Optimal rotations of deformable bodies and orbits in magnetic fields

J.E. Avron, O. Gat, O. Kenneth and U. Sivan
Department of Physics, Technion, Haifa 32000, Israel *
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Deformations can induce rotation with zero angular momentum where dissipation is a natural "cost function". This gives rise to an optimization problem of finding the most effective rotation with zero angular momentum. For certain plastic and viscous media in two dimensions the optimal path is the orbit of a charged particle on a surface of constant negative curvature with magnetic field whose total flux is half a quantum unit.

Rotations with zero angular momentum are intriguing. The most celebrated phenomenon of this kind is the rotation of a falling cat. A mechanical model [1] replacing the cat by two rigid bodies that can rotate relative to each other, has been extensively studied, see [2, 3] and references therein. Here we address rotations with zero angular momentum under linear deformations. Our motivation comes from nano-mechanics: Imagine an elastic or plastic material with its own energy source, and ask what is the most efficient way of turning it through an appropriate sequence of autonomous deformations without external torque.

Deformations can generate rotations because order matters: A cycle of deformations will, in general, result in a rotation. The ratio between the rotation and the (infinitesimal) area of the controls can be interpreted as curvature [4, 5]. Consequently, small cycles are ineffective since a cycle of length ε in the controls leads to a rotation of order ε^2. The search for optimal paths forces one to mind deformations that are not small.

The problem we address has three parts. The first part is to determine the rotation for a given path of the controls. We solve this problem for general linear deformations. In two dimensions this leads to curvature on the space of the controls which is exponentially localized. If one thinks of the curvature as a magnetic field, then the total magnetic flux is that of half a unit of quantum flux. The second part is to set up a model for the cost function. We choose the cost function to be a measure of dissipation and focus on two settings, one where the dissipation is rate independent, as is the case in certain plastics, and the other where it is rate dependent as in liquids. Both cost functions lead to the same metric on the space of deformations. The third part is to pose and solve the problem of finding the path of minimal dissipation for a given rotation. In two dimensions and for either model of dissipation, the problem maps to finding the shortest path that starts at a given point and encloses a given amount magnetic flux. Optimal paths tend to linger near the circle in the space of controls where the ratio of eigenvalues of the quadrupole moment of the body is $(\phi + \sqrt{\phi^2 - 1})^2$. $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

Deformations generate rotations *because* angular momentum is conserved [5]. Consider a collection of point masses m_α at positions x_α. Internal forces may deform the body, but there are no external forces. Suppose that the center of mass of the body is at rest at the origin and that the body has zero angular momentum. The total angular momentum, L_{ij}, must then stay zero for all times.

A linear deformation is represented by a matrix M that sends $x \to Mx$. The i, j component of the angular momentum is

$$L_{ij} = \mathrm{Tr}\,(\dot{M} Q M^t\, \ell_{ij}), \tag{1}$$

where Q is the quadrupole moment of the body

$$Q_{ij} = \sum_\alpha m_\alpha (x_\alpha)_i (x_\alpha)_j \tag{2}$$

and $\ell_{ij},\ i < j$ are the $\binom{n}{2}$ generators of rotations in n dimensions, *i.e.* $(x, \ell_{ij} y) = x_i y_j - x_j y_i$. Since ℓ_{ij} span the anti-symmetric matrices, the set of $\binom{n}{2}$ equations $L_{ij} = 0$ imply that the matrix $(dM)QM^t$ is symmetric.

Two immediate consequences of this symmetry are:

- Isotropic bodies: With $Q = 1$, and M close to the identity, the symmetry of $(dM)QM^t$ reduces to dM being symmetric: the linear transformation must be a strain [6].

- Pointers: Pointers are bodies with large aspect ratios, such as needles and discs. In the limit of infinite aspect ratio, Q may be identified with a projection where $\dim Q$ is the dimension of the pointer. With M near the identity, the symmetry of $(dM)Q$ implies that $(1 - Q)(dM)Q = (1 - Q)Q(dM)^t = 0$. Since Q does not acquire a component in the normal direction, $1 - Q$, under dM a pointer keeps its orientation.

We now derive the fundamental relation between the response (rotation) and the controls (deformations). To this end we use the polar decomposition $M = RS$ with R a rotation and S a positive matrix. Assuming S positive is a choice of a gauge which makes the representation unique with $S=\sqrt{M^tM}$. The symmetry of $(dM)QM^t$ gives

$$\{\mathcal{A},SQS\}=SQ(dS)-(dS)QS,\quad \mathcal{A}=R^{-1}\text{đ}R. \tag{3}$$

Eq. (3) determines the differential rotation, $\mathcal{A}$, in terms of the variation of the controls dS. The symbol đ stresses that the differential will not, in general, integrate to a function on the space of deformations. Geometrically, the differential rotation is the connection 1-form which fixes a notion of parallel transport.

Eq. (3) can be interpreted in terms of a variational principle: The motion is such that the kinetic energy is minimal for a given deformation. To see this, let $M=1$ and $\dot M=\dot R+\dot S$ with $\dot R$ antisymmetric (*i.e.* a rotation) and $\dot S$ symmetric (*i.e.* a strain). The kinetic energy is

$$E=\frac{1}{2}\mathrm{Tr}\left(\dot MQ\dot M^t\right)=\frac{1}{2}\,\mathrm{Tr}\left((\dot R+\dot S)Q(-\dot R+\dot S)\right). \tag{4}$$

Minimizing with respect to $\dot R$ gives

$$0=\delta E=\frac{1}{2}\mathrm{Tr}\left(\delta\dot R\big(-\{Q,\dot R\}+[Q,\dot S]\right). \tag{5}$$

The trace is of a product of antisymmetric matrices, and its vanishing for an arbitrary antisymmetric $\delta\dot R$ implies $\{\dot R,Q\}=[Q,\dot S]$ which is Eq. (3) for $M=1$.

One readily sees that if $\mathcal{A}$ is the solution of Eq. (3) given S and Q, then it is also a solution for λS and Q for λ a scalar valued function. Hence scaling does not drive rotations and we may restrict ourselves to volume (or area) preserving deformations with $\det S=1$.

Since any Q is obtainable by a linear deformation of the identity, we may assume without loss of generality that $Q=1$. Eq. (3) reduces to

$$\{\mathcal{A},S^2\}=[S,dS]. \tag{6}$$

Eq. (6) is conveniently solved in a basis where S is diagonal. Let s_j denote the eigenvalues of S then

$$\mathcal{A}_{ij}=\frac{s_i-s_j}{s_i^2+s_j^2}\,(dS)_{ij}. \tag{7}$$

Normally, the curvature F is more interesting than the connection. It is defined by $F=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$. Calculation gives

$$F_{ij}=2\sum_k\frac{s_i\,s_j\,(s_i+s_k)(s_j+s_k)}{(s_i^2+s_j^2)(s_j^2+s_k^2)(s_i^2+s_k^2)}\,(dS)_{ik}\wedge(dS)_{kj}. \tag{8}$$

The situation is particularly simple in two dimensions. We use the Pauli matrices $\sigma_x,\sigma_z,i\sigma_y$ and the identity as basis for the real 2×2 matrices. Positive unimodular matrices correspond to points on the hyperboloid $t^2-x^2-z^2=1,\ t>0$ shown in Fig. 1 where

$$S^2(t,x,z)=t+x\sigma_x+z\sigma_z. \tag{9}$$

Natural coordinates are (τ,θ)

$$(t,x,z)=(\cosh\tau,\sinh\tau\,\cos\theta,\sinh\tau\,\sin\theta), \tag{10}$$

with $S=\cosh(\tau/2)+\sinh(\tau/2)(\sigma_x\cos\theta+\sigma_z\sin\theta)$. Let $R^t\text{đ}R=-i\,\sigma_y\,\text{đ}\varphi$. Eqs. (7) and (8) give

$$\text{đ}\varphi=a(\tau)\sinh\tau\,d\theta,\quad a(\tau)=\frac{\cosh\tau-1}{\sinh2\tau}, \tag{11}$$

and

$$F(\tau)=f(\tau)\sinh\tau\,d\tau\wedge d\theta,\quad f(\tau)=\frac{1}{2\cosh^2\tau}. \tag{12}$$

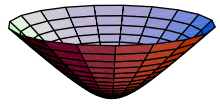

FIG. 1: Positive, area preserving linear transformations may be identified with the hyperboloid of revolution. The induced metric obtained by embedding the hyperboloid in Minkowski space gives it the structure of the Lobachevsky plane with constant negative curvature.

The total curvature is easily computed from Eq. (11) and one finds $\int F=\pi$. This, together with the positivity of f, implies that in any single closed cycle (*i.e.* one without self intersections), the angle of rotation is at most π. When interpreted as magnetic flux, π corresponds to half a unit of quantum flux [7].

The cost function must include some measure of dissipation. For, without dissipation energy is a function of the controls and no change in energy is associated with a closed loop. We consider two models of dissipation in isotropic media. Both lead to the same metric on the space of controls, namely

$$(d\ell)^2=\ \mathrm{Tr}\left(S^{-2}\,d(S^2)\otimes S^{-2}d(S^2)\right). \tag{13}$$

The metric is distinguished by symmetry: It is invariant under congruence, $g\to AgA^t$, for an arbitrary invertible matrix A.

Consider a medium with viscosity tensor η. The power due to dissipation is

$$P=\frac{1}{2}\sum\eta^{ijkl}\dot u_{ij}\dot u_{kl}, \tag{14}$$

where u is the strain tensor and $\dot u$ the strain rate. S is related to u by $2u=S^2-1$. To see this recall that the strain is defined as the change in the distance between two neighboring points caused by a deformation. If the deformation is described by a metric g then $g=1+2u$, where 1 is the metric associated with the undeformed reference system [6]. When considering *linear* deformation described by a symmetric matrix S, the resulting metric is $g = S^2$ (regarding the covariant components of g as the elements of a positive matrix). This establishes the claimed connection between u and S.

The space of (symmetric) 4-th rank isotropic tensors is two dimensional and spanned by the two viscosity coefficients η and η' [6]

$$\eta^{ijkl} = \eta g^{ik} g^{jl} + \eta' g^{ij} g^{kl} \ . \tag{15}$$

We therefore find that in an isotropic medium

$$P = 2\eta \, Tr\left(g^{-1}\dot{g}g^{-1}\dot{g}\right) + 2\eta' \, Tr\left(g^{-1}\dot{g}\right) Tr\left(g^{-1}\dot{g}\right) \tag{16}$$

For volume preserving transformations the term multiplying η' vanishes and one is left with the first term alone. By choosing the unit of time appropriately one can take $\eta = 1$. This leads to the metric of Eq. (13).

The dissipation in certain plastic materials can be rate independent. This is the continuum mechanics analog of the dissipation due to friction when one body slides on another [9]. In plastics, rate independence is a consequence of the Lévy-Mise constitutive relation: $s\,\delta\lambda = \delta u$, where s is the stress and $\delta\lambda$ a scalar valued function [8]. The constitutive relation is formally the same as for fluids [8], and by isotropy, the dissipation must be a function of $d\ell$ of Eq. (13). If the material is memoryless, dissipation is additive with respect to concatenating paths and must be proportional to $d\ell$.

In two dimensions, with the parametrization of Eq. (9), the metric Eq. (13) gives

$$(d\ell)^2 = (dx)^2 + (dz)^2 - (dt)^2 = (d\tau)^2 + \sinh^2\tau (d\theta)^2 \ . \tag{17}$$

This is the metric on the hyperboloid induced from Minkowski space. It gives the hyperboloid the geometry of the pseudo-sphere i.e. it makes it into a surface of constant negative curvature -1 [4].

The metric enables us to assign a scalar to the curvature 2-form F of Eq. (12) as the ratio between F and the area form of the pseudo-sphere, $\sinh\tau \, d\tau \wedge d\theta$. This ratio is $f(\tau)$ and is plotted in Fig. 2, along with $a(\tau)$ which serves as the θ component of a vector potential for f. The curvature is everywhere positive and it is concentrated near the origin, $\tau = 0$. It decays exponentially with τ. This means that large deformations are ineffective. We already know that small deformations are ineffective. This bring us to the optimization problem.

The control problem is to find a closed path γ in the space of deformations, starting at $S_0 = \sqrt{Q_0}$, (Q_0 is the initial quadrupole), which rotates the quadrupole by Φ radians, with minimal dissipation. If the dissipation is rate dependent, one adds a constraint that the time of traversal is 1.

For viscous media $\gamma = \big(\tau(t), \theta(t)\big)$ is then the solution of the variational problem

$$\delta \int_0^1 \left(\frac{\dot\tau^2 + \dot\theta^2 \sinh^2\tau}{2} - \lambda a(\tau)\sinh(\tau)\dot\theta \right) dt = 0 \tag{18}$$

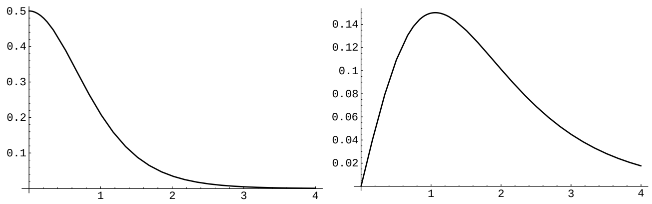

FIG. 2: The scalar curvature, $f(\tau)$, which can be interpreted as a magnetic field (left) and the θ component of the vector potential $a(\tau)$. f is exponentially localized near the origin of the control space (τ, θ), while a has a maximum on the circle $\cosh\tau = \phi$, ϕ the golden ratio. $a^2/2$ appears also as a potential in the effective 1-dimensional dynamics of the optimization problem.

where λ is a Lagrange multiplier and $\gamma(0) = \gamma(1) = S_0 = S(\tau_0, \theta_0)$. This is the (variation of the) action of a classical particle with charge λ and unit mass moving on the hyperbolic plane (i.e. the pseudo-sphere) in the presence of a magnetic field $f(\tau)$ given in Eq. (12).

Since motion in a magnetic field conserves kinetic energy the particle moves at constant speed and the dissipation, $\frac{1}{2}|\gamma|^2$, depends only on the length of the path. The variational problem can therefore be cast in purely geometric terms: Find the shortest closed path starting at a given point which encloses a given amount of magnetic flux. The shortest path is evidently also the solution in the case that the dissipation is rate independent.

Consider the family of isospectral deformations $\tau = const$ which keep the the eigenvalues of S (or Q) constant while rotating its eigenvectors. We call these "stirring", see Fig. 3. One can not stir an isotropic body, since its eigenvectors don't have well defined directions.

Among the stirring cycles there is an optimal one which maximizes the rotation per unit length. From Eqs. (11,17) the flux to length ratio for stirring cycles is $a(\tau)$ of Eq. (11). The function a takes its maximum at $\cosh\tau_s = \phi, \tau_s \approx 1.061$, see Fig. 2. Every cycle of the controls rotates by $\Phi_s = (2-\phi)\pi \approx .382\pi$ radians, somewhat less than a quarter turn.

To make full use of the optimal stirring cycle the initial conditions must be right. This is the case for a quadrupole with $Q = S(2\tau_s, \theta)$, θ arbitrary. With other initial conditions and for large angles of rotations, the optimal paths approach the optimal stirring cycle, linger near it, eventually returning to the initial point. This is shown in Fig. 4 for a half turn of an isotropic body.

Since the magnetic field $F(\tau)$ is rotationally invariant, the angular momentum

$$J = \sinh\tau \left(\dot\theta \sinh\tau - \lambda a(\tau)\right) \tag{19}$$

is conserved. Conservation of energy gives

$$0 \le \dot\tau^2 = 2E - \left(\frac{J}{\sinh\tau} + \lambda a(\tau)\right)^2 , \tag{20}$$

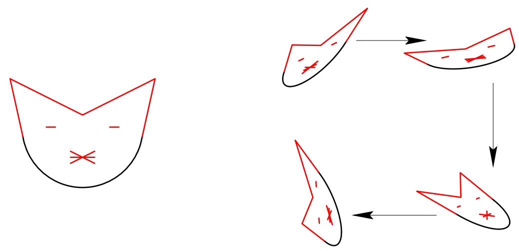

FIG. 3: A reference shape (left), assumed to have $Q = 1$, and four instances from the optimal stirring cycle (right). The configurations on the right are ordered clockwise with increasing θ, starting with the top left. Both the first and the last correspond to $\theta = 0$ and are therefore related by a pure rotation.

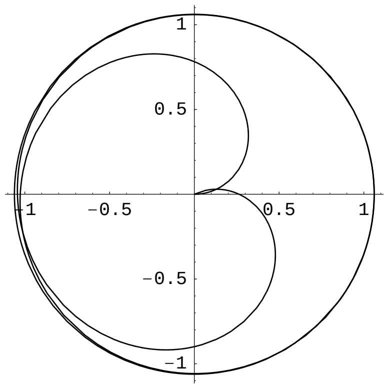

FIG. 4: The optimal path in the space of shapes for a rotation by π of a body with initial quadrupole moment $Q = 1$. Polar coordinates (τ, θ) are used to parametrize the plane. The path reaches exponentially close to the optimal stirring cycle at $\cosh\tau = \phi$, and winds around it twice before returning to the initial configuration. The optimal stirring cycle is not distinguishable from the envelope of the orbit in the scale of this figure.

which reduces the problem to quadrature. The three conditions, $\tau(1) = \tau(0)$, $\theta(1) \equiv \theta(0) \bmod 2\pi$, and the constraint of enclosed flux Φ, determine the three parameters E, J and λ.

The initial condition $\tau(0) = 0$ is special: By Eq. (19) the angular momentum J is forced to have the value 0. In turn, there is also one less condition to satisfy, as the value of θ when $\tau = 0$ is meaningless. It then follows from Eqs. (19) and (20) that the optimal orbit, $\tau(\theta)$, depends only on the ratio E/λ^2. Rescaling time properly, we can achieve $\lambda = 1$ by relaxing the constraint that the time to complete a cycle should be 1.

The key equation controlling the dynamics is Eq. (20), which describes effective one-dimensional motion in the potential $a^2(\tau)/2$. As shown above, $a(\tau)$ has a maximum at τ_s. Therefore, closed orbits which correspond to optimal paths have energy values $E < a(\tau_s)^2/2$. The motion is quite simple: Trajectories leave the origin with a positive $\dot{\tau}$, reach the turning point $\tau_t = a^{-1}(\sqrt{2E})$, and return symmetrically to the origin, thereby completing a cycle. The flux accumulated during a cycle is

$$\Phi = \int_0^{t_{\text{cycle}}} a(\tau)\sinh(\tau)\dot{\theta}dt \ , \tag{21}$$

where $\dot{\theta}$ is obtained from Eq. (19).

Increasingly longer orbits are obtained when E approaches the separatrix energy $a(\tau_s)^2/2$. The orbit corresponding to the unstable equilibrium point at $\tau = \tau_s$ is the optimal stirring cycle, which is a $J = 0$ orbit. The flux accumulated during a complete turn in parameter space, where θ increases by 2π, is hence bounded by Φ_s. Large values Φ require many turns during which the orbit approaches exponentially the optimal stirring cycle, see Fig. 4. A circular disc of playdough would therefore rotate by π, with no angular momentum, in about three cycles.

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* Electronic address: `avron@physics.technion.ac.il`

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[7] It is noteworthy that the total curvature of the hyperboloid is π and not an integer multiple of 2π as one would expect from Gauss Bonnet Chern theorem. The reason is the restriction to the hyperboloid with $t > 0$. If both signs of t are taken, the two hyperboloids can be glued at infinity to give a sphere with total curvature 2π.

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[9] We are indebted to Amos Ori for this observation.