \sum_{\delta=\pm(2s+1)^{-d}} |\mu^{\alpha,\lambda(\alpha,\Lambda_{x,s},m_{x,s}+\delta)}(g) - \mu^{\alpha,\lambda(\alpha,\Lambda_{x,s},m_{x,s})}(g)| \le C(g)s^{-d}$$

which, thanks to (5.16) with g_s replaced with $\hat{g}_s$, implies that $V_m(\mathcal{L}g_s) = O(s^{-2})$.

□

5.3 Forms on Ω

Given a sequence of bonds $\underline{b} = (b_1, \ldots, b_n)$ we set

$$S_{\underline{b}} \equiv S_{b_n} \circ \cdots \circ S_{b_1}$$

with S_b defined as in (1.1).

A **form** on Ω is a family $\xi = \{\xi_b\}_{b\in\mathbb{Z}^d}$ of measurable functions $\xi_b : \Omega \to \mathbb{R}$.

Let $\eta \in \Omega$ and let $b_1, b_2, \ldots, b_n$ be some bonds in $\mathbb{Z}^d$. We write $(\eta; b_1, b_2, \ldots, b_n)$ for the path γ equal to $(\eta_0, \eta_1, \ldots, \eta_n)$ where

$$\eta_0 \equiv \eta \quad \text{and} \quad \eta_k \equiv S_{b_1,b_2,\ldots,b_k}\eta \ \ \forall k = 1, \ldots, n.$$

η_0 and η_n are called respectively the initial and final point of γ.

We define the integral of the form ξ on the path γ as

$$\int_\gamma \xi \equiv \sum_{i=1}^{n} \xi_{b_i}(\eta_{i-1}).$$

If $n = 0$ we set $\int_\gamma \xi = 0$.

Let $\Delta \subset \mathbb{Z}^d$. Then a form ξ is said **closed form** in Δ if, given η and $b_1, b_2, \ldots, b_n \in \Delta$ such that $\eta = S_{b_1,b_2,\ldots,b_n}\eta$,

$$\int_\gamma \xi = 0 \quad \text{with } \gamma \equiv (\eta; b_1, b_2, \ldots, b_n).$$

If $\Delta = \mathbb{Z}^d$ we call ξ simply a closed form.
It's easy to prove that ξ is closed in Δ if and only if

$$\int_\gamma \xi = \int_{\gamma'} \xi$$

for any couple of paths $\gamma = (\eta; b_1, \ldots, b_n)$ and $\gamma' = (\eta; b'_1, \ldots b'_s)$ having the same initial and final points and with $b_1, \ldots, b_n \in \Delta$ and $b'_1, \ldots b'_s \in \Delta$.

The following lemma states that for any form, closed in a box Δ, there is a function which is its primitive w.r.t. the bonds in Δ.

Lemma 5.4. *Let $\xi = \{\xi_b\}_{b \in \mathbb{Z}^d}$ be a closed form in a box Δ.*
For any $n : 0 \leq n \leq |\Delta|$ we fix a configuration σ_n in $\{0,1\}^\Delta$ having n particles.
For any $\omega \in \{0,1\}^\Delta$ let $\underline{b}(\omega)$ be a fixed sequence of bonds $b_1, b_2, \ldots, b_k \in \Delta$ such that $\omega = S_{\underline{b}(\omega)}\sigma_{N_\Delta(\omega)}$.
For any configuration $\eta \in \Omega$ let $T\eta$ be the configuration

$$(T\eta)(x) \equiv \begin{cases} \sigma_{N_\Delta(\eta)}(x) & \text{if } x \in \Delta \\ \eta(x) & \text{otherwise} \end{cases}.$$

Then the function

$$f(\eta) \equiv \xi(T\eta) + \int_{\gamma\left(T\eta, \underline{b}(\eta_\Delta)\right)} \xi$$

is measurable and $\xi_b = \nabla_b f$ for any bond $b \in \Delta$.

Proof. We prove that, given $b \in \Delta$, $\xi_b(\eta) = \nabla_b f(\eta)$. Let $n \equiv N_\Delta(\eta) = N_\Delta(\eta^b)$. Since $T\eta = T(\eta^b)$ it's enough to prove that for any couple of paths γ_1 and γ_2 respectively from $T\eta$ to η and from $T\eta$ to η^b of the form $(T\eta; b_1, \ldots, b_s)$ with $b_i \in \Delta$ for $i = 1, \ldots, s$ then

$$\xi_b(\eta) = \int_{\gamma_2} \xi - \int_{\gamma_1} \xi.$$

If $\gamma_1 \equiv (T\eta; b_1, \ldots, b_r)$ we define $\gamma_2' \equiv (T\eta; b_1, \ldots, b_r, b)$.
Since ξ is closed in Δ, $\int_{\gamma_2'} \xi = \int_{\gamma_2} \xi$ and by definition $\xi_b(\eta) = \int_{\gamma_2'} \xi - \int_{\gamma_1} \xi$.

□

Lemma 5.5. *Let Δ be a finite box or $\Delta = \mathbb{Z}^d$.*
A form ξ is closed in Δ if and only if it satisfies the following properties $P1$, $P2$ and $P3$.

P.1 . *Let $a, v, w \in \mathbb{Z}^d$ with $|v| = |w| = 1$ and $v \pm w \neq 0$. We define*

$$c \equiv a+v+w, \quad x \equiv a+v, \quad x' \equiv a+w, \quad b_1 \equiv \{a, x\}, \quad b_2 \equiv \{x, c\}, \quad b_1' \equiv \{a, x'\}, \quad b_2' \equiv \{x', c\}.$$

If $a, x, x', c \in \Delta$ then

$$\xi_{b_1} \circ S_{b_2} \circ S_{b_1} + \xi_{b_2} \circ S_{b_1} + \xi_{b_1} = \xi_{b_1'} \circ S_{b_2'} \circ S_{b_1'} + \xi_{b_2'} \circ S_{b_1'} + \xi_{b_1'}. \tag{5.17}$$

P.2 . *For any couple of bonds $b_1, b_2 \in \Delta$ such that $b_1 \cap b_2 = \emptyset$*

$$\xi_{b_2} \circ S_{b_1} + \xi_{b_1} = \xi_{b_1} \circ S_{b_2} + \xi_{b_2}. \tag{5.18}$$

P.3 . *For any bond $b \in \Delta$*

$$\xi_b \circ S_b + \xi_b = 0. \tag{5.19}$$

Proof. It's trivial to prove that a closed form in Δ satisfies **P.1** , **P.2** and **P.3** .
In order to prove the inverse implication we first reduce the problem to a one-dimensional problem and then we consider the case $d = 1$. For simplicity of notation we assume $\Delta = \mathbb{Z}^d$. It's simple to verify that the following arguments are valid for general Δ.

Let $\bar\eta \in \Omega$ and $\bar b_1, \ldots, \bar b_s$ such that $\bar\eta = S_{\bar b_1, \bar b_2, \ldots, \bar b_s} \bar\eta$. We want to prove

$$\int_{\gamma^*} \xi = 0 \quad \text{where } \gamma^* \equiv (\bar\eta; \bar b_1, \ldots, \bar b_s). \tag{5.20}$$

Let $d \geq 2$. For simplicity of notation we consider only the case $d = 2$. We fix an integer l such that $\bar b_1, \ldots, \bar b_s \in \Lambda_l$ and we set $I \equiv \{0, \ldots, (2l+1)^d - 1\}$. We enumerate Λ_l as $\Lambda_l = \{x_i\}_{i \in I}$ where $x_i = (a_1, a_2)$ if

$$i = \begin{cases} (2l+1)(a_2+l) + (a_1+l) & \text{if } a_2+l \text{ is even} \\ (2l+1)(a_2+l) + (-a_1+l) & \text{if } a_2+l \text{ is odd} \end{cases} \tag{5.21}$$

We observe that x_i and x_{i+1} are adjacent for any i. We write $x_i = (a_{i,1}, a_{i,2})$ and set $\Theta \equiv \{\ \{x_i, x_{i+1}\}\ \}_{i \in I}$.

Let b be a generic bond in Λ_l. Then $b = \{x_k, x_n\}$ with $k < n$. We assert that there exist $b_1, b_2, \ldots, b_r \in \Theta$ such that

$$S_b = S_{b_1,\ldots,b_r} \quad \text{and} \quad \xi_b \equiv \int_{\gamma(\cdot)} \xi \quad \text{where } \gamma(\eta) \equiv (\eta; b_1, \ldots, b_r). \tag{5.22}$$

We prove (5.22) by induction on $n-k$.
The case $n-k=1$ is trivial. Let $n-k \geq 2$ and the thesis be true for smaller values of $n-k$. We observe that, since $|x_k - x_n| = 1$, $n-k \geq 3$ and $x_n = x_k + e_2$.
As example let $a_{k,2}+l$ be even. Then $a_{k,1} < l$ otherwise $x_n = x_{k+1}$. We obtain $x_{k+1} = x_k + e_1$ and $x_{n-1} = x_n + e_1$.
We set $b' \equiv \{x_{k+1}, x_{n-1}\}$, $b_+ \equiv \{x_k, x_{k+1}\}$ and $b_- \equiv \{x_{n-1}, x_n\}$.
By inductive hypothesis $\exists \underline{b} = (b_1, b_2, \cdots, b_r)$ such that

$$S_{b'} = S_{\underline{b}} \tag{5.23}$$

$$\xi_{b'} = \int_{\gamma'(\cdot)} \xi \quad \text{where } \gamma'(\eta) \equiv (\eta; \underline{b}). \tag{5.24}$$

We assert

$$\xi_b = \int_{\gamma_0(\cdot)} \xi \quad \text{where } \gamma_0(\eta) \equiv (\eta; b_+, b', b_-, b', b_+). \tag{5.25}$$

In order to prove (5.25) we set

$$\gamma_1(\eta) \equiv (\eta; b', b_-, b') \qquad \gamma_2(\eta) \equiv (\eta; b_+, b, b_+).$$

Then we can write

$$\int_{\gamma_0(\cdot)} \xi = \xi_{b_+} + \int_{\gamma_1(S_{b_+}(\cdot))} \xi + \xi_{b_+} \circ S_{b_+,b',b_-,b'}. \tag{5.26}$$

Since $S_{b',b_-,b'} = S_{b_+,b,b_+}$ and, by **P.1** $\int_{\gamma_1(\cdot)} \xi = \int_{\gamma_2(\cdot)} \xi$, the r.h.s. of (5.26) can be written as

$$\int_{\gamma_3(\cdot)} \xi \quad \text{where} \quad \gamma_3(\eta) = (\eta; b_+, b_+, b, b_+, b_+).$$

By **P.3**, $\int_{\gamma_0(\cdot)} \xi = \xi_b$ and (5.25) is proved.

(5.25) corresponds to

$$\xi_b = \xi_{b_+} + \xi_{b'} \circ S_{b_+} + \xi_{b_-} \circ S_{b_+,b'} + \xi_{b'} \circ S_{b_+,b',b_-} + \xi_{b_+} \circ S_{b_+,b',b_-,b'}. \tag{5.27}$$

Let $\bar{\gamma}(\cdot)$ be defined as

$$\bar{\gamma}(\eta) \equiv (\eta; b_+, \underline{b}, b_-, \underline{b}, b_+),$$

then by (5.23)

$$\int_{\bar{\gamma}(\eta)} \xi = \xi_{b_+}(\eta) + \int_{\gamma'(S_{b_+}\eta)} \xi + \xi_{b_-}(S_{b_+,b'}\eta) + \int_{\gamma'(S_{b_+,b',b_-}\eta)} \xi + \xi_{b_+}(S_{b_+,b',b_-,b'}\eta). \tag{5.28}$$

By (5.24), (5.27) and (5.28) we obtain $\xi_b = \int_{\bar{\gamma}(\eta)} \xi$.
It's simple to verify that $S_b = S_{b_+,\underline{b},b_-,\underline{b},b_+}$ therefore (5.22) is satisfied if we take

$$(b_+, \underline{b}, b_-, \underline{b}, b_+)$$

as sequence of bonds.

We come back to (5.20). By applying (5.22) to every bond $\bar{b}_1, \dots, \bar{b}_s$ we obtain that there exists a sequence $\underline{b} = (b_1, \dots, b_r)$ of bonds in Θ such that

$$\bar{\eta} = S_{\underline{b}}\eta \quad \text{and} \quad \int_{\gamma^*} \xi = \int_\gamma \xi \;\text{ where }\; \gamma \equiv (\bar{\eta}; \underline{b})$$

By the above result and the particular enumeration $\{x_i\}_{i\in I}$ it's enough to consider the case $d = 1$.

We prove the thesis by induction on s, with s as in (5.20). If $s = 1$ then $\bar{\eta} = S_{\bar{b}_1}\eta$ and $\int_{\gamma^*} \xi = \xi_{\bar{b}_1}(\bar{\eta})$. By **P.3** we obtain the thesis.

Let $s > 1$ and the thesis be true for smaller values of s. By **P.2** , **P.3** and the inductive hypothesis we can suppose $\bar{b}_i \cap \bar{b}_{i+1} \neq \emptyset$ and $b_i \neq b_{i+1}$ for $i = 1, \dots, s-1$.
Let $\eta_0 \equiv \bar{\eta}$ and $\eta_k \equiv S_{\bar{b}_1,\bar{b}_2,\dots,\bar{b}_k}\bar{\eta}$ for any $k = 1, \dots, s$.
If $S_{\bar{b}_k}\eta_{k-1} = \eta_{k-1}$ for some $k : 1 \le k \le s$ then $\bar{\eta} = S_{\underline{b}}\bar{\eta}$ with

$$\underline{b} \equiv (\bar{b}_1, \dots, \bar{b}_{k-1}, \bar{b}_{k+1}, \dots, \bar{b}_s),$$

that is we omit the bond $\bar{b}_k$, and by **P.3**

$$\int_{\gamma^*} \xi = \int_\gamma \xi \quad \text{where} \quad \gamma \equiv (\bar{\eta}; \underline{b}).$$

By the inductive hypothesis $\int_{\gamma^*} \xi = 0$. Our last step is then to prove that such a k exists. For absurd, we can assume

$$S_{\bar{b}_k}\eta_{k-1} \neq \eta_{k-1} \quad \forall k = 1, \dots, s. \tag{5.29}$$

For any $k = 2, \ldots, s$ we write $\bar{b}_k > 0$ if $\bar{b}_{k-1} \cap \bar{b}_k = z + 1$ where z is defined by $\bar{b}_{k-1} = \{z, z+1\}$, otherwise we write $\bar{b}_k < 0$. We suppose, as example, that $\bar{b}_2 > 0$. We assert that implies $\bar{b}_k > 0$ for any $k \geq 2$.
In fact, we assume there exists $k > 2$ with $\bar{b}_k < 0$. Then for some $z \in \mathbb{Z}$

$$\bar{b}_r = \{z + r - 1, z + r\} \ \forall r = 1, \ldots, k-1 \quad \text{and} \quad \bar{b}_k = \{z + k - 3, z + k - 2\}.$$

Let, as example, $\bar{\eta}_z = 0$. Since $S_{\bar{b}_r} \eta_{r-1} \neq \eta_{r-1}$ for any $r = 1, \ldots, k-1$,

$$\bar{\eta}_{z+i} = 1 \quad \forall i = 1, \ldots, k-1$$

and therefore $S_{\bar{b}_k} \eta_{k-1} = \eta_{k-1}$ in contradiction with the assumptions.
We have proved $\bar{b}_k > 0 \ \forall k = 2, \ldots, s$. If $\bar{b}_1 = \{z, z+1\}$ and as example $\eta_z = 0$ then hypothesis (5.29) implies

$$\bar{\eta}_z = 0 \quad \text{and} \quad \bar{\eta}_{z+i} = 1 \ \forall i = 1, \ldots, s$$

and in this case $S_{\bar{b}_1, \ldots, \bar{b}_k} \bar{\eta} \neq \bar{\eta}$ in contradiction with the assumptions.

□

5.4 Forms on $\tilde{\Omega}$

A family of functions $\xi = \{\xi_b\}_{b \in \mathbb{Z}^d}$ with $\xi_b \in L^2(\mathbb{P}^*)$ is called a **form** in $L^2(\mathbb{P}^*)$.
In this section, when there is no explicit mention, a form have to be thought in $L^2(\mathbb{P}^*)$ and not on Ω.

Let Δ be a finite box or $\mathbb{Z}^d$. Then a form ξ is called **closed** in Δ if it satisfies properties **P.1** , **P.2** and **P.3** of lemma 5.5 w.r.t. Δ where the equalities (5.17), (5.18) and (5.19) have to be intended in $L^2(\mathbb{P}^*)$.
If $\Delta = \mathbb{Z}^d$ ξ is called simply "a closed form".

The form ξ is called **translationally covariant** if $\tau_x \xi_b = \xi_{b+x}$ for any x, b.

A family of functions $\xi = \{\xi_e\}_{e>0}$ with $\xi_e \in L^2(\mathbb{P}^*)$ is called a **germ** of a closed form if the form $\xi' \equiv \{\xi'_b\}_{b \in \mathbb{Z}^d}$ with $\xi'_{x,x+e} \equiv \tau_x \xi_e$ is a closed form.
We observe ξ' is automatically translationally covariant. ξ' is said to be generated by ξ.

Given $f \in L^2(\mathbb{P}^*)$ we can define the form $\xi = \{\xi_b\}_{b \in \mathbb{Z}^d}$ with $\xi_b = \nabla_b f$. Then ξ is closed but it's translationally covariant only if $\xi_b = 0$ for any bond b.

Given $u \in \mathbb{G}$ the form $\{\xi_b\}_{b \in \mathbb{Z}^d}$ with $\xi_b = \nabla_b \underline{u}$ is closed and translationally covariant. It is called **exact** .

For any $e > 0$ we define $\mathfrak{U}^e$ to be the form $\{\mathfrak{U}^e_b\}_{b \in \mathbb{Z}^d}$ with $\mathfrak{U}^e_{x,x+e'}(\eta) \equiv \delta_{e,e'}(\eta_{x+e} - \eta_x)$.
Then for any $e > 0$, $\mathfrak{U}^e$ is closed and translationally covariant. Easily it can be proved it isn't exact.

We define

$$\Xi_C \equiv \{\xi = \{\xi_e\}_{e>0} \in L^2(\otimes^d \mathbb{P}^*) \,:\, \xi \text{ is germ of a closed form}\}$$
$$\Xi_0 \equiv \{\xi = \{\xi_e\}_{e>0} \in L^2(\otimes^d \mathbb{P}^*) \,:\, \exists a \in \mathbb{R}^d, u \in \mathbb{G} \text{ with } \xi_e = a_e \mathfrak{U}^e + \nabla_e \underline{u} \;\; \forall e > 0\}$$

By the previous observations $\Xi_0 \subset \Xi_C$ and it's simple to check that Ξ_C is a closed subspace in $L^2(\otimes^d \mathbb{P}^*)$.
The following theorem asserts that actually any germ in Ξ_C can be approximated arbitrarily well by a germ in Ξ_C.

Theorem 5.6.
$$\Xi_C = \overline{\Xi_0} \qquad \textit{in } L^2(\otimes^d \mathbb{P}^*)$$

The proof of theorem 5.6 follows strongly the standard proof for non disordered models. In particular, we will often refer to the proof of theorem 4.14 in appendix 3 of [13].
There are two fundamental differences. In the standard proof, the bound of the L^2 norm of the so-called "boundary terms" is obtained thanks to the property that the spectral gap of the dynamics in a cube of side L is inferiorly bounded by CL^{-2} and this is not true in general for disordered models. Although this bound is valid in the model under consideration, in lemma 5.7 we explain how to develop the proof using only the estimates on the spectral gap given in chapter 2, which, for disordered systems, are more general than the diffusive scaling (for example they are valid in the dilute Ising model with Kawasaki dynamics under the percolation threshold as discussed in [7]).
Moreover, differently from the non disordered case, the limit points of the so-called "boundary terms" generally do not belong to the space $\{\sum_{e>0} a_e \mathfrak{U}^e \,:\, a \in \mathbb{R}^d\}$ but we prove they are in $\overline{\Xi_0}$. This result is obtained by lemma 5.8.

Let us remember that, for any $\Lambda \subset \mathbb{Z}^d$,

$$\mathbb{K}_\Lambda \equiv \sigma(\alpha_x, \eta_x \,:\, x \in \Lambda).$$

Lemma 5.7. *Let $f \in L^2(\mathbb{P}^*)$ $\mathbb{K}_{3n}-$measurable such that $\forall b \in \Lambda_{3n}$*

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big((\nabla_b f)^2\big)\Big] \le c_0 \qquad \forall b \in \Lambda_{3n}. \tag{5.30}$$

Let $h \equiv \mathbb{E}^*[f \,|\, \mathbb{K}_n]$. Then, for any $y \in \Lambda_n$,

$$\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big((\nabla_y h)^2\big)\Big] \le C(c_0)\, n^2.$$

Proof. We set $\Delta \equiv \Lambda_{3n} \setminus \Lambda_n$.
We write α_0 $[\alpha_1]$ for the disorder configuration in Λ_n $[\Delta]$. We write ζ $[\omega]$ for the particle configuration in Λ_n $[\Delta]$.
We write $f_{\alpha_0,\zeta}(\alpha_1,\omega)$ for $f(\alpha_0,\alpha_1,\zeta,\omega)$. When α_0,α_1 are fixed we simply write $f_\zeta(\omega)$.
With the above notation

$$h(\alpha_0,\zeta^y) - h(\alpha_0,\zeta) = \int \mathbb{P}(d\alpha_1)\mu_\Delta^{\alpha_1,\lambda_0(m)}\big(f_{\alpha_0,\zeta^y} - f_{\alpha_0,\zeta}\big).$$

In what follows we fix the disorder configurations α_0, α_1 and we write μ or μ_{α_1} for the grand canonical measure $\mu_\Delta^{\alpha_1,\lambda_0(m)}$.

We fix a cube Λ with side n such that $\Lambda \subset \Delta$. Moreover, for any $z \in \Delta$ we define $g_z(\omega) \equiv (1-\omega_z)e^{-\nabla_z H}$ where $H = -\sum_{x\in\mathbb{Z}^d} \alpha_x \eta_x$.
By this choice $\mu(f_{\zeta^y}\omega_z) = \mu(f_{\zeta^y}(\omega^z)g_z)$ and then

$$\begin{aligned}\mu(f_{\zeta^y}) - \mu(f_\zeta) = \mathrm{Av}_{z\in\Lambda}\mu\big([f_{\zeta^y}(\omega^z) - f_\zeta(\omega)]\frac{g_z}{\mu(\omega_z)}\big) + \mu\Big(f_\zeta \cdot \mathrm{Av}_{z\in\Lambda}\Big[\frac{g_z}{\mu(\omega_z)} - 1\Big]\Big)\\ + \mu\Big(f_{\zeta^y}\cdot \mathrm{Av}_{z\in\Lambda}\Big[1 - \frac{\omega_z}{\mu(\omega_z)}\Big]\Big).\end{aligned} \tag{5.31}$$

We observe $\mu(g_z) = \mu(\omega_z)$ and therefore the second and third addendum in the r.h.s. of (5.31) are some covariances.
By Schwarz inequality, the Moving Particle Lemma and hypothesis (5.30),

$$\mathbb{E}^*\Big[\Big(\int \mathbb{P}(d\alpha_1)\mathrm{Av}_{z\in\Lambda}\mu^{\alpha_1}\big(|f_{\zeta^y}(\omega^z) - f_\zeta(\omega)|\big)\Big)^2\Big] \le$$
$$\mathrm{Av}_{z\in\Lambda}\mathbb{E}^*\Big[\int \mathbb{P}(d\alpha_1)\mu^{\alpha_1}\big((f_{\zeta^y}(\omega^z) - f_\zeta(\omega))^2\big)\Big] = \mathrm{Av}_{z\in\Lambda}\mathbb{E}^*\big((\nabla_{y,z}f)^2\big) \le C(c_0)n^2.$$

Therefore, the first addendum of (5.31) gives the right contribution to $\mathbb{E}^*\big((\nabla_y h)^2\big)$.

Let us consider the contribution of the second addendum of (5.31). For simplicity of notation we consider only the second addendum. It's module is bounded by

$$Cn^{-\frac{d}{2}}\mathrm{Var}_\mu(f_\zeta)^{\frac12} \le Cn^{-\frac{d}{2}+1}\gamma_{\alpha_1}(\Delta)^{\frac12}\mathcal{D}_\Delta(f_\zeta;\mu)^{\frac12}$$

where, with a slight abuse of notation w.r.t. chapter 2, $\gamma_{\alpha_1}(\Delta)$ is defined as

$$\gamma_{\alpha_1}(\Delta) \equiv n^{-2}\sup_{N\in[0,|\Delta|]}\Big(gap(\mathcal{L}^{\alpha_1}_{\Delta,N})\Big)^{-1}.$$

By the same arguments of chapter 2 it can be proved

$$\sup_n \mathbb{E}\big(\gamma_{\alpha_1}(\Delta)\big) < \infty.$$

Therefore, by Schwarz inequality,

$$\begin{aligned}\mathbb{E}^*\Big[\Big(\int \mathbb{P}(d\alpha_1)n^{-\frac{d}{2}+1}\gamma_{\alpha_1}(\Delta)^{\frac12}\mathcal{D}_\Delta(f_\zeta;\mu)^{\frac12}\Big)^2\Big] \le\\ n^{-d+2}\mathbb{E}^*\Big[\int \mathbb{P}(d\alpha_1)\big(\gamma_{\alpha_1}(\Delta)\big)\cdot\int \mathbb{P}(d\alpha_1)\big(\mathcal{D}_\Delta(f_\zeta;\mu)\big)\Big] =\\ n^{-d+2}\mathbb{E}\big(\gamma_{\alpha_1}(\Delta)\big)\mathbb{E}\Big[\mathcal{D}_\Delta(f;\mu^{\alpha,\lambda_0(m)})\Big] \le C(c_0)n^2.\end{aligned}$$

We have proved that the second addendum in (5.31) gives the right contribution to $\mathbb{E}^*\big((\nabla_y h)^2\big)$. The third addendum can be treated in a similar way.

□

Lemma 5.8. *Let $\xi \in \Xi_C$ such that, for any $e > 0$, $\xi_e = F_e(\alpha)(\eta_e - \eta_0)$ for some function $F_e \in L^2(\mathbb{P})$. Then $\xi \in \overline{\Xi_0}$.*

Proof. We denote by the same symbol ξ the form generated by the germ ξ.
By subtracting the germ $\{\mathbb{E}(F_e)(\eta_e - \eta_0)\}_{e>0}$, we can assume $\mathbb{E}(F_e) = 0$ for any $e > 0$.
We first define for any $x \in \mathbb{Z}^d$ a function $g_x \in L^2(\mathbb{P})$ such that $\forall n \in \mathbb{N}$

$$\nabla_b h_n = \xi_b \quad \forall b \in \Lambda_n \qquad \text{where } \; h_n = -\sum_{x\in\Lambda_n} g_x(\alpha)\eta_x \tag{5.32}$$

It's simple to verify that (5.32) is true if and only if $g_{x+e} - g_x = \tau_x F_e$ for any $x \in \mathbb{Z}^d$ and $e > 0$.
In order to define g_x for any $x \in \mathbb{Z}^d$ we define $\eta^{(x)} \in \Omega$ by

$$\eta^{(x)}(z) \equiv \begin{cases} 1 & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

and we fix a path γ_x from $\eta^{(0)}$ to $\eta^{(x)}$, that is

$$\gamma_x = (\eta_{(0)}; b_1, \dots, b_s) \quad \text{and} \quad \eta^{(x)} = S_{b_1,\dots,b_s}\eta^{(0)}.$$

If $x = 0$ we can chose $\gamma_0 = \emptyset$.

We define

$$g_x(\alpha) \equiv \int_{\gamma_x} \xi(\alpha, \cdot) \quad \forall x \in \mathbb{Z}^d$$

where $\xi(\alpha, \cdot)$ denotes the form of Ω given by $\{\xi_b(\alpha, \cdot)\}_{b\in\mathbb{Z}^d}$.

We assert $g_{x+e} - g_x = \tau_x F_e$ for any x and $e > 0$. In fact, since ξ is closed by lemma 5.5 $\xi(\alpha, \cdot)$ is a closed form on Ω for almost any disorder configuration α. Let α be such a configuration. If $\gamma_x = (\eta^{(0)}; b_1, \dots, b_s)$ we define $\gamma'_{x+e} \equiv (\eta^{(0)}; b_1, \dots, b_s, \{x, x+e\})$. Then γ'_{x+e} is a path from $\eta^{(0)}$ to $\eta^{(x+e)}$.
Since $\xi(\alpha, \cdot)$ is closed $g_{x+e} = \int_{\gamma'_{x+e}} \xi(\alpha, \cdot)$ and the thesis comes easily.

Let $\xi_b^{(n)} \equiv \frac{1}{(2n)^d}\nabla_b \underline{h_n}$. Then $\xi_b^{(n)} \in \overline{\Xi_0}$. In order to conclude the proof it's enough to show

$$\lim_{n\uparrow\infty} \xi_e^{(n)} = \xi_e \quad \text{in } \; L^2(\mathbb{P}^*).$$

To this aim we observe that (5.32) and the rule $\nabla_{0,e}\tau_x = \tau_x\nabla_{-x,-x+e}$ imply that

$$\nabla_{0,e}\tau_x h_n = \xi_e \quad \text{if } \; -x, -x+e \in \Lambda_n.$$

Therefore we can write for $i = 1, \dots, d$

$$\xi_{e_i}^{(n)} = \frac{(2n+1)^{d-1}}{(2n)^{d-1}}\xi_{e_i} + \frac{1}{(2n)^d}\sum_{x\in\Lambda_n : x_i = n} \tau_{-x}\nabla_{x,x+e_i}h_n + \frac{1}{(2n)^d}\sum_{\substack{x\in\Lambda_n - e_i \\ x_i = -n-1}} \tau_{-x}\nabla_{x,x+e_i}h_n$$

It's enough to prove the second and third addenda in the r.h.s. goes to 0 in $L^2(\mathbb{P}^*)$. We consider for example the second addendum.
By Schwarz inequality and the equality

$$\nabla_{x,x+e_i} h_n = -g_x(\alpha)(\eta_{x+e_i} - \eta_x) \quad \forall x \in \Lambda_n : x_i = n,$$

it's enough to prove

$$\lim_{n\uparrow\infty} n^{-d-1} \sum_{x\in\Lambda_n : x_i = n} \mathbb{E}(g_x^2) = 0.$$

To this aim we exhibit a particular choice of $\{\gamma_x\}_x$ in the definition of g_x.
Let $x = (x_1, \dots, x_d)$. We set $\gamma_x \equiv (\eta^{(y_0)}, \eta^{(y_1)}, \dots, \eta^{(y_r)})$ where y_0 is the origin of $\mathbb{Z}^d$, $y_r \equiv x$ and $y_0, y_1, \dots, y_r$ are the points encountered by moving in $\mathbb{Z}^d$ first from $(0, \dots, 0)$ to $(x_1, 0, \dots, 0)$ in the first direction, then from $(x_1, 0, \dots, 0)$ to $(x_1, x_2, 0, \dots, 0)$ in the second direction and so on until arriving to x.

Then it's simple to verify that for any $x \in \Lambda_n$ there are $z_1, \dots, z_d \in \Lambda_n$ and integers $k_1, \dots, k_d \in [0, n]$ (depending on x) such that

$$g_x^2 \le C(d) \sum_{i=1}^{d} \Big(\sum_{s=0}^{k_i} \tau_{z_i + s e_i} F_{e_i} \Big)^2.$$

By translation invariance of $\mathbb{P}$ it's enough to prove $\forall e > 0$

$$\lim_{n\uparrow\infty} \sup_{k\in[0,n]} n^{-2} \mathbb{E}\Big(\big(\sum_{s=0}^{k} \tau_{se} F_e\big)^2 \Big).$$

For simplicity of notation we fix such a direction $e > 0$ and we write F_s in place of $\tau_{se} F_e$. We fix a positive integer r and we define $F_s^{(r)} \equiv \mathbb{E}[F_s \,|\, \alpha_{\Lambda_{se,r}}]$. Then $F_s^{(r)} = \tau_{se} F_0^{(r)}$ and $\mathbb{E}(F_s^{(r)}) = 0$ (since we have assumed that $\mathbb{E}(F_e) = 0$). Therefore

$$n^{-2} \mathbb{E}\Big(\big(\sum_{s=0}^{k} F_s\big)^2 \Big) \le 2n^{-2} \mathbb{E}\Big(\big(\sum_{s=0}^{k} [F_s - F_s^{(r)}]^2\big)^2 \Big) + 2n^{-2} \mathbb{E}\Big(\big(\sum_{s=0}^{k} F_s^{(r)}\big)^2 \Big) \le$$
$$2\mathbb{E}\Big((F_0 - F_0^{(r)})^2 \Big) + C(r) n^{-1}.$$

By taking $n \uparrow \infty$ and then $r \uparrow \infty$ we obtain the thesis.

□

Proof of theorem 5.6. We strongly follow the proof of theorem 4.14 in appendix 3 of [13] to which we refer for details.
Since $\Xi_0 \subset \Xi_C$ and since for a convex set its strong closure is equal to its weak closure, it's enough to prove that Ξ_C is included in the weak closure of Ξ_0.
To this aim let $\xi = \{\xi_b\}_{b\in\mathbb{Z}^d}$ be a closed and translationally covariant form on $\tilde{\Omega}$.
For any $n \in \mathbb{N}$ we define a new form $\xi^{(n)}$ as

$$\xi_b^{(n)} \equiv \mathbb{E}^*[\xi_b \,|\, \mathbb{K}_n].$$

Then $\xi^{(3n)}$ is closed w.r.t. Λ_{3n}. By lemma 5.5 for almost any configuration α the form $\{\xi^{(3n)}(\alpha,\cdot)\}_{b\in\mathbb{Z}^d}$ is a form on Ω closed w.r.t. Λ_{3n}, therefore we can apply lemma 5.4 which implies $\exists f_n \in L^2(P^*)$ with support in Λ_{3n} and $\xi_b^{(n)} = \nabla_b f_n$ for any $b \in \Lambda_{3n}$.
We define $g_n \equiv f_n - \mu^\alpha[f_n \,|\, m_{3n}]$ then

$$\mu^\alpha[g_n \,|\, m_{3n}] \equiv 0 \quad \xi_b^{(n)} = \nabla_b f_n \quad \forall b \in \Lambda_{3n}.$$

We set $\psi_n \equiv \mathbb{E}^*[g_n \,|\, \mathbb{K}_n]$.
For any $i = 1,\dots,d$ we set $R_n^i \equiv \frac{1}{(2n)^d}\nabla_{0,e_i}\underline{\psi_n}$. Then we can write

$$R_n^i = \frac{1}{(2n)^d}\sum_{\substack{y\in\Lambda_n\\ -n<y_i\le n}} \tau_y \xi_{-y,-y+e_i} + \sigma_n^{i,-} + \sigma_n^{i,-}$$

where

$$\sigma_n^{i,-} \equiv \frac{1}{(2n)^d}\nabla_{0,e_i}\sum_{\substack{y\in\Lambda_n\\ y_i=-n}}\tau_y\psi_n \quad \text{and} \quad \sigma_n^{i,+} \equiv \frac{1}{(2n)^d}\nabla_{0,e_i}\sum_{\substack{y\in e_i+\Lambda_n\\ y_i=n+1}}\tau_y\psi_n.$$

We set $\sigma_n^{\pm} = (\sigma_n^{1,\pm},\dots,\sigma_n^{i,\pm})$.
As in [13] it can be proved that for any $i = 1,\dots,d$

$$\lim_{n\uparrow\infty}\frac{1}{(2n)^d}\sum_{\substack{y\in\Lambda_n\\ -n<y_i\le n}}\tau_y\xi_{-y,-y+e_i} = \xi_{0,e_i} \quad \text{ in } L^2(\otimes^d\mathbb{P}^*).$$

Therefore, for any i, $R_n^i - (\sigma_n^{i,-} + \sigma_n^{i,+})$ converges to ξ_{e_i} in $L^2(\mathbb{P}^*)$ and since $R_n^i \in \overline{\Xi_0}$ it's enough to prove $\{\sigma_n^{\pm}\}_n$ admits a subsequence weakly convergent to some vector-function $b_{\pm}$ and that $b_- + b_+ \in \overline{\Xi_0}$.
We observe that in any case, since $\lim_{n\uparrow}\operatorname{dist}(\sigma_n^- + \sigma_n^+, \Xi_C) = 0$, $b_- + b_+ \in \Xi_C$.

By Schwarz inequality

$$\mathbb{E}^*\big((\sigma_n^{i,-})^2\big) \le Cn^{-d-1}\sum_{\substack{y\in\Lambda_n\\ y_i=n}}\mathbb{E}^*\big((\nabla_{y,y+e_i}\psi_n)^2\big) \le Cn^{-d-1}\sum_{\substack{y\in\Lambda_n\\ y_i=n}}\mathbb{E}^*\big((\nabla_y\psi_n)^2\big).$$

Since for any $b \in \Lambda_{3n}$ $\mathbb{E}^*\big((\nabla_b g)^2\big) = \mathbb{E}^*\big((\xi_b^{(3n)})^2\big) \le \mathbb{E}^*(\xi_b^2) \le C$ we can apply lemma 5.7 and prove $\sup_n \mathbb{E}^*\big((\sigma_n^{i,-})^2\big) < \infty$ and, by similar arguments, that the same is true for $\sigma_n^{i,+}$.
By Banach-Alaoglu theorem we can assume, without loss of generality, that $\sigma_n^{i,\pm}$ weakly converges to some $b_{\pm}^i \in \Lambda^2(P^*)$ as $n \uparrow \infty$. We write $b_{\pm}$ for $(b_{\pm}^1,\dots,b_{\pm}^n)$.

By the same arguments as in [13] it can be proved that $b_{\pm}^i$ if measurable w.r.t. $\sigma(\eta_0,\eta_{e_i},\alpha)$.
We write $b_{\pm}^i(\alpha,a,c)$ for the function $b_{\pm}^i(\alpha,\eta)$ with $\eta_0 = a$ and $\eta_{e_i} = c$.

It's simple to compute $\nabla_{e_i,2e_i}b_-^i$ and $\nabla_{-e_i,0}b_+^i$. The result implies

$$b_-^i(\alpha,\eta_0,\eta_{2e_i}) - b_-^i(\alpha,\eta_0,\eta_{e_i}) = b_-^i(\alpha,\eta_{e_i},\eta_{2e_i})$$
$$b_+^i(\alpha,\eta_{-e_i},\eta_{e_i}) - b_+^i(\alpha,\eta_0,\eta_{e_i}) = b_+^i(\alpha,\eta_{-e_i},\eta_0)$$

which implies that $b^i_\pm = F^i_\pm(\alpha)(\eta_{e_i} - \eta_0)$ for some $F^i_\pm \in L^2(\mathbb{P})$. Since $b_- + b_+ \in \Xi_C$ we can apply lemma 5.8 and obtain the thesis.

□

5.5 Connection between V_m and closed forms

Proposition 5.9. *Given $f \in \mathcal{G}$, for any $e > 0$ we can define a function $\phi_e \in \mathbb{G}$ such that*

$$V^*(f) \leq \sup_{\xi \in \Xi_C} \Theta_f(\xi) \tag{5.33}$$

$$V_*(f) \geq \sup_{\xi \in \Xi_0} \Theta_f(\xi). \tag{5.34}$$

where

$$\Theta_f(\xi) \equiv 2\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\sum_{e>0} c_e\phi_e\xi_e\big)\Big] - \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(\sum_{e>0} c_e\xi_e^2\big)\Big].$$

Moreover, given $a \in \mathbb{R}^d$ and $u \in \mathbb{G}$,

(5.35)
$$\Theta_f\big(\sum_{e>0} -a_e\mathfrak{U}^e + \nabla_e\underline{u}\big) = \sum_{e>0} 2a_e t_e(f) + 2(f,u)_0 - \sum_{e>0}\frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(c_e(a_e\nabla_e\eta_0 - \nabla_e\underline{u})^2\big)\Big]$$

Proof. Let $f \in \mathcal{G}_s$ and $k \geq s^2$.
For simplicity of notation in what follows we set $\lambda \equiv \lambda_0(m)$.
We assert that

$$\mathbb{E}\Big[V_k(f,\mu^{\alpha,\lambda})\Big] = \sup_{h \in \mathbb{G}_k} F_k(h) \tag{5.36}$$

where

$$F_k(h) \equiv (2k)^{-d}\Big(2\mathbb{E}\Big[\mu^{\alpha,\lambda}\big(\sum_{|x|\leq k_1}\tau_x f, h\big)\Big] - \mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(h;\mu^{\alpha,\lambda})\Big]\Big).$$

In fact, the l.h.s. of (5.36) is equal to $F_k(h)$ with $h \in \mathbb{G}_k$ such that $\sum_{|x|\leq k_1}\tau_x f = -\mathcal{L}_k(h)$ and for any $h \in \mathbb{G}_k$

$$\mathbb{E}\Big[V_k(f,\mu^{\alpha,\lambda})\Big] - F_k(h) = (2k)^{-d}\mathbb{E}\Big[\mu^{\alpha,\lambda}\Big(\big[\sum_{|x|\leq k_1}\tau_x f + \mathcal{L}_k h\big], -\mathcal{L}_k^{-1}\big[\sum_{|x|\leq k_1}\tau_x f + \mathcal{L}_k h\big]\Big)\Big] \geq 0.$$

In (5.36) we can restrict to functions $h \in \mathbb{G}_k$ such that $F_k(h) \geq 0$. Let $\mathbb{G}_k^*$ be the set of functions in $\mathbb{G}_k$ such that

$$\mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(h;\mu^{\alpha,\lambda})\Big] \leq C(f)k^d. \tag{5.37}$$

Then, by lemma 1.11, in (5.36) we can restrict to functions $h \in \mathbb{G}_k^*$ with a particular choice of $C(f)$.

i) Proof of (5.33).

Let $f = \mathcal{L}_s g$ with $g \in \mathbb{G}_s$. By the translation invariance of the measure $\mathbb{P}^*$ and the reversibility of measure $\mu^{\alpha,\lambda}$,

$$\mathbb{E}\Big[\mu^{\alpha,\lambda}\big(\sum_{|x|\le k_1} \tau_x f, h\big)\Big] = \sum_{b\in\Lambda_k} \mathbb{E}\Big[\mu\big(c_b \phi_b' \nabla_b h\big)\Big] \quad \text{with } \phi_b' \equiv \sum_{\substack{x:|x|\le k_1\\ b\in\Lambda_{x,s}}} \tau_x g. \tag{5.38}$$

We set

$$\phi_b \equiv \sum_{x:\, b\in\Lambda_{x,s}} \tau_x g.$$

Since $\phi_b = \phi_b'$ if the distance between b and $\Lambda_{k_1}^c$ is more than $10s$, by Schwarz inequality and by the property $h \in \mathbb{G}_k^*$

$$\mathbb{E}\Big[\mu\big(c_b(\phi_b' - \phi_n)\nabla_b h\big)\Big]^2 \le C(s)k^{d-\frac{1}{2}}\mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(h;\mu^{\alpha,\lambda})\big] \le C(f)k^{2d-\frac{1}{2}}.$$

By the above estimate and (5.38),

$$F_k(h) \equiv (2k)^{-d}\Big(2\sum_{b\in\Lambda_k}\mathbb{E}\Big[\mu\big(c_b\phi_b\nabla_b h\big)\Big] - \mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(h;\mu^{\alpha,\lambda})\Big]\Big) + O(k^{-\frac{1}{4}}).$$

We fix a sequence of functions $h_k \in \mathbb{G}_k^*$ such that

$$\varlimsup_{k\uparrow\infty}\mathbb{E}\Big[V_k(f,\mu^{\alpha,\lambda})\Big] = \varlimsup_{k\uparrow\infty} F(h_k).$$

Similarly to [13], we set

$$R_{i,k}(h) \equiv (2k)^{-d}\nabla_{0,e_i}\sum_{x:\,x,x+e_i\in\Lambda_k}\tau_{-x}h \quad \text{and} \quad R_{i,k} \equiv R_{i,k}(h_k),$$

then

$$(2k)^{-d}\sum_{b\in\Lambda_k}\mathbb{E}\Big[\mu\big(c_b\phi_b\nabla_b h\big)\Big] = \sum_{i=1}^d \mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}\phi_{0,e_i}R_{i,k}(h))\Big] \tag{5.39}$$

and by Schwarz inequality, since $h \in \mathbb{G}_k^*$,

$$\sum_{i=1}^d \mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}R_{i,k}(h)^2)\Big] \le 2(2k)^{-d}\mathcal{D}_{\Lambda_k}(h_k;\mu^{\alpha,\lambda}) + O(k^{-1}) \le C'(f).$$

Therefore, for a sequence c_k with $\lim_{k\uparrow\infty} c_k = 0$ and for any $h \in \mathbb{G}_k^*$,

$$F_k(h) \le \sum_{i=1}^{d}\Big\{2\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}\phi_{0,e_i}R_{i,k}(h))\Big] - \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}R_{i,k}(h)^2)\Big]\Big\} + c_k. \tag{5.40}$$

Since the sequence $\{R_{i,k}\}_k$ is bounded in $L^2(\mathbb{P}^*)$, we can extract a subsequence weakly converging to a function $R_i \in L^2(\mathbb{P}^*)$. Without loss of generality, we assume that $\forall_i$ $R_{i,k}$ weakly converges to R_i as $k \uparrow \infty$.
Then, by (5.36) and (5.40),

$$\varlimsup_{k\uparrow\infty} \mathbb{E}\Big[V_k(f;\mu^{\alpha,\lambda})\Big] \le \sum_{i=1}^{d}\Big\{2\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}\phi_{0,e_i}R_i)\Big] - \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}R_i^2)\Big]\Big\}. \tag{5.41}$$

$\{R_i\}_i$ is a germ of a closed form in $L^2(\mathbb{P}^*)$. We isolate the proof of this statement in lemma 5.10. By assuming this result (5.33) is proved.

ii) Proof of (5.34).

In order to prove (5.34), let $a \in \mathbb{R}^d$ and $u \in \mathbb{G}$ and $\forall k$ let

$$v_k \equiv -\sum_{|y|\le k}(a,y)\eta_y - \sum_{|y|\le k_1}\tau_y u,$$

then

$$\varliminf_{k\uparrow\infty} \mathbb{E}\Big[V_k(f;\mu^{\alpha,\lambda})\Big] \ge \varliminf_{k\uparrow\infty} F_k(v_k).$$

It's simple to check that

$$R_i^{(k)}(v_k) = a_{e_i}(\eta_{e_i} - \eta_0) - \nabla_{0,e_i}\underline{u} + O(k^{-1})$$

and that

$$\sum_{i=1}^{d}\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}R_{i,k}(h)^2)\Big] = 2(2k)^{-d}\mathcal{D}_{\Lambda_k}(v_k;\mu^{\alpha,\lambda}) + c'_k$$

with $\lim_{k\uparrow\infty} c'_k = 0$.
Let us remember that (5.39) remains valid since

$$\sup_k k^{-d}\mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(v_k;\mu^{\alpha,\lambda})\Big] < \infty. \tag{5.42}$$

Thanks to (5.42), for the same arguments of the first part of the proof,

$$F_k(v_k) \geq \sum_{i=1}^{d}\Big\{2\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}\phi_{0,e_i}R_{i,k}(h))\Big] - \frac{1}{2}\mathbb{E}\Big[\mu^{\alpha,\lambda}(c_{0,e_i}R_{i,k}(h)^2)\Big]\Big\} - c_k$$

with $\lim_{k\uparrow\infty} c_k = 0$.

i) Proof of (5.35). The thesis follows by computing directly $\lim_{k\uparrow\infty} F_k(v_k)$.

$\square$

Lemma 5.10. *By assuming the same setting of the proof of lemma* 5.9*,* $\{R_i\}_{i=1}^d$ *is a germ of a closed form in* $L^2(\mathbb{P}^*)$*.*

Proof. We observe that for any $x \in \mathbb{Z}^d$ and bond b, τ_x and S_b are continuous functions on $L^2(\mathbb{P}^*)$ endowed of the weak topology.
We define the forms ξ and ξ^k as

$$\xi_{x,x+e_i} \equiv \tau_x R_i \qquad \xi^k_{x,x+e_i} \equiv \tau_x R_{i,k} \qquad \forall x \in \mathbb{Z}^d,\ \ \forall i = 1,\ldots,d.$$

Then, for any bond b, ξ^k_b weakly converges to ξ_b by taking the limit $k \uparrow \infty$.

We have to prove that ξ satisfies properties **P.1** , **P.2** and **P.3** of lemma 5.5 where (5.17), (5.18) and (5.19) have to be thought as equalities in $L^2(\mathbb{P}^*)$.
Let

$$A_{i,k} \equiv \{\, x \in \Lambda_k \,:\, x + e_i \in \Lambda_k \,\} \qquad\qquad F_{i,k} \equiv (2k)^{-d} \sum_{x\in A_{i,k}} \tau_{-x} h_k$$

$$A_k \equiv \{\, x \in \Lambda_k \,:\, x + e_i \in \Lambda_k \, \forall i = 1,\ldots,d\} \qquad\qquad F_k \equiv (2k)^{-d} \sum_{x\in A_k} \tau_{-x} h_k.$$

Then, by the definition of $R_{i,k}$, $\xi^k_{y,y+e_i} = \nabla_{y,y+e_i}\tau_y F_{i,k}$. By Schwarz inequality and the property $h_k \in \mathbb{G}^*_k$

$$\mathbb{E}^*\Big(\,\big(\nabla_{y,y+e_i}[\tau_y F_{i,k} - F_k]\big)^2\Big) = (2k)^{-2d}\mathbb{E}^*\Big(\,\big(\sum_{x\in(A_{i,k}-y)\Delta A_k} \pm\tau_{-x}\nabla_{x+y,x+y+e_i} h_k\big)^2\Big)$$
$$\leq Ck^{-d-1}|y|\mathbb{E}\Big[\mathcal{D}_{\Lambda_k}(h;\mu^{\alpha,\lambda})\Big] \leq C(f)k^{-1}|y|.$$

Therefore, given a positive integer s,

$$\|\xi^k_b - \nabla_b F_k\|_{L^2(\mathbb{P}^*)} \leq C\sqrt{s/k} \qquad \text{if } \ b \in \Lambda_s.$$

Since $\{\nabla_b F_k\}_{b\in\mathbb{Z}^d}$ is a closed form, ξ^k satisfies (5.17), (5.18) and (5.19) for bonds in Λ_s within an error of order $\sqrt{(s/k)}$. By taking the weak limit we obtain the thesis.

$\square$

5.6 Cauchy sequences

In this section we prove that, in dimension $d \geq 3$, for any $m \in (0,1)$ and $e > 0$, the sequence

$$n^{-1}\big(\, m_n^{1,e} - m_n^{2,e} - \mu^{\alpha}[m_n^{1,e} - m_n^{2,e}|m_n^e]\,\big)$$

is Cauchy for the semi-norm $V_m^{1/2}$ and that its limit point in $\bar{\mathcal{G}}$, the completion of $\mathcal{G}$ w.r.t. the semi-norm $V_m^{1/2}$, is orthogonal to the subspace $\mathcal{L}\mathbb{G}$ defined as

$$\mathcal{L}\mathbb{G} \equiv \{\mathcal{L}g \,:\, g \in \mathbb{G}\}.$$

We fix $e > 0$, which will be often understood, and we use the same notation of chapter 4, since the proof is based on the results of sections 4.2 and 4.3 and on lemma 4.15. In particular, if $0 < n \leq s$, then

$$\begin{aligned}
\phi_{n,s} &\equiv \mu^{\alpha}[m_n^{1,e} - m_n^{2,e}|m_s^e],\\
\hat{\phi}_{n,s} &\equiv \mu^{\alpha,\lambda(\alpha,\Lambda_s^e,m_s^e)}(m_n^{1,e} - m_n^{2,e}),\\
\psi_{n,s} &\equiv m_n^{1,e} - m_n^{2,e} - \phi_{n,s}.
\end{aligned}$$

By the density statement of theorem 5.2, $\mathcal{L}\mathbb{G}^{\perp}$ is finite dimensional and is generated by $\{(1-P)j_e\}_{e>0}$, where P is the orthogonal projection on $\mathcal{L}\mathbb{G}$.
The fundamental problem is the behavior of $\{n^{-1}P\psi_{n,n}\}_n$ as sequence in a infinite dimensional space. If we know that this sequence converges to zero then by theorem 3.9 we can compute directly the limit $V_m(n^{-1}\psi_{n,n}, Pj_e)$ for any $e > 0$ and this computation gives a complete description of the behavior of $n^{-1}(1-P)\psi_{n,n}$.
We observe that in the study of the sequence $\{n^{-1}P\psi_{n,n}\}_n$ the crucial problem is given by term $n^{-1}\phi_{n,n}$. In fact, let us remember that $\forall g \in \mathbb{G}$

$$V_m(f, \mathcal{L}g) = -\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}(f, \tau_x g)\Big]. \tag{5.43}$$

Since $\overline{\mathcal{L}\mathbb{G}} = \overline{\mathcal{L}\mathcal{G}}$ we restrict to $g \in \mathcal{L}\mathcal{G}$. In this case, the r.h.s. of (5.43) with $f = m_n^{1,e} - m_n^{2,e}$, which is meaningful though $f \notin \mathcal{G}$, is null by translation invariance.

Since $\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}(\phi_{n,s})$ decreases as $s \uparrow \infty$, it's convenient to study first, for any given n, the sequence $\{n^{-1}\psi_{n,s}\}_{s\geq n}$.
We prove that one of its subsequences is Cauchy and that its limit point $\bar{\psi}_n$ is in $\mathcal{L}\mathbb{G}^{\perp}$. At this point it's enough to prove the distance between $\bar{\psi}_n$ and $n^{-1}\psi_{n,n}$ decreases by taking the limit $n \uparrow \infty$.

Lemma 5.11. *Let $f \in \mathcal{G}_s$ and k, R be positive integers such that $100k \geq R \geq k + s$. We set $\mathcal{F}_{\Lambda_R} \equiv \sigma(m_R, ; \eta_x$ with $x \notin \Lambda_R)$.*
Then, it can be proved by the diffusive scaling of the spectral gap that for any $m \in [0,1]$

$$V_m(f) \leq C(d,B) R^{d+2}\, \mathbb{E}\left(\mu^{\alpha,\lambda_0(m)}\big(\mathrm{Var}_{\mu^{\alpha}}(\mathrm{Av}_{x\in\Lambda_k}\tau_x f \,|\, \mathcal{F}_{\Lambda_R})\big)\right). \tag{5.44}$$

Moreover, it can be proved by the estimates of the spectral gap of chapter 2 that for any $m \in [0,1]$

$$V_m(f) \le C(d)R^{d+2}\Big[\,\mathbb{E}\,\Big(\mu^{\alpha,\lambda_0(m)}\big(\mathrm{Var}_{\mu^\alpha}(\mathrm{Av}_{x\in\Lambda_k}\tau_x f\,|\,\mathcal{F}_{\Lambda_R})\big)^2\Big)\,\Big]^{\frac{1}{2}}. \tag{5.45}$$

Proof. The thesis follows from lemma 4.15.
Given an odd integer j, let $l \in \mathbb{N}$ such that $2l_1+1 = j(2k+1)$ (for simplicity of notation we assume this equation has an integer solution, but it's simple to check that this is not a restriction).
We subdivide the cube Λ_{l_1} in j^d cubes of side $2k+1$:

$$\Lambda_{l_1} = \sqcup_{i\in I}\Lambda_{x_i,k}.$$

Then we set $A \equiv \Lambda_l$ and $A_i \equiv \Lambda_{x_i,R}$. We define the functions F, F_i as

$$F \equiv \mathrm{Av}_{x\in\Lambda_{l_1}}\tau_x f \qquad \text{and} \qquad F_i \equiv \mathrm{Av}_{x\in\Lambda_{x_i,k}}\tau_x f \quad \forall i\in I.$$

Since A, A_i, F, F_i satisfy the conditions of lemma 4.15,

$$\begin{gathered} l^d\mu^{\alpha,\lambda_0(m)}(\mathrm{Av}_{x\in\Lambda_{l_1}}\tau_x f,\, \frac{1}{-\mathcal{L}_l}\,\mathrm{Av}_{x\in\Lambda_{l_1}}\tau_x f) \le \\ C(d)R^{d+2}\mathrm{Av}_i\,\Big[\,\gamma(\Lambda_{x_i,R})\,\mu^{\alpha,\lambda_0}\Big(\mathrm{Var}_{\mu^\alpha}\big(\,\mathrm{Av}_{x\in\Lambda_{x_i,k}}\tau_x f\,|\,\mathcal{F}_i\,\big)\Big)\Big]\end{gathered}$$

where $\mathcal{F}_i$ is as in lemma 4.15.
By taking the expectation w.r.t. the disorder α and then the limit $j \uparrow \infty$, we obtain (5.44) if we take advantage of the diffusive scaling of the spectral gap for the model under consideration. Using only the estimate of chapter 2, and in particular proposition 2.21, we obtain (5.45) thanks to Schwarz inequality.

□

We give a crucial application of lemma 5.11 and proposition 4.9.

Lemma 5.12. *Let $0 < \gamma < d$. Let n, s, k be integers such that*

$$n_0(\gamma) \le n \le s \le k \le 10^{12}s,$$

then

$$V_m\Big(\frac{\phi_{n,s}}{n} - \frac{\phi_{n,k}}{n}\Big) \le C(d)n^{-2}s^{2-d+\gamma}. \tag{5.46}$$

Proof. The function $\phi_{n,s} - \phi_{n,k} \in \mathcal{G}_k$. By (5.45) of lemma 5.11, the l.h.s. of (5.46) can be bounded by $C(d)\mathbb{E}\big(\Psi(\alpha)^2\big)^{\frac{1}{2}}$ where

$$\begin{gathered}\Psi(\alpha) \equiv n^{-2}k^{d+2}\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\mathrm{Av}_{x\in\Lambda_k}\tau_x[\phi_{n,s} - \phi_{n,k}]\right) \le \\ n^{-2}k^{d+2}\Big(\mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}\right) + \mathrm{Var}_{\mu^{\alpha,\lambda_0(m)}}\left(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,k}\right)\Big).\end{gathered}$$

The thesis follows from proposition 4.9.

□

Corollary 5.13. *Let $d \geq 3$. For any integer $\mathfrak{a} \geq 1$ and for any $m \in [0,1]$*

$$\lim_{n\uparrow\infty} V_m\Big(\frac{\phi_{n,n}}{n} - \frac{\phi_{n,n^{\mathfrak{a}}}}{n}\Big) = 0.$$

Proof. Given n let $R = R(n)$ be such that $2^R n \leq n^{\mathfrak{a}} < 2^{R+1} n$.
We first prove that

$$\lim_{n\uparrow\infty} V_m\Big(\frac{\phi_{n,n}}{n} - \frac{\phi_{n,2^R n}}{n}\Big) = 0. \tag{5.47}$$

Let $\gamma, \delta > 0$. If γ is small enough and $n \geq n_0(\gamma)$, then by (4.50) and lemma 5.12

$$V_m\Big(\frac{\phi_{n,n}}{n} - \frac{\phi_{n,2^R n}}{n}\Big) \leq n^{-2} C_\delta \sum_{k=1}^{R} k^{1+\delta} V_m\big(\phi_{n,2^{k-1}n} - \phi_{n,2^k n}\big) \leq$$
$$n^{-2} C'_\delta \sum_{k=1}^{R} k^{1+\delta} 2^{k(2-d+\gamma)} \leq C(\delta, \mathfrak{a}) n^{-2} (\ln n)^{1+\delta}.$$

In order to conclude the proof it's enough to observe that by lemma 5.12

$$V_m\Big(\frac{\phi_{n,2^R n}}{n} - \frac{\phi_{n,n^{\mathfrak{a}}}}{n}\Big) \leq C n^{-2+\mathfrak{a}(2-d+\gamma)}.$$

□

Remark 5.14. *By corollary 5.13, for any $m \in [0,1]$ the sequences*

$$\Big\{\frac{\psi_{n,n}}{n}\Big\}_{n\geq 1} \quad \textit{and} \quad \Big\{\frac{\psi_{n,n^4}}{n}\Big\}_{n\geq 1}$$

have the same asymptotic behavior in $\mathcal{G}$ endowed of the semi-inner product V_m.

Lemma 5.15. *Let $d \geq 3$ and $m \in [0,1]$.*
Then for n large enough the sequence $\{\psi_{n,i^4}\}_{i\geq n}$ is Cauchy in i w.r.t. the semi-norm $V_m^{1/2}$. We call $\bar{\psi}_n$ its limit in the completion $\bar{\mathcal{G}}$. Then

$$\lim_{n\uparrow\infty} V_m\Big(\frac{\bar{\psi}_n}{n} - \frac{\psi_{n,n^4}}{n}\Big) = 0. \tag{5.48}$$

Moreover, $\bar{\psi}_n \in \mathcal{L}\mathbb{G}^\perp$.

Proof. Let $0 < \gamma < d$ small enough. By lemma 5.12 if $n \geq n_0(\gamma)$

$$\sum_{i=n}^{\infty} V_m^{1/2}(\phi_{n,i^4} - \phi_{n,(i+1)^4}) \leq C \sum_{i=n}^{\infty} i^{4-2d+2\gamma} < \infty.$$

The above estimate proves the Cauchy property of $\{\psi_{n,i^4}\}_{i\geq n}$ and (5.48).

We prove $\bar{\psi}_n \in \mathcal{L}\mathbb{G}^{\perp}$.
For simplicity of notation we write $\lambda = \lambda_0(m)$.
By lemma 5.3 the subspace $\{\mathcal{L}g \,:\, g \in \mathcal{G}\}$ is dense in $\mathcal{L}\mathbb{G}$. Therefore it's enough to show

$$V_m(\bar{\psi}_n, \mathcal{L}g) = 0 \quad \forall g \in \mathcal{G}. \tag{5.49}$$

By definition of $\bar{\psi}_n$, it's enough to show $V_m(\psi_{n,i^4}, \mathcal{L}g)$ goes to 0 if $i \uparrow \infty$. To this aim let $s \geq n$, then by proposition 5.1, translation invariance and the property $g \in \mathcal{G}$,

$$V_m(\psi_{n,s}, \mathcal{L}g) = -\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big(\mu^{\alpha,\lambda}(\psi_{n,s}; \tau_x g)\Big) = \sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big(\mu^{\alpha,\lambda}(\phi_{n,s}; \tau_x g)\Big). \tag{5.50}$$

We define

$$\partial_s \equiv \{x \in \mathbb{Z}^d \ : \ (x+\Delta_g)\cap\Lambda_s \neq \emptyset \text{ and } (x+\Delta_g)\cap\Lambda_s^c \neq \emptyset\ \}.$$

Then

$$\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big(\mu^{\alpha,\lambda}(\phi_{n,s}; \tau_x g)\Big) = \mathbb{E}\Big(\mu^{\alpha,\lambda}(\phi_{n,s}; \sum_{x\in\partial_s}\tau_x g)\Big) \leq \mathbb{E}\Big(\mathrm{Var}_{\mu^{\alpha,\lambda}}(\phi_{n,s})^{\frac{1}{2}}\mathrm{Var}_{\mu^{\alpha,\lambda}}(\sum_{x\in\partial_s}\tau_x g)^{\frac{1}{2}}\Big) \tag{5.51}$$

Since $\mathrm{Var}_{\mu^{\alpha,\lambda}}(\sum_{x\in\partial_s}\tau_x g) \leq C_g|\partial_s| = C_g' s^{d-1}$, it remains to prove that

$$\mathrm{Var}_{\mu^{\alpha,\lambda}}(\phi_{n,s}) \leq C s^{-d}. \tag{5.52}$$

By the equivalence of ensembles we can substitute $\phi_{n,s}$ with $\hat{\phi}_{n,s}$ with an error of order Cs^{-2d}. Let $\lambda_s(\eta) \equiv \lambda(\alpha, \Lambda_s^e, m_s^e(\eta))$. Then, by Glauber inequality (proposition $A.9$),

$$\mathrm{Var}_{\mu^{\alpha,\lambda}}(\hat{\phi}_{n,s}) \leq C s^d \mu^{\alpha,\lambda}\Big([\nabla_0(\,\mu^{\alpha,\lambda_s}(m_n^{(1)} - m_n^{(2)})\,)]^2\Big). \tag{5.53}$$

By lemma $A.3$, $\nabla_0(\,\mu^{\alpha,\lambda_s}(m_n^{(1)} - m_n^{(2)})\,) \leq C s^{-d}$. Therefore (5.53) implies (5.52).

□

Theorem 5.16. *Let $d \geq 3$, $m \in (0,1)$ and $\bar{\mathcal{G}}$ be the completion of the space $\mathcal{G}$ with the semi-norm $V_m^{1/2}$.*
Then for any $e > 0$ the sequence $\{\frac{\psi^e_{n,n}}{n}\}_{n\geq 1}$ is Cauchy and its limit point is orthogonal to the subspace $\{\mathcal{L}g \,|\, g \in \mathbb{G}\}$.

Proof. Let P be the orthogonal projection on the closure of $\mathcal{L}\mathbb{G} \equiv \{\mathcal{L}g \,|\, g \in \mathbb{G}\}$. By remark 5.14 and lemma 5.15

$$\lim_{n\uparrow\infty} P\frac{\psi^e_{n,n}}{n} = 0. \tag{5.54}$$

By theorem 5.2 the subspace orthogonal to $\mathcal{L}\mathbb{G}$ is given by

$$\mathcal{H} \equiv (1-P)\{\sum_e a_e j_e \;:\; a \in \mathbb{R}^d\}.$$

By (5.54), for any $e' > 0$

$$V_m\Big((1-P)\frac{\psi^e_{n,n}}{n}, j_{e'}\Big) = V_m\Big(\frac{\psi^e_{n,n}}{n}, j_{e'}\Big) + E_n \quad \text{with} \quad \lim_{n\uparrow\infty} E_n = 0.$$

Then, by theorem 3.9 and proposition 5.1, $(1-P)\frac{\psi^e_{n,n}}{n}$ is weakly convergent in $\mathcal{H}$ which is finite dimensional and therefore it is strongly convergent in $\mathcal{H}$.
This result and (5.54) imply that $\frac{\psi^e_{n,n}}{n}$ is strongly convergent in $\bar{\mathcal{G}}$ and its limit point is orthogonal to $\mathcal{L}\mathbb{G}$.

□

5.7 Decomposition of currents

For simplicity of notation we write ψ^e_n in place of $\psi^e_{n,n}$.

Lemma 5.17. *There exists a constant C such that $\forall m \in (0,1)$, $\forall n \in \mathbb{N}$ and $\forall e > 0$*

$$V_m\Big(\frac{\psi^e_n}{n}\Big) \le Cm(1-m).$$

Proof. By (1.47),

$$V_l\Big(\frac{\psi^e_n}{n}; \mu^{\alpha,\lambda_0(m)}\Big) \le Cn^d \mathrm{Av}_{|x|\le l_1} \gamma_\alpha(\Lambda^e_{x,n}) \mu^{\alpha,\lambda_0(m)}\Big(\mathrm{Var}_{\mu^\alpha}\big(\tau_x[m^{1,e}_n - m^{2,e}_n] \,|\, m^e_{x,n}\big)\Big).$$

By lemma 2.6, setting $\mu \equiv \mu^{\alpha,\lambda(\alpha,\Lambda^e_{x,n},m^e_{x,n})}$,

$$\mathrm{Var}_{\mu^\alpha}\big(\tau_x[m^{1,e}_n - m^{2,e}_n] \,|\, m^e_{x,n}\big) \le \mathrm{Var}_\mu\big(\tau_x[m^{1,e}_n - m^{2,e}_n]\big) \le Cm(1-m)n^{-d}.$$

By the previous estimates and proposition 2.21

$$\mathbb{E}\Big[V_l\Big(\frac{\psi^e_n}{n}; \mu^{\alpha,\lambda_0(m)}\Big)\Big] \le Cm(1-m)\mathbb{E}\Big(\gamma_\alpha(\Lambda^e_n)\Big) \le C'm(1-m).$$

□

We fix $m \in (0,1)$ and $d \ge 3$. In what follows we work in the space $\bar{\mathcal{G}}$, the completion of $\mathcal{G}$ w.r.t. the semi-norm $V_m^{1/2}$.

Let P^* and P be respectively the orthogonal projection of $\bar{\mathcal{G}}$ on $(\mathcal{L}\mathbb{G})^\perp$ and on $\overline{\mathcal{L}\mathbb{G}}$.
By theorems 3.9 and 5.16, for any $e > 0$ the sequence $\{\frac{W^e_n}{n}\}_{n\ge 1}$ is Cauchy and its limit, W_e, belongs to $(\mathcal{L}\mathbb{G})^\perp$. By proposition 5.1, for any $e, e' > 0$, $V(j_e, W_{e'}) = \delta_{e,e'} 2m(1-m)$ (in particular, for any $e > 0$ $P^* j_e$ and W_e aren't 0).

Lemma 5.18. *$\{W_e\}_{e>0}$ is a basis of $(\mathcal{L}\mathbb{G})^\perp$.*

Proof. We define

$$Y \equiv \{\sum_{e>0} a_e P^* j_e \ : \ a \in \mathbb{R}^d\} \qquad Z \equiv \{\sum_{e>0} a_e W_e \ : \ a \in \mathbb{R}^d\}.$$

By theorem 5.2 $(\mathcal{L}\mathbb{G})^\perp = Y$ and therefore it's enough to prove $Z = Y$.
We have already observed $Z \subset Y$. Since $V_m(j_e, W_{e'}) = \delta_{e,e'} 2m(1-m)$, the vectors $\{W_e\}_{e>0}$ are linearly independent and $\dim(Z) = d$. Since Y is generated by d vectors $\dim(Y) \leq d$ and therefore $Y = Z$.

□

By lemma 5.18, for some $d \times d$ matrix Q we can write

$$j_e = \sum_{e'} Q_{e,e'}(m) W_{e'} + P j_e \quad \forall e > 0 \tag{5.55}$$

By proposition 5.1,

$$V(j_{e'}, j_e) = 2m(1-m) Q_{e,e'} + V(j_{e'}, P j_e) \quad \forall e, e' > 0. \tag{5.56}$$

By the symmetry of V and the property $V(j_{e'}, P j_e) = V(j_e, P j_{e'})$, (5.56) implies that Q is symmetric.
Moreover, by (5.56) for any $a \in \mathbb{R}^d$

$$2m(1-m)(a, Qa) = V_m\Big(\sum_e a_e j_e, \sum_e a_e (j_e - P j_e)\Big) = V_m\Big(P^*\big(\sum_e a_e j_e\big)\Big). \tag{5.57}$$

Proposition 5.1 implies

$$V_m\Big(P^*\big(\sum_e a_e j_e\big)\Big) = \frac{1}{2} \inf_{g \in \mathbb{G}} \sum_{e>0} \mathbb{E}\Big[\mu^{\alpha, \lambda_0(m)}\big(c_e (a_e \nabla_{0,e} \eta_0 - \nabla_{0,e} \sum_{x \in \mathbb{Z}^d} \tau_x g)^2\big)\Big].$$

Therefore the symmetric matrix $\sigma(m)$ given by (1.4) is well-defined and

$$Q(m) = \frac{\sigma(m)}{2m(1-m)}.$$

Proposition 5.19. $\exists C > 1$ *such that*

$$C^{-1}\mathbb{I} \leq Q(m) \leq C\mathbb{I} \quad \forall m \in (0,1).$$

The same property is valid for the diffusion matrix $D(m) \equiv \lambda_0'(m)\sigma(m)$.

Proof. We first prove that $Q(m) \geq C\mathbb{I}$ for some positive constant C and $\forall m \in (0,1)$.
We fix $a \in \mathbb{R}^d \setminus \{0\}$. Let $v \equiv \sum_{e>0} a_e P^* j_e$ and $w \equiv \sum_{e>0} a_e W_e$.
Then $V_m(v, w) = 2m(1-m)\|a\|^2$. Moreover, by lemma 5.17 and theorem 3.9,

$$V_m(w,w) \le d \sum_{e>0} |a_e|^2 V_m(W_e, W_e) \le Cm(1-m)\|a\|^2.$$

By Schwarz inequality

$$(a, Qa) = \frac{V_m(v,v)}{2m(1-m)} \ge \frac{1}{2m(1-m)} \frac{V_m(v,w)^2}{V_m(w,w)} \ge C\|a\|^2.$$

We prove that $Q(m) \le C\mathbb{I}$ for some positive constant C and $\forall m \in (0,1)$.
By (5.57)

$$(a, Qa) \le \frac{V_m(\sum_e a_e j_e)}{2m(1-m)} = \sum_{e>0} \frac{a_e^2}{4m(1-m)} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(m)}\big(c_e(\eta_e - \eta_0)^2\big) \le C\|a\|^2.$$

The last statement follows from lemma 3.10. □

By theorem 3.9

$$W_e = \lim_{n\uparrow\infty} 2m(1-m)\lambda_0'(m)\psi_n^e. \tag{5.58}$$

Therefore, by proposition 5.19, for any $m \in (0,1)$ and any $e > 0$,

$$j_e = \sum_{e'>0} D_{e,e'}(m)\psi_{n,n}^{e'} + Pj_e + \xi_{e,n}$$

where $\lim_{n\uparrow\infty} V_m(\xi_{e,n}) = 0$.
By some arguments as in the proof of proposition 1.12 and by following [13], chapter 7, we can prove that for any $f \in \mathcal{G}$ the map $[0,1] \to V_m(f)$ is continuous and then that $Q(m)$ is continuous on $(0,1)$.

• At the moment we are not able to prove that D can be continuously extended to $0, 1$. We assume the continuity. We remark that generally the diffusion matrix has no pathological behavior. Recently, the C^∞ regularity of the diffusion matrix has been proved for some models by duality methods, we refer to [2], [16] and [17].
We observe that if, as in other non gradient systems,

$$(a, D(m)a) \approx \lambda_0'(m) V_m\Big(\sum_e a_e j_e\Big) \tag{5.59}$$

with $m \downarrow 0$ and with $m \uparrow 1$, then by direct computation it would be simple to exhibit some dynamics for which the diffusion equation is non linear because $D(0) \ne D(1)$.

By the continuity of D, with the methods explained in [13], chapter 7, we obtain

$$\inf_{g\in\mathbb{G}} \sup_{m\in[0,1]} V_m\Big(j_e + \mathcal{L}g - \sum_{e'>0} Q_{e,e'}(m) W_{e'}\Big) = 0.$$

By (5.58), the above expression implies

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty} \sup_{m\in[0,1]} V_m\Big(j_e + \mathcal{L}g - \sum_{e'>0} D_{e,e'}(m)\frac{\psi^{e'}_{n,n}}{n}\Big) = 0.$$

Chapter 6

Comments on [21]

In this chapter we show that the method indicated in [21] for studying the hydrodynamic behavior of model under consideration seems to give some problems already in the case of the one dimensional symmetric simple exclusion process.

We assume the dimension to be 1 and the probability $\mathbb{P}$ to be concentrated on the disorder configuration α with $\alpha_x = 0$ for any site x. We choose transition rates $c_{x,y} = 2$. Unlike our standard notation we set $w_{x,y} \equiv c_{x,y}(\eta_y - \eta_x) = 2(\eta_y - \eta_x)$ following in [21]. Let us set

$$\frac{W_k}{k} = \frac{2}{k^2}(N_{2,k} - N_{1,k})$$

where $N_{i,k} = \sum_{x \in \Lambda_{i,k}} \eta_x$ and

$$\Lambda_{1,k} = \{-k, -k+1, \dots, -1\} \qquad \Lambda_{2k} = \{0, 1, \dots, k-1\}.$$

Given λ let μ^λ be the product probability measure such that $\mu^\lambda(\eta_x = 1) = \frac{e^\lambda}{1+e^\lambda}$. Given $m \in (0,1)$ let $\lambda(m)$ be such that $\mu^{\lambda(m)}(\eta_x) = m$, that is $\lambda(m) = \ln \frac{m}{1-m}$.

We remember the identity

$$\mu^\lambda(w_{xy} f) = -\mu^\lambda(\ (\eta_y - \eta_x)\nabla_{xy} f). \tag{6.1}$$

It's simple to check that

$$\sigma(m) = 2m(1-m). \tag{6.2}$$

In fact, by definition

$$\sigma(m) = \inf_g \mu^{\lambda(m)}\big((\nabla_{0,e}\eta_0 - \nabla_{0,e}\underline{g})^2\big)$$

where the infimum is taken among all local functions g. This infimum is realized by the null function (and corresponds to the r.h.s. of (6.2)) since $\mu^\lambda(\nabla_{0,e}\eta_0 \nabla_{0,e}\underline{g}) = 0$ for any g. In fact, by lemma 5.3 it's enough to prove this statement for $g \in \mathcal{G}$. Then by (6.1) and by translation invariance

$$\mu^\lambda(\nabla_{0,e}\eta_0\nabla_{0,e}\underline{g}) = -2\sum_{x\in\mathbb{Z}}\mu^\lambda(\nabla_{0,e}\eta_0;\tau_x g) = 0.$$

We give a more direct proof of (6.2). It is well-known that the hydrodynamic equation of the system is given by $\partial_t m = 2\partial_\theta^2 m$. Since the nongradient version corresponds to $\partial_t m = \partial_\theta\big(D(m)\partial_\theta m\big)$, it must be $D(m)\equiv 2$ and $\sigma(m) = 2m(1-m)$.

Warning. In this chapter (unlike our standard notation) we write $\mathcal{L}_k$ for the infinitesimal generator of the dynamics referred to $\Lambda_k \equiv \Lambda_{1,k}\cup\Lambda_{2,k}$, that is

$$\mathcal{L}_k \equiv 2\sum_{x=-k}^{k-2}\nabla_{x,x+1}.$$

Moreover, we denote $\mu_{k,m}$ the canonical measure on the volume Λ_k with density m.

Proposition 6.1. *For any density m*

(6.3)
$$\lim_{k\uparrow\infty} k\mu^{\lambda(m)}\Big(\frac{W_k}{k},\frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = \lim_{k\uparrow\infty} k\mu_{k,m}\Big(\frac{W_k}{k},\frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = \frac{16}{3}\frac{m^2(1-m)^2}{\sigma(m)} = \frac{8}{3}m(1-m).$$

Proof. We first prove the limit for grand canonical expectations.
Given an integer n, let Δ_{n+1} be the discrete Laplacian defined on the functions g on $\{0,1,2,\dots,n\}$ given by

$$\Delta_{n+1}g(x) = \begin{cases} g(1)-g(0) & \text{if } x=0\\ g(x+1)-2g(x)+g(x-1) & \text{if } x=1,\dots,n-1\\ -g(n)+g(n-1) & \text{if } x=n.\end{cases}$$

Given a positive integer n and a vector $a\in\mathbb{R}^{n+1}$, let

$$\mathcal{L}'_{n+1} = \nabla_{01}+\nabla_{12}+\cdots+\nabla_{n-1,n} \qquad a\cdot\eta = \sum_{s=0}^{n} a_s\eta_s.$$

Then

$$\mathcal{L}'_{n+1}(a\cdot\eta) = (\Delta_{n+1}a)\cdot\eta. \tag{6.4}$$

In order to prove the proposition it's convenient to identify $\Lambda_{1,k}$ with $\{0,1,\dots,k-1\}$ and $\Lambda_{2,k}$ with $\{k,k+1,\dots,2k-1\}$. Then $W_k(\eta) = 2k^{-1}a\cdot\eta$ with $a(x) = -1$ for $x = 0,1,\dots,k-1$ and $a(x) = 1$ for $x = k,\dots,2k-1$ and by (6.4)

$$k\mu^{\lambda(m)}\Big(\frac{W_k}{k},\frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = 4k^{-3}\mu^{\lambda(m)}(a\cdot\eta;(-\Delta_{2k}^{-1}a)\cdot\eta) = 4k^{-3}m(1-m)(a,-\Delta_{2k}^{-1}a).$$

Therefore the thesis with the grand canonical expectations corresponds to

$$\lim_{k\uparrow\infty} k^{-3}(a, -\Delta_{2k}^{-1}a) = \frac{2}{3}. \tag{6.5}$$

In order to prove (6.5) let b such that $a = \Delta_{2k} b$ with $b_0 = 0$.

We set $B_x \equiv b_{x+1} - b_x$ for $x = 0, \dots, 2k-2$, then

$$\begin{cases} (\Delta_{2k} b)_0 = B_0 \\ (\Delta_{2k} b)_x = B_x - B_{x-1} & \text{if } x = 1, \dots, 2k-2 \\ (\Delta_{2k} b)_{2k-1} = -B_{2k-2}. \end{cases}$$

From the equality $a = \Delta_{2k} b$ it's simple to derive $\{B_x\}_x$:

$$\begin{cases} B_x = -(x+1) & \text{if } x = 0, \dots, k-1 \\ B_{k+r} = -k + r + 1 & \text{if } r = 0, \dots, k-2. \end{cases}$$

Setting $\Sigma_n \equiv 1 + 2 + \dots + n$ if n is a positive integer and $\Sigma_0 \equiv 0$,

$$\begin{cases} b_x \equiv -\Sigma_x & \text{if } x = 0, \dots, k \\ b_{k+r} \equiv -\Sigma_k - rk + \Sigma_r & \text{if } r = 1, \dots, k-1. \end{cases}$$

In particular if $k \geq 2$

$$(a, -\Delta_{2k}^{-1}a) = \sum_{x=0}^{k-1} b_x - \sum_{x=k}^{2k-1} b_x = -2(\Sigma_1 + \dots + \Sigma_{k-1}) + k^3.$$

It can be verified that $\Sigma_1 + \dots + \Sigma_{k-1} = 6^{-1}(k^3 - k)$ and we obtain (6.5).

We now prove the thesis for the canonical expectations. By (6.4)

$$k\mu_{k,m}\Big(\frac{W_k}{k}, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = 4k^{-3}\mu_{k,m}(a \cdot \eta; (-\Delta_{2k}^{-1}a)\cdot \eta) = 4k^{-3}\sum_{x,y} a_x(-\Delta_{2k}^{-1}a)_y \mu_{k,m}(\eta_x;\eta_y).$$

Since

$$\begin{aligned} &\mu_{k,m}(\eta_x;\eta_x) = m(1-m) \quad \forall x \in \Lambda_k, \\ &\mu_{k,m}(\eta_x;\eta_y) = -(2k-1)^{-1}\mu_{k,m}(\eta_x;\eta_x) \quad \forall x \neq y \in \Lambda_k, \end{aligned}$$

and $\sum_{x:x\neq y} a_x = -a_y$, we obtain

$$\sum_{x,y} a_x(\Delta_{2k}^{-1}a)_y \mu_{k,m}(\eta_x;\eta_y) = m(1-m)\big\{(a, \Delta_{2k}^{-1}a) - \frac{1}{2k-1}\sum_{x,y:x\neq y} a_x(\Delta_{2k}^{-1}a)_y\big\} =$$
$$m(1-m)(a, \Delta_{2k}^{-1}a)(1 + \frac{1}{2k-1}).$$

By the previous computations we obtain the thesis.

□

Proposition 6.1 gives a counterexample to the statement at the end of page 76 in [21] but, independent of this particular statement, proposition 6.1 seems to be non compatible with some ideas in [21]. In what follows we explain the reason.

We assert that

$$\lim_{k\uparrow\infty} 2k\mu^{\lambda(m)}\Big(\mathrm{Av}_{\substack{x\in\Lambda_k:\\ x+1\in\Lambda_k}} w_{x,x+1}, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = 2m(1-m). \tag{6.6}$$

Let us prove (6.6). It's simple to check that

$$\sum_{\substack{x\in\Lambda_k:\\ x+1\in\Lambda_k}} w_{x,x+1} = -\mathcal{L}_k\big(\sum_{x\in\Lambda_k} x\eta_x\big).$$

Therefore

$$2k\mu^{\lambda(m)}\Big(\mathrm{Av}_{\substack{x\in\Lambda_k:\\ x+1\in\Lambda_k}} w_{x,x+1}, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = \frac{2k}{2k-1}\sum_{x\in\Lambda_k} x\mu^{\lambda(m)}\Big(\eta_x, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big).$$

and (6.6) follows by rule (6.1).
Since by proposition 6.1

$$k\mu^{\lambda(m)}\big(\frac{W_k}{k}, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\big)$$

is bounded uniformly in k, it's simple to verify that in (6.6) we can substitute the average of the currents by $\mathrm{Av}_{x\in\Lambda_{k_1}} w_{x,x+1}$ with $k_1 = k-\sqrt{k}$.
Thanks to (5.2) and the integration by parts property (1.3) it is simple to check that for any local function g

$$\lim_{k\uparrow\infty} 2k\mu^{\lambda(m)}\Big(\mathrm{Av}_{x\in\Lambda_{k_1}}\tau_x\mathcal{L}g, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = 0. \tag{6.7}$$

We assert that we cannot substitute the microscopic current by a linear combination of a microscopic fluctuation $\mathcal{L}g$ and of the mesoscopic function $\frac{W_k}{k}$ since the approximation statement

$$\inf_{a\in\mathbb{R},g\in\mathbb{G}}\lim_{k\uparrow\infty} 2k\mu^{\lambda(m)}\Big(\mathrm{Av}_{x\in\Lambda_{k_1}}\tau_x\big(aw_{0,e}+\mathcal{L}g\big)-\frac{W_k}{k}, \frac{1}{-\mathcal{L}_k}\big[\mathrm{Av}_{x\in\Lambda_{k_1}}\tau_x\big(aw_{0,e}+\mathcal{L}g\big)-\frac{W_k}{k}\big]\Big) = 0 \tag{6.8}$$

is incompatible with proposition 6.1. In fact, by (6.6) and (6.7), this statement corresponds to

$$\lim_{k\uparrow\infty} 2k\mu^{\lambda(m)}\Big(\frac{W_k}{k}, \frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big) = \sup_{a\in\mathbb{R},g\in\mathbb{G}}\Big\{4m(1-m)a - V_m(aw_{0,e}+\mathcal{L}g)\Big\},$$

where V_m is the usual semi-inner product, that is

$$V_m(aw_{0,e}+\mathcal{L}g)=\lim_{k\uparrow\infty}2k\mu^{\lambda(m)}\Big(\mathrm{Av}_{x\in\Lambda_{k_1}}\tau_x\big(aw_{0,e}+\mathcal{L}g\big),\frac{1}{-\mathcal{L}_k}\mathrm{Av}_{x\in\Lambda_{k_1}}\tau_x\big(aw_{0,e}+\mathcal{L}g\big)\Big).$$

Since given $A>0$

$$\sup_{\gamma\in\mathbb{R}}\Big(2B\gamma-A\gamma^2\Big)=\frac{B^2}{A},$$

$$\sup_{a\in\mathbb{R},g\in\mathbb{G}}\Big\{4m(1-m)a-V_m(aw_{0,e}+\mathcal{L}g)\Big\}=\sup_{g\in\mathbb{G}}\sup_{a\in\mathbb{R}}\Big\{4m(1-m)a-a^2V_m(w_{0,e}+\mathcal{L}g)\Big\}=$$
$$=\sup_g\frac{4m^2(1-m)^2}{V_m(w_{0,e}+\mathcal{L}g)}=\frac{4m^2(1-m)^2}{\sigma(m)}.$$

Therefore, proposition (6.8) implies

$$\lim_{k\uparrow\infty}2k\mu^{\lambda(m)}\Big(\frac{W_k}{k},\frac{1}{-\mathcal{L}_k}\frac{W_k}{k}\Big)=\frac{4m^2(1-m)^2}{\sigma(m)}$$

which is in contradiction with proposition 6.1.

Appendix A

A.1 Technical tools

Proposition A.1. *Let $f = f(\alpha_0)$ be a bounded measurable function with $\mathbb{E}(f) = 0$. Then $\forall \Lambda \in \mathbb{F}$ and $\forall \delta > 0$*

$$\mathbb{P}[|\mathrm{Av}_{x\in\Lambda} f_x| \geq \delta] \leq 2e^{-\frac{\delta^2|\Lambda|}{4\|f\|_\infty^2}}. \tag{A.1}$$

Proof. Given $t \geq 0$,

$$\mathbb{E}(e^{tf}) = 1 + \sum_{n=2}^{\infty} \frac{t^n}{n!}\mathbb{E}(f^n) \leq 1 + \sum_{n=2}^{\infty} \frac{t^n}{n!}\|f\|_\infty^n = e^{t\|f\|_\infty} - \|f\|_\infty t.$$

Since $e^x - x \leq e^{x^2}$ $\forall x \geq 0$, $\mathbb{E}(e^{tf}) \leq e^{K_f t^2}$ with $K_f \equiv \|f\|_\infty^2$.
By Chebyshev inequality, for any $t > 0$,

$$\mathbb{P}[\mathrm{Av}_{x\in\Lambda} f_x \geq \delta] \leq e^{-t\delta}\mathbb{E}[e^{t\mathrm{Av}_{x\in\Lambda} f_x}] = e^{-t\delta}[\mathbb{E}(e^{tf|\Lambda|^{-1}})]^{|\Lambda|} \leq e^{-t\delta + K_f t^2|\Lambda|^{-1}}.$$

Taking $t = \frac{\delta|\Lambda|}{2K_f}$, $\mathbb{P}[\mathrm{Av}_{x\in\Lambda} f_x \geq \delta] \leq e^{-\frac{\delta^2|\Lambda|}{4K_f}}$. By repeating the same argument with $-f$ we obtain the thesis.

□

Lemma A.2. *Let $\Lambda \in \mathbb{F}$, α a disorder configuration, $\Delta \subset \Lambda$ and $\lambda \in \mathbb{R}$. Let f be a local function on $\{0,1\}^{\mathbb{Z}^d}$ with support $\Delta_f \subset \Lambda$.*
We set $m \equiv \mu^{\alpha,\lambda}(m_\Lambda)$, $a_m \equiv \min(m, 1-m)$ and $\mu = \mu^{\alpha,\lambda}$. Then

a) $\mu(\eta_x) \leq e^{4B}\mu(\eta_y)$ *and* $\mu(1-\eta_x) \leq e^{4B}\mu(1-\eta_y)$ $\quad \forall x, y \in \mathbb{Z}^d$,

b) $e^{-4B}|\Delta|m \leq \mu(N_\Delta) \leq e^{4B}|\Delta|m$, $\quad e^{-4B}|\Delta|(1-m) \leq \mu(|\Delta| - N_\Delta) \leq e^{4B}|\Delta|(1-m)$,

c) $C(B)^{-1}|\Delta|a_m \leq \mu(N_\Delta; N_\Delta) \leq C(B)|\Delta|a_m$ *with* $C(B) > 1$,

d) $|\mu(f; N_\Lambda)| \leq C(B)\|f\|_\infty \min\Big(|\Delta_f|a_m, \sqrt{|\Delta_f|a_m}\,\Big)$,

e) $|\mu(N_\Delta; N_\Delta; N_\Delta)| \leq \mu(N_\Delta; N_\Delta)$.

Proof.

a) $e^{-2B}\frac{e^\lambda}{1+e^\lambda} \le \mu(\eta_x) \le e^{2B}\frac{e^\lambda}{1+e^\lambda}$ and $e^{-B}\frac{1}{1+e^\lambda} \le \mu(1-\eta_x) \le e^{B}\frac{1}{1+e^\lambda}$ for any $x \in \mathbb{Z}^d$.

b) follows by *a)*.

c) Thanks to *a)* and *b)*

$$\mu(N_\Delta; N_\Delta) = \sum_{x\in\Delta} \mu(\eta_x;\eta_x) \le \sum_{x\in\Delta} \mu(\eta_x) \le e^{4B} m|\Delta|,$$

$$\mu(N_\Delta; N_\Delta) = \mu(|\Delta| - N_\Delta; |\Delta| - N_\Delta) = \sum_{x\in\Delta} \mu(1-\eta_x; 1-\eta_x) \le e^{4B}(1-m)|\Delta|.$$

Therefore, $\mu(N_\Delta; N_\Delta) \le e^{4B} a_m |\Delta|$.
We first prove that $\mu(N_\Lambda; N_\Lambda) \ge (4e^{4B})^{-1}|\Lambda| a_m$. As example, let $m \in [0, \frac{1}{2}]$. Then $|W| \ge \frac{1}{2}|\Lambda|$ where $W \equiv \{x \in \Lambda : \mu(\eta_x) \le \frac{1}{2}\}$ and

$$\mu(N_\Lambda; N_\Lambda) \ge \sum_{x\in W} \mu(\eta_x;\eta_x) \ge \frac{1}{2}\sum_{x\in W} \mu(\eta_x) \ge \frac{1}{2}|W| e^{-4B} m \ge \frac{e^{-4B}}{4}|\Lambda| m.$$

Given $\Delta \subset \Lambda$ let $m' \equiv \frac{\mu(N_\Delta)}{|\Delta|}$. By *b)* and the previous result,

$$\mu(N_\Delta; N_\Delta) \ge (4e^{4B})^{-1}|\Delta| \min(m', 1-m') \ge (4e^{8B})^{-1}|\Delta| \min(m, 1-m).$$

d) By Schwarz inequality $|\mu(f; N_\Lambda)| \le \mu(f;f)^{\frac{1}{2}} \mu(N_{\Delta_f}; N_{\Delta_f})^{\frac{1}{2}}$. By *c)*, $\mu(N_{\Delta_f}; N_{\Delta_f}) \le C(B)|\Delta_f| a_m$. It remains to prove that

$$\mu(f;f)^{\frac{1}{2}} \le C(B)\|f\|_\infty \sqrt{|\Delta_f| a_m}.$$

To this aim let 0_{Δ_f} $[1_{\Delta_f}]$ be a configuration which is constantly 0 [1] on Δ_f. Then by *b)*

$$\mu(f;f) \le \mu\big((f - f(0_{\Delta_f}))^2\big) \le \|f\|_\infty^2 \mu(N_{\Delta_f}) \le \|f\|_\infty^2 e^{4B}|\Delta_f| m$$

and

$$\mu(f;f) \le \mu\big((f - f(1_{\Delta_f}))^2\big) \le \|f\|_\infty^2 \mu(|\Delta_f| - N_{\Delta_f}) \le \|f\|_\infty^2 e^{4B}|\Delta_f|(1-m).$$

e) We set $\bar\eta_x = \eta_x - \mu(\eta_x)$. Then

$$\mu(N_\Delta; N_\Delta; N_\Delta) = \sum_{x\in\Delta} \mu((\bar\eta_x)^3) \le \mu(N_\Delta; N_\Delta).$$

□

Lemma A.3. *Given $\Lambda \in \mathbb{F}$, $\lambda, \lambda' \in \bar{\mathbb{R}}$, a local function f on $\{0,1\}^{\mathbb{Z}^d}$ with support $\Delta_f \subset \Lambda$*

$$|\mu^{\alpha,\lambda'}(f) - \mu^{\alpha,\lambda}(f)| \le C(B)\|f\|_\infty |\Delta_f| |\mu^{\alpha,\lambda'}(m_\Lambda) - \mu^{\alpha,\lambda}(m_\Lambda)| \tag{A.2}$$

Proof. For simplicity of notation we write $\lambda(s)$ for $\lambda_\Lambda(\alpha, s)$ and we set $m' = \mu^{\alpha,\lambda'}(m_\Lambda)$ and $m = \mu^{\alpha,\lambda}(m_\Lambda)$.
By the continuity in $\lambda, \lambda' \in \bar{\mathbb{R}}$ of both members of $(A.2)$, we can consider only the case $\lambda, \lambda' \in \mathbb{R}$.

$$\mu^{\alpha,\lambda'}(f) - \mu^{\alpha,\lambda}(f) = \int_m^{m'} \frac{\partial}{\partial s}\mu^{\alpha,\lambda(s)}(f)ds = \int_m^{m'} \mu^{\alpha,\lambda(s)}(f; N_{\Delta_f})\lambda'(s)ds.$$

By lemma $A.2$

$$|\mu^{\alpha,\lambda(s)}(f; N_{\Delta_f})| \leq C(B)\|f\|_\infty |\Delta_f| \big(s \wedge (1-s)\big).$$

Since $s = \mu^{\alpha,\lambda(s)}(N_\Lambda/|\Lambda|)$

$$\lambda'(s) = \frac{|\Lambda|}{\mu^{\alpha,\lambda(s)}(N_\Lambda; N_\Lambda)}.$$

By lemma $A.2$, $\lambda'(s) \geq C(B)\big(s \wedge (1-s)\big)^{-1}$. Therefore, for any s,

$$|\mu^{\alpha,\lambda(s)}(f; N_{\Delta_f})|\lambda'(s) \leq C(B)\|f\|_\infty |\Delta_f|.$$

□

Lemma A.4. *Let $h = h(\alpha, \eta)$ be a measurable function depending only on α_0 and η_0. $\forall x \in \mathbb{Z}^d$ and $\forall \Lambda \in \mathbb{F}$ we set $h_x = \tau_x h$ and $H_\Lambda = \sum_{x \in \Lambda} h_x$.*
Then, given $\Lambda, \Delta \in \mathbb{F}$ with $\Lambda \subset \Delta$ and $\lambda, \lambda' \in \bar{\mathbb{R}}$,

$$|\mu^{\alpha,\lambda'}(H_\Lambda; N_\Lambda) - \mu^{\alpha,\lambda}(H_\Lambda; N_\Lambda)| \leq C(B)\|h\|_\infty |\Lambda| |\mu^{\alpha,\lambda'}(m_\Delta) - \mu^{\alpha,\lambda}(m_\Delta)| \tag{A.3}$$

and (with $\lambda, \lambda' \in \mathbb{R}$)

$$|\mu^{\alpha,\lambda'}(H_\Lambda; N_\Lambda) - \mu^{\alpha,\lambda}(H_\Lambda; N_\Lambda)| \leq C(B)\|h\|_\infty \mu^{\alpha,\lambda\vee\lambda'}(N_\Delta)|\lambda' - \lambda| \tag{A.4}$$

Proof. We first prove $(A.3)$.
For simplicity of notation we write $\lambda(s)$ for $\lambda_\Delta(\alpha, s)$ and we set $m' = \mu^{\alpha,\lambda'}(m_\Delta)$ and $m = \mu^{\alpha,\lambda}(m_\Delta)$. Therefore $\lambda' = \lambda(m')$ and $\lambda = \lambda(m)$.
By the continuity in $\lambda, \lambda' \in \bar{\mathbb{R}}$ of both members of $(A.3)$, we can consider only the case $\lambda, \lambda' \in \mathbb{R}$. Then

$$\mu^{\alpha,\lambda'}(H_\Lambda; N_\Lambda) - \mu^{\alpha,\lambda}(H_\Lambda; N_\Lambda) =$$
$$\int_m^{m'} \frac{\partial}{\partial s}\mu^{\alpha,\lambda(s)}(H_\Lambda; N_\Lambda)ds = \int_m^{m'} \mu^{\alpha,\lambda(s)}(H_\Lambda; N_\Lambda; N_\Lambda)\lambda'(s)ds.$$

Since

$$|\mu^{\alpha,\lambda(s)}(H_\Lambda;N_\Lambda;N_\Lambda)| = \sum_{x\in\Lambda}\mu^{\alpha,\lambda(s)}(h_x;\eta_x;\eta_x)| \le 2\|h\|_\infty \mu^{\alpha,\lambda(s)}(N_\Lambda;N_\Lambda),$$

by lemma $A.2$

$$|\mu^{\alpha,\lambda(s)}(H_\Lambda;N_\Lambda;N_\Lambda)| \le C(B)\|h\|_\infty|\Lambda|\big(s\wedge(1-s)\big). \tag{A.5}$$

As already proved for lemma $A.3$, $\lambda'(s)\ge C(B)\big(s\wedge(1-s)\big)^{-1}$. Therefore, for any s

$$|\mu^{\alpha,\lambda(s)}(H_\Lambda;N_\Lambda;N_\Lambda)\lambda'(s)| \le C(B)\|h\|_\infty|\Lambda|.$$

We prove $(A.4)$. Since

$$|\mu^{\alpha,\lambda'}(H_\Lambda;N_\Lambda) - \mu^{\alpha,\lambda}(H_\Lambda;N_\Lambda)| = |\int_\lambda^{\lambda'}\frac{d}{d\tilde\lambda}\mu^{\alpha,\tilde\lambda}(H_\Lambda;N_\Lambda)d\tilde\lambda| = |\int_\lambda^{\lambda'}\mu^{\alpha,\tilde\lambda}(H_\Lambda;N_\Lambda;N_\Lambda)d\tilde\lambda|,$$

by an estimate similar to $(A.5)$ and lemma $A.2$

$$|\mu^{\alpha,\tilde\lambda}(H_\Lambda;N_\Lambda;N_\Lambda)| \le C(B)\|h\|_\infty\mu^{\alpha,\tilde\lambda}(N_\Lambda) \le C(B)\|h\|_\infty\mu^{\alpha,\lambda\vee\lambda'}(N_\Delta).$$

□

Lemma A.5. *Given $y\in\mathbb{Z}^d$, $m,m'\in[0,1]$*

$$|\mu^{\alpha,\lambda_0(m')}(\eta_y) - \mu^{\alpha,\lambda_0(m)}(\eta_y)| \le e^{6B}|m'-m|, \tag{A.6}$$

$$|\mu^{\alpha,\lambda_0(m')}(\eta_y;\eta_y) - \mu^{\alpha,\lambda_0(m)}(\eta_y;\eta_y)| \le 3e^{6B}|m'-m|. \tag{A.7}$$

In general, given a function f with support in the cube Λ_k,

$$|\mu^{\alpha,\lambda_0(m')}(f) - \mu^{\alpha,\lambda_0(m)}(f)| \le C(B,k)\|f\|_\infty|m'-m| \tag{A.8}$$

Proof. We prove $(A.6)$. By continuity it is not restrictive to consider $m,m'\in(0,1)$ with $m\le m'$. Then

$$\mu^{\alpha,\lambda_0(m')}(\eta_y) - \mu^{\alpha,\lambda_0(m)}(\eta_y) = \int_m^{m'}\frac{d}{ds}\mu^{\alpha,\lambda_0(s)}(\eta_y)ds = \int_m^{m'}\mu^{\alpha,\lambda_0(s)}(\eta_y;\eta_y)\lambda_0'(s)ds.$$

Since $s=\mathbb{E}\Big[\mu^{\alpha,\lambda_0(s)}(\eta_y)\Big]$,

$$\mu^{\alpha,\lambda_0(s)}(\eta_y;\eta_y)\lambda_0'(s) = \frac{\mu^{\alpha,\lambda_0(s)}(\eta_y;\eta_y)}{\mathbb{E}\Big[\mu^{\alpha,\lambda_0(s)}(\eta_y;\eta_y)\Big]}.$$

It's simple to verify that for any couple of disorder configurations α and α'

$$\frac{\mu^{\alpha,\lambda_0(s)}(\eta_y;\eta_y)}{\mu^{\alpha',\lambda_0(s)}(\eta_y;\eta_y)} \le e^{6B}.$$

This allows to conclude the proof of $(A.6)$.
$(A.7)$ is a simple consequence of $(A.6)$ while $(A.8)$ follows from $(A.6)$ thanks to the estimate

$$\begin{aligned}|\mu^{\alpha,\lambda_0(m')}(f) - \mu^{\alpha,\lambda_0(m)}(f)| &\le \sum_{\zeta\in\{0,1\}^{\Lambda_k}} |f(\zeta)|\, |\mu^{\alpha,\lambda_0(m')}(\eta_{\Lambda_k}=\zeta) - \mu^{\alpha,\lambda_0(m)}(\eta_{\Lambda_k}=\zeta)| \\ &\le C(k,B)\|f\|_\infty |m'-m|.\end{aligned}$$

□

Lemma A.6. *For any* $m\in(0,1)$

$$\lambda_\Lambda(\alpha,m) - \lambda_0(m) = [m - \mu^{\alpha,\lambda_0(m)}(m_\Lambda)]\frac{|\Lambda|}{\mu^{\alpha,\tilde\lambda}(N_\Lambda;N_\Lambda)} \tag{A.9}$$

where $\tilde\lambda$ *is in the interval with extremes* $\lambda_\Lambda(\alpha,m)$ *and* $\lambda_0(m)$.
In particular

$$|\lambda_\Lambda(\alpha,m) - \lambda_0(m)| \le \frac{C(B)}{m(1-m)}|m - \mu^{\alpha,\lambda_0(m)}(m_\Lambda)|. \tag{A.10}$$

Proof. We consider the function $\lambda(s) \equiv s\lambda_\Lambda(\alpha,m) + (1-s)\lambda_0(m)$ defined on $[0,1]$. By Lagrange theorem

$$m = \mu^{\alpha,\lambda_\Lambda(\alpha,m)}(m_\Lambda) = \mu^{\alpha,\lambda_0(m)}(m_\Lambda) + |\Lambda|^{-1}\mu^{\alpha,\tilde\lambda}(N_\Lambda;N_\Lambda)\big(\lambda_\Lambda(\alpha,m) - \lambda_0(m)\big)$$

for a particular $\tilde\lambda$ in the interval with extremes $\lambda_\Lambda(\alpha,m)$ and $\lambda_0(m)$. This proves $(A.9)$.

By lemma $A.2$,

$$|\Lambda|^{-1}\mu^{\alpha,\tilde\lambda}(N_\Lambda;N_\Lambda) \ge C(B)\mu^{\alpha,\tilde\lambda}(m_\Lambda)\big(1-\mu^{\alpha,\tilde\lambda}(m_\Lambda)\big).$$

This and the observation that

$$\lambda_\Lambda(\alpha,m), \lambda_0(m) \in [\ln\frac{m}{1-m} - B, \ln\frac{m}{1-m} + B]$$

give

$$|\Lambda|^{-1}\mu^{\alpha,\tilde\lambda}(N_\Lambda;N_\Lambda) \ge C(B)m(1-m).$$

$(A.10)$ follows now from $(A.9)$.

□

Lemma A.7. *For any $m \in [0,1]$ and any local function f on $\{0,1\}^{\mathbb{Z}^d}$ with support in Λ*

$$|\mu^{\alpha,\lambda_\Lambda(\alpha,m)}(f) - \mu^{\alpha,\lambda_0(m)}(f)| \le C(B)\|f\|_\infty |\Delta_f||m - \mu^{\alpha,\lambda_0(m)}(m_\Lambda)| \tag{A.11}$$

Proof. It's a consequence of lemma $A.3$ with $\lambda' = \lambda_\Lambda(\alpha,m)$ and $\lambda = \lambda_0(m)$. □

Lemma A.8. *Let V, W be two disjoint cubes of side l. Then, given a disorder configuration α, for any density m on $V \cup W$*

$$\mu^\alpha_{V\cup W,m}\big((m_V - m_W)^2\big) \le C(B)\Big(l^{-d} + \big(m - \mu^{\alpha,\lambda_0(m)}(m_V)\big)^2 + \big(m - \mu^{\alpha,\lambda_0(m)}(m_W)\big)^2\Big).$$

Proof. We set $\nu \equiv \mu^\alpha_{V\cup W,m}$. Then

$$\nu\big((m_V - m)^2\big) \le 2\mathrm{Var}_\nu(m_V) + 2(\nu(m_V) - m)^2.$$

By lemma 2.6, $\mathrm{Var}_\nu(m_V) \le C(B)\mathrm{Var}_\mu(m_V) \le C'(B)l^{-d}$. By the equivalence of ensembles and lemma $A.7$, $(\nu(m_V) - m)^2$ is bounded by

$$C(B)\Big(l^{-2d} + (m - \mu^{\alpha,\lambda_0(m)}(m_V))^2 + (m - \mu^{\alpha,\lambda_0(m)}(m_W))^2\Big).$$

By a similar estimate for $\nu\big((m_W - m)^2\big)$ we obtain the thesis.

□

We conclude this section with an estimate of the spectral gap for Glauber dynamics.

Proposition A.9. *Let μ be a product measure on $\{0,1\}^{\mathbb{Z}}$. We set*

$$C = \sup_{x\in\mathbb{Z}} \mu(\eta_x)\big(1 - \mu(\eta_x)\big).$$

Then for any local function f

$$\mathrm{Var}_\mu(f) \le C \sum_{x\in\mathbb{Z}} \mu\big((\nabla_x f)^2\big).$$

Proof. We prove the proposition by induction on the size n of the support Δ_f of f.
Let $n = 1$. Then $f = a\eta_x + b(1-\eta_x)$ for some $x \in \mathbb{Z}$ and $\mathrm{Var}_\mu(f) = (a-b)^2\mu(\eta_x)\big(1-\mu(\eta_x)\big)$.
We assume the thesis valid for functions with support of size non larger than n and we prove it for a function f with $|\Delta_f| \le n+1$. By a permutation, it is not restrictive to consider $f = f(\eta_0, \eta_1, \ldots, \eta_n)$. The thesis follows by inductive hypothesis and the identity

$$Var_\mu(f) = \mu\big(\mathrm{Var}_\mu(f\,|\,\eta_n)\big) + \mathrm{Var}_\mu\big(\mu(f\,|\,\eta_n)\big).$$

□

A.2 Some theorems on Markov processes

In this section we collect some theorems and remarks about Markov processes. Proofs can be found in [13].

In what follows E is a finite set and L is an infinitesimal generator on E.
We indicate by $D\big([0,T],E)$ the space of right continuous paths with left limits $X:[0,T]\to E$ and by $\mathcal{F}_t$ the $\sigma-$algebra $\sigma(X_s : 0\le s\le t)$.

Proposition A.10. .
Let $F:[0,T]\times E\to\mathbb{R}$ *be such that*

$$F(\cdot,x)\in C^2([0,T])\ \ \forall x\in E\ \ \textit{and}\ \sup_{(s,x)}\big|(\partial_s^j F)(s,x)\big|<\infty\quad\forall j=0,1,2.$$

Let $\mathbb{P}$ *be a probability measure on* $D([0,T],E)$ *induced by the Markov process with generator* L *and an arbitrary initial distribution.*
For any $t\in[0,T]$ *we define*

$$M^F(t)=F(t,X_t)-F(0,X_0)-\int_0^t ds(\partial_s+L)F(s,X_s)$$
$$N^F(t)=\big(M^F(t)\big)^2-\int_0^t ds\{(LF^2)(s,X_s)-2F(LF)(s,X_s)\}.$$

Then $M^F(t)$ *and* $N^F(t)$*, with* $t\in[0,T]$*, are* $\mathcal{F}_t$*-martingales w.r.t.* $\mathbb{P}$.

Lemma A.11. *Let* $\mathcal{P}(E)$ *be the set of probability measures on* E.
For any $\nu\in\mathcal{P}(E)$ *we indicate by* $\mathbb{P}^\nu$ *the probability measure on* $D([0,T],E)$ *determined by the Markov process with generator* L *and initial distribution* ν.
Let $\mu\in\mathcal{P}(E)$ *such that* $\exists A$ *with* $\mu(x)\ge e^{-A}\quad\forall x\in E$.
Then $\forall\nu\in\mathcal{P}(E)$

$$\ln\Big\|\frac{d\mathbb{P}^\nu}{d\mathbb{P}^\mu}\Big\|_\infty\le A\qquad\textit{and}\qquad H[\mathbb{P}^\nu|\mathbb{P}^\mu]\le A.$$

Proof.
The second inequality follows from the first one by the identity $H[\mathbb{P}^\nu|\mathbb{P}^\mu]=\mathbb{E}^\nu(\ln\frac{d\mathbb{P}^\nu}{d\mathbb{P}^\mu})$.
Since

$$\mathbb{P}^\mu=\sum_{x\in E}\mu(x)\mathbb{P}^{\delta_x}\qquad\mathbb{P}^\nu=\sum_{x\in E}\nu(x)\mathbb{P}^{\delta_x}$$

and the measures $\mathbb{P}^{\delta_x}$, with $x\in E$, have disjoint supports, we have

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}^\mu}=\sum_{x\in E}\frac{\nu(x)}{\mu(x)}1_{\{\eta_\cdot|\eta_0=x\}}\quad\text{and}\quad\Big\|\frac{d\mathbb{P}^\nu}{d\mathbb{P}^\mu}\Big\|_\infty\le\sup_{x\in E}\frac{\nu(x)}{\mu(x)}\le\sup_{x\in E}\frac{1}{\mu(x)}\le e^A.$$

That proves the first estimate.

□

Proposition A.12. *(Application of the Feynman-Kac Formula)*
Let ν be a measure on E, reversible w.r.t. the infinitesimal generator L, and $V : \mathbb{R}_+ \times E \to \mathbb{R}$ be a bounded measurable function.
Then, for any $t \geq 0$,

$$\mathbb{E}_\nu\Big[\exp\Big\{\int_0^t V(s, X_s)ds\Big\}\Big] \leq \exp\Big\{\int_0^t \Gamma_s ds\Big\}$$

where

$$\Gamma_s \equiv \sup spec_{L^2(\nu)}\{L + V(s, \cdot)\}.$$

Proposition A.13. *(Perturbation Theory)*
Let ν be a measure on E, reversible w.r.t. the infinitesimal generator L.
We supposed that L has simple eigenvalue 0 and spectral gap u^{-1}.
Given a function V on E and $\beta > 0$ we define

$$\lambda_\beta \equiv \sup spec_{L^2(\nu)}\{L + \beta V\}.$$

If $4u\|V\|_\infty\beta < 1$ then

$$\beta\nu(V) \leq \lambda_\beta \leq \beta\nu(V) + \frac{\beta^2}{1 - 4u\|V\|_\infty\beta}\nu(\overline{V}, -L^{-1}\overline{V})$$

where $\overline{V} \equiv V - \nu(V)$.

A.3 Space $D([0,1], \mathcal{M}_1)$

In this section we collect some technical results concerning the space $D([0,1], \mathcal{M}_1)$ endowed of the Skohorod topology. In what follows we will denote d the standard metric on $\mathcal{M}_1$ inducing the weak topology. Les us recall the definition.
Given a dense contable family $\{f_k;\, k \geq 1\}$ of continuous functions on $\mathbb{T}^d$, then

$$d(\mu, \nu) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\mu(f_k) - \nu(f_k)|}{1 + |\mu(f_k) - \nu(f_k)|} \qquad \mu, \nu \in \mathcal{M}_1.$$

Let us introduce also the Skohorod metric d_0 on $D([0,1], \mathcal{M}_1)$ (see also [4] or [13], chapter 4). Let Λ be the set of strictly increasing continuous functions λ of $[0,T]$ into itself. We define then

$$\|\lambda\| = \sup_{s \neq t}\Big|\ln \frac{\lambda(t) - \lambda(s)}{t - s}\Big|$$

and

$$d_0(\mu, \nu) = \inf_{\lambda \in \Lambda} \max\Big\{\|\lambda\|_\infty, \sup_{0 \leq t \leq T} d(\mu_t, \nu_{\lambda(t)})\Big\} \qquad \forall \mu, \nu \in D([0,1], \mathcal{M}_1).$$

In what follows, μ, ν will be generic elements of $D([0,1], \mathcal{M}_1)$.

Lemma A.14. *For any disorder configuration α, given a sequence $\{\nu_\epsilon\}_{\epsilon > 0}$ of probability measures on Ω_ϵ then any limit point Q^α of Q^α_ϵ belongs to $C([0,T], \mathcal{M}_2)$.*

Proof. We first observe that for any $H \in C(\mathbb{T}^d)$ and $b \in \mathbb{T}^d_\epsilon$

$$\text{(A.12)} \qquad \big|\, \pi_\epsilon(\eta)[H] \,\big| \le \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} |H(\epsilon x)| \quad \text{and} \quad \big|\, \pi_\epsilon(\eta^b)[H] - \pi_\epsilon(\eta)[H] \,\big| \le 2\, \|H\|_\infty \epsilon^d.$$

Let us prove that $Q^\alpha(\mu \,:\, \mu_t \ll dx \;\forall t \in [0,T]) = 1$. Let $H \in C(\mathbb{T}^d)$. Then thanks to (A.12), given $\delta > 0$ for $\epsilon \le \epsilon_0(\delta, H)$ we have $Q^\alpha_\epsilon(A_{\delta,H}) = 1$ where

$$A_{\delta,H} := \{\mu \,:\, \mu_t(H) \le \delta + \int_{\mathbb{T}^d} H(\theta) d\theta \;\; \forall t \in [0,T]\}.$$

We observe that $A_{\delta,H}$ is closed. In fact, let $\mu \in A^c_{\delta,H}$, i.e. $\exists t \in [0,T]$ such that $\mu_t(H) > \delta + \int_{\mathbb{T}^d} H(\theta)d\theta$. We want to prove that, given β small enough, $d_0(\nu,\mu) < \beta$ implies that $\nu \in A^c_{\delta,H}$. By definition, there exists $\lambda \in \Lambda$ with $d(\nu_{\lambda(s)}, \mu_s) < \beta$ forall $s \in [0,T]$. Let us choose $\gamma > 0$ such that $\mu_t(H) > \delta + \gamma + \int_{\mathbb{T}^d} H(\theta)d\theta$. Then, if β is small enough, $|\nu_{\lambda(t)}(H) - \mu_t(H)| \le \gamma$ and $\nu \in A^c_{\delta,H}$. Because of weak convergence, $\forall H \in C(\mathbb{T}^d)$, $\cap_{\delta>0} Q^\alpha(A_{\delta,H}) = 1$, thus implying that

$$Q^\alpha(\mu : \mu_t(H) \le \int_{\mathbb{T}^d} H(\theta)d\theta \;\forall t \in [0,T]) = 1 \qquad \forall H \in C(\mathbb{T}^d).$$

It is enough to consider a countable base of $C(H)$ in order to obtain the thesis.
Let us prove continuity of the paths almost surely. Given $\delta > 0$ let

$$A_\delta := \{\mu \,:\, \exists n \in \mathbb{N} \text{ such that } d(\mu_{t_1}, \mu_{t_2}) \le \delta \;\forall t_1, t_2 \text{ with } |t_1 - t_2| \le \frac{1}{n}\}.$$

It is enough to show that $Q^\alpha(\cap_{\delta>0} A_\delta) = 1$. Because of (A.12), we have that

$$\lim_{\epsilon\downarrow 0} \sup_{\eta\in\Omega_\epsilon} d(\pi_\epsilon(\eta^b), \pi_\epsilon(\eta)) = 0$$

and therefore, if $\epsilon \le \epsilon_0(\delta)$, $Q^\alpha_\epsilon(\cap_{s>\delta} A_s) = 1$. Now it is enough to prove that $\cap_{s>\delta} A_s$ is closed. To this aim fix $s > \delta$, $1 > \gamma > 0$ such that $s - 2\gamma > \delta$, $\mu \in A^c_s$ and choose ν such that $d_0(\mu,\nu) < \gamma$. We claim that $\nu \in A^c_{s-2\gamma}$ thus proving that $\cup_{s>\delta} A^c_s$ is open. Since $\mu \in A^c_s$, given $n \in \mathbb{N}$ there exist $t_1 > t_2$ such that $t_1 - t_2 \le \frac{1}{n}$ and $d(\mu_{t_1}, \mu_{t_2}) > s$. Since $d_0(\mu,\nu) < \gamma$, there exists $\lambda \in \Lambda$ such that

$$\sup_{0\le s<t\le 1} \Big|\ln \frac{\lambda(t) - \lambda(s)}{t-s}\Big| < \gamma \quad \text{and} \quad \sup_{0\le t\le 1} d(\nu_t, \mu_{\lambda(t)}) < \gamma.$$

In particular, $d(\nu_{\lambda(t_1)}, \nu_{\lambda(t_2)}) > s - 2\gamma$ and $\lambda(t_1) - \lambda(t_2) \le e^\gamma(t_1 - t_2) \le \frac{e}{n}$. Therefore, $\nu \in A^c_{s-2\gamma}$. □

Proposition A.15. *Given $m \in D\big([0,T], \mathcal{M}_2\big)$ there exists a measurable function $m(t,\theta)$ such that $0 \le m(t,\theta) \le 1$ for any t, θ and $m_t = m(t,\theta)d\theta$ for any $t \in [0,T]$.*

Proof. For any t, let $\rho_t(\theta)$ such that $m_t = \rho_t(\theta)d\theta \in \mathcal{M}_2$. Fix a sequence of C^∞–mollifiers ϕ_ϵ and define $m_\epsilon(t,\theta) := (\rho_t \star \phi_\epsilon)(\theta)$ (where $\star$ denotes convolution). Then for any t, $m_\epsilon(t,\theta) \to \rho_t(\theta)$ a.s. on $\mathbb{T}^d$ as $\epsilon \downarrow 0$.

Let us prove that m_ϵ is measurable on $[0,T]\times\mathbb{T}^d_\epsilon$. To this aim let us observe that

$$\begin{aligned} a_k^\epsilon(t) : &= \int_{\mathbb{T}^d} m_\epsilon(t,\theta)e^{-ik\theta}d\theta = \int_{\mathbb{T}^d} d\theta \int_{\mathbb{T}^d} d\theta' m(t,\theta+\theta')\phi_\epsilon(\theta')^{-ik\theta} \\ &= \int_{\mathbb{T}^d} d\theta\, m(t,\theta) \int_{\mathbb{T}^d} d\theta' \phi_\epsilon(\theta')e^{-ik(\theta-\theta')}. \end{aligned}$$

Since $m\in D([0,T],\mathcal{M}_2)$, $a_k^\epsilon(t)$ is measurable in t. Define

$$s_n^\epsilon(t,\theta) := \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d,\,|k|\le n} a_k^\epsilon(t)e^{ik\theta}.$$

Then s_n^ϵ is measurable on $[0,T]\times\mathbb{T}^d$ and by lemma A.16

$$\lim_{n\uparrow\infty} s_n^\epsilon(t,\theta) = m_\epsilon(t,\theta) \quad \forall t,\theta.$$

Therefore m_ϵ is measurable on $[0,T]\times\mathbb{T}^d$.
Define

$$A := \{(t,\theta) : \lim_{\epsilon\downarrow 0} m_\epsilon(t,\theta) \text{ exists finite }\},$$

then A is measurable. Indicating by $|\cdot|$ the Lebesgue measure of a set, by Fubini theorem $|A| = \int_0^T |A_t| = 0$. Therefore, we can define a measurable function $m(t,\theta)$ such that $lim_{\epsilon\downarrow 0} m_\epsilon(t,\theta) = m(t,\theta)$ a.s. on $[0,T]\times\mathbb{T}^d$. Since, for any t, $lim_{\epsilon\downarrow 0} m_\epsilon(t,\theta) = \rho(t,\theta)$ a.s. on $\mathbb{T}^d$, we get the thesis. □

Lemma A.16. *Let $f\in C^\infty(\mathbb{T}^d)$ and set, for $k\in\mathbb{Z}^d$ and $n\in\mathbb{N}$,*

$$a_k := \int_{\mathbb{T}^d} f(\theta)e^{-ik\theta}d\theta, \qquad s_n(\theta) := \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d,\,|k|\le n} a_k e^{ik\theta}.$$

Then $\lim_{n\uparrow\infty}\|f-s_n\|_\infty = 0$.

Proof. Let us observe that, for any $j=1,\dots,d$ and $s\in\mathbb{N}$,

$$e^{-ik\theta} = (-ik_j)^{-s}\frac{\partial^s}{\partial\theta_j^s}e^{-ik\theta}.$$

By integrating by parts

$$a_k = \pm(ik_j)^{-s}\int_{\mathbb{T}^d}\frac{\partial^s f}{\partial\theta_j^s}e^{-ik\theta},$$

thus implying that

$$|a_k| \le C_s(f)\min\{|k_1|^{-s},|k_2|^{-s},\dots,|k_d|^{-s}\}.$$

By chosing s large enough (depending on d) we can prove $\sum_{k\in\mathbb{Z}^d}|a_k| \le C(f)$. Since for $n'>n$ $\|s_{n'}-s_n\|_\infty \le \sum_{k\in\mathbb{Z}^d,|k|\ge n}|a_k|$ we get that $\{s_n\}_{n>0}$ converges uniformly on $\mathbb{T}^d$ to some $g\in C(\mathbb{T}^d)$. Since $s_n\to f$ in $L^2(\mathbb{T}^d)$ there is a subsequence $\{s_{n_k}\}_{k>0}$ converging to f a.s.. therefore $g=f$. □

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