Local exponents and infinitesimal generators of canonical transformations on Boson Fock spaces

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Abstract

A one-parameter symplectic group $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$ derives proper canonical transformations on a Boson Fock space. It has been known that the unitary operator U_t implementing such a proper canonical transformation gives a projective unitary representation of $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$ and that U_t can be expressed as a normal-ordered form. We rigorously derive the self-adjoint operator $\Delta(\widehat{A})$ and a phase factor $e^{i\int_0^t \tau_{\widehat{A}}(s)ds}$ with a real-valued function $\tau_{\widehat{A}}$ such that $U_t = e^{i\int_0^t \tau_{\widehat{A}}(s)ds} e^{it\Delta(\widehat{A})}$.

Key words: Canonical transformations(Bogoliubov transformations), symplectic groups, projective unitary representations, one-parameter unitary groups, infinitesimal self-adjoint generators, local factors, local exponents, normal-ordered quadratic expressions.

1 Introduction

This paper is motivated by a close resemblance between infinitesimal generators of rotation groups in *white noise analysis* and those of *proper canonical transformations* (Bogoliubov transformations) on a Boson Fock space $\mathcal{F}$. In white noise analysis rotation groups $\{g_\theta\}_{\theta\in\mathbb{R}}$ acting on (S') have been studied so far by many authors, e.g., see Hida [7]. Here (S') is the dual of a subspace (S) of $\mathcal{F}$. Such rotation groups are induced from e.g., shifts, dilations, $SO(n)$, special conformal transformations and the Lévy group, etc. Their infinitesimal generators are defined by

$$\lim_{\theta\to 0}\frac{g_\theta - 1}{\theta} \quad in\ (S')$$

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and it is established that generators consist of infinite dimensional Laplacians, e.g., the Gross Laplacian, the Lévy Laplacian, the Beltrami Laplacian, etc. These Laplacians are expressed in the form of polynomials of second degree of the annihilation operators and the creation operators in (S'). See e.g. Obata [14].

Meanwhile, it is known that a proper canonical transformation

$$U_{\mathrm{B}}(A) : \mathcal{F} \to \mathcal{F}, \quad A \in \Sigma_2,$$

gives a *projective* unitary representation of a symplectic group Σ_2 ([20]). In this paper from $U_{\mathrm{B}}(\cdot)$ a unitary representation $\widehat{U}_t := e^{-i\theta(t)}U_{\mathrm{B}}(e^{t\widehat{A}})$ of a one-parameter symplectic group $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}} \subset \Sigma_2$ is constructed with some real-valued function $\theta(t)$. Classically this is known as the Bargmann theorem [5]. The unitary representation $\widehat{U}_t$ induces a one-parameter unitary group $\{\widehat{U}_t\}_{t\in\mathbb{R}}$ on $\mathcal{F}$. Regarding $\mathcal{F}$ as $\mathcal{F} \subset (S')$, we recognize that the self-adjoint infinitesimal generator $\Delta(\widehat{A})$ of $\widehat{U}_t$ may correspond to a Laplacian in $\mathcal{F}$. The purpose of this paper is to give $\Delta(\widehat{A})$ and $\theta(t)$ *rigorously*.

$$\begin{array}{ccccc} \mathcal{F} & & \subset & & (S') \\ \Big\downarrow{\scriptstyle \widehat{U}_t} & & & & \Big\downarrow{\scriptstyle g_\theta} \\ \mathcal{F} & & \subset & & (S') \end{array}$$

Figure 1: One-parameter unitary group $\widehat{U}_t$ and rotation group g_θ

Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$ endowed with an antiunitary involution Γ with $\Gamma^2 = 1$. Set

$$\overline{X} = \Gamma X \Gamma$$

for an operator acting on $\mathcal{H}$. Let Σ be a symplectic group, i.e., $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$ if and only if bounded operators S, T on a Hilbert space $\mathcal{H}$ satisfy

$$A^* J A = A J A^* = J,$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The linear canonical transformation on $\mathcal{F}$ over $\mathcal{H}$ associated with $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$ is given as the map

$$\begin{aligned} a(f) &\longmapsto b_A(f) := a(Sf) + a^*(Tf) \\ a^*(f) &\longmapsto b_A^*(f) := a(\overline{T}f) + a^*(\overline{S}f), \end{aligned}$$

where $a^*(f)$ and $a(f)$, $f \in \mathcal{H}$, are the creation operator and the annihilation operator on $\mathcal{F}$ smeared by f, respectively. Let us define subgroup Σ_2 by

$$\Sigma_2 = \left\{ A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma \middle| \, T \text{ a Hilbert-Schmidt operator} \right\}.$$

It had been established that the linear canonical transformation associated with $A \in \Sigma$ can be implemented by a unitary operator $U_{\rm B}(A)$ on $\mathcal{F}$ if and only if $A \in \Sigma_2$, i.e.,

$$A \in \Sigma_2 \Leftrightarrow U_{\rm B}(A) a^\sharp(f) U_{\rm B}(A)^{-1} = b_A^\sharp(f), \tag{1.1}$$

where $a^\sharp = a, a^*$ and $b_A^\sharp = b_A, b_A^*$. The correspondence between a, a^* and b_A, b_A^* arising as in (1.1) is called a proper canonical transformation. Without effort one can show that if $\dim\mathcal{H} < \infty$, all linear canonical transformations are proper. It is known that $U_{\rm B}(A)$ is equal to

$$U(A) := \det(1 - K_1^* K_1)^{1/4} e^{-\frac{1}{2}\Delta_{K_1}^\dagger} :e^{-N_{K_2}}: e^{-\frac{1}{2}\Delta_{K_3}} \tag{1.2}$$

up to a phase factor. Here $:X:$ denotes the normal ordering of X,

$$K_1 = TS^{-1}, \quad K_2 = 1 - \overline{(S^{-1})^*}, \quad K_3 = -S^{-1}\overline{T},$$

and the operators $\Delta_X^\dagger$, Δ_X, N_X are quadratic operators of a^*a^* type, a^*a type and aa type respectively, which are constructed by operator X. $U(A)$ is called the "normal-ordered quadratic expression" of $U_{\rm B}(A)$.

$$U(\cdot) : \Sigma_2 \longrightarrow \text{unitary operators on } \mathcal{F}$$

induces a projective unitary representation of Σ_2, i.e.,

$$U(A)U(B) = \omega(A,B)U(AB), \quad A, B \in \Sigma_2,$$

with local factor $\omega(A,B) \in \{e^{i\psi} | \psi \in \mathbb{R}\}$. We are interested in one-parameter symplectic groups in Σ_2. Let

$$\sigma_2 := \left\{ \widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \Big| \widehat{A}J + J\widehat{A} = \widehat{A}^* J + J\widehat{A}^* = 0, T \text{ a Hilbert-Schmidt operator} \right\}.$$

A one-parameter symplectic group $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$ belongs to Σ_2 if and only if $\widehat{A} \in \sigma_2$. When $\widehat{A} \in \sigma_2$,

$$e^{t\widehat{A}} = \begin{pmatrix} (e^{t\widehat{A}})_{11} & (e^{t\widehat{A}})_{12} \\ (e^{t\widehat{A}})_{21} & (e^{t\widehat{A}})_{22} \end{pmatrix}$$

induces the proper canonical transformation

$$\begin{aligned} a(f) &\longmapsto b_t(f) := a((e^{t\widehat{A}})_{11} f) + a^*((e^{t\widehat{A}})_{21} f), \\ a^*(f) &\longmapsto b_t^*(f) := a((e^{t\widehat{A}})_{12} f) + a^*((e^{t\widehat{A}})_{22} f). \end{aligned}$$

Set

$$\begin{aligned} U_t &:= U(e^{t\widehat{A}}), \quad t \in \mathbb{R}, \\ e^{i\rho(t,s)} &:= \omega(e^{t\widehat{A}}, e^{s\widehat{A}}), \quad t, s \in \mathbb{R}, \end{aligned}$$

where $\rho(t,s)$ is called a local exponent. Then U_t satisfies

$$U_t U_s = e^{i\rho(t,s)} U_{t+s}, \quad t, s \in \mathbb{R}.$$

In this paper we shall show that

$$\begin{aligned} \widehat{U}_t &:= e^{-i\int_0^t \tau_{\widehat{A}}(s)ds} U_t, \\ \tau_{\widehat{A}}(s) &:= \frac{1}{2} \mathrm{Im}\mathrm{Tr}(T^*(e^{s\widehat{A}})_{21}(e^{s\widehat{A}})_{11}^{-1}), \end{aligned} \tag{1.3}$$

satisfies

$$\widehat{U}_t \widehat{U}_s = \widehat{U}_{t+s}.$$

Namely local exponent $\rho(t,s)$ is equivalent to zero, i.e., $\widehat{U}_t$ furnishes a unitary representation of $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$. Moreover we derive the infinitesimal generator of U_t, i.e., we show that

$$\widehat{U}_t = e^{it\Delta(\widehat{A})}, \quad t \in \mathbb{R}, \tag{1.4}$$

where

$$\Delta(\widehat{A}) := \frac{i}{2}(\Delta_T^\dagger - \Delta_{\overline{T}}) - iN_{\overline{S}},$$

and we prove that $\Delta(\widehat{A})$ is essentially self-adjoint on a certain domain. (1.4) also implies that $\widehat{U}_t$ gives the normal-ordered quadratic expression of $e^{it\widehat{A}}$.

In Berezin [6, Chapter 3] these kinds of argument exist, however it is not rigorous at a few places. Quadratic operators such as $\Delta(A)$ has been studied in e.g., Araki [1], Araki-Shiraishi [3] and Araki-Yamagami [4]. Langmann [12] calculated the local exponent in (1.3) in the different way as ours. Fermionic cases for our discussion are established by e.g., Lundberg [13], Carey and Ruijsenaars [8] and Araki [2].

Next issue will be to study infinitesimal generators in the case where T is *not* a Hilbert-Schmidt operator, i.e.,

$$A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \notin \Sigma_2. \tag{1.5}$$

We, however, do not consider this problem here. Actually in the case where T is not a Hilbert-Schmidt operator, we can not define $\Delta_T^\dagger$ as an operator acting in $\mathcal{F}$, and as was mentioned above the linear canonical transformation is not implemented by a unitary operator. So, in the case of (1.5) we may have to shift our argument to white noise analysis. See [9, 10, 11] to this direction.

We organize this paper as follows. In Section 2 we review fundamental facts on the Fock space, quadratic operators. In Section 3 we introduce one-parameter symplectic groups and main theorems. In Section 4 we show the weak differentiability of $U_t\Omega$ in t. In Section 5 we give proofs of the main theorems. In Section 6 we give some examples.

2 Fundamental facts

2.1 Boson Fock spaces

Let $\mathcal{F} = \mathcal{F}(\mathcal{H})$ denote the Boson Fock space over $\mathcal{H}$ defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)},$$

where $\mathcal{F}^{(n)} = \mathcal{H}^{\otimes_s^n}$ is the n-fold symmetric tensor product of $\mathcal{H}$ with $\mathcal{H}^{\otimes_s^0} := \mathbb{C}$. Vector Ψ of $\mathcal{F}$ is written as $\Psi = \left\{\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, \cdots\right\}$ with $\Psi^{(n)} \in \mathcal{F}^{(n)}$. The vacuum $\Omega \in \mathcal{F}$ is defined by

$$\Omega := \{1, 0, 0, \cdots\}.$$

The creation operator $a^*(f) : \mathcal{F} \to \mathcal{F}$ smeared by $f \in \mathcal{H}$ is given by

$$\left(a^*(f)\Psi\right)^{(n)} := S_n(f \otimes \Psi^{(n-1)}),$$

where S_n denotes the symmetrizer of n-degree. Let

$$\mathcal{F}_0 := \text{the linear hull of } \{a^*(f_1)\cdots a^*(f_n)\Omega | f_j \in \mathcal{H}, j = 1, ..., n, n \geq 0\}.$$

It is known that $\mathcal{F}_0$ is dense in $\mathcal{F}$. Simply for $f \in \mathcal{H}$, we write as $\overline{f}$ for Γf. The annihilation operator $a(f)$ is defined by

$$a(f) := \left(a^*(\overline{f})\Big\lceil_{\mathcal{F}_0}\right)^*.$$

Since $a^\sharp(f)\lceil_{\mathcal{F}_0}$ is closable, we denote its closed extension by the same symbol $a^\sharp(f)$. It holds that

$$(\Psi, a^*(f)\Phi)_{\mathcal{F}} = (a(\overline{f})\Psi, \Phi)_{\mathcal{F}}, \quad \Psi, \Phi \in \mathcal{F}_0.$$

where $(f, g)_{\mathcal{K}}$ denotes the scalar product on Hilbert space $\mathcal{K}$, which is linear in g and antilinear in f. In addition, we denote by $\|f\|_{\mathcal{K}}$ the associated norm. If no confusions arise, we omit $\mathcal{K}$ of $\|\cdot\|_{\mathcal{K}}$ and $(\cdot,\cdot)_{\mathcal{K}}$. The creation operator and the annihilation operator satisfy canonical commutation relations:

$$[a(f), a^*(g)] = (\overline{f}, g)_{\mathcal{H}},$$
$$[a(f), a(g)] = 0,$$
$$[a^*(f), a^*(g)] = 0$$

on $\mathcal{F}_0$. The field operator is defined by

$$\phi(f) := \frac{1}{\sqrt{2}}(a^*(\overline{f}) + a(f)).$$

The following proposition is known.

Proposition 2.1 *(1) Suppose that a bounded operator K commutes with $e^{i\phi(f)}$ for all $f \in \mathcal{H}$. Then K is a multiple of the identity.*

(2) Let $\mathcal{G}$ be a closed subspace of $\mathcal{F}$ such that $e^{i\phi(f)}\mathcal{G} \subset \mathcal{G}$ for all $f \in \mathcal{H}$. Then $\mathcal{G} = \mathcal{F}$.

2.2 Symplectic groups

Let $B = B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$ and $B_2 = B_2(\mathcal{H})$ Hilbert-Schmidt operators. We denote the norm (resp. Hilbert-Schmidt norm) of a bounded operator X on $\mathcal{H}$ by $\|X\|$ (resp. $\|X\|_2$). For $S, T \in \mathrm{B}$ we define

$$A := \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

by

$$A(\phi \oplus \psi) := (S\phi + \overline{T}\psi) \oplus (T\phi + \overline{S}\psi).$$

Let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}.$$

We define the symplectic group Σ and a subgroup Σ_2 of Σ as follows.

Definition 2.2 *(1)*

$$\Sigma := \left\{ A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} \middle| \, AJA^* = A^*JA = J \right\},$$

where $A^* = \begin{pmatrix} S^* & T^* \\ \overline{T^*} & \overline{S^*} \end{pmatrix}$.

(2)

$$\Sigma_2 := \left\{ A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma \middle| \, T \in B_2 \right\}.$$

Note that $\overline{(K^*)} = (\overline{K})^*$ and that the inverse of $A \in \Sigma$ is given by

$$A^{-1} = JA^*J = \begin{pmatrix} S^* & -T^* \\ -\overline{T^*} & \overline{S^*} \end{pmatrix}. \tag{2.1}$$

2.3 Quadratic operators

The number operator N is defined by

$$D(N) := \left\{ \{\Psi^{(n)}\} \in \mathcal{F} \Big| \sum_{n=0}^{\infty} n^2 \|\Psi^{(n)}\|^2_{\mathcal{F}^{(n)}} < \infty \right\},$$
$$(N\Psi)^{(n)} := n\Psi^{(n)}.$$

Now we introduce fundamental facts.

Proposition 2.3 *(1) Let $f_1, ..., f_m \in \mathcal{H}$. Then there exists a constant $c_m(f_1, ..., f_m)$ such that for $\Psi \in D(N^{m/2})$,*

$$\|a^*(f_1) \cdots a^*(f_m)\Psi\| \leq c_m(f_1, ..., f_m)\|(N+1)^{m/2}\Psi\|.$$

Moreover suppose that

$$f_j^{(n)} \to f_j, \quad j = 1, ..., m,$$

strongly in $\mathcal{H}$ as $n \to \infty$. Then for $\Psi \in D(N^{m/2})$,

$$s - \lim_{n\to\infty} a^\sharp(f_1^{(n)}) ... a^\sharp(f_m^{(n)})\Psi = a^\sharp(f_1) ... a^\sharp(f_m)\Psi. \tag{2.2}$$

(2) Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system. Then

$$(\Psi, N\Phi) = \sum_{n=1}^{\infty} (a(e_n)\Psi, a(e_n)\Phi). \tag{2.3}$$

Proof: See e.g., [15, Section X.7]. □

Let $K \in B_2$. Then there exist two orthonormal systems $\{\psi_n\}$, $\{\phi_n\}$ in $\mathcal{H}$, and a positive sequence $\lambda_1 \geq \lambda_2 \geq ... > 0$ such that

$$Kf = \sum_{n=0}^{\infty} \lambda_n (\psi_n, f)\phi_n, \quad f \in \mathcal{H},$$

with $\sum_{n=0}^{\infty} \lambda_n^2 = \|K\|_2^2$.

Lemma 2.4 *Let $K \in B_2$ and $S \in B$. Then for $\Psi \in \mathcal{F}_0$,*

$$(1)\|\sum_{n=1}^{M} \lambda_n a^*(\overline{\psi}_n)a^*(\phi_n)\Psi\| \leq \sqrt{6}\|K\|_2\|(N+1)\Psi\| \tag{2.4}$$

$$(2)\|\sum_{n=1}^{M} \lambda_n a(\overline{\psi}_n)a(\phi_n)\Psi\| \leq \|K\|_2\|N\Psi\|, \tag{2.5}$$

$$(3)\|\sum_{n=1}^{M} a^*(e_n)a(\overline{S^*e_n})\Psi\| \leq \|S\|\|N\Psi\|, \tag{2.6}$$

where $\{e_n\}$ is a complete orthonormal system of $\mathcal{H}$.

Proof: We have

$$\begin{aligned}\|\sum_{n=1}^{M} \lambda_n a^*(\overline{\psi}_n)a^*(\phi_n)\Psi\|^2 &= \sum_{n,m}^{M} \lambda_n\lambda_m(a^*(\overline{\psi}_m)a^*(\phi_m)\Psi, a^*(\overline{\psi}_n)a^*(\phi_n)\Psi) \\ &= A+B+C+D+E,\end{aligned}$$

where

$$\begin{aligned}A &= \sum_{n}^{M} \lambda_n^2(\Psi, a(\overline{\psi}_n)a^*(\psi_n)\Psi), \\ B &= \sum_{n}^{M} \lambda_n^2(\Psi, a^*(\overline{\phi}_n)a(\phi_n)\Psi), \\ C &= \sum_{n}^{M} \lambda_n(\Psi, a(\overline{K}\psi_n)a^*(\overline{\phi}_n)\Psi), \\ D &= \sum_{n}^{M} \lambda_n(\Psi, a^*(\psi_n)a(K^*\overline{\phi}_n)\Psi), \\ E &= \sum_{n}^{M} \lambda_n(a(\overline{\psi}_n)a(\phi_n)\Psi, a(\overline{\psi}_m)a(\phi_m)\Psi).\end{aligned}$$

We have

$$|A| \leq \sum_{n}^{M} \lambda_n^2\|a^*(\psi_n)\Psi\|^2 \leq \sum_{n}^{M} \lambda_n^2\|(N+1)^{1/2}\Psi\|^2 \leq \|K\|_2^2\|(N+1)^{1/2}\Psi\|^2,$$

and

$$|B| \leq \sum_{n}^{M} \lambda_n^2\|a(\phi_n)\Psi\|^2 \leq \sum_{n}^{M} \lambda_n^2\|N^{1/2}\Psi\|^2 \leq \|K\|_2^2\|N^{1/2}\Psi\|^2.$$

We have

$$C = \sum_{n}^{M} \lambda_n(K\overline{\psi}_n, \overline{\phi}_n)\|\Psi\|^2 + \sum_{n}^{M} \lambda_n(\Psi, a^*(\overline{\phi}_n)a(\overline{K}\psi_n)\Psi)$$

We estimate the right-hand side above such as

$$\begin{aligned}\left|\sum_n^M \lambda_n (K\overline{\psi}_n, \overline{\phi}_n)\|\Psi\|^2\right| &\leq \sum_n^M \lambda_n \|K\overline{\psi}_n\|\|\Psi\|^2 \\ &\leq \left(\sum_n^M \lambda_n^2\right)^{1/2} \left(\sum_n^M \|K\overline{\psi}_n\|^2\right)^{1/2} \|\Psi\|^2 \\ &\leq \|K\|_2^2 \|\Psi\|^2,\end{aligned}$$

and

$$\begin{aligned}|\sum_n^M \lambda_n (\Psi, a^*(\overline{\phi}_n) a(\overline{K}\psi_n)\Psi)| &\leq \left(\sum_n^M \lambda_n^2\right)^{1/2} \left(\sum_n^M \|a(\phi_n)\Psi\|^2 \|K\|^2 \|N^{1/2}\Psi\|^2\right)^{1/2} \\ &\leq \|K\|_2 \|K\| \|N^{1/2}\Psi\|^2.\end{aligned}$$

Hence

$$|C| \leq \|K\|_2^2 \|\Psi\|^2 + \|K\|_2 \|K\| \|N^{1/2}\Psi\|^2.$$

We have

$$\begin{aligned}|D| &\leq \left(\sum_n^M \lambda_n^2\right)^{1/2} \left(\sum_n^M \|a(\overline{\psi}_n)\Psi\|^2 \|K\|^2 \|N^{1/2}\Psi\|^2\right)^{1/2} \\ &\leq \|K\|_2 \|K\| \|N^{1/2}\Psi\|^2.\end{aligned}$$

Finally we see that

$$\begin{aligned}|E| &= \|\sum_{n=1}^M \lambda_n a(\overline{\phi}_n) a(\psi_n)\Psi\|^2 \\ &\leq \sum_{n=1}^M \lambda_n^2 \sum_{n=1}^M \|a(\overline{\phi}_n) a(\psi_n)\Psi\|^2 \\ &\leq \|K\|_2^2 \sum_{n=1}^M \|N^{1/2} a(\psi_n)\Psi\|^2 \\ &= \|K\|_2^2 \sum_{n=1}^M (a(\psi_n)\Psi, N a(\psi_n)\Psi) \\ &= \|K\|_2^2 \sum_{n=1}^M (a(\psi_n)\Psi, a(\psi_n) N\Psi) - (a(\psi_n)\Psi, a(\psi_n)\Psi) \\ &\leq \|K\|_2^2 (\Psi, N^2\Psi) \\ &= \|K\|_2^2 \|N\Psi\|^2.\end{aligned}$$

Combining estimates from A to E above, we obtain that

$$\begin{aligned}
&\|\sum_{n=1}^{M}\lambda_n a^*(\overline{\psi}_n)a^*(\phi_n)\Psi\|^2\\
&\le \|K\|_2^2\left(\|(N+1)^{1/2}\Psi\|^2+3\|N^{1/2}\Psi\|^2+\|\Psi\|^2+\|N\Psi\|^2\right)\\
&\le 6\|K\|_2^2\|(N+1)\Psi\|^2.
\end{aligned}$$

Thus (1) follows. From the estimate of E in (1) we have

$$\|\sum_{n=1}^{M}\lambda_n a(\overline{\phi}_n)a(\psi_n)\Psi\|^2 \;\le\; \|K\|_2^2\|N\Psi\|^2.$$

Thus (2) follows. Finally we estimate (3). It is proven on $\mathcal{F}_0$ that

$$s-\lim_{M\to\infty}\sum_{n=1}^{M}a^*(e_n)a(\Gamma S^*e_n)\Psi=\bigoplus_{n=1}^{\infty}\left[\sum_{j=1}^{n}1\otimes\cdots\otimes\overset{j}{S}\otimes\cdots\otimes 1\right]\Psi^{(n)}.$$

Since

$$\|\bigoplus_{n=1}^{\infty}\left[\sum_{j=1}^{n}1\otimes\cdots\otimes\overset{j}{S}\otimes\cdots\otimes 1\right]\Psi^{(n)}\|\le\|S\|\|N\Psi\|,$$

we obtain (3). □

From Lemma 2.4 we can define for $\Psi\in\mathcal{F}_0$

$$\Delta_K^\dagger\Psi:=s-\lim_{M\to\infty}\sum_{n=1}^{M}\lambda_n a^*(\overline{\psi}_n)a^*(\phi_n)\Psi, \tag{2.7}$$

$$\Delta_K\Psi:=s-\lim_{M\to\infty}\sum_{n=1}^{M}\lambda_n a(\overline{\psi}_n)a(\phi_n)\Psi, \tag{2.8}$$

$$N_S\Psi:=s-\lim_{M\to\infty}\sum_{n=1}^{M}a^*(e_n)a(\overline{S^*e_n})\Psi. \tag{2.9}$$

Let $\Psi=a^*(f_1)\cdots a^*(f_n)\Omega$. Then it is seen that

$$\Delta_K\Psi=\sum_{i\neq j}^{n}(\bar{f}_i,(K+\overline{K^*})f_j)a^*(f_1)\cdots\widehat{a^*}(f_i)\cdots\widehat{a^*}(f_j)\cdots a^*(f_n)\Omega, \tag{2.10}$$

$$N_S\Psi=\sum_{j=1}^{n}a^*(f_1)\cdots a^*(Sf_j)\cdots a^*(f_n)\Omega. \tag{2.11}$$

The following lemma is inherited from Lemma 2.4.

Lemma 2.5 *Let $K \in B_2$ and $S \in B$. Then for $\Psi \in \mathcal{F}_0$,*

$$\|\Delta_K^\dagger \Psi\| \leq \sqrt{6}\|K\|_2\|(N+1)\Psi\|, \tag{2.12}$$

$$\|\Delta_K \Psi\| \leq \|K\|_2\|N\Psi\|, \tag{2.13}$$

$$\|N_S\Psi\| \leq \|S\|\|N\Psi\|. \tag{2.14}$$

From this lemma it is shown that the domains of $\Delta_K^\dagger$, Δ_K and N_S can be extended to $D(N)$. We directly see the following lemma.

Lemma 2.6 *(1) We have $(\Delta_K)^* = \Delta_{K^*}^\dagger$ and $(N_S)^* = N_{S^*}$ on $D(N)$.*

(2)

$$[\Delta_K^\dagger, a(f)] = -a^*((K + \overline{K^*})f), \tag{2.15}$$

$$[\Delta_K, a^*(f)] = a((K + \overline{K^*})f), \tag{2.16}$$

$$[N_S, a(f)] = -a(\overline{S^*}f), \tag{2.17}$$

$$[N_S, a^*(f)] = a^*(Sf). \tag{2.18}$$

(3) Let $K, L \in B_2$ and $S, T \in B$. Then on $D(N)$,

$$\Delta_{K+L}^\dagger = \Delta_K^\dagger + \Delta_L^\dagger,$$
$$\Delta_{K+L} = \Delta_K + \Delta_L,$$
$$N_{S+T} = N_S + N_T.$$

From above commutation relations it follows that

$$\|\Delta_K^\dagger \Omega\|^2 = \mathrm{Tr}[K^*(K + \overline{K^*})]. \tag{2.19}$$

We set

$$\mathcal{D}_\infty := \bigcap_{k=1}^{\infty} D(N^k).$$

Lemma 2.7 *(1) Suppose that*

$$(i)\ K \in B_2, \quad (ii)\ \overline{K^*} = K, \quad (iii)\ \|K\| < 1.$$

Then for $\Psi \in \mathcal{F}_0$,

$$e^{-\frac{1}{2}\Delta_K^\dagger}\Psi := s - \lim_{M\to\infty} \sum_{n=0}^{M} \frac{1}{n!} \left(-\frac{1}{2}\Delta_K^\dagger\right)^n \Psi$$

exists and $e^{-\frac{1}{2}\Delta_K^\dagger}\Psi \in \mathcal{D}_\infty$.

(2) Suppose that $S \in \mathrm{B}$ *and* $K \in B_2$. *Then for* $\Psi \in \mathcal{F}_0$,

$$:e^{-N_S}:\Psi := s - \lim_{M\to\infty} \sum_{n=0}^{M} \frac{1}{n!} :\left(-\frac{1}{2}N_S\right)^n: \Psi \tag{2.20}$$

$$e^{-\frac{1}{2}\Delta_K}\Psi := s - \lim_{M\to\infty} \sum_{n=0}^{M} \frac{1}{n!} \left(-\frac{1}{2}\Delta_K\right)^n \Psi \tag{2.21}$$

exist, and $:e^{-N_S}:\Psi \in \mathcal{F}_0$ *and* $e^{-\frac{1}{2}\Delta_K}\Psi \in \mathcal{F}_0$.

Proof: We shall prove (1). It is enough to show the lemma for $\Psi = a^*(f_1)\cdots a^*(f_N)\Omega$. Let

$$a_n = \left(\frac{1}{2^n}\frac{1}{n!}\right)^2 \|(\Delta_K^\dagger)^n\Omega\|^2.$$

It is proven in [18] that $f(z) = \sum_{m=0}^{\infty} a_m z^m$ exists for $|z| < 1/\|K\|^2$ and

$$f(z) = \det(1 - zK^*K)^{-1/2}.$$

In particular $f(z)$ is analytic for $|z| < 1/\|K\|^2$. We have

$$\begin{aligned} \|\sum_{m=0}^{\infty} \frac{1}{m!}(-\frac{1}{2}\Delta_K^\dagger)^m\Psi\|^2 \;&\le\; \|f_1\|^2\cdots\|f_N\|^2 \\ &\quad\times \sum_{m=0}^{\infty}(N-1+2m)(N-2+2m)\cdots 2m a_m < \infty. \end{aligned}$$

Thus (1) is proven. (2) follows from the fact that $:N_S^n:\Psi = 0$ and $\Delta_K^n\Psi = 0$ for a sufficiently large n. □

From (2.15)-(2.18) the following commutation relations hold on $\mathcal{F}_0$:

$$[e^{-\frac{1}{2}\Delta_K^\dagger}, a(f)] = \frac{1}{2}a^*((K+\overline{K^*})f)e^{-\frac{1}{2}\Delta_K^\dagger}, \tag{2.22}$$

$$[e^{-\frac{1}{2}\Delta_K}, a^*(f)] = -\frac{1}{2}a((K+\overline{K^*})f)e^{-\frac{1}{2}\Delta_K}, \tag{2.23}$$

$$[:e^{-N_S}:, a(f)] = \frac{1}{2}a(\overline{S^*}f)\;:e^{-N_S}:, \tag{2.24}$$

$$[:e^{-N_S}:, a^*(f)] = -\frac{1}{2}a^*(Sf)\;:e^{-N_S}:\,. \tag{2.25}$$

2.4 Proper canonical transformations

Let $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}$. Then A induces the following maps:

$$a(f) \longmapsto b_A(f) := a(Sf) + a^*(Tf), \tag{2.26}$$

$$a^*(f) \longmapsto b_A^*(f) := a(\overline{T}f) + a^*(\overline{S}f). \tag{2.27}$$

Formally we may write

$$(b_A(f) \;\; b_A^*(f)) = (a(f) \;\; a^*(f)) \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}.$$

Then canonical commutation relations

$$[b_A(f), b_A^*(g)] = (\overline{f}, g)_{\mathcal{H}}, \tag{2.28}$$

$$[b_A(f), b_A(g)] = 0, \tag{2.29}$$

$$[b_A^*(f), b_A^*(g)] = 0, \tag{2.30}$$

hold on $\mathcal{F}_0$ and

$$(\Psi, b_A^*(f)\Phi)_{\mathcal{F}} = (b_A(\overline{f})\Psi, \Phi)_{\mathcal{F}}, \quad \Psi, \Phi \in \mathcal{F}_0,$$

follows. Since $b_A^\sharp(f)\lceil_{\mathcal{F}_0}$ is closable, we denote its closed extension by the same symbol $b_A^\sharp(f)$. The following proposition is well known.

Proposition 2.8 *There exists a unitary operator $U_{\rm B}(A)$ on $\mathcal{F}$ such that*

$$U_{\rm B}(A) : D(a^\sharp(f)) \to D(b_A^\sharp(f))$$

with

$$U_{\rm B}(A) a^\sharp(f) U_{\rm B}(A)^{-1} = b_A^\sharp(f) \tag{2.31}$$

if and only if $A \in \Sigma_2$.

Proof: See [6, 18]. □

If $U'_{\rm B}(A)$ also implements such as (2.31), then it holds that

$$U_{\rm B}(A)^{-1} U'_{\rm B}(A) a^\sharp(f) = a^\sharp(f) U_{\rm B}(A)^{-1} U'_{\rm B}(A).$$

Hence

$$U_{\rm B}(A)^{-1}U'_{\rm B}(A)e^{i\phi(f)} = e^{i\phi(f)}U_{\rm B}(A)^{-1}U'_{\rm B}(A),$$

which implies that

$$U'_{\rm B}(A) = \omega U_{\rm B}(A)$$

with some $\omega \in \{e^{i\psi}|\psi \in \mathbb{R}\}$ by Proposition 2.1.

Now we construct a unitary operator $U_{\rm B}(A)$ concretely. The condition $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$ is equivalent with the following algebraic relations:

$$S^*S - T^*T = 1, \tag{2.32}$$

$$\overline{S^*}T - \overline{T^*}S = 0, \tag{2.33}$$

$$SS^* - \overline{TT^*} = 1, \tag{2.34}$$

$$TS^* - \overline{ST^*} = 0. \tag{2.35}$$

Lemma 2.9 *Let* $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$. *Then (1)* $S^{-1} \in B$, *(2)* $\|TS^{-1}\| < 1$, *(3)* $\overline{(TS^{-1})^*} = TS^{-1}$, *(4)* $\overline{(S^{-1}\overline{T})^*} = S^{-1}\overline{T}$.

Proof: From (2.32) it follows that

$$S^*S = 1 + T^*T \geq 1. \tag{2.36}$$

Thus $\|S\|^2 \geq 1$, and (1) follows. By (2.36) we have $TS^{-1} = T(1+T^*T)^{-1}S^*$, which implies that

$$(TS^{-1})(TS^{-1})^* = T(1+T^*T)^{-1}S^*S(1+T^*T)^{-1}T^* = T(1+T^*T)^{-1}T^*.$$

Thus

$$\begin{aligned} \|TS^{-1}\|^2 &= \|T(1+T^*T)^{-1}T^*\| \\ &= \|T(1+T^*T)^{-1/2}(1+T^*T)^{-1/2}T^*\| \\ &= \|(1+T^*T)^{-1/2}T^*T(1+T^*T)^{-1/2}\| < 1. \end{aligned}$$

Thus (2) follows. By (2.33) we have $\overline{S^*}TS^{-1} = \overline{T^*}$. Then $S^*\overline{TS^{-1}} = T^*$ follows. Note that $(S^*)^{-1} = (S^{-1})^*$. It is obtained that

$$TS^{-1} = \overline{(S^*)^{-1}T^*} = \overline{(S^{-1})^*T^*} = \overline{(TS^{-1})^*}.$$

Hence (3) follows. Similarly (4) is obtained from (2.35). □

Let $A = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_2$. We set

$$K_1 := TS^{-1}, \quad K_2 := 1 - \overline{(S^{-1})^*}, \quad K_3 := -S^{-1}\overline{T}.$$

Since $K_1 \in B_2$, $\overline{K_1^*} = K_1$ and $\|K_1\| < 1$, we see that by Lemma 2.7,

$$U(A) := \det(1 - K_1^* K_1)^{1/4} e^{-\frac{1}{2}\Delta_{K_1}^\dagger} :e^{-N_{K_2}}: e^{-\frac{1}{2}\Delta_{K_3}} \tag{2.37}$$

is well defined on $\mathcal{F}_0$. Moreover it is seen that

$$U(A)\Psi \in \mathcal{D}_\infty, \quad \Psi \in \mathcal{F}_0.$$

Lemma 2.10 *Let $A \in \Sigma_2$. Then $U(A)$ can be uniquely extended to a unitary operator on $\mathcal{F}$.*

Proof: By the commutation relations (2.22)-(2.23) it is seen that

$$U(A)a^\sharp(f)U(A)^{-1}\Psi = b_A^\sharp(f)\Psi \tag{2.38}$$

for $\Psi \in \mathcal{F}_0$. From this, and the canonical commutation relations (2.28) and (2.29), it follows that

$$\begin{aligned}
&\|U(A)a^*(f_1)\cdots a^*(f_n)\Omega\|^2 \\
&= \|b_A^*(f_1)\cdots b_A^*(f_n)U(A)\Omega\|^2 \\
&= \det(1 - K_1^* K_1)^{1/2} \|b_A^*(f_1)\cdots b_A^*(f_n) e^{-\frac{1}{2}\Delta_{K_1}^\dagger}\Omega\|^2 \\
&= \det(1 - K_1^* K_1)^{1/2} \sum_{\pi\in\mathcal{P}_n} (\overline{f}_1, f_{\pi(1)})\cdots(\overline{f}_n, f_{\pi(n)}) \|e^{-\frac{1}{2}\Delta_{K_1}^\dagger}\Omega\|^2 \\
&= \|a^*(f_1)\cdots a^*(f_n)\Omega\|^2,
\end{aligned}$$

where $\mathcal{P}_n$ denotes the set of permutations of n degree, and we used that

$$\|e^{-\frac{1}{2}\Delta_{K_1}^\dagger}\Omega\|^2 = \det(1 - K_1^* K_1)^{-1/2}$$

and

$$b_A(f)e^{-\frac{1}{2}\Delta_{K_1}^\dagger}\Omega = 0, \quad f \in \mathcal{H}.$$

Then $U(A)$ is an isometry from $\mathcal{F}_0$ onto $\mathcal{E}$, where

$$\mathcal{E} := \text{the linear hull of } \times \\ \times\{b_A^*(f_1)\cdots b_A^*(f_n)U(A)\Omega, U(A)\Omega | f_j \in \mathcal{H}, j = 1, ..., n, n \geq 1\}.$$

From (2.1) it follows that

$$(a(f) \;\; a^*(f)) = (b_A(f) \;\; b_A^*(f)) \begin{pmatrix} S^* & -T^* \\ -\overline{T^*} & \overline{S^*} \end{pmatrix}. \tag{2.39}$$

By this we see that

$$a^\sharp(f)\mathcal{E} \subset \mathcal{E}, \quad f \in \mathcal{H}. \tag{2.40}$$

Let $\Psi \in \mathcal{E}$ and

$$\Psi_N := \{\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, ..., \Psi^{(N)}, 0, 0, ...\}.$$

Since $\Psi \in \mathcal{F}_0$, we see that Ψ_N is an analytic vector of $\phi(f)$, i.e.,

$$e^{i\phi(f)}\Psi_N = \sum_{n=0}^{\infty} \frac{1}{n!}(i\phi(f))^n \Psi_N, \tag{2.41}$$

which implies, together with (2.40), that $e^{i\phi(f)}\Psi_N \in \overline{\mathcal{E}}$, and by a limiting argument $e^{i\phi(f)}\Psi \in \overline{\mathcal{E}}$. Thus $e^{i\phi(f)}\mathcal{E} \subset \overline{\mathcal{E}}$ follows. By a limiting argument we have

$$e^{i\phi(f)}\overline{\mathcal{E}} \subset \overline{\mathcal{E}}.$$

Thus $\overline{\mathcal{E}} = \mathcal{F}$ by Proposition 2.1. Hence we conclude that $U(A)$ can be uniquely extended to a unitary operator on $\mathcal{F}$. The lemma follows. □

We denote the unitary extension of $U(A)$ by the same symbol $U(A)$.

Proposition 2.11 *Let $A \in \Sigma_2$. Then we can choose $U(A)$ as $U_{\rm B}(A)$ in (2.31).*

Proof: Note that $D(a^\sharp(f)) \supset \mathcal{D}_\infty \supset \mathcal{F}_0$. In particular $\mathcal{D}_\infty$ is a core of $b_A^\sharp(f)$. Since $U(A) : \mathcal{F}_0 \to \mathcal{D}_\infty$, we see that $U(A)$ maps a core of $a^\sharp(f)$ to a core of $b_A^\sharp(f)$. By (2.38) we conclude that $U(A)$ maps $D(a^\sharp(f))$ to $D(b_A^\sharp(f))$ with

$$a^\sharp(f) = U(A)^{-1} b_A^\sharp(f) U(A).$$

Thus the proposition follows. □

$U(\cdot)$ gives a projective unitary representation of Σ_2, i.e.,

$$U(A)U(B) = \omega(A,B)U(AB)$$

with a local factor $\omega(A,B) \in \{e^{i\psi} | \psi \in \mathbb{R}\}$.

3 One-parameter unitary groups

3.1 One-parameter symplectic groups

Definition 3.1

$$\begin{aligned} \sigma_2 : &= \left\{ \widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \Big| \widehat{A}J + J\widehat{A} = \widehat{A}^*J + J\widehat{A}^* = 0, T \in B_2 \right\} \\ &= \left\{ \widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \Big| S^* = -S, \overline{T^*} = T, T \in B_2 \right\}. \end{aligned}$$

Lemma 3.2 *Let $\widehat{A} \in \sigma_2$. Then*

$$e^{t\widehat{A}} \in \Sigma_2, \quad t \in \mathbb{R}.$$

In order to prove Lemma 3.2 we need lemmas.

Lemma 3.3 *We have $e^{t\widehat{A}} \subset \Sigma$ if and only if $\widehat{A}J + J\widehat{A} = \widehat{A}^*J + J\widehat{A}^* = 0$.*

Proof: Assume that $e^{t\widehat{A}} \in \Sigma$. Namely

$$e^{t\widehat{A}} J e^{tA^*} = e^{tA^*} J e^{t\widehat{A}} = J, \quad t \in \mathbb{R}. \tag{3.1}$$

Take a strong derivative at $t = 0$ on the both sides of (3.1). Then it follows that

$$AJ + JA^* = A^*J + JA = O. \tag{3.2}$$

Conversely assume (3.2). Then we have

$$\frac{d}{dt} e^{t\widehat{A}} J e^{tA^*} = 0.$$

Hence it holds that

$$J(t) := e^{t\widehat{A}} J e^{tA^*} = J(0) = J, \quad t \in \mathbb{R}.$$

In the similar manner

$$e^{tA^*} J e^{t\widehat{A}} = J, \quad t \in \mathbb{R}$$

is proven. Thus the lemma follows. □

For $\widehat{A} \in \sigma_2$, we set

$$e^{t\widehat{A}} = \begin{pmatrix} (e^{t\widehat{A}})_{11} & (e^{t\widehat{A}})_{12} \\ (e^{t\widehat{A}})_{21} & (e^{t\widehat{A}})_{22} \end{pmatrix} = \begin{pmatrix} (e^{t\widehat{A}})_{11} & \overline{(e^{t\widehat{A}})_{21}} \\ (e^{t\widehat{A}})_{21} & \overline{(e^{t\widehat{A}})_{11}} \end{pmatrix}.$$

In what follows we set $\mathbf{S} = \begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 0 & \overline{T} \\ T & 0 \end{pmatrix}$. Moreover $B^2_{\oplus}$ denotes the set of Hilbert-Schmidt operators on $\mathcal{H} \oplus \mathcal{H}$, and $\|X\|_{B^2_{\oplus}}$ the Hilbert-Schmidt norm of $X \in B^2_{\oplus}$.

Lemma 3.4 *Let* $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \sigma_2$. *Then*

$$e^{t\widehat{A}} - e^{t\mathbf{S}} \in B^2_{\oplus}, \quad t \in \mathbb{R},$$

and

$$\left\| e^{t\widehat{A}} - e^{t\mathbf{S}} \right\|_{B^2_{\oplus}} \leq \frac{\|\mathbf{T}\|_{B^2_{\oplus}}}{\|T\|} \left(e^{t\|T\|} - 1 \right), \quad t \in \mathbb{R}.$$

Proof: Let

$$Y(t) = e^{t\widehat{A}} e^{-t\mathbf{S}}, \quad T(t) = e^{t\mathbf{S}} \mathbf{T} e^{-t\mathbf{S}} = \begin{pmatrix} 0 & e^{tS} \overline{T} e^{-t\overline{S}} \\ e^{t\overline{S}} T e^{-tS} & 0 \end{pmatrix}. \tag{3.3}$$

We see that

$$\frac{d}{dt} Y(t) = Y(t) T(t)$$

in the operator norm on $\mathcal{H} \oplus \mathcal{H}$, which implies that

$$Y(t) = E + \sum_{n=1}^{\infty} Y_n(t), \tag{3.4}$$

where

$$Y_n(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} T(t_1)T(t_2)\cdots T(t_n)dt_1dt_2\cdots dt_n.$$

We have

$$\begin{aligned}\|Y_n(t)\|_{B^2_\oplus} \le \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|T(t_1)\|_{B^2_\oplus} \times \\ \times \|T(t_2)\|_{\mathcal{H}\oplus\mathcal{H}} \cdots \|T(t_n)\|_{\mathcal{H}\oplus\mathcal{H}} dt_1dt_2\cdots dt_n,\end{aligned}$$

where $\|X\|_{\mathcal{H}\oplus\mathcal{H}}$ denotes the norm of bounded operator X on $\mathcal{H}\oplus\mathcal{H}$. Since $e^{t\mathbf{S}}$ is a unitary operator in $\mathcal{H}\oplus\mathcal{H}$, we have

$$\begin{aligned}&\|T(t_j)\|_{\mathcal{H}\oplus\mathcal{H}} \le \|T\|, \quad j = 2, 3, ..., n, \\ &\|T(t_1)\|_{B^2_\oplus} \le \|\mathbf{T}\|_{B^2_\oplus}.\end{aligned}$$

Then it follows that

$$\|Y_n(t)\|_{B^2_\oplus} \le \frac{\|T\|^{n-1}}{n!} t^n \|\mathbf{T}\|_{B^2_\oplus}, \tag{3.5}$$

which yields that $Y(t) - E \in B^2_\oplus$ and

$$\|Y(t) - E\|_{B^2_\oplus} \le \frac{\|\mathbf{T}\|_{B^2_\oplus}}{\|T\|} \left(e^{t\|T\|} - 1\right). \tag{3.6}$$

From (3.6) it follows that

$$\begin{aligned}\left\|e^{t\widehat{A}} - e^{t\mathbf{S}}\right\|_{B^2_\oplus} &= \|(Y(t) - E)e^{t\mathbf{S}}\|_{B^2_\oplus} \\ &\le \|(Y(t) - E)\|_{B^2_\oplus} \\ &\le \frac{\|\mathbf{T}\|_{B^2_\oplus}}{\|T\|} \left(e^{t\|T\|} - 1\right).\end{aligned}$$

Hence the lemma follows. □

Proof of Lemma 3.2

The fact $e^{t\widehat{A}} \in \Sigma$ follows from Lemma 3.3. We shall prove $(e^{t\widehat{A}})_{21} \in B_2$. By Lemma 3.4, $e^{t\widehat{A}} - e^{t\mathbf{S}} \in B^2_\oplus$. It implies

$$(e^{t\widehat{A}} - e^{t\mathbf{S}})_{21} \in B_2.$$

Since $(e^{t\widehat{A}})_{21} = (e^{t\widehat{A}} - e^{t\mathbf{S}})_{21}$, $e^{t\widehat{A}} \in \Sigma_2$ follows. □

Remark 3.5 *In [6, Lemma 6.3] it has been proven that if* $S, T \in B$, $S^* = -S$, $\overline{T^*} = T$, *and*

$$F(t) = \int_0^t e^{\tau \overline{S}} T e^{-\tau S} d\tau \in B_2$$

with $\|F(t)\|_2 \in L^1_{\mathrm{loc}}(\mathbb{R}, dt)$. *Then* $e^{t\widehat{A}} \in \Sigma_2$.

3.2 The main theorems

For $A \in \sigma_2$, we set

$$U_t := U(e^{t\widehat{A}}), \quad t \in \mathbb{R}.$$

U_t gives a projective unitary representation of $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$, i.e.,

$$U_t U_s = e^{i\rho(t,s)} U_{t+s},$$

where we set

$$e^{i\rho(t,s)} := \omega(e^{t\widehat{A}}, e^{s\widehat{A}}).$$

For $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}$, let

$$\Delta(\widehat{A}) := \frac{i}{2}(\Delta_T^* - \Delta_{\overline{T}}) - iN_{\overline{S}}.$$

The main theorems in this paper are as follows.

Theorem 3.6 *Let* $\widehat{A} \in \sigma_2$. *Then* $\Delta(\widehat{A})$ *is essentially self-adjoint on* $\mathcal{F}_0$.

Theorem 3.7 *Let* $\widehat{A} \in \sigma_2$. *Then*

$$e^{it\Delta(\widehat{A})} a^\sharp(f) e^{-it\Delta(\widehat{A})} = b^\sharp(f), \quad t \in \mathbb{R}.$$

For notational convenience we set

$$K_t := (e^{t\widehat{A}})_{21} (e^{t\widehat{A}})_{11}^{-1}.$$

Theorem 3.8 *Let* $\widehat{A} \in \sigma_2$. *Then*

$$U_t = e^{i\int_0^t \tau_{\widehat{A}}(s) ds} e^{it\Delta(\widehat{A})}, \quad t \in \mathbb{R},$$

where

$$\tau_{\widehat{A}}(s) := \frac{1}{2} \mathrm{Im} \mathrm{Tr}(T^* K_s).$$

Corollary 3.9 *Let $\widehat{A} \in \sigma_2$. $\int_0^t \mathrm{ImTr}(T^* K_s) ds = 0$ if and only if*

$$U_t = e^{it\Delta(\widehat{A})}, \quad t \in \mathbb{R}.$$

Namely in the case of $\int_0^t \mathrm{ImTr}(T^* K_s) ds = 0$, U_t gives a unitary representation of $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$.

Proof: It follows from Theorem 3.8. □

Corollary 3.10 *Let $\widehat{A} \in \sigma_2$. Then the local exponent $\rho(\cdot,\cdot)$ is given by*

$$\rho(t,s) = \int_0^t \tau_{\widehat{A}}(r)dr + \int_0^s \tau_{\widehat{A}}(r)dr - \int_0^{t+s} \tau_{\widehat{A}}(r)dr.$$

Proof: We directly see that

$$U_t U_s = e^{i\left\{\int_0^t \tau_{\widehat{A}}(r)dr + \int_0^s \tau_{\widehat{A}}(r)dr\right\}} e^{i(t+s)\Delta(\widehat{A})} = e^{i\left\{\int_0^t \tau_{\widehat{A}}(r)dr + \int_0^s \tau_{\widehat{A}}(r)dr - \int_0^{t+s} \tau_{\widehat{A}}(r)dr\right\}} U_{t+s}.$$

Then the corollary follows. □

Corollary 3.11 *Let $\widehat{A} \in \sigma_2$. Then*

$$\widehat{U}_t := e^{-i\int_0^t \tau_{\widehat{A}}(s)ds} U_t$$

gives a unitary representation of $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$, and $\widehat{U}_t$ is a normal ordered quadratic expression of $e^{it\Delta(A)}$. In particular it follows that

$$(\Omega, e^{it\Delta(A)}\Omega) = \det(1 - K_t^* K_t)^{1/4} e^{-i\int_0^t \tau_{\widehat{A}}(s)ds}.$$

Proof: Since we actually have $\widehat{U}_t = e^{it\Delta(A)}$, the corollary follows. □

Example 3.12 *Let*

$$\overline{S} = S = -S^*, \quad \overline{T} = T = T^*. \tag{3.7}$$

Then $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}$ satisfies the assumptions of Corollary 3.9.

Let $\mathcal{H} = L^2(\mathbb{R}, dx)$ and Γ be the complex conjugation. Define the Hilbert-Schmidt operator T by

$$T\phi(x) = \int_{\mathbb{R}} g(x,y)\phi(y)dy$$

with a real-valued function $g(x,y) \in L^2(\mathbb{R}\times\mathbb{R})$. Let f be a real-valued measurable function such that $f \in L^\infty(\mathbb{R})$ and $f(-x) = -f(x)$. Define S by

$$S = f(\frac{d}{dx}).$$

Then S and T satisfy (3.7).

4 Weak differentiability of U_t

In this section we shall prove the weak differentiability of $U_t\Omega$ in t. Throughout this section we assume that $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \sigma_2$. The next lemma is fundamental.

Lemma 4.1 *Let $T_t \in B_2$ and $S_t \in B$. Assume that T_t is differentiable in t in B_2, and S_t in B with $\frac{d}{dt}T_t = T_t'$ and $\frac{d}{dt}S_t = S_t'$. Then T_tS_t is differentiable in t in B_2 with*

$$\frac{d}{dt}T_tS_t = T_t'S_t + T_tS_t'.$$

Proof: We have

$$\begin{aligned}
&\left\|\frac{T_{t+\epsilon}S_{t+\epsilon} - T_tS_t}{\epsilon} - T_t'S_t - T_tS_t'\right\|_2 \\
&\leq \left\|\frac{T_{t+\epsilon} - T_t}{\epsilon} - T_t'\right\|_2 \|S_{t+\epsilon}\| + \|T_t'\|_2\|S_{t+\epsilon} - S_t\| + \|T_t\|_2\left\|\frac{S_{t+\epsilon} - S_t}{\epsilon} - S_t'\right\| \\
&\to 0
\end{aligned}$$

as $\epsilon \to 0$. Hence the lemma follows. □

Lemma 4.2 *It follows that $(e^{t\widehat{A}})_{ij}$, $i \neq j$ (resp. $i = j$), is differentiable in t in B_2 (resp. B) with*

$$\frac{d}{dt}(e^{t\widehat{A}})_{ij} = (\widehat{A}e^{t\widehat{A}})_{ij}.$$

Proof: Since

$$\frac{d}{dt}e^{t\widehat{A}} = \widehat{A}e^{t\widehat{A}} \quad in\ B(\mathcal{H}\oplus\mathcal{H}),$$

it follows that

$$\frac{d}{dt}(e^{t\widehat{A}})_{ij} = (\widehat{A}e^{t\widehat{A}})_{ij} \quad in\ B.$$

Let $i = 2, j = 1$. Then we have

$$\begin{aligned}
&\left\|\left\{\frac{1}{\epsilon}\left(e^{(t+\epsilon)\widehat{A}} - e^{t\widehat{A}}\right) - \widehat{A}e^{t\widehat{A}}\right\}_{21}\right\|_2 \\
&= \left\|\sum_{k=1,2}(e^{t\widehat{A}})_{2k}\left\{\frac{1}{\epsilon}\left(e^{\epsilon\widehat{A}} - E\right) - A\right\}_{k1}\right\|_2 \\
&\le \|(e^{t\widehat{A}})_{21}\|_2\left\|\left\{\frac{1}{\epsilon}\left(e^{\epsilon\widehat{A}} - E\right) - A\right\}_{11}\right\| + \|(e^{t\widehat{A}})_{22}\|\left\|\left\{\frac{1}{\epsilon}\left(e^{\epsilon\widehat{A}} - E\right) - A\right\}_{21}\right\|_2.
\end{aligned}$$

Since

$$\left\|\left\{\frac{1}{\epsilon}\left(e^{\epsilon\widehat{A}} - E\right) - A\right\}_{11}\right\| \to 0, \quad \|(e^{\epsilon\widehat{A}})_{22}\| \to 1$$

as $\epsilon \to 0$, it is enough to prove that

$$\left\|\left\{\frac{1}{\epsilon}\left(e^{\epsilon\widehat{A}} - E\right) - A\right\}_{21}\right\|_2 = \left\|\frac{1}{\epsilon}(e^{\epsilon\widehat{A}})_{21} - T\right\|_2 \to 0$$

as $\epsilon \to 0$. By (3.3) and (3.4) we see that

$$\begin{pmatrix} 0 & \overline{(e^{t\widehat{A}})_{21}} \\ (e^{t\widehat{A}})_{21} & 0 \end{pmatrix} = \sum_{m=0}^{\infty} Y_{2m+1}(t)e^{t\mathbf{S}}.$$

Hence

$$\begin{pmatrix} 0 & \overline{(e^{t\widehat{A}})_{21}} - t\overline{T} \\ (e^{t\widehat{A}})_{21} - tT & 0 \end{pmatrix} = \sum_{m=0}^{\infty} Y_{2m+1}(t)e^{t\mathbf{S}} - t\mathbf{T}.$$

Since

$$\left\|\begin{pmatrix} 0 & \overline{(e^{t\widehat{A}})_{21}} - t\overline{T} \\ (e^{t\widehat{A}})_{21} - tT & 0 \end{pmatrix}\right\|^2_{B^2_\oplus} = 2\|(e^{t\widehat{A}})_{21} - tT\|_2^2,$$

it is enough to show that

$$\lim_{t\to 0}\frac{1}{|t|}\left\|\sum_{m=0}^{\infty} Y_{2m+1}(t)e^{t\mathbf{S}} - t\mathbf{T}\right\|_{B^2_\oplus} = 0. \tag{4.1}$$

Since $\|e^{t\mathbf{S}} - E\|_{\mathcal{H}\oplus\mathcal{H}} = \|e^{tS} - 1\|$, we have

$$\left\|\sum_{m=0}^{\infty} Y_{2m+1}(t)e^{t\mathbf{S}} - t\mathbf{T}\right\|_{B^2_\oplus} \leq \left\|\sum_{m=0}^{\infty} Y_{2m+1}(t) - t\mathbf{T}\right\|_{B^2_\oplus} + |t|\|\mathbf{T}\|_{B^2_\oplus}\|e^{tS} - 1\|.$$

It is obvious that

$$\lim_{t\to 0}\frac{1}{|t|}|t|\|\mathbf{T}\|_{B^2_\oplus}\|e^{tS} - 1\| = 0.$$

We have

$$\frac{1}{|t|}\left\|\sum_{m=0}^{\infty} Y_{2m+1}(t) - t\mathbf{T}\right\|_{B^2_\oplus} \leq \frac{1}{|t|}\|Y_1(t) - t\mathbf{T}\|_{B^2_\oplus} + \frac{1}{|t|}\left\|\sum_{m=1}^{\infty} Y_{2m+1}(t)\right\|_{B^2_\oplus}.$$

Since

$$\|e^{t\mathbf{S}}\mathbf{T}e^{-t\mathbf{S}} - \mathbf{T}\|_{B^2_\oplus} \leq \|\mathbf{T}\|_{B^2_\oplus}(\|e^{-tS} - 1\| + \|e^{tS} - 1\|),$$

$\|e^{t\mathbf{S}}\mathbf{T}e^{-t\mathbf{S}} - \mathbf{T}\|_{B^2_\oplus}$ is continuous at $t = 0$. Then we have

$$\begin{aligned}\lim_{t\to 0}\frac{1}{|t|}\|Y_1(t) - t\mathbf{T}\|_{B^2_\oplus} &= \lim_{t\to 0}\frac{1}{|t|}\|\int_0^t (e^{t_1\mathbf{S}}\mathbf{T}e^{-t_1\mathbf{S}} - \mathbf{T})dt_1\|_{B^2_\oplus} \\ &\leq \lim_{t\to 0}\frac{1}{|t|}\int_0^t \|e^{t_1\mathbf{S}}\mathbf{T}e^{-t_1\mathbf{S}} - \mathbf{T}\|_{B^2_\oplus}\,dt_1 = 0.\end{aligned}$$

Moreover by (3.5),

$$\begin{aligned}\lim_{t\to 0}\frac{1}{|t|}\left\|\sum_{m=1}^{\infty} Y_{2m+1}(t)\right\|_{B^2_\oplus} &\leq \lim_{t\to 0}\frac{1}{|t|}\sum_{m=1}^{\infty}\frac{\|T\|^{2m}}{(2m+1)!}|t|^{2m+1}\|\mathbf{T}\|_{B^2_\oplus} \\ &= \lim_{t\to 0}\|\mathbf{T}\|_{B^2_\oplus}\frac{\sinh(|t|\|T\|) - |t|\|T\|}{|t|\|T\|} = 0.\end{aligned}$$

Hence we conclude (4.1). In the case of $i = 1, j = 2$, the lemma is similarly proven.
□

Lemma 4.3 *K_t is differentiable in t in B_2 with*

$$\frac{d}{dt}K_t = K_t' := (\widehat{A}e^{t\widehat{A}})_{21}(e^{t\widehat{A}})^{-1}_{11} - (e^{t\widehat{A}})_{21}(e^{t\widehat{A}})^{-1}_{11}(\widehat{A}e^{t\widehat{A}})_{11}(e^{t\widehat{A}})^{-1}_{11}. \tag{4.2}$$

Proof: By Lemma 4.2 we see that

$$\begin{aligned}&\frac{d}{dt}(e^{t\widehat{A}})_{21} = (\widehat{A}e^{t\widehat{A}})_{21} \ \ \textit{in } B_2, \\ &\frac{d}{dt}(e^{t\widehat{A}})^{-1}_{11} = (e^{t\widehat{A}})^{-1}_{11}(\widehat{A}e^{t\widehat{A}})_{11}(e^{t\widehat{A}})^{-1}_{11} \ \ \textit{in } B.\end{aligned}$$

Then by Lemma 4.1 we obtain (4.2). □

Lemma 4.4 *Let $\Psi \in \mathcal{F}_0$. Then $(\Delta^\dagger_{K_t})^n\Psi$ is strongly differentiable in t with*

$$\frac{d}{dt}(\Delta^\dagger_{K_t})^n\Psi = n\Delta^\dagger_{K'_t}(\Delta^\dagger_{K_t})^{n-1}\Psi.$$

Proof: Using Lemma 2.6, we have

$$\frac{1}{\epsilon}\left(\Delta^\dagger_{K_{t+\epsilon}} - \Delta^\dagger_{K_t}\right)\Psi - \Delta^\dagger_{K'_t}\Psi = \Delta^\dagger_{\frac{1}{\epsilon}(K_{t+\epsilon}-K_t)-K'_t}\Psi.$$

Then by Lemmas 2.5 and 4.3, we have

$$\left\|\frac{1}{\epsilon}\left(\Delta^\dagger_{K_{t+\epsilon}} - \Delta^\dagger_{K_t}\right)\Psi - \Delta^\dagger_{K'_t}\Psi\right\| \leq \sqrt{6}\left\|\frac{1}{\epsilon}\left(K_{t+\epsilon} - K_t\right) - K'_t\right\|_2 \|(N+1)\Psi\| \to 0$$

as $\epsilon \to 0$. Then

$$\frac{d}{dt}\Delta^\dagger_{K_t}\Psi = \Delta^\dagger_{K'_t}\Psi.$$

We have

$$\begin{aligned}
&\frac{1}{\epsilon}\left\{(\Delta^\dagger_{K_{t+\epsilon}})^n - (\Delta^\dagger_{K_t})^n\right\}\Psi - n\Delta^\dagger_{K'_t}{\Delta^\dagger_{K_t}}^{n-1}\Psi \\
&= \sum_{j=0}^{n-1}(\Delta^\dagger_{K_{t+\epsilon}})^j\left\{\frac{1}{\epsilon}(\Delta^\dagger_{K_{t+\epsilon}} - \Delta^\dagger_{K_t}) - \Delta^\dagger_{K'_t}\right\}(\Delta^\dagger_{K_t})^{n-j-1}\Psi \\
&\qquad + \sum_{j=0}^{n-1}\sum_{i=0}^{j-1}(\Delta^\dagger_{K_{t+\epsilon}})^i\left(\Delta^\dagger_{K_{t+\epsilon}} - \Delta^\dagger_{K_t}\right)\Delta^\dagger_{K'_t}(\Delta^\dagger_{K_t})^{n-i-2}\Psi.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
&\left\|\frac{1}{\epsilon}((\Delta^\dagger_{K_{t+\epsilon}})^n - (\Delta^\dagger_{K_t})^n)\Psi - n\Delta^\dagger_{K'_t}{\Delta^\dagger_{K_t}}^{n-1}\Psi\right\| \\
&\leq (\sqrt{6})^n\sum_{j=0}^{n-1}(2n-1)!!\|K_{t+\epsilon}\|^j\|K_t\|^{n-j-1}\left\|\frac{1}{\epsilon}(K_{t+\epsilon} - K_t) - K'_t\right\|_2\|\Psi\| \\
&+(\sqrt{6})^n\sum_{j=0}^{n-1}\sum_{i=0}^{j-1}(2n-1)!!\|K_{t+\epsilon}\|_2^i\|K'_t\|_2\|K_t\|^{n-i-2}\|K_{t+\epsilon} - K_t\|_2\|\Psi\|. \quad (4.3)
\end{aligned}$$

As $\epsilon \to 0$, the right-hand side of (4.3) goes to zero. Then the lemma follows. □

Lemma 4.5 *Let $\Psi \in \mathcal{F}$. Then $(\Psi, U_t\Omega)$ is differentiable in t.*

Proof: We have

$$(\Psi, U_t\Omega) = \det(1 - K_t^* K_t)^{1/4} (\Psi, e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega).$$

We shall show that $\det(1 - K_t^* K_t)^{1/4}$ and $(\Psi, e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega)$ are differentiable in t. Note that

$$\det(1 - K_t^* K_t)^{-1/2} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\left(-\frac{1}{2}\right)^n\right)^2 \|(\Delta^\dagger_{K_t})^n \Omega\|^2.$$

Since

$$\frac{d}{dt}\|(\Delta^\dagger_{K_t})^n\Omega\|^2 = 2\mathrm{Re}((\Delta^\dagger_{K_t})^n\Omega, n\Delta^\dagger_{K_t'}(\Delta^\dagger_{K_t})^{n-1}\Omega)$$

by Lemma 4.4, and

$$\left|\frac{d}{dt}\|(\Delta^\dagger_{K_t})^n\Omega\|^2\right| \leq 2n(2n-1)\sqrt{6}\|K_t'\|_2 \|(\Delta^\dagger_{K_t})^n\Omega\| \|(\Delta^\dagger_{K_t})^{n-1}\Omega\|,$$

we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left|\left(\frac{1}{n!}\left(-\frac{1}{2}\right)^n\right)^2 \frac{d}{dt}\|(\Delta^\dagger_{K_t})^n\Omega\|^2\right| \\
&\leq \sqrt{6}\left(\sum_{n=0}^{\infty}\left(\frac{1}{n!}\left(-\frac{1}{2}\right)^n\right)^2 \|(\Delta^\dagger_{K_t})^n\Omega\|^2\right)^{1/2} \\
&\qquad \times \left(\sum_{n=0}^{\infty}(2n+1)^2\left(\frac{1}{n!}\left(-\frac{1}{2}\right)^n\right)^2 \|(\Delta^\dagger_{K_t})^n\Omega\|^2\right)^{1/2} < \infty.
\end{aligned}$$

Thus $\det(1 - K_t^* K_t)^{-1/2}$ is continuously differentiable in t with

$$\frac{d}{dt}\det(1 - K_t^* K_t)^{-1/2} = 2\mathrm{Re}(e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega, -\frac{1}{2}\Delta^\dagger_{K_t'} e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega).$$

In particular $\det(1 - K_t^* K_t)^{1/4}$ is continuously differentiable in t. Next we estimate $(\Psi, e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega)$. We have

$$(\Psi, e^{-\frac{1}{2}\Delta^\dagger_{K_t}}\Omega) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n (\Psi^{(2n)}, (\Delta^\dagger_{K_t})^n\Omega).$$

Since by Lemma 4.4,

$$\frac{d}{dt}(\Psi^{(2n)}, (\Delta^\dagger_{K_t})^n\Omega) = (\Psi^{(2n)}, n\Delta^\dagger_{K_t'}(\Delta^\dagger_{K_t})^{n-1}\Omega),$$

and

$$\left|\frac{d}{dt}(\Psi^{(2n)}, (\Delta^\dagger_{K_t})^n\Omega)\right| \leq \|\Psi^{(2n)}\| n(2n-1)\sqrt{6}\|K_t'\|_2 \|(\Delta^\dagger_{K_t})^{n-1}\Omega\|,$$

it follows that

$$\sum_{n=1}^{\infty}\left|\frac{d}{dt}(\Psi^{(2n)},(\Delta_{K_t}^{\dagger})^n\Omega)\right|$$
$$\leq \|\Psi\|\sqrt{6}\|K_t'\|_2\left(\sum_{n=0}^{\infty}(2n+1)^2\frac{1}{2^2}\left(\frac{1}{n!}\left(-\frac{1}{2}\right)^n\right)^2\|(\Delta_{K_t}^{\dagger})^n\Omega\|^2\right)^{1/2}<\infty,$$

which implies that $(\Psi, e^{-\frac{1}{2}\Delta_{K_t}^{\dagger}}\Omega)$ is continuously differentiable in t with

$$\frac{d}{dt}(\Psi,e^{-\frac{1}{2}\Delta_{K_t}^{\dagger}}\Omega)=(\Psi,-\frac{1}{2}\Delta_{K_t'}^{\dagger}e^{-\frac{1}{2}\Delta_{K_t}^{\dagger}}\Omega).$$

Hence the lemma follows. □

5 Proof of Theorems

5.1 Proof of Theorem 3.6

Lemma 5.1 *Let $\widehat{A}\in\sigma_2$. Then*

$$\|\Delta(\widehat{A})\Psi\|\leq\alpha\|(N+1)\Psi\|,$$

where $\alpha:=\max\{\sqrt{6}\|T\|_2,\|S\|\}$.

Proof: We obtain from (2.12)–(2.14) that

$$\begin{aligned}\|\Delta(\widehat{A})\Psi\| &\leq \frac{1}{2}\left(\|\Delta_T^{\dagger}\Psi\|+\|\Delta_{\overline{T}}\Psi\|\right)+\|\widehat{A\overline{S}}\Psi\|,\\ &\leq \frac{1}{2}\left(\sqrt{6}\|T\|_2\|(N+1)\Psi\|+\|T\|_2\|N\Psi\|\right)+\|S\|\|N\Psi\|,\\ &\leq \max\{\sqrt{6}\|T\|_2,\|S\|\}\|(N+1)\Psi\|.\end{aligned}$$

Thus the lemma follows. □

Lemma 5.2 *Let $\Psi\in\mathcal{F}_0$. Then, for $t\in(-1/2\alpha,1/2\alpha)$,*

$$\sum_{n=0}^{\infty}\frac{\|\Delta(\widehat{A})^n\Psi\|t^n}{n!}<\infty.$$

Proof: It is enough to prove the lemma for $\Psi \in \mathcal{F}^{(n)}$. Note that

$$\Delta(\widehat{A})^k \Psi \in \bigoplus_{m=0}^{n+2k} \mathcal{F}^{(m)}.$$

By Lemma 5.1 it follows that

$$\|\Delta(\widehat{A})^k \Psi\| \leq \alpha^k (n+2k-1)(n+2k-3)\cdots(n+1)\|\Psi\|.$$

Hence we have

$$\sum_{n=0}^{\infty} \frac{\|\Delta(\widehat{A})^n \Psi\| t^n}{n!} \leq \sum_{n=0}^{\infty} \frac{(n+2k-1)(n+2k-3)\cdots(n+1)}{n!} \alpha^k t^k \|\Psi\|.$$

The right-hand side above converges for $t\alpha \in (-1/2, 1/2)$, thus the lemma follows. □

Proof of Theorem 3.6

By Lemma 2.6 and the assumption such that $\overline{T^*} = T$, $S^* = -S$, it follows that

$$(\Delta_T^\dagger)^* = \Delta_{\overline{T}}, \quad N_S^* = -N_{\overline{S}}$$

on $\mathcal{F}_0$. In particular $D(\widehat{A})$ is a symmetric operator on $\mathcal{F}_0$. By Lemma 5.2, we see that $\mathcal{F}_0$ is a set of analytic vectors for $\Delta(\widehat{A})$. Hence the Nelson analytic vector theorem [15, Theorem X.39] yields that $\Delta(\widehat{A})$ is essentially self-adjoint on $\mathcal{F}_0$. □

5.2 Proof of Theorem 3.7

Let $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^k(B) = [A, \mathrm{ad}_A^{k-1}(B)]$. It is well known that

$$A^n B = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}_A^k(B) A^{n-k}.$$

Lemma 5.3 *We have on* $\mathcal{F}_0$

$$\begin{aligned}
&(1)\ \mathrm{ad}_{i\Delta(\widehat{A})}^k(a(f)) = a((\widehat{A}^k)_{11} f) + a^*((\widehat{A}^k)_{21} f), \\
&(2)\ \mathrm{ad}_{i\Delta(\widehat{A})}^k(a^*(f)) = a((\widehat{A}^k)_{12} f) + a^*((\widehat{A}^k)_{22} f).
\end{aligned}$$

Proof: We prove the lemma through an induction. We directly see that

$$[i\Delta(\widehat{A}), a(f)] = a(Sf) + a^*(Tf),$$
$$[i\Delta(\widehat{A}), a^*(f)] = a(\overline{T}f) + a^*(\overline{S}f).$$

Thus the lemma follows for $k = 1$. Assume that

$$\mathrm{ad}^k_{i\Delta(\widehat{A})}(a^*(f)) = a((\widehat{A}^k)_{12}f) + a^*((\widehat{A}^k)_{22}f).$$

Then we have

$$\begin{aligned}
\mathrm{ad}^k_{i\Delta(\widehat{A})}(a^*(f)) &= [i\Delta(\widehat{A}), +a((\widehat{A}^k)_{12}f) + a^*((\widehat{A}^k)_{22}f)] \\
&= a(\overline{T}(\widehat{A}^k)_{22}f + S(\widehat{A}^k)_{12}f) + a^*(\overline{S}(\widehat{A}^k)_{22}f + T(\widehat{A}^k)_{12}f) \\
&= a((\widehat{A}^{k+1})_{12}f) + a^*((\widehat{A}^{k+1})_{22}f).
\end{aligned}$$

Thus (1) follows. (2) is proven in the similar manner. □

Proof of Theorem 3.7

Let $\Psi, \Phi \in \mathcal{F}_0$. We define for $z \in \mathbb{C}$

$$F_1(z) = (\Phi, e^{iz\Delta(\widehat{A})}a^*(f)\Psi),$$
$$F_2(z) = (b_{\overline{z}}(\overline{f})\Phi, e^{iz\Delta(\widehat{A})}\Psi),$$

where

$$b_{\overline{z}}(\overline{f}) = a((e^{\overline{z}\widehat{A}})_{11}\overline{f}) + a^*((e^{\overline{z}\widehat{A}})_{21}\overline{f}).$$

Since, by Lemma 5.2, Ψ and Φ are analytic vectors for $\Delta(\widehat{A})$, for $|z| < 1/2\alpha$, $F_j(z)$ are analystic. We have

$$\begin{aligned}
\frac{d^n F_1(z)}{dz^n}\Big|_{z=0} &= (\Phi, (i\Delta(\widehat{A}))^n a^*(f)\Psi) \\
&= \sum_{k=0}^{n} \binom{n}{k} (\Phi, \mathrm{ad}^k_{i\Delta(\widehat{A})}(a^*(f))(i\Delta(\widehat{A}))^{n-k}\Psi) \\
&= \sum_{k=0}^{n} \binom{n}{k} (\Phi, [a((\widehat{A}^k)_{12}f) + a^*((\widehat{A}^k)_{22}f)](i\Delta(\widehat{A}))^{n-k}\Psi).
\end{aligned}$$

On the other side we see that

$$\begin{aligned}\frac{d^n F_2(z)}{dz^n}\lceil_{z=0} &= \sum_{k=0}^{n}\binom{n}{k}([a((\widehat{A}^k)_{11}\overline{f})+a^*((\widehat{A}^k)_{21}\overline{f})]\Phi,(i\Delta(\widehat{A}))^{n-k}\Psi)\\ &= \sum_{k=0}^{n}\binom{n}{k}(\Phi,[a((\widehat{A}^k)_{12}f)+a^*((\widehat{A}^k)_{22}f)](i\Delta(\widehat{A}))^{n-k}\Psi).\end{aligned}$$

Here we used

$$\overline{(\widehat{A}^k)_{21}}=(\widehat{A}^k)_{12},\quad \overline{(\widehat{A}^k)_{11}}=(\widehat{A}^k)_{22}.$$

Hence we obtain

$$F_1(z)=F_2(z),\quad |z|<1/2\alpha,$$

which implies that

$$e^{-iz\widehat{A}}\Phi\in D(a(\overline{f}))$$

with

$$a(\overline{f})e^{-iz\widehat{A}}\Phi=e^{-iz\widehat{A}}b_{\overline{z}}(\overline{f})\Phi,\quad |z|<1/2\alpha.$$

In particular we have

$$e^{it\widehat{A}}a(f)e^{-it\widehat{A}}\Phi=b_t(f)\Phi$$

for all $t\in\mathbb{R}^d$ by the group property. The identity

$$e^{it\widehat{A}}a^*(f)e^{-it\widehat{A}}\Phi=b_t^*(f)\Phi$$

can be also proven similarly. Then the theorem follows. □

5.3 Proof of Theorem 3.8

By the uniqueness of the proper canonical transformation, we have

$$U_t=e^{i\theta(t)}e^{it\widehat{A}},$$

with some function $\theta(\cdot):\mathbb{R}\to\mathbb{R}$.

Lemma 5.4 *We have $\theta\in C(\mathbb{R})$.*

Proof: Note that

$$e^{i\theta(t)} = \frac{(\Omega, U_t\Omega)}{(\Omega, e^{it\Delta(A)}\Omega)}.$$

Since $(\Omega, U_t\Omega)$ and $(\Omega, e^{it\Delta(A)}\Omega)$ are continuously differentiable in t by Lemma 4.5, the lemma follows. □

Proof of Theorem 3.8

Since

$$\mathbb{R} \ni \det(1 - K_t^* K_t)^{-1/2} = (\Omega, U_t\Omega) = (\Omega, e^{i\theta(t)} e^{it\Delta(A)}\Omega),$$

we see that

$$\frac{d}{dt}(\Omega, e^{i\theta(t)} e^{it\Delta(A)}\Omega) = i\theta'(t)(\Omega, U_t\Omega) + (\Omega, i\Delta(A)U_t\Omega) \in \mathbb{R}.$$

Moreover $(\Omega, U_t\Omega) \in \mathbb{R}$ employs that

$$\theta'(t)(\Omega, U_t\Omega) = -\mathrm{Im}(\Omega, i\Delta(A)U_t\Omega) = -\mathrm{Im}(\frac{1}{2}\Delta_T^\dagger\Omega, U_t\Omega).$$

Then we have

$$\theta'(t) = -\mathrm{Im}\frac{1}{2}\frac{(\Delta_T^\dagger\Omega, U_t\Omega)}{(\Omega, U_t\Omega)} = -\frac{1}{2}\mathrm{Im}(\Delta_T^\dagger\Omega, e^{-\frac{1}{2}\Delta_{K_t}^\dagger}\Omega) = \frac{1}{4}\mathrm{Im}(\Delta_T^\dagger\Omega, \Delta_{K_t}^\dagger\Omega).$$

It can be directly proven that

$$(\Omega, [\Delta_{T^*}, \Delta_{K_t}^\dagger]\Omega) = 2\mathrm{Tr}(T^* K_t).$$

Thus we have

$$\theta'(t) = \frac{1}{2}\mathrm{ImTr}(T^* K_t).$$

Hence we conclude that

$$\theta(t) = \int_0^t \frac{1}{2}\mathrm{ImTr}(T^* K_s)ds,$$

and then the theorem follows. □

6 Examples

6.1 Diagonal cases

Lemma 6.1 *Let* $\widehat{A} = \begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix} \in \sigma_2$. *Then*

$$U_t = e^{it\Delta(A)}. \tag{6.1}$$

Proof: $\tau_{\widehat{A}}(s) \equiv 0$ implies (6.1) by Corollary 3.9. □

Theorem 6.2 *Let* $S^* = -S$. *Then*

$$:\exp\left(-N_{(1-e^{+t\overline{S}})}\right):= \exp\left(tN_{\overline{S}}\right). \tag{6.2}$$

Proof: Let $\widehat{A} = \begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix}$. Since

$$\begin{aligned} U_t &= :\exp\left(-N_{(1-e^{+t\overline{S}})}\right): \\ \exp\left(it\Delta(A)\right) &= \exp\left(tN_{\overline{S}}\right), \end{aligned}$$

the lemma follows from Lemma 6.1.[1] □

In particular setting $S = i1$ in Lemma 6.2,[2] we have

$$:\exp\left(-(1-e^{it})N\right): \Psi = \exp\left(itN\right)\Psi. \tag{6.3}$$

Since $\Psi \in \mathcal{F}_0$ is an analytic vector for $\exp(itN)$, (6.3) can be extended to $t \to -i\beta$, $\beta \in \mathbb{R}$, i.e., we have for $\Psi \in \mathcal{F}_0$,

$$:\exp\left[(e^{\beta}-1)N\right]: \Psi = \exp\left(\beta N\right)\Psi.$$

In particular we obtain for $\Psi \in \mathcal{F}_0$,

$$:\exp\left(tN\right): \Psi = \exp\left(\log(t+1)\ N\right)\Psi, \quad t \geq 0. \tag{6.4}$$

[1] (6.2) is shown in [18, Corollary 5.4].
[2] e^{itN} can be regarded as the Fourier trasformation on $\mathcal{F}$. See [19].

6.2 Integral formulae

Let $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}))$. Let $a(k)$ be the kernel of $a(f)$ defined by

$$(a(k)\Psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1} \int \Psi^{(n+1)}(k, k_1, \cdots, k_n) dk_1 \cdots dk_n.$$

It is well known that for $\Psi \in \mathcal{F}_0$,

$$\int_{\mathbb{R}^n} \|a(k)\Psi\|^2 dk = \|N^{1/2}\Psi\|^2.$$

This formula can be extended as follows.

Corollary 6.3 *Let $\Psi \in \mathcal{F}_0$ and $t \geq 0$. Then*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^n} \|a(k_1) \cdots a(k_n)\Psi\|^2 dk_1 \cdots dk_n = \|e^{(1/2)\log(t+1)\ N}\Psi\|^2.$$

Proof: We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^n} \|a(k_1) \cdots a(k_n)\Psi\|^2 dk_1 \cdots dk_n = (\Psi, :e^{tN}: \Psi)$$

Then from (6.4) the lemma follows. □

6.3 Commutative cases

Theorem 6.4 *Let $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \sigma_2$ and $[S, T] = 0$. Then*

$$\begin{aligned} &\exp\left(it\left\{\frac{i}{2}(\Delta_T^\dagger - \Delta_{\overline{T}}) - iN_{\overline{S}}\right\}\right) \\ &= C \exp\left(-\frac{1}{2}\Delta^\dagger_{\tanh tT}\right) :\exp\left(-N_{(1-e^{+t\overline{S}}\cosh^{-1}tT)}\right): \exp\left(+\frac{1}{2}\Delta_{\tanh tT}\right), \quad (6.5) \end{aligned}$$

where

$$\begin{aligned} &C := e^{-i\int_0^t \frac{1}{2}\mathrm{Im}\mathrm{Tr}(T^* \tanh sT)ds} \det(1 - |\tanh tT|^2)^{1/4}, \\ &|\tanh tT|^2 = (\tanh tT)^* \tan htT. \end{aligned}$$

Proof: It follows that

$$\left[\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\right] = 0.$$

Hence we see that

$$\begin{aligned} \exp(tA) &= \exp\left(t\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\right)\exp\left(t\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} e^{tS}\cosh tT & e^{tS}\sinh tT \\ e^{tS}\sinh tT & e^{tS}\cosh tT \end{pmatrix}. \end{aligned} \tag{6.6}$$

Since

$$\begin{aligned} &(e^{t\widehat{A}})_{21}(e^{t\widehat{A}})_{11}^{-1} = (e^{t\widehat{A}})_{11}^{-1}(e^{t\widehat{A}})_{21} = \tanh tT \\ &1 - \overline{((e^{t\widehat{A}})_{11}^{-1})^*} = 1 - \overline{(e^{-tS}\cosh^{-1} tT)^*} = 1 - e^{+t\overline{S}}\cosh^{-1} tT, \end{aligned}$$

(6.5) follows from Theorem 3.7. □

Corollary 6.5 *Assume the same assumptions as in Theorem 6.4. Then*

$$(\Omega, e^{it\left\{\frac{i}{2}(\Delta_T^\dagger - \Delta_{\overline{T}}) - iN_{\overline{S}}\right\}}\Omega) = e^{-i\int_0^t \frac{1}{2}\mathrm{Im}\mathrm{Tr}(T^*\tanh sT)ds}\det(1 - |\tanh tT|^2)^{1/4}.$$

A Appendix

We demonstrate a direct proof of the following theorem in this Appendix.

Theorem A.1 *Let* $\widehat{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \sigma_2$. *Then for* $\Psi \in \mathcal{F}_0$,

$$s - \lim_{t\to 0}\frac{U_t - 1}{t}\Psi = i\Delta(\widehat{A})\Psi.$$

Before going to a proof of Theorem A.1, we show several lemmas. For notational convenience we set

$$\Delta_t^\dagger := -\frac{1}{2}\Delta_{K_t}^\dagger, \quad \Delta^\dagger = -\frac{1}{2}\Delta_T^\dagger.$$

Lemma A.2 *We have*

$$\lim_{t\to 0}\mathrm{Tr}\left[\frac{1}{t^2}\log(1 - {K_t}^* K_t)\right] = -\|T\|_2^2.$$

Proof: We shall prove

$$\lim_{t\to 0} \mathrm{Tr}\left[\frac{1}{t^2}\log(1-{K_t}^*K_t)+T^*T\right]=0.$$

Note that $\|K_t\|<1$. Then we have

$$\log(1-{K_t}^*K_t)=-\sum_{n=1}^{\infty}\frac{1}{n}({K_t}^*K_t)^n$$

in the operator norm, from which it follows that

$$\begin{aligned}\mathrm{Tr}\left[\frac{1}{t^2}\log(1-{K_t}^*K_t)+T^*T\right] &= \mathrm{Tr}\left[T^*T-\frac{1}{t^2}{K_t}^*K_t\right]-\mathrm{Tr}\left[\frac{1}{t^2}\sum_{n=2}^{\infty}({K_t}^*K_t)^n\right]\\ &= \left(\|T\|_2^2-\left\|\frac{K_t}{t}\right\|_2^2\right)-\mathrm{Tr}\left[\frac{1}{t^2}\sum_{n=2}^{\infty}({K_t}^*K_t)^n\right].\end{aligned}$$

By Lemma 4.3, $K_t/t\to T$ in B_2 as $t\to 0$. Then

$$\lim_{t\to 0}\left(\|T\|_2^2-\left\|\frac{K_t}{t}\right\|_2^2\right)=0.$$

Let $\{e_j\}$ be a complete orthonormal system of $\mathcal{H}$. We have

$$\begin{aligned}|\mathrm{Tr}\left[\frac{1}{t^2}\sum_{n=2}^{\infty}\frac{1}{n}({K_t}^*K_t)^n\right]| &= |\sum_{j=1}^{\infty}(e_j,\frac{1}{t^2}\sum_{n=2}^{\infty}({K_t}^*K_t)^ne_j)|\\ &\le \sum_{j=1}^{\infty}\frac{1}{t^2}\sum_{n=2}^{\infty}\frac{1}{n}|(K_te_j,K_t({K_t}^*K_t)^{n-2}{K_t}^*K_te_j)|\\ &\le \sum_{j=1}^{\infty}\frac{1}{t^2}\sum_{n=2}^{\infty}\frac{1}{n}\|K_te_j\|^2\|K_t\|^{2n-2}\\ &= \sum_{j=1}^{\infty}\frac{\|K_te_j\|^2}{t^2}\left(\frac{-\log(1-\|K_t\|^2)-\|K_t\|^2}{\|K_t\|^2}\right)\\ &= \left\|\frac{K_t}{t}\right\|_2^2\left(\frac{-\log(1-\|K_t\|^2)-\|K_t\|^2}{\|K_t\|^2}\right).\end{aligned}$$

Since $\|K_t/t\|_2^2\to\|T\|_2^2$ as $t\to 0$ and $\|K_t\|\to 0$ as $t\to 0$, we obtain that

$$\lim_{t\to 0}\left\|\frac{K_t}{t}\right\|_2^2\left(\frac{-\log(1-\|K_t\|^2)-\|K_t\|^2}{\|K_t\|^2}\right)=0.$$

Hence the lemma follows. □

Lemma A.3 *We have*

$$\lim_{t\to 0}\frac{1}{t^2}\left\{\det(1-{K_t}^*K_t)^{1/4}-1\right\}=-\frac{1}{4}\|T\|_2^2.$$

Proof: Since $\det(1-{K_t}^*K_t)^{1/4}\lceil_{t=0}=1$, it follows that

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t^2}\left\{\det(1-{K_t}^*K_t)^{1/4}-1\right\} &= \frac{d}{d(t^2)}\det(1-{K_t}^*K_t)^{1/4}\lceil_{t=0}\\ &= \frac{d}{d(t^2)}\log\det(1-{K_t}^*K_t)^{1/4}\lceil_{t=0}.\end{aligned}$$

Note that

$$\log\det(1-{K_t}^*K_t)^{1/4}=\frac{1}{4}\mathrm{Tr}\log(1-{K_t}^*K_t).$$

Then we have by Lemma A.2

$$\lim_{t\to 0}\frac{1}{t^2}\left\{\det(1-{K_t}^*K_t)^{1/4}-1\right\}=\frac{1}{4}\lim_{t\to 0}\frac{1}{t^2}\mathrm{Tr}\log\left(1-{K_t}^*K_t\right)=-\frac{1}{4}\|T\|_2^2.$$

The proof is complete. □

Lemma A.4 *We have*

$$\lim_{t\to 0}\left\|\frac{U_t-1}{t}\Omega\right\|=\|\Delta^{\dagger}\Omega\|.$$

Proof: We have

$$\lim_{t\to 0}\left\|\frac{U_t-1}{t}\Omega\right\|^2=\lim_{t\to 0}\frac{2}{t^2}\left\{1-\mathrm{Re}(U_t\Omega,\Omega)\right\}.$$

Note that

$$(U_t\Omega,\Omega)=\det(1-{K_t}^*K_t)^{1/4}.$$

Thus by Lemma A.3 we have

$$\lim_{t\to 0}\frac{2}{t^2}(1-\det(1-{K_t}^*K_t)^{1/4})=\frac{1}{2}\|T\|_2^2=\|\Delta^{\dagger}\Omega\|^2.$$

Thus the lemma follows. □

Lemma A.5 *Let $k\geq 1$. Then*

$$\lim_{t\to 0}\frac{1}{t}\|N^kU_t\Omega\|=\|N^k\Delta^{\dagger}\Omega\|.$$

Proof: We note that

$$\frac{1}{t^2}\|N^k U_t\Omega\|^2 = \sum_{n=1}^{\infty} \frac{1}{(n!)^2}(2n)^{2k} \left\| \frac{(\Delta_t^\dagger)^n}{t}\Omega \right\|^2 .$$

By Lemma 4.4 we have

$$\lim_{t\to 0} \left\| \frac{(\Delta_t^\dagger)^n}{t}\Omega \right\|^2 = \begin{cases} \|\Delta^\dagger\Omega\|^2, & n=1, \\ 0, & n \geq 2, \end{cases}$$

and $\lim_{t\to 0} \|(\Delta_t^\dagger)^n\Omega\|^2/t^2 \to 0$ uniformly in $n \geq 2$. Then by the Lebesgue dominated convergence theorem yields that

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t^2}\|N^k U_t\Omega\|^2 &= \lim_{t\to 0} \sum_{n=1}^{\infty} \frac{1}{(n!)^2}(2n)^{2k} \left\| \frac{(\Delta_t^\dagger)^n}{t}\Omega \right\|^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{(n!)^2}(2n)^{2k} \lim_{t\to 0} \left\| \frac{(\Delta_t^\dagger)^n}{t}\Omega \right\|^2 = 2^{2k}\|\Delta^\dagger\Omega\|^2 = \|N^k\Delta^\dagger\Omega\|^2. \end{aligned}$$

Thus the lemma follows. □

Lemma A.6 *We have*

$$s - \frac{d}{dt} N^k U_t\Omega\lceil_{t=0} = N^k\Delta^\dagger\Omega.$$

Proof: (In the case of $k = 0$)

We see that

$$\left\| \frac{U_t - 1}{t}\Omega - \Delta^\dagger\Omega \right\|^2 = \left\| \frac{U_t - 1}{t}\Omega \right\|^2 - 2\mathrm{Re}\left(\frac{U_t - 1}{t}\Omega, \Delta^\dagger\Omega \right) + \left\| \Delta^\dagger\Omega \right\|^2 . \tag{1.1}$$

From Lemma 4.5 it follows that

$$\lim_{t\to 0} \left(\frac{U_t - 1}{t}\Omega, \Delta^\dagger\Omega \right) = \|\Delta^\dagger\Omega\|^2,$$

and from Lemma A.4

$$\lim_{t\to 0} \left\| \frac{U_t - 1}{t}\Omega \right\|^2 = \|\Delta^\dagger\Omega\|^2.$$

Thus

$$\lim_{t\to 0} \left\| \frac{U_t - 1}{t}\Omega - \Delta^\dagger\Omega \right\| = 0$$

holds and the lemma follows for $k = 0$.

(In the case of $k \geq 1$)

We have

$$\begin{aligned}&\left\| N^k \frac{U_t - 1}{t}\Omega - N^k \Delta^\dagger \Omega \right\|^2 \\ &= \left\| N^k \frac{U_t - 1}{t}\Omega \right\|^2 - 2\mathrm{Re}\left(\frac{U_t - 1}{t}\Omega, N^{2k}\Delta^\dagger \Omega \right) + \left\| N^k \Delta^\dagger \Omega \right\|^2 .\end{aligned}$$

From Lemma 4.5 it follows that

$$\lim_{t\to 0}\left(\frac{U_t - 1}{t}\Omega, N^{2k}\Delta^\dagger \Omega \right) = \|N^k \Delta^\dagger \Omega\|^2,$$

and from Lemma A.5

$$\lim_{t\to 0} \frac{1}{t^2}\left\| N^k \frac{U_t - 1}{t}\Omega \right\|^2 = \lim_{t\to 0}\frac{1}{t^2}\|N^k U_t \Omega\|^2 = \|N^k \Delta^\dagger \Omega\|^2.$$

Thus

$$\lim_{t\to 0}\left\| N^k \frac{U_t - 1}{t}\Omega - N^k \Delta^\dagger \Omega \right\| = 0$$

follows. Hence the proof is complete. □

Proof of Theorem A.1

Let $\Phi = a^*(f_1)\cdots a^*(f_n)\Omega$. Then

$$\frac{U_t - 1}{t}\Phi = I(t) + II(t),$$

where

$$\begin{aligned}I(t) &= \sum_{j=1}^{n} b_t^*(f_1)\cdots b_t^*(f_{j-1})\frac{1}{t}\left(b_t^*(f_j) - a^*(f_j)\right) a^*(f_{j+1})\cdots a^*(f_n)\Omega, \\ II(t) &= b_t^*(f_1)\cdots b_t^*(f_n)\left(\frac{U_t - 1}{t}\right)\Omega, \\ b_t^*(f_j) &= a^*((e^{t\widehat{A}})_{22} f_j) + a((e^{t\widehat{A}})_{12} f_j).\end{aligned}$$

Note that

$$\begin{aligned}s - \lim_{t\to 0}(e^{t\widehat{A}})_{22} f_j &= f_j, \\ s - \lim_{t\to 0}(e^{t\widehat{A}})_{12} f_j &= 0.\end{aligned}$$

Moreover

$$\begin{aligned}
&\frac{1}{t}\left(b_t^*(f_j)-a^*(f_j)\right)=a^*(\frac{1}{t}(e^{t\widehat{A}}-E)_{22}f_j)+a(\frac{1}{t}(e^{t\widehat{A}})_{12}f_j),\\
&s-\lim_{t\to 0}\frac{1}{t}(e^{t\widehat{A}}-E)_{22}f_j=\overline{S}f_j,\\
&s-\lim_{t\to 0}\frac{1}{t}(e^{t\widehat{A}})_{12}f_j=s-\lim_{t\to 0}\frac{1}{t}(e^{t\widehat{A}}-E)_{12}f_j=\overline{T}f_j.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
s-\lim_{t\to 0}I(t) &= \sum_{j=1}^{n}a^*(f_1)\cdots\Big(a(\overline{T}f_j)+a^*(\overline{S}f_j)\Big)\cdots a^*(f_n)\Omega\\
&= \left(\frac{1}{2}\Delta_{\overline{T}}+N_{\overline{S}}\right)a^*(f_1)\cdots a^*(f_n)\Omega. \qquad (1.2)
\end{aligned}$$

Here we used the assumption $\overline{T^*}=T$. Next we shall estimate $II(t)$. We have

$$\begin{aligned}
&\left\|b_t^*(f_1)\cdots b_t^*(f_n)\left\{\frac{U_t-1}{t}-(-\frac{1}{2}\Delta_T^\dagger)\right\}\Omega\right\|\\
&\le C\left\|(N+1)^{n/2}\left\{\frac{U_t-1}{t}-(-\frac{1}{2}\Delta_T^\dagger)\right\}\Omega\right\|
\end{aligned}$$

with some constant C independent of t. By Lemma A.6 it follows that

$$\lim_{t\to 0}\left\|(N+1)^{n/2}\left\{\frac{U_t-1}{t}-(-\frac{1}{2}\Delta_T^\dagger)\right\}\Omega\right\|=0.$$

Hence we obtain that

$$\begin{aligned}
s-\lim_{t\to 0}II(t) &= s-\lim_{t\to 0}b_t^*(f_1)\cdots b_t^*(f_n)(-\frac{1}{2}\Delta_T^\dagger)\Omega\\
&= a^*(f_1)\cdots a^*(f_n)(-\frac{1}{2}\Delta_T^\dagger)\Omega\\
&= -\frac{1}{2}\Delta_T^\dagger a^*(f_1)\cdots a^*(f_n)\Omega.
\end{aligned}$$

By (1.2) and (1.3) the theorem follows. □

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