On the Riesz Basisness of the Root Functions of the Nonselfadjoint Sturm-Liouville Operator

Neşe Dernek, O.A. Veliev
Depart. of Math, Fen-Ed. Fak, Marmara Univ.,
Göztepe Kampüsü, 81040, Kadiköy,
Istanbul, Turkey.
e-mail: ndernek@marmara.edu.tr
oveliev@marmara.edu.tr

Abstract

In this article we obtain the asymptotic formulas of arbitrary order for eigenfunctions and eigenvalues of the nonselfadjoint Sturm-Liouville operators with periodic and antiperiodic boundary conditions, when the potential is a summable function. Then using these asymptotic formulas, we find the conditions on Fourier coefficients of the potential for which the eigenfunctions and associated functions of these operators form a Riesz basis in $L_2(0,1)$.

Let $L_t(q)$ be the operator generated in $L_2[0,1]$ by the expression

$$-y^{''}+q(x)y, \tag{1}$$

and the boundary conditions

$$y(1)=e^{it}y(0),\ y^{'}(1)=e^{it}y^{'}(0), \tag{2}$$

where $t=0,\pi$ and $q(x)$ is a complex-valued summable function.

In this article we obtain the asymptotic formulas of order $O(n^{-l})$ (for all $l>0$) for n-th eigenvalue and corresponding eigenfunction of the operator $L_t(q)$ with $q(x)\in L_1(0,1)$, for $t=0,\pi$, that is, for periodic and antiperiodic boundary conditions. Then using these asymptotic formulae, we find the conditions on Fourier coefficients $q_n = (q(x), e^{i2\pi nx})$, where $(.,.)$ denotes inner product in $L_2(0,1)$, of $q(x)$ for which the root functions (the eigenfunctions and associated functions) of $L_t(q)$, for $t = 0, \pi$, form a Riesz basis in $L_2(0,1)$. Note that the periodic and antiperiodic boundary conditions are regular boundary conditions, but are not strongly regular boundary conditions. Therefore, in general, the eigenfunctions and associated functions of $L_0(q)$ and $L_\pi(q)$ do not form a Riesz basis, they form a basis with brachet (see [4],[5]). In this paper we prove that if

$$\lim_{n\to\infty} \frac{ln\,|n|}{nq_{2n}} = 0, \quad q_{2n} \sim q_{-2n}, \tag{3}$$

where $a_n \sim b_n$ means that $c_1 \mid b_n \mid < \mid a_n \mid < c_2 \mid b_n \mid$, for all $n = 1, 2, ...$,(here and in forthcoming relations we denote by c_m, for $m = 1, 2, ...$, the positive constants whose exact value are inessential) then the eigenfunctions and associated functions of $L_0(q)$ form a Riesz basis. Similarly if

$$\lim_{n\to\infty} \frac{ln\,|n|}{nq_{2n+1}} = 0, \quad q_{2n+1} \sim q_{-2n-1}, \tag{4}$$

then the eigenfunctions and associated functions of $L_\pi(q)$ form a Riesz basis.

It is well-known that (see formulas (47a), (47b)) in page 65 of [1]) the eigenvalues of the operator $L_0(q)$ and $L_\pi(q)$ consists of the sequences $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}$ and $\{\mu_{n,1}\}, \{\mu_{n,2}\}$ satisfying

$$\lambda_{n,j} = (2n\pi)^2 + O(n^{\frac{1}{2}}), \ \forall j = 1, 2 \tag{5}$$

and

$$\mu_{n,j} = (2n\pi + \pi)^2 + O(n^{\frac{1}{2}}), \ \forall j = 1, 2, \tag{6}$$

respectively, for $n \geq N$. Here and in forthcoming relations by N we denote a big positive integer, that is, $N \gg 1$. From these formulas one can easily obtain the following inequalities from which we use essentially:

$$\begin{aligned} \left|\lambda_{n,j} - (2\pi k)^2\right| &> |2(n-k)\pi|\,|2(n+k)\pi| - c_3 n^{\frac{1}{2}} > c_4 n, \\ \forall\ j &= 1, 2,\ \forall k \neq n,\ k = 0, 1, ..., \end{aligned} \tag{7}$$

$$\begin{aligned} \left|\mu_{n,j} - (2\pi k + \pi)^2\right| &> |2(n-k)\pi|\,|2(n+k+1)\pi| - c_5 n^{\frac{1}{2}} > c_6 n \\ \forall\ j &= 1, 2,\ \forall k \neq n, , \forall k = 0, 1, ... \end{aligned} \tag{8}$$

for $n \geq N$. To obtain the asymptotic formula for eigenvalues $\lambda_{n,j}$ and corresponding normalized eigenfunctions $\Psi_{n,j}(x)$ of $L_0(q)$ we use (7) and the following well-known relation

$$(\lambda_{N,j} - (2\pi n)^2)(\Psi_{N,j}(x), e^{i2\pi nx}) = (q(x)\Psi_{N,j}(x), e^{i2\pi nx}). \tag{9}$$

Moreover, we use the following relations, which can be easily proved arguing as in the proof of Lemma 1 of [6],

$$(q(x)\Psi_{N,j}(x), e^{i2\pi nx}) = \sum_{m=-\infty}^{\infty} q_{-m}(\Psi_{N,j}(x), e^{i2\pi(n+m)x}) \tag{10}$$

$$\left|(q(x)\Psi_{N,j}(x), e^{i2\pi nx})\right| < 4M, \ \forall n, \ \forall N \gg 1 \tag{11}$$

where $M = \sup\limits_{n\in Z} |q_n|$. Note that $q_n \to 0$ as $|n| \to \infty$ and, without loss of generality, we assume that $q_0 = 0$.

Using (10) in (9) we get

$$(\lambda_{n,j} - (2\pi n)^2)(\Psi_{n,j}(x), e^{i2\pi nx}) = \sum_{\substack{n_1=-\infty, \\ n_1\neq 0}}^{\infty} q_{n_1}(\Psi_{n,j}(x), e^{i2\pi(n-n_1)x}). \tag{12}$$

Now isolating the term in right-hand side of (12) containing the multiplicant $(\Psi_{n,j}(x), e^{-i2\pi nx})$ (i.e., case $n_1 = 2n$), replacing

$(\Psi_{n,j}(x), e^{i2\pi(n-n_1)x})$ by $\frac{(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1)x}))}{\lambda_{n,j}-(2\pi(n-n_1))^2}$, for $n_1 \neq 2n$

(this replacement can be obtained from (9) by taking n and $n-n_1$ instead of N and n) and using (10) for the numerator of the last fraction we obtain

$$(\lambda_{n,j} - (2\pi n)^2)(\Psi_{n,j}(x), e^{i2\pi nx}) - q_{2n}(\Psi_{n,j}(x), e^{-i2\pi nx}) =$$
$$\sum_{\substack{n_1,n_2=-\infty \\ n_1\neq 2n}}^{\infty} \frac{q_{n_1}q_{n_2}(\Psi_{n,j}(x), e^{i2\pi(n-n_1-n_2)x})}{\lambda_{n,j} - (2\pi(n-n_1))^2}. \tag{13}$$

Note that, since $q_0 = 0$, here and in the forthcoming relations the sums are taken under conditions $n_1, n_2, ..., \neq 0$. Now we isolate the terms in right-hand side of (13) containing one of the multiplicants $(\Psi_{n,j}(x), e^{i2\pi nx})$,

$(\Psi_{n,j}(x), e^{-i2\pi nx})$ (i.e., cases $n_1+n_2=0,2n$) and apply the above replacement to other terms (i.e., cases $n_1+n_2\neq 0,2n$) to get

$$(\lambda_{n,j}-(2\pi n)^2)(\Psi_{n,j}(x), e^{i2\pi nx}) - q_{2n}(\Psi_{n,j}(x), e^{-i2\pi nx}) -$$

$$\sum_{\substack{n_1=-\infty \\ n_1\neq 2n}}^{\infty} \frac{q_{n_1}q_{-n_1}(\Psi_{n,j}(x), e^{i2\pi nx})}{\lambda_{n,j}-(2\pi(n-n_1))^2} - \sum_{\substack{n_1=-\infty \\ n_1\neq 2n}}^{\infty} \frac{q_{n_1}q_{2n-n_1}(\Psi_{n,j}(x), e^{-i2\pi nx})}{\lambda_{n,j}-(2\pi(n-n_1))^2} =$$

$$\sum_{\substack{n_1,n_2=-\infty \\ n_1\neq 2n, n_1+n_2\neq 0,2n}}^{\infty} \frac{q_{n_1}q_{n_2}(q(x)\Psi_{n,j}(x), e^{i(2\pi(n-n_1-n_2)x})}{\lambda_{n,j}-(2\pi(n-n_1))^2(\lambda_{n,j}-(2\pi(n-n_1-n_2))^2)} \tag{14}$$

Repeating this process $m-$times (i.e. applying (10) to the numerator of the fractions in the rigth-hand side of (14) and isolating the terms containing one of the multiplicants $(\Psi_{n,j}(x), e^{i2\pi nx})$, $(\Psi_{n,j}(x), e^{-i2\pi nx})$ etc.) we get

$$(\lambda_{n,j}-(2\pi n)^2 - A_m(\lambda_{n,j}))(\Psi_{n,j}(x), e^{i2\pi nx}) - \tag{15}$$

$$(q_{2n}+B_m(\lambda_{n,j})(\Psi_{n,j}(x), e^{-i2\pi nx}) = R_m,$$

where $A_m(\lambda_{n,j}) = \sum_{k=1}^m a_k(\lambda_{n,j}),\ B_m(\lambda_{n,j}) = \sum_{k=1}^m b_k(\lambda_{n,j})$

$$a_k(\lambda_{n,j}) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}...q_{n_k}q_{-n_1-n_2-\dots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]...[\lambda_{n,j}-(2\pi(n-n_1-...-n_k))^2]},$$

$$b_k(\lambda_{n,j}) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}...q_{n_k}q_{2n-n_1-n_2-\dots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]...[\lambda_{n,j}-(2\pi(n-n_1-..-n_k))^2]},$$

$$R_m = \sum_{n_1,n_2,\dots,n_{m+1}} \frac{q_{n_1}q_{n_2}...q_{n_m}q_{n_{m+1}}(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1-\dots-n_{m+1})x})}{[\lambda_{n,j}-(2\pi(n-n_1))^2]...[\lambda_{n,j}-(2\pi(n-n_1-...-n_{m+1}))^2]},$$

Here the sums are taken under the conditions

$$n_j\neq 0, \forall j\ ;\ \sum_{j=1}^k n_j \neq 0, 2n,\ \forall k=1,2,...,m+1.$$

Using (7) and arguing as in the proof of equalities (27), (28) of [6] we get the following relations

$$a_k = O\left((\frac{ln\,|n|}{n})^k \right), b_k = O\left((\frac{ln\,|n|}{n})^k \right), R_m = O\left((\frac{ln\,|n|}{n})^{m+1} \right). \tag{16}$$

In the same way one can obtain the relations

$$(\lambda_{n,j}-(2\pi n)^2-A_m^{'}(\lambda_{n,j}))(\Psi_{n,j}(x),e^{-i2\pi nx})- \tag{17}$$

$$(q_{-2n}+B_m^{'}(\lambda_{n,j}))(\Psi_{n,j}(x),e^{i2\pi nx})=R_m^{'},$$

where $A_m^{'}(\lambda_{n,j})=\sum_{k=1}^m a_k^{\prime}(\lambda_{n,j})$, $B_m^{'}(\lambda_{n,j})=\sum_{k=1}^m b_k^{\prime}(\lambda_{n,j})$,

$$a_k^{'}(\lambda_{n,j})=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-n_1-n_2-\ldots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]...[\lambda_{n,j}-(2\pi(n+n_1+...+n_k))^2]},$$

$$b_k^{'}(\lambda_{n,j})=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-2n-n_1-n_2-\ldots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]...[\lambda_{n,j}-(2\pi(n+n_1+...+n_k))^2]},$$

$$a_k^{'}=O\left((\frac{ln\,|n|}{n})^k\right),b_k^{'}=O\left((\frac{ln\,|n|}{n})^k\right),R_m^{'}=O\left((\frac{ln\,|n|}{n})^{m+1}\right). \tag{18}$$

Here the sums are taken under the conditions

$$n_j\neq 0,\forall j\ ;\ \sum_{j=1}^k n_j\neq 0,-2n,\forall k=1,2,...,m+1.$$

It follows from (7), (9) and (11) that

$$\sum_{k\in Z,k\neq n,-n}\left|(\Psi_{n,j}(x),e^{i2\pi kx})\right|^2=\sum_{k\in Z,k\neq n,-n}\frac{(4M)^2}{[\lambda_{n,j}-(2\pi k)^2]^2}=O(\frac{1}{n^2}).$$

Therefore $\Psi_{n,t}(x)$ has an expansion of the form

$$\Psi_{n,j}(x)=u_{n,j}e^{i2\pi nx}+v_{n,j}e^{-i2\pi nx}+h(x), \tag{19}$$

where $u_{n,j}=(\Psi_{n,j}(x),e^{i2\pi nx})$, $v_{n,j}=(\Psi_{n,j}(x),e^{-i2\pi nx})$, $\|h(x)\|=O(\frac{1}{n})$,

$$|u_{n,j}|^2+|v_{n,j}|^2=1+O(\frac{1}{n^2}) \tag{20}$$

Using (16), (18), (20) from the system of equations (15), (17) we can find asymptotic formulas of order $O(n^{-l})$ (for all $l>0$) for eigenfunctions $\Psi_{n,j}(x)$ and eigenvalues $\lambda_{n,j}$ of the operator $L_0(q)$ with $q(x)\in L_1(0,1)$. We do it under conditions (3), since in these cases the asymptotic formulas give the Riesz basisness.

Theorem 1 *Let the conditions (3) hold. Then:*

(a) The big eigenvalues of the operator L_0 are simple. They consists of two sequences $\{\lambda_{n,1} : n > N\}$ and $\{\lambda_{n,2} : n > N\}$ satisfying

$$\lambda_{n,j} = (2\pi n)^2 + (-1)^j p_n + O\left(\frac{ln\,|n|}{n}\right), \forall j = 1, 2, \tag{21}$$

where $p_n = (q_{2n}q_{-2n})^{\frac{1}{2}}$.

The corresponding eigenfunction $\varphi_{n,j}(x)$ satisfies

$$\varphi_{n,j}(x) = e^{i2\pi nx} + \alpha_{n,j}e^{-i2\pi nx} + O(\frac{1}{n}), \forall j = 1, 2, \tag{22}$$

where

$$\alpha_{n,j} \sim 1, \forall j = 1, 2, \tag{23}$$

$$\alpha_{n,j} = \frac{(-1)^j p_n}{q_{2n}} + O\left(\frac{ln\,|n|}{nq_{2n}}\right), \forall j = 1, 2, \tag{24}$$

(b) The root functions of $L_0(q)$ form a Riesz basis in $L_2(0, 1)$.

Proof. First we prove that if the conditions (3) hold then

$$u_{n,j} \sim v_{n,j} \sim 1, \forall j = 1, 2 \tag{25}$$

(for the definitions of $u_{n,j}$ and $v_{n,j}$ see (19)) and

$$\lambda_{n,j} - (2\pi n)^2 = a_j p_n + O\left(\frac{ln\,|n|}{n}\right). \tag{26}$$

where a_j is either 1 or -1. From (15)- (18) we obtain the system of equations

$$(\lambda_{n,j} - (2\pi n)^2)u_{n,j} = q_{2n}v_{n,j} + O\left(\frac{ln\,|n|}{n}\right), \tag{27}$$

$$(\lambda_{n,j} - (2\pi n)^2)v_{n,j} = q_{-2n}u_{n,j} + O\left(\frac{ln\,|n|}{n}\right). \tag{28}$$

Without loss of generality we assume that $|v_{n,j}| > \frac{1}{2}$ (see (20)) and hence $v_{n,j} \sim 1$. Then from (28) and (3) we obtain $|\lambda_{n,j} - (2\pi n)^2| < c_7\,|q_{-2n}|$. Moreover the inequalities $|u_{n,j}| < 2$ (see (20)), $|v_{n,j}| > \frac{1}{2}$, and (27) imply that $|\lambda_{n,j} - (2\pi n)^2| > c_8\,|q_{2n}|$. Therefore using (3) from these inequalities we get

$$\lambda_{n,j} - (2\pi n)^2 \sim q_{2n} \sim q_{-2n} \sim p_n, \lim_{n\to\infty} \frac{ln\,|n|}{np_n} = 0. \tag{29}$$

This with (27) implies that $u_{n,j} \sim v_{n,j}$. Thus (25) is proved. Now dividing both sides of (27) and (28) by $u_{n,j}\ q_{2n}$ and $(\lambda_{n,j} - (2\pi n)^2)u_{n,j}$ respectively we get

$$\frac{v_{n,j}}{u_{n,j}} = \frac{\lambda_{n,j} - (2\pi n)^2}{q_{2n}} + O\left(\frac{ln\,|n|}{nq_{2n}}\right) = \frac{q_{-2n}}{\lambda_{n,j} - (2\pi n)^2} + O\left(\frac{ln\,|n|}{nq_{2n}}\right), \quad (30)$$

from which using (29) by simple calculation we get (26).

The proof of (a). Suppose $\lambda_{n,j}$, where $n > N, N \gg 1$, is multiple. There are two cases:

1. There is associated functions $\Psi^1_{n,j}(x)$ corresponding to the eigenfunctions $\Psi_{n,j}(x)$, that is,

$$(L_0 - \lambda_{n,j})\Psi^1_{n,j}(x) = \Psi_{n,j}(x). \quad (31)$$

Since the boundary condition (2) for $t = 0$ is selfadjoint $\overline{\lambda_{n,j}}$ and $\overline{\Psi_{n,j}}$ are eigenvalue and eigenfunction of adjoint operator L_0^*. Therefore multiplying both sides of (31) by $\overline{\Psi_{n,j}}$ we get $(\Psi_{n,j}(x), \overline{\Psi_{n,j}}) = 0$. This with (19) give the equality $u_{n,j}v_{n,j} = O(\frac{1}{n})$ which contradicts (25).

2. There are two eigenfunctions corresponding $\lambda_{n,j}$. Then all solutions of the equation

$$-y'' + q(x)y = \lambda_{n,j}y$$

are eigenfunctions. In particular, the solution

$$y(x, \lambda_{n,j}) = e^{\rho_{n,j}x} + O(\frac{1}{n}) = e^{2\pi nx} + O(\frac{1}{n}),$$

where $\rho_{n,j} = (\lambda_{n,j})^{\frac{1}{2}} = 2\pi n + (\frac{1}{n})$, (see page 52 of [1]), and (26)) is eigenfunction. In (27) instead of using $\Psi_{n,j}$ taking this eigenfunction we get

$$\lambda_{n,j} = (2\pi n)^2 + O\left(\frac{ln\,|n|}{n}\right),$$

which contradicts (29). Thus the eigenvalues $\lambda_{n,j}$ for $n > N, j = 1, 2$ are simple and satisfy (26). Now, in order to obtain (21) from (26) it remains to prove that the numbers a_1, a_2 in (26) are different, say $a_1 = -1, a_2 = 1$. Suppose both a_1 and a_2 are -1. Then (26), (27) yield

$$-p_n u_{n,1} = q_{2n}v_{n,1} + O\left(\frac{ln\,|n|}{n}\right), -p_n u_{n,2} = q_{2n}v_{n,2} + O\left(\frac{ln\,|n|}{n}\right), \quad (32)$$

Multiplying both sides of first and second equalities by $-v_{n,2}$ and $v_{n,1}$ respectively and then summing the obtained equalities we get

$$u_{n,1}v_{n,2}-u_{n,2}v_{n,1}=O\left(\frac{ln\,|n|}{np_n}\right). \tag{33}$$

On the other hand, since $\overline{\lambda_{n,j}}$ and $\overline{\Psi_{n,j}}$ are eigenvalue and eigenfunction of adjoint operator L_0^* and $\lambda_{n,1}\neq\lambda_{n,2}$ we have

$$0=(\Psi_{n,1},\overline{\Psi_{n,2}})=u_{n,1}v_{n,2}+u_{n,2}v_{n,1}+O(\frac{1}{n}) \tag{34}$$

(see (19)). It follows from (33), (34), (29) that $u_{n,1}v_{n,2}=O\left(\frac{ln|n|}{np_n}\right)=o(1)$ which contradicts (25). If we suppose that both a_1 and a_2 are 1 then, in the same way, we obtain the same contradiction. Thus one of these numbers, say a_1 is -1 and the other a_2 is 1. The formulas (21) are proved. Since $\lambda_{n,1}$ satisfies (21), for $j=1$, first equality in (32) holds. Therefore

$$\frac{v_{n,1}}{u_{n,1}}=\frac{-p_n}{q_{2n}}+O\left(\frac{ln\,|n|}{nq_{2n}}\right) \tag{35}$$

Denoting $\alpha_{n,j}=\frac{v_{n,j}}{u_{n,j}}$, $\varphi_{n,j}(x)=\frac{\Psi_{n,j}(x)}{u_{n,j}}$ from (19), (25) and (35) we get (22), (23), (24) for $j=1$. Taking into account that $\lambda_{n,2}$ satisfies (21) for $j=2$ in the same way we obtain (22), (23), (24) for $j=2$.

The proof of (b). Now we prove that the eigenfunctions $\varphi_{n,j}\equiv\varphi_{n,j}^0$and associated functions $\varphi_{n,j}^k$, where $n=1,2,...,$ and $k=1,2,,,,s(n,j)$, of $L_0(q)$ form a Riesz basis in $L_2(0,1)$. For every $f(x)\in L_2(0,1)$, the asymptotic formulas (22), (23) yield

$$\sum_{n=1}^{N}(\sum_{k=0}^{s(n,j)}\sum_{j=1}^{2}\left|(f,\varphi_{n,j}^k)\right|^2)+\sum_{j=1}^{2}\sum_{n=N+1}^{\infty}\left|(f,\varphi_{n,j})\right|^2<\infty. \tag{36}$$

Let $\chi_{n,j}\equiv\chi_{n,j}^0$ and $\chi_{n,j}^k$, where $k=1,2,,,,t(n,j)$, be the biorthonormal system of eigenfunctions and associated functions of L_0^*. Clearly $\chi_{n,j}(x)=\frac{\overline{\varphi_{n,j}(x)}}{(\varphi_{n,j},\overline{\varphi_{n,j}})}$ for $n>N$. Therefore using (22), (23) we get

$$\sum_{n=1}^{N}(\sum_{k=0}^{t(n,j)}\sum_{j=1}^{2}\left|(f,\chi_{n,j}^k)\right|^2)+\sum_{j=1}^{2}\sum_{n=N+1}^{\infty}\left|(f,\chi_{n,j})\right|^2<\infty. \tag{37}$$

Since $\{\varphi_{n,j}^k : k = 0, 1, 2, , , , s(n, j); j = 1, 2; n = 1, 2, ...\}$, and the biorthonormal system is total (see chap.1, sec.3 of [2]) by the well-known theorem of Bari (see [3], chap. 6) for the Riesz basis, (36) and (37) imply that the system of the eigenfunctions and associated functions of $L_0(q)$ forms a Riesz basis in $L_2(0, 1)$ .The theorem is proved■

Instead of (3), (7), and (9) using (4), (8) and

$$(\mu_{N,j} - (2\pi n + \pi)^2)(\Phi_{N,j}(x), e^{i\pi(2n+1)x}) = (q(x)\Phi_{N,j}(x), e^{i\pi(2n+1)x}), \quad (38)$$

where $\Phi_{N,j}(x)$ is normalized eigenfunction of $L_\pi(q)$, and arguing as in proof of Theorem 1 we obtain

Theorem 2 *Let the conditions (4) hold. Then:*

(a) *The big eigenvalues of the operator L_π are simple. They consist of two sequences $\{\mu_{n,1} : n > N\}$ and $\{\mu_{n,2} : n > N\}$ satisfying*

$$\mu_{n,j} = (2\pi n + \pi)^2 + (-1)^j p_n^{'} + O\left(\frac{ln\,|n|}{n}\right), \ \forall j = 1, 2, \quad (39)$$

where $p_n^{'} = (q_{2n+1}q_{-2n-1})^{\frac{1}{2}}$. The corresponding eigenfunction $\phi_{n,j}(x)$ satisfies

$$\phi_{n,j}(x) = e^{i(2\pi n+\pi)x} + \alpha_{n,j}^{'} e^{-i(2\pi n+\pi)x} + O(\frac{1}{n}), \ \forall j = 1, 2, \quad (40)$$

where $\alpha_{n,j}^{'} \sim 1,\ \alpha_{n,j}^{'} = \frac{(-1)^j p_n^{'}}{q_{2n+1}} + O\left(\frac{ln|n|}{nq_{2n+1}}\right), \forall\ j = 1, 2.$

(b) *The root functions of $L_\pi(q)$ form a Riesz basis in $L_2(0, 1)$.*

Now we prove asymptotic formulas of arbitrary order for eigenfunctions and eigenvalues of the operators $L_0(q)$, $L_\pi(q)$ with $q(x) \in L_1(0, 1)$.

Theorem 3 *For the eigenvalue $\lambda_{n,j}$ satisfying (21), the following formulas hold*

$$\begin{aligned} \lambda_{n,j} &= (2\pi n)^2 + F_{m,j} + (-1)^j G_{m,j} + O\left((\frac{ln\,|n|}{n})^{m+1} \right), \quad (41) \\ \forall m &= 0, 1, 2, ..., \forall j = 1, 2, \end{aligned}$$

where $F_{0,j} = 0, G_{0,j} = p_n,\ F_{k,j} = \frac{1}{2}(A_k((2\pi n)^2 + F_{k-1,j} + (-1)^j G_{k-1,j})+$

$A_k'((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j})),$
$G_{k,j}=[\frac{1}{4}(A_k((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j})-A_k'((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j}))^2+(B_k((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j})+q_{2n})(B_k'((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j})+q_{-2n})]^{\frac{1}{2}}$, $\forall k=1,2,...,$

For the eigenfunction $\varphi_{n,j}(x)$ satisfying (23) the following formulas hold

$$\begin{aligned}\varphi_{n,j}(x) &= e^{i2\pi nx}+\alpha_{n,j}e^{-i2\pi nx}+A_m^*((2\pi n)^2+F_{m,j}+(-1)^jG_{m,j}) \\ &+ (B_m^*((2\pi n)^2+F_{m,j}+(-1)^jG_{m,j}))\alpha_{n,j}+O((\frac{\ln n}{n})^{m+1}),\end{aligned}\tag{42}$$

where A_m^* , B_m^* are obtained from A_m, B_m respectively by replacing q_{n_1} with $e^{i2\pi(n-n_1)x}$ and

$$\begin{aligned}\alpha_{n,j} &= \frac{v_{n,j}}{u_{n,j}}=\frac{F_{m,j}+(-1)^jG_{m,j}-A_m((2\pi n)^2+F_{m,j}+(-1)^jG_{m,j})}{q_{2n}+B_m((2\pi n)^2+F_{m,j}+(-1)^jG_{m,j})}+ \\ & O\left(\frac{1}{q_{2n}}(\frac{ln\,|n|}{n})^{m+1}\right).\end{aligned}$$

For the eigenvalue $\mu_{n,j}$ and eigenfunction $\phi_{n,j}(x)$ of $L_\pi(q)$ satisfying (39), (40) similar formulas hold.

Proof. Arguing as in the proof of (30), using (15), (17) in place of (27), (28), and denoting $\lambda=\lambda_{n,j}-(2\pi n)^2$ we get

$$\begin{aligned}\frac{v_{n,j}}{u_{n,j}} &= \frac{\lambda-A_m(\lambda_{n,j})}{(q_{2n}+B_m(\lambda_{n,j}))}+O\left(\frac{1}{q_{2n}}(\frac{ln\,|n|}{n})^{m+1}\right) \\ &= \frac{(q_{-2n}+B_m'(\lambda_{n,j}))}{\lambda-A_m'(\lambda_{n,j})}+O\left(\frac{1}{q_{2n}}(\frac{ln\,|n|}{n})^{m+1}\right),\end{aligned}\tag{43}$$

Solving this square equation with respect to λ we obtain

$$\lambda_{n,j}-(2\pi n)^2=\frac{1}{2}(A_m(\lambda_{n,j})+A_m'(\lambda_{n,j}))\mp[\frac{1}{4}(A_m(\lambda_{n,j})-A_m'(\lambda_{n,j}))^2+ \\ (B_m(\lambda_{n,j})+q_{2n})(B_m'(\lambda_{n,j})+q_{-2n}))+O\left(q_{2n}(\frac{ln\,|n|}{n})^{m+1}\right)]^{\frac{1}{2}}.\tag{44}$$

Here upper and lower sign is taken for $j=1$, and $j=2$ respectively, since the expression in the square bracket is

$$q_{2n}q_{-2n}+O(q_{2n}\frac{ln\,|n|}{n})=q_{2n}q_{-2n}(1+o(1))$$

(see (3), (16), (18)), and (21) holds. Thus in (44) the $\mp$ can be replaced by $(-1)^j$. Now we prove (41) by induction. It is proved for $m=1$ (see (21)). Assume that it is true for $m=k-1$. Substituting the value of $\lambda_{n,j}$, given by this formula for $m=k-1$, in (44) for $m=k$ and taken into account that

$$\begin{aligned} & A_k((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j}+O\left((\frac{ln\,|n|}{n})^k\right)) \\ = \; & A_k((2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j})+O\left((\frac{ln\,|n|}{n})^{k+1}\right), \qquad (45) \end{aligned}$$

and the same formulas hold for $A_k^{'}$, B_k, $B_k^{'}$ we get the proof of (41) for $m=k$. Note that the validity of (45) can be easily verified by using the obvious relation

$$\sum_{\substack{n_1=-\infty,\\ n_1\neq 0,2n}}^{\infty} \left| \frac{1}{(2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j}+O\left((\frac{ln|n|}{n})^k\right)-(2\pi(n-n_1)^2}- \right.$$

$$\left. \frac{1}{(2\pi n)^2+F_{k-1,j}+(-1)^jG_{k-1,j}-(2\pi(n-n_1)^2}\right| = O\left((\frac{ln\,|n|}{n})^{k+1}\right).$$

Now we prove (42). Writing the decomposition of $\Psi_{n,j}(x)$ by $\{e^{i2\pi(n-n_1)x}:n_1\in Z\}$ we obtain

$$\Psi_{n,j}(x)-u_{n,j},e^{i2\pi nx}=\sum_{\substack{n_1=-\infty,\\ n_1\neq 0}}^{\infty}(\Psi_{n,j}(x),e^{i2\pi(n-n_1)x})e^{i2\pi(n-n_1)x}. \qquad (46)$$

The right side of this formula can be obtained from the right side of (12) by replacing q_{n_1} with $e^{i2\pi(n-n_1)x}$. Therefore doing the iteration which was dane in order to obtain (15) from (12) (namely, isolate the terms in right-hand side of (46) containing the multiplicant $(\Psi_{n,j}(x),e^{-i2\pi nx})$ (i.e., case $n_1=2n$) and apply (9), (10) to other terms (i.e., cases $n_1\neq 2n$), then isolate the terms containing one of the multiplicants $(\Psi_{n,j}(x),e^{-i2\pi nx})$, $(\Psi_{n,j}(x),e^{-i2\pi nx})$ and apply (9), (10) to other terms , ets.) and the estimation as in (16) we get

$$\begin{aligned} \Psi_{n,j}(x) \;=\; & u_{n,j}e^{i2\pi nx}+v_{n,j}e^{-i2\pi nx}+ \qquad (47) \\ & u_{n,j}A_m^*(\lambda_{n,j})+v_{n,j}B_m^*(\lambda_{n,j})+O\left((\frac{ln\,|n|}{n})^{m+1}\right) \end{aligned}$$

Dividing both sides of (47) by $u_{n,j}$ and using (43), (41), (45) we get (42).

Instead of (3), (7), (9) and Theorem 1 using (4), (8), (38) and Theorem 2 in the same way we get the similar asymptotic formulas for eigenvalues and eigenfunctions of $L_{\pi}(q)$■

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