Dynamical zeta functions for analytic surface diffeomorphisms with dominated splitting

Viviane Baladi, Enrique R. Pujals, and Martín Sambarino

August 2003

Abstract. We consider a real-analytic compact surface diffeomorphism f, for which the tangent space over the nonwandering set Ω admits a dominated splitting. We study the dynamical determinant $d_f(z) = \exp - \sum_{n\ge 1} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}^* f^n} |\mathrm{Det}\,(Df^n(x) - \mathrm{Id}\,)|^{-1}$, where $\mathrm{Fix}^* f^n$ denotes the set of fixed points of f^n with no zero Lyapunov exponents. By combining previous work of Pujals and Sambarino [PS1, PS2] on C^2 surface diffeomorphisms with, on the one hand, results of Rugh [Ru1] on hyperbolic analytic maps, and on the other, our two-dimensional version of the same author's [Ru3] analysis of one-dimensional analytic dynamics with neutral fixed points, we prove that $d_f(z)$ is either an entire function or a holomorphic function in a (possibly multiply) slit plane.

1. Introduction

Let f be a real-analytic diffeomorphism of a compact two-dimensional analytic riemannian manifold M. Our dynamical assumption is that the tangent space over the nonwandering set Ω of f admits a dominated splitting, i.e., $T_\Omega M = E \oplus F$, and there are $C > 0$ and $0 < \lambda < 1$ so that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \le C\lambda^n \,, \forall x \in \Omega \,, n \ge 0 \,. \tag{1.1}$$

We study the dynamical determinant

$$d_f(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}^* f^n} \frac{1}{|\mathrm{Det}\,(Df^n(x) - \mathrm{Id}\,)|} \,, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 37C30 37D30 37E30.

V.B. is grateful to PUC-Rio and IMPA for their hospitality at several stages of this work and thanks the organisers of 2001 l'Odyssée Dynamique in CIRM/Luminy and the PRODYN programme of the European Science Foundation for support. Visits to Orsay of E.R.P. in 2000 and of E.R.P. and M.S. in 2001 were instrumental in the preparation of this article, we are grateful to Université de Paris-Sud and the Brazilian-French cooperation program. E.R.P. is partially supported by FAPERJ-Brazil and the Guggenheim Foundation, and M.S. is partially supported by the University of Maryland and CNPq/PRONEX.

where $\mathrm{Fix}^* f^n$ denotes the (finite) set of hyperbolic fixed points of f^n, i.e., those with no zero Lyapunov exponents. (Note that the dominated splitting assumption implies that each fixed point of f^n has at most one zero Lyapunov exponent.)

In order to state our theorem, we define the *$\Omega \setminus P$-isolated periodic points:* they are the elements p of the set Per of periodic points of f which are not in $\overline{(\Omega \setminus \mathrm{Per})}$, i.e., which admit a neighbourhood U_p with $U_p \cap \Omega \subset \mathrm{Per}$. (Besides isolated hyperbolic sinks, sources, and saddles, this set only contains periodic points which do not contribute to the chaotic dynamics.) In Section 4, we shall recall the decomposition of Ω into periodic, quasi-periodic, and almost hyperbolic components from [PS2]. This will be our starting point in the analysis of f. We just mention here the fact that the set $\mathcal{N}$ of nonhyperbolic periodic points of f which are *not* $\Omega \setminus P$-isolated periodic points is (empty or) finite. If $p \in \mathrm{Per}$, we write $P = P(p) \geq 1$ for its minimal period and $\lambda_E = \lambda_E(p)$, $\lambda_F = \lambda_F(p)$ for its multipliers, i.e., eigenvalues of $Df^P(p)$ with $|\lambda_E| < |\lambda_F|$ (both multipliers are real because of the dominated splitting). We associate to each $p \in \mathcal{N}$ the following subset of $\mathbb{C}$:

$$\Sigma(p) = \begin{cases} \{z \mid z^P \in [-1,1]\}\,, & \text{if } \lambda_F = -1 \text{ and } |\lambda_E| < 1, \\ \{z \mid z^P \in [\min(0,\lambda_E),1]\}\,, & \text{if } \lambda_F = +1 \text{ and } |\lambda_E| < 1, \\ \{z \mid z^P \in |\lambda_F|^{-1} \cdot [-1,1]\}\,, & \text{if } \lambda_E = -1 \text{ and } |\lambda_F| > 1, \\ \{z \mid z^P \in |\lambda_F|^{-1} \cdot [\min(0,\lambda_F^{-1}),1]\}\,, & \text{if } \lambda_E = +1 \text{ and } |\lambda_F| > 1. \end{cases} \tag{1.3}$$

Our main result can now be summarized as follows (see Section 4 for more):

Theorem A. *Let $f : M \to M$ be a real-analytic diffeomorphism of a compact analytic riemannian surface. Assume that f admits a dominated splitting* (1.1) *over its nonwandering set Ω. Let $\mathcal{N}$ be the empty or finite set of nonhyperbolic periodic points which are not $\Omega \setminus P$-isolated periodic points. Then $d_f(z)$ is is holomorphic and nonzero in the open unit disc and admits a holomorphic extension to the plane, slit plane, or multiply slit plane defined by*

$$\{z \in \mathbb{C} \mid 1/z \notin \cup_{p \in \mathcal{N}} \Sigma(p)\}\,. \tag{1.4}$$

We conjecture that the endpoints of the slits are nonpolar singularities. It is an open question whether $d(z)$ may be analytically continued across the open slits (to different sheets of a Riemann surface). See below for more involved conjectures and questions.

Theorem A immediately implies:

(1) If $\mathcal{N}$ is empty, i.e., if all nonhyperbolic points are $\Omega \setminus P$ isolated, then $d_f(z)$ is an entire function with no zeroes in the open unit disc.
(2) If there exist points in $\mathcal{N}$ with $|\lambda_F| = 1$, then $d_f(z)$ is analytic and nonzero in the disc of radius 1, with a possibly nonpolar singularity at $z = 1$ or -1, and it admits an analytic extension to a (possibly multiply) slit plane.
(3) If there exist points in $\mathcal{N}$ with $|\lambda_E| = 1$, but no points in $\mathcal{N}$ with $|\lambda_F| = 1$, letting $|\lambda_F|^{1/P}$ be the smallest modulus of P-th roots of F-multipliers in $\mathcal{N}$, then $d_f(z)$ is analytic and nonzero in the unit disc, and it may be analytically

extended to the disc of radius $|\lambda_F|^{1/P} > 1$, with (finitely many) possibly non polar singularities on its boundary, and a further analytic extension to a (possibly multiply) slit plane.

We next say a few words about the proof of Theorem A, sketching the contents of the paper. If $\mathcal{N}$ is empty, we shall see in Section 4 that f is uniformly hyperbolic on a compact invariant subset Λ of its wandering set which contains all the nonisolated hyperbolic periodic points. The results of Rugh [Ru1, Ru2] on the dynamical determinants of hyperbolic analytic maps immediately imply that $d_f(z)$ is an entire function. The key point in Rugh's analysis, inspired by Ruelle's [Rue1] seminal study (Ruelle only considered the case when the dynamical foliations are analytic), was to express $d_f(z)$ as a quotient of the Grothendieck-Fredholm determinants of two nuclear operators, proving also that zeros in the denominator are always cancelled by the numerator. The nonanalyticity of the dynamical foliations can be disregarded by working with two contracting and analytic half-inverses of f, in appropriate coordinates. In practice, Rugh [Ru2] constructs a symbolic model for a real-analytic hyperbolic map, starting from a Markov partition.

If $\mathcal{N}$ is not empty we must modify Rugh's model to investigate $d_f(z)$. The description of the corresponding *almost hyperbolic real-analytic* symbolic model $\hat{f}$ is carried out in Sections 2 and 3, while Section 4 discusses how to reduce from our surface diffeomorphism f to $\hat{f}$. In a nutshell, we discuss in Section 4 Markov partitions for f, describing how they contain both "good" (i.e., of hyperbolic type) and "bad" rectangles (those which contain an element of $\mathcal{N}$). The dynamical determinant $d_f(z)$ is morally the (regularised) determinant of a transfer operator $\widehat{\mathcal{L}}$ analysed in Sections 2–3. The building blocks of $\widehat{\mathcal{L}}$ are either "good," and of the type studied in [Ru1, Ru2], or "bad" and approximate direct products of a one-dimensional hyperbolic operator with a one-dimensional parabolic operator, studied in another work of Rugh [Ru3]. For the parabolic operator, we use a normal form [H] to adapt the analysis of one-dimensional analytic dynamics with neutral fixed points in [Ru3], to our setting. (In [Ru3], the nondiscrete spectrum of the operator was the compact interval $[0, 1]$.)

More precisely, we describe in Section 2 the almost hyperbolic model and introduce the building blocks $\widehat{\mathcal{L}}_{kj}$ of the symbolic transfer operators as well as the Banach spaces $\mathcal{B}_k$ they act on. In Section 3.A, we analyse the spectrum of the "bad" $\widehat{\mathcal{L}}_{kk}$. The crucial tool to do this is an approximate Fatou coordinate for parabolic points. Section 3.B contains a complete description of the spectrum and of the regularised determinant of the symbolic model. We combine Sections 2–3 wih Section 4 in Section 5.A, using a sequence of Markov partitions with diameter going to zero, to show that, for every neighbourhood of the "slit plane" (1.4) in Theorem A, $d_f(z)$ is analytic outside of this neighbourhood.

To keep the paper reasonably short, we do not reproduce the arguments of Rugh when they can be used without nontrivial modifications. Note also that for the sake of simplicity, in Sections 2–4, we (mostly) restrict to the case where all elements of $\mathcal{N}$ are *parabolic fixed* points in a strict sense (i.e., the order of $f - \mathrm{Id}$ is equal to two) with a nonhyperbolic multiplier equal to $+1$ (and not -1). The reduction from the general case to this setting is explained in Section 5.B.

The appendix is devoted to the construction of adapted metrics in our setting.

Finally, in order to state a conjecture motivated by our result, we recall some definitions. A u-Gibbs state is an ergodic invariant probability measure, whose induced measures along the Pesin unstable manifolds are absolutely continuous with respect to Lebesgue (see [PeS]). An invariant probability measure μ is called a physical measure if there is a set of positive Lebesgue measure of points x so that $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}$ weak-$*$ converges to μ as $n\to\infty$. We say that a compact invariant set Λ is an attracting set if there is an open neigbourhood $\widehat{\Lambda}$ such that $\Lambda=\cap_{n\geq 0}f^n(\widehat{\Lambda})$. Let Λ_j be a basic set in the decomposition of Ω from [PS1, PS2] detailed in Section 4, and assume that Λ_j is an attracting set which does not contain any periodic point with $|\lambda_F|=1$. Using the information in the appendix below — in particular Lemma A.5 for F-hyperbolic points which gives the "mostly contracting" condition; density of the strong unstable leaves follows from the fact that each mixing basic subset is a homoclinic class $\overline{W^s(p)\pitchfork W^u(p)}$ — the results of Bonatti-Viana [BoVi] may be adapted to $f|\Lambda_j$, proving that it enjoys a single u-Gibbs state which is also a physical measure. Let us call SRB measure a u-Gibbs state which is also a physical measure. (In particular, the Dirac mass at a hyperbolic sink is an SRB measure.)

By "exponential rate of mixing" for an f-invariant probability measure μ, we mean that there is $\tau<1$ so that the correlation function satisfies

$$|\rho_{\varphi,\psi}(k)|=|\int\varphi\circ f^k\psi\,d\mu-\int\varphi\,d\mu\int\psi\,d\mu|\leq C_{\varphi\psi}\tau^{|k|}\,, \tag{1.5}$$

for all $k\in\mathbb{Z}$ and all Lipschitz φ, ψ, with $C_{\varphi\psi}$ depending on the Lipschitz norms of φ and ψ. Let Λ_{i_j} be a (topologically mixing for an iterate f^{n_i}) basic subset (from the decomposition of Ω recalled in Section 4), which is attracting and does not contain any periodic point with $|\lambda_F|=1$. The results of Castro [Cas] (see also Dolgopyat [Do]) indicate that the unique SRB measure μ for f^{n_i} on Λ_{i_j} furnished by Bonatti-Viana [BoVi] has exponential rates of mixing (for Lipschitz observables). Define the *analytic correlation spectrum* of such an attracting basic subset Λ_{i_j} and its SRB measure μ to be the union over all pairs of analytic observables φ, ψ (extending holomorphically on a fixed complex neighbourhood of M) of the singular set of the Fourier transform of the correlation function:

$$\hat\rho_{\varphi\psi}(\omega)=\sum_{k\in\mathbb{Z}}e^{i\omega k}\rho_{\varphi\psi}(n_ik)\,.$$

Exponential decay of the correlation function implies that $\hat\rho$ is analytic in the strip $|\Im\omega|<\log 1/\tau$.

If $\mathcal{N}=\emptyset$, the order $D\geq 0$ of 1 as a zero of $d_f(z)$ coincides with the number of SRB measures of f. (As observed above, we are in a hyperbolic situation. The statement for the multiplicity of the pole at 1 of the dynamical zeta function $\zeta_f(z)=\exp\sum_{n\geq 1}\frac{z^n}{n}\sum_{x\in\mathrm{Fix}\,f^n}\lambda_F(x)^{-n/P(x)}$, weighted with $\lambda_F^{-1/P}$, follows e.g. from [Rue2].

Then use that $d_f(z)\zeta_f(z)$ is holomorphic and nonvanishing in a disc of radius larger than 1.)

Conjecture B. *Assume that* $\min_{\mathcal{N}} |\lambda_F| > 1$. *We conjecture that the the order* $D \geq 0$ *of* 1 *as a zero of* $d_f(z)$ *coincides with the number of (ergodic) SRB measures of* f *(which coincides with the number of attracting basic sets added to the number of hyperbolic sinks). Furthermore, under the same assumption, let* μ *be the SRB measure of an attracting basic set* Λ_j *in* Λ. *We conjecture that* $\omega \neq 0$ *is in the analytic correlation spectrum only if: either* $d_{f|\Lambda_j}(z)$ *is not holomorphic at* $z = \exp(i\omega)$ *or it vanishes there. We ask whether the other implication holds. In particular, we conjecture that the SRB measure* μ *has exponential rates of mixing (if and only if* Λ_j *is mixing), if and only if* $z = 1$ *is the only zero of modulus one of* $d_{f|\Lambda_j}(z)$.

Conjecture B states in particular that the presence of a "gap" in the dynamical determinant (of a transitive component) reflects exponential mixing of the SRB measure in the setting of (analytic) surface diffeomorphisms enjoying dominated splitting. The only setting where we know a (proved) analogue of this statement is S-unimodal interval maps [Ke].

Acknowledgements: V.B. is grateful to Artur Avila, Sébastien Gouëzel, and Hans Henrik Rugh for useful comments.

2. The symbolic maps and their transfer operators

2.A Almost hyperbolic analytic maps. The key is to reduce (using suitable coordinate charts on Markov covers close to small enough Markov partitions) the problem to a variant of the symbolic model introduced in [Ru1]: (real)-analytic hyperbolic maps. We shall call "almost hyperbolic analytic maps" our variant, where some of the building blocks are associated to periodic points with a neutral multiplier. It is convenient to use the following open "petals" in $\mathbb{C}$, associated to real numbers $r_0 > 0$, $\theta_0 \in (0, \pi)$ by

$$\mathcal{U}(\theta_0, r_0) = \{re^{i\theta} \in \mathbb{C} \mid 0 < r < r_0 \,, \, -\theta_0 < \theta < \theta_0\} \,. \tag{2.1}$$

Remark 2.1 (Fixed points or periodic points). The symbolic model of this section is adapted to the situation where all nonhyperbolic periodic points are *fixed* points with an eigenvalue $+1$ (but no eigenvalue -1). It is not difficult, although cumbersome, to lift this restriction, we shall do this Section 5.B.

Definition 2.2 (The model: almost hyperbolic analytic surface map). An almost hyperbolic analytic surface map $\hat{f}$ consists in a finite set $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, with $\mathcal{S}_0$, $\mathcal{S}_1$ nonempty and disjoint, and data

$$[(\mathcal{D}^1_i \times \mathcal{D}^2_i \subset \mathbb{C} \times \mathbb{C}) \,, (t_{ij} \in \{0, 1\}) \,, (\hat{f}_{ij} : \mathcal{D}^1_i \times \mathcal{D}^2_i \to \mathcal{D}^1_j \times \mathbb{C} \,, \forall \, t_{ij} = 1) \mid i, j \in \mathcal{S}] \,,$$

satisfying the following assumptions: The $\mathcal{S} \times \mathcal{S}$ matrix $(t_{ij})_{i,j \in I}$ is irreducible with no wandering states, and $t_{ij} = 1$ with $(i, j) \in \mathcal{S}_0 \times \mathcal{S}_0$ if and only if $i = j$. Each $\hat{f}_{ij}$ admits a real-analytic extension $(\hat{f}^1_{ij}, \hat{f}^2_{ij})$ to a neighbourhood of $\mathcal{D}^1_i \times \mathcal{D}^2_i$ and

(H) For all $i \in \mathcal{S}_1$, $\mathcal{D}_i^1$ and $\mathcal{D}_i^2$ are connected compact subsets of the complex plane with C^1 boundaries. If i or $j \in \mathcal{S}_1$ then:

(H.1) $\hat{f}_{ij}$ is contracting in the first coordinate, i.e., $\hat{f}_{ij}^1(\mathcal{D}_i^1 \times \mathcal{D}_i^2) \subset \mathrm{Int}(\mathcal{D}_j^1)$.

(H.2) $\hat{f}_{ij}$ is expanding in the second coordinate, i.e., there is a real-analytic function $\phi_{ij,s}$ (the "partial inverse") defined on a neighbourhood of $\mathcal{D}_i^1 \times \mathcal{D}_j^2$, such that for each $(w_1, z_2) \in \mathcal{D}_i^1 \times \mathcal{D}_j^2$ the image $w_2 = \phi_{ij,s}(w_1, z_2)$ lies in the interior of $\mathcal{D}_i^2$ and is the unique solution of

$$\hat{f}_{ij}^2(w_1, w_2) = z_2 \,. \tag{2.2}$$

(P) If $i = j \in \mathcal{S}_0$ there are an integer $\nu_i \geq 1$, and real numbers $\pi/(2\nu_i) < \theta_i < \tilde{\theta}_i < \pi/\nu_i$ and $\tilde{r}_i > r_i > 0$ so that $\hat{f}_{ii}$ is described in one of the two "partially hyperbolic" forms:

(P.a) $\mathcal{D}_i^1$ is a closed disc centered at the origin. The boundary of the compact connected simply connected set $\mathcal{D}_i^2$ is C^1 except at one point which is assumed to be the origin. $\mathcal{D}_i^2$ contains $\mathcal{U}(\theta_i, r_i)$ and is contained in the closure of $\mathcal{U}(\tilde{\theta}_i, \tilde{r}_i)$. The map $\hat{f}_{ii}$ fixes the origin and is contracting in the first coordinate in the sense of (H.1). There is a real-analytic function $\phi_{ii,s}(w_1, z_2)$ defined on a neighbourhood of $\mathcal{D}_i^1 \times \mathcal{D}_i^2$, which is the unique solution of $\hat{f}_{ii}^2(w_1, \cdot) = z_2$ there, and which has the normal form:

$$\phi_{ii,s}(w_1, z_2) = z_2 - z_2^{1+\nu_i} + z_2^{2+\nu_i}\tilde{\phi}_{ii,s}(w_1, z_2) \,. \tag{2.3}$$

(P.b) $\mathcal{D}_i^2$ is a closed disc centered at the origin. The boundary of the compact connected simply connected set $\mathcal{D}_i^1$ is C^1 except at one point which is assumed to be the origin. $\mathcal{D}_i^1$ contains $\mathcal{U}(\theta_i, r_i)$ and is contained in the closure of $\mathcal{U}(\tilde{\theta}_i, \tilde{r}_i)$. The map $\hat{f}_{ii}$ fixes the origin and is expanding in the second coordinate in the sense of (H.2). The (real-analytic) map $\phi_{ii,u}(w_1, z_2) = \hat{f}_{ii}^1(w_1, \phi_{ii,s}(w_1, z_2))$ enjoys the normal form

$$\phi_{ii,u}(w_1, z_2) = w_1 - w_1^{1+\nu_i} + w_1^{2+\nu_i}\tilde{\phi}_{ii,u}(w_1, z_2) \,. \tag{2.4}$$

Remark 2.3.

(1) **(Admissible sequences – $\mathcal{I}_j^i$)** A symbol sequence $\vec{\imath} = (i_1, \dots, i_n) \in \mathcal{S}^n$ is called admissible if $t_{i_k, i_{k+1}} = 1$ for every $k = 1, \dots, n-1$. Write $\mathcal{I}_i^j = \mathbb{R} \cap \mathcal{D}_i^j$ for $j = 1, 2$, $i \in \mathcal{S}$.

(2) **(Attracting and repelling petals)** In case (P.a), up to shrinking the domains, the normal form implies that for each $w_1 \in \mathcal{D}_i^1$ the map $\phi_{ii,s}(w_1, \cdot)$ sends the "attracting petal" $\mathcal{D}_i^2$ injectively into $\mathrm{int}\, \mathcal{D}_i^2 \cup \{0\}$. In case (P.b), for each $z_2 \in \mathcal{D}_i^2$ the map $\phi_{ii,u}(\cdot, z_2)$ sends the "repelling petal" $\mathcal{D}_i^1$ injectively into $\mathrm{int}\, \mathcal{D}_i^1 \cup \{0\}$.

(3) **(Invertibility)** If $|\det D\hat{f}_{ij}|$ does not vanish on $\mathcal{I}_i^1 \times \phi_{ij,s}(\mathcal{I}_i^1 \times \mathcal{I}_j^2)$ we say that $\hat{f}$ is invertible. To fix ideas, and since this will be the case in our application to surface diffeomorphisms, we assume from now on that $\hat{f}$ is invertible.

Pinning coordinates (half-inverses)

To define the transfer operator, Rugh introduces "pinning coordinates:"

Definition 2.4 (Pinning coordinates/half inverses). Let $\hat{f}$ be almost hyperbolic analytic. For each pair (i,j) with $t_{ij}=1$, the *pinning coordinates* are the two real-analytic maps $\phi_{ij,s}$, $\phi_{ij,u}$ defined on a neighbourhood of $\mathcal{D}_i^1 \times \mathcal{D}_j^2$ by

$$\begin{cases} \phi_{ij,s}(w_1,z_2) \in \mathcal{D}_i^2\,, \\ \phi_{ij,u}(w_1,z_2) = \hat{f}_{ij}^1(w_1,\phi_{ij,s}(w_1,z_2)) \in \mathcal{D}_j^1\,. \end{cases} \tag{2.5}$$

In the hyperbolic case (H), the pinning coordinate maps $\mathcal{D}_j^2$ for each $w_1 \in \mathcal{D}_i^1$, respectively $\mathcal{D}_i^1$ for each $z_2 \in \mathcal{D}_j^2$, injectively into the interior of $\mathcal{D}_i^2$, respectively $\mathcal{D}_j^1$. In the parabolic case (P.a) $\phi_{ii,u}(\cdot,z_2)$ maps $\mathcal{D}_i^1$ injectively into the interior of $\mathcal{D}_i^1$, while $\phi_{ii,s}(w_1,\cdot)$ maps $\mathcal{D}_i^2$ injectively into $\operatorname{int}\mathcal{D}_i^2 \cup \{0\}$. Case (P.b) has the natural symmetric characteristics.

It is easy to check that Definition 2.4 implies

$$\hat{f}_{ij}(w_1,\phi_{ij,s}(w_1,z_2)) = (\phi_{ij,u}(w_1,z_2),z_2)\,, \quad \forall (w_1,z_2) \in \mathcal{D}_i^1 \times \mathcal{D}_j^2\,. \tag{2.6}$$

Proposition 2.6 will show that pinning coordinates allow to "pin down" the whole orbit up to time n by knowing only the first coordinate of the initial position and the second coordinate of the final position. They are in some sense "half-inverses" for the map. They have Fried's [Fr] "cross maps" as an avatar. See also the "implicit (or macroscopic) coordinates" used by Palis-Yoccoz [PY].

A word about terminology: By definition of an almost hyperbolic analytic map, each $\phi_{ij,s}(w_1,\cdot)$ is a diffeomorphism onto its image for fixed w_1, so that $\phi_{ij,s}$ deserves to be called a "coordinate." If $\hat{f}$ is invertible, then each $\phi_{ij,u}(\cdot,z_2)$ is a diffeomorphism onto its image for fixed z_2 and both pinning maps deserve to be called "coordinates." Finally, the notation $\phi_{ij,u}$ for the (weakly) contracting direction of $\hat{f}_{ij}$ and $\phi_{ij,s}$ for its (weakly) expanding direction is used, not only for the sake of compatibility with published literature, but because the transfer operator is (at least morally, see Remark 3.6) associated to the inverse dynamics, which exchanges stable and unstable directions. This remark also applies to the "attracting" vs. "repelling petal" terminology (the adjectives refer to the inverse map) in Remark 2.3 and the following definition:

Definition 2.5 (Complements to Definition 2.2).

(1) **(Convenient extensions)** The reader is invited to check (Schwarz lemma) that Definition 2.2 implies (up to slightly changing the domains $\mathcal{D}_i^{1,2}$) that there are compact connected $\widetilde{\mathcal{D}}_i^{1,2}$ with $\mathcal{D}_i^1 \subset \operatorname{int}\widetilde{\mathcal{D}}_i^1$ and $\mathcal{D}_i^2 \subset \operatorname{int}\widetilde{\mathcal{D}}_i^2$ so that, if $(i,j) \notin \mathcal{S}_0 \times \mathcal{S}_0$ then

(1.a) $\hat{f}_{ij}$ may be extended analytically to $\widetilde{\mathcal{D}}_i^1 \times \widetilde{\mathcal{D}}_i^2$, with $\hat{f}_{ij}^1(\widetilde{\mathcal{D}_i^1 \times \mathcal{D}_i^2}) \subset \operatorname{int}(\mathcal{D}_j^1)$ in (H.1);

(1.b) $\phi_{ij,s}$ is defined on $\widetilde{\mathcal{D}_i^1} \times \widetilde{\mathcal{D}_j^2}$, and for each (w_1,z_2) there, $w_2 = \phi_{ij,s}(w_1,z_2)$ lies in the interior of $\mathcal{D}_i^2$ in (H.2);

(1.c) if $i=j\in\mathcal{S}_0$, the hyperbolic conditions of (P.a), (P.b) hold for the extended domain, and we may assume that $\phi_{ii,s}(\mathcal{D}^1_i\times\mathcal{D}^2_i)\subset \operatorname{int}\widetilde{\mathcal{D}}^2_i$ in case (P.a) and $\phi_{ii,u}(\mathcal{D}^1_i\times\mathcal{D}^2_i)\subset \operatorname{int}\widetilde{\mathcal{D}}^1_i$ in case (P.b).

(2) **(Case (P.b) – Attracting petals)** In case (P.b), it follows from the assumptions (up to slightly changing $\mathcal{D}^{1,2}_i$, $\widetilde{\mathcal{D}}^{1,2}_i$) that for each $z_2\in\mathcal{D}^2_i$, the map $\phi_{ii,u}(\cdot,z_2)$ sends an "attracting petal" $\mathcal{D}^{1,-}_i := e^{i\pi/\nu_i}\mathcal{D}^1_i$ injectively on a domain containing $\mathcal{D}^{1,-}_i$, and the inverse transformation $\phi^{-1}_{ii,u}(\cdot,z_2)$ maps $\mathcal{D}^{1,-}_i$ into $\mathcal{D}^{1,-}_i\cup\{0\}$. Up to taking a smaller attracting petal, we can assume additionally that $\phi_{ij,u}(\mathcal{D}^1_i,\mathcal{D}^2_j)\cap\mathcal{D}^{1,-}_j=\emptyset$ for all $j\in\mathcal{S}_1$ with $t_{ji}\neq 0$.

From now on we assume that $\nu_i=1$ if $i\in\mathcal{S}_0$, i.e., that all nonhyperbolic transitions are parabolic in the strict sense. We write $-\mathcal{D}^1_i$ for the attracting petal $\mathcal{D}^{1,-}_i$. (We shall explain in Section 5.B how to reduce to this situation from the general case.)

The following result is the key to iterating the "half-inverses." It hinges heavily on the analytic structure.

Proposition 2.6 (Iterating pinning coordinates in $\mathcal{S}^{n+1}\setminus\mathcal{S}^{n+1}_0$). *For each $n\geq 1$ and any admissible symbol sequence $\vec{\imath}\in\mathcal{S}^{n+1}\setminus\mathcal{S}^{n+1}_0$, there are uniquely defined iterated pinning maps, real-analytic in a neighbourhood of $\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}$*

$$\phi^{(n)}_{\vec{\imath},s}:\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}\to\mathcal{D}^2_{i_1}\,,\quad \phi^{(n)}_{\vec{\imath},u}:\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}\to\mathcal{D}^1_{i_{n+1}}\,, \tag{2.7}$$

mapping $\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}$ into the interior of $\mathcal{D}^2_{i_1}$, respectively $\mathcal{D}^1_{i_{n+1}}$, in such a way that

$$\begin{aligned}\hat f^{(n)}_{\vec{\imath}}(w_1,\phi^{(n)}_{\vec{\imath},s}(w_1,z_2)) &:= \hat f_{i_ni_{n+1}}\circ\cdots\circ\hat f_{i_1i_2}(w_1,\phi^{(n)}_{\vec{\imath},s}(w_1,z_2))\\ &= (\phi^{(n)}_{\vec{\imath},u}(w_1,z_2),z_2)\quad \forall (w_1,w_2)\in\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}\,.\end{aligned} \tag{2.8}$$

Proof of Proposition 2.6. If there are no consecutive symbols in $\mathcal{S}_0$ in $\vec{\imath}$, case (P) never occurs, and we are in the setting of [Ru1]. Let us recall his proof for completeness: set $\phi^{(1)}_{i_1i_2,u}=\phi_{i_1i_2,u}$ and $\phi^{(1)}_{i_1i_2,s}=\phi_{i_1i_2,s}$. For $n\geq 2$, assume by induction that the maps

$$\begin{cases}\phi^{(1)}_{i_ni_{n+1},s}:\mathcal{D}^1_{i_n}\times\mathcal{D}^2_{i_{n+1}}\to\mathcal{D}^2_{i_n}\,,\\ \phi^{(n-1)}_{i_1\ldots i_n,u}:\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_n}\to\mathcal{D}^1_{i_n}\,,\end{cases}$$

have been defined and satisfy the required properties. Then, for each fixed (w_1,z_2) in $\mathcal{D}^1_{i_1}\times\mathcal{D}^2_{i_{n+1}}$, one easily checks that the (real-analytic) map

$$\Phi_{w_1,z_2}(\xi_1,\xi_2)=(\phi^{(n-1)}_{i_1\ldots i_n,u}(w_1,\xi_2),\phi^{(1)}_{i_ni_{n+1},s}(\xi_1,z_2)) \tag{2.9}$$

is contracting in the sense that $\Phi_{w_1,z_2}(\mathcal{D}^1_{i_n}\times\mathcal{D}^2_{i_n})\subset\operatorname{Int}(\mathcal{D}^1_{i_n}\times\mathcal{D}^2_{i_n})$. One deduces from this (Lemma 1 in [Ru1], see also [Rue1]) that Φ_{w_1,z_2} possesses a unique fixed point

$(\xi_1^*, \xi_2^*) \in \mathrm{Int}(\mathcal{D}^1_{i_n} \times \mathcal{D}^2_{i_n})$, which depends analytically on w_1 and z_2. Finally, since $\phi^{(n-1)}_{i_1\ldots i_n,s}$ and $\phi^{(1)}_{i_n i_{n+1},u}$ also exist by induction, define the pinning maps by

$$\begin{cases} \phi^{(n)}_{\vec{\imath},u}(w_1, z_2) & = \phi^{(1)}_{i_n i_{n+1},u}(\xi_1^*(w_1, z_2), z_2)\,, \\ \phi^{(n)}_{\vec{\imath},s}(w_1, z_2) & = \phi^{(n-1)}_{i_1\ldots i_n,s}(w_1, \xi_2^*(w_1, z_2))\,. \end{cases} \tag{2.10}$$

Indeed, the induction assumption together with the fixed point property imply that

$$\begin{aligned} \hat{f}_{i_n i_{n+1}} \hat{f}^{(n-1)}_{i_1\ldots i_n}(w_1, \phi^{(n-1)}_{i_1\ldots i_n,s}(w_1, \xi_2^*)) &= \hat{f}_{i_n i_{n+1}}(\phi^{(n-1)}_{i_1\ldots i_n,u}(w_1, \xi_2^*), \xi_2^*)\,, \\ &= \hat{f}_{i_n i_{n+1}}(\xi_1^*, \phi^{(1)}_{i_n i_{n+1},s}(\xi_1^*, z_2)) \\ &= (\phi^{(1)}_{i_n i_{n+1},u}(\xi_1^*, z_2), z_2)\,. \end{aligned}$$

Uniqueness follows by induction and uniqueness of ξ_1^*, ξ_2^*.

In fact, the above argument shows that the iterated pinning coordinates map $\widetilde{\mathcal{D}}^1_{i_1} \times \widetilde{\mathcal{D}}^2_{i_{n+1}}$ into the interior of $\mathcal{D}^2_{i_1}$, respectively $\mathcal{D}^1_{i_{n+1}}$.

Next, assume for the moment that there is a single occurrence of $i_T = i_{T+1}$ in $\mathcal{S}_0$, at time $T \in \{1, \ldots, n\}$. The structure of the induction means we only need to consider $T = 1$ or n: There are thus four possibilities and we consider first the case $T = n$ and type (P.a). Our starting point is then the pair

$$\phi^{(1)}_{i_n i_{n+1},s} : \mathcal{D}^1_{i_n} \times \mathcal{D}^2_{i_{n+1}} \to \widetilde{\mathcal{D}}^2_{i_n}\,, \qquad \phi^{(n-1)}_{i_1\ldots i_n,u} : \mathcal{D}^1_{i_1} \times \mathcal{D}^2_{i_n} \to \mathcal{D}^1_{i_n}\,.$$

For each (w_1, z_2) in $\mathcal{D}^1_{i_1} \times \mathcal{D}^2_{i_{n+1}}$, the map (2.9) is contracting in the sense that

$$\Phi_{w_1,z_2}(\mathcal{D}^1_{i_n} \times \widetilde{\mathcal{D}}^2_{i_n}) \subset \mathrm{Int}(\mathcal{D}^1_{i_n} \times \widetilde{\mathcal{D}}^2_{i_n})\,.$$

Therefore, Φ_{w_1,z_2} possesses a unique fixed point $(\xi_1^*, \xi_2^*) \in \mathcal{D}^1_{i_n} \times \widetilde{\mathcal{D}}^2_{i_n}$, depending analytically on w_1 and z_2. Using the inductive assumption (see also the remark at the end of the hyperbolic argument) we may again define the pinning maps by (2.10). The case where $T = 1 = n - 1$ and (P.b) hold is similar.

If $T = n$ and (P.b), or $T = 1 = n-1$ and (P.a), hold then we need to use the inclusion from (H.1), respectively (H.2), at $i_{n-1} i_n$.

If there are never more than two consecutive symbols in $\mathcal{S}_0$ but possibly several such pairs (separated by symbols in $\mathcal{S}_1$), the argument just described also applies.

Finally, the case where there are (possibly more than one group of) three or more consecutive occurrences of symbols in $\mathcal{S}_0$ may be dealt with by an easy induction on the number of such occurrences (using again the property at the end of the first part of the proof). The idea is to first consider an $\mathcal{S}_0\mathcal{S}_0\mathcal{S}_1$ (or $\mathcal{S}_1\mathcal{S}_0\mathcal{S}_0$) event (as explained above), and then add one by one the preceding (or following) symbols in $\mathcal{S}_0$. $\square$

Remark 2.7 (Pinning coordinates as coordinates). For each $n \geq 1$ and each admissible $\vec{\imath} \in \mathcal{S}^{n+1} \setminus \mathcal{S}_0^{n+1}$, Proposition 2.6 produces a parametrisation of the subset $\mathcal{D}_{\vec{\imath}}$ of $\mathcal{D}^1_{i_1} \times \mathcal{D}^2_{i_1}$

consisting of those points (w_1, w_2) such that $\hat{f}_{i_n i_{n+1}} \circ \cdots \circ \hat{f}_{i_1 i_2}(w_1, w_2)$ is well-defined and belongs to $\mathcal{D}^1_{i_{n+1}} \times \mathcal{D}^2_{i_{n+1}}$. More precisely, $\mathcal{D}_{\vec{\imath}}$ is the isomorphic image of $\mathcal{D}^1_{i_1} \times \mathcal{D}^2_{i_{n+1}}$ under the transformation $(w_1, z_2) \mapsto (w_1, \phi^{(n)}_{\vec{\imath},s}(w_1, z_2))$. (Real-analyticity implies that this transformation maps $\mathcal{I}^1_{i_1} \times \mathcal{I}^2_{i_{n+1}}$ into $\mathcal{I}^1_{i_1} \times \mathcal{I}^2_{i_1} \subset \mathbb{R} \times \mathbb{R}$.)

An important consequence of Proposition 2.6 is the following statement (see also [Ru1]):

Corollary 2.8 (Iterating hyperbolic analytic maps). *For each $n \geq 1$ and every admissible symbol sequence $\vec{\imath} \in \mathcal{S}^{n+1} \setminus \mathcal{S}_0^{n+1}$, there exists an almost hyperbolic analytic map $\hat{f}^{(n)}_{\vec{\imath}} : \mathcal{D}_{i_1 \dots i_{n+1}} \to \mathcal{D}^1_{i_{n+1}} \times \mathcal{D}^2_{i_{n+1}}$ such that*

$$\hat{f}^{(n)}_{\vec{\imath}}|_{\mathcal{D}_{i_1 \dots i_{n+1}}} = \hat{f}_{i_{n-1} i_n} \circ \cdots \circ \hat{f}_{i_1 i_2}|_{\mathcal{D}_{i_1 \dots i_n}} .$$

The map $\hat{f}^{(n)}_{\vec{\imath}}$ has a fixed point in $\mathcal{D}_{i_1 \dots i_n}$ if and only if $i_{n+1} = i_1$, the fixed point is then unique. It follows that the "hyperbolic points of period n for $\hat{f}$," i.e., those $(w_1, w_2) \in \mathcal{D}_{\vec{\imath}}$, for some $\vec{\imath} \in \mathcal{S}^{n+1} \setminus \mathcal{S}_0^{n+1}$ such that $f^{(n)}_{\vec{\imath}}(w_1, w_2) = (w_1, w_2)$, are in bijection with the hyperbolic symbolic "cycles" of length n, i.e., admissible sequences $\vec{\jmath} \in \mathcal{S}^n \setminus \mathcal{S}_0^n$ such that $t_{j_n j_1} = 1$.

2.B Banach spaces and elementary transfer operators.

The Banach spaces $\mathcal{B}_k$, $\mathcal{B}'_k$

We now associate Banach spaces of complex functions to each $k \in \mathcal{S}_0 \cup S_1$. First, for each $k \in \mathcal{S}_1$ we set $\mathcal{B}'_k = \mathcal{B}_k$ to be the Banach space of holomorphic functions on the interior of $(\overline{\mathbb{C}} \setminus \mathcal{D}^1_k) \times \mathcal{D}^2_k$, vanishing at $\{\infty\} \times \mathcal{D}^2_k$, and which extend continuously to the boundary, with the supremum norm.

For $k \in \mathcal{S}_0$, we distinguish between cases (P.a) and (P.b). In case (P.a), we shall define below an open simply connected subset U^2_k of $\mathcal{D}^2_k$, containing the compact set

$$K^2_k := \cup_{j \in \mathcal{S}_1} \phi_{kj,s}(\mathcal{D}^1_k, \mathcal{D}^2_j) , \tag{2.11}$$

and such that $\phi_{kk,s}(w_1, U^2_k) \subset U^2_k$ for all $w_1 \in \partial \mathcal{D}^1_k$. The space $\mathcal{B}_k$ will be a subset of the space of analytic functions in $(\bar{\mathbb{C}} \setminus \mathcal{D}^1_k) \times U^2_k$, vanishing at $\{\infty\} \times U^2_k$, and extending continuously to $\partial \mathcal{D}^1_k \times U^2_k$, endowed with a norm to be introduced below, using Fatou coordinates. We shall use in the proof of Lemma 3.8 that the Banach space $\mathcal{B}'_k$ of holomorphic functions in $(\bar{\mathbb{C}} \setminus \mathcal{D}^1_k) \times U^2_k$, which vanish at $\{\infty\} \times U^2_k$ and extend continuously to the boundary, is continuously embedded in $\mathcal{B}_k$.

In case (P.b), we shall introduce below an open simply connected subset U^1_k of the attracting petal $-\mathcal{D}^1_k$ which does not intersect the compact set

$$G^1_k := \cup_{j \in \mathcal{S}_1} (\phi_{jk,u}(\mathcal{D}^1_j, \mathcal{D}^2_k)) , \tag{2.12}$$

and such that $\phi^{-1}_{kk,u}(U^1_k, z_2) \subset U^1_k$ for all $z_2 \in \mathcal{D}^2_k$ (recall Definition 2.5). The Banach space $\mathcal{B}_k$ will be a subset of the analytic functions in the interior of $U^1_k \times \mathcal{D}^2_k$, extending continuously to $U_k^1 \times \partial\mathcal{D}_k^2$, endowed with a norm to be defined below. We shall use in Lemma 3.8 that the Banach space $\mathcal{B}'_k$ of holomorphic functions in $U_k^1 \times \operatorname{int}\mathcal{D}_k^2$ which extend continuously to the boundary is continuously embedded in $\mathcal{B}_k$. Note that case (P.b) did not occur in the one-dimensional situation studied by Rugh [Ru3].

The elementary transfer operators $\widehat{\mathcal{L}}_{kj}$

Next, to each j,k with $t_{kj} = 1$, we associate an *elementary transfer operator.* If $(k,j) \in \mathcal{S}_1 \times (\mathcal{S}_1 \cup \mathcal{S}_0)$, we set for $\psi \in \mathcal{B}_k$:

$$\widehat{\mathcal{L}}_{kj}\psi\,(z_1,z_2) = \oint_{\partial\mathcal{D}_k^1}\oint_{\partial\mathcal{D}_k^2} \frac{dw_1}{2i\pi}\frac{dw_2}{2i\pi}\frac{s_{\phi'_{kj,s}}\partial_2\phi_{kj,s}(w_1,z_2)}{w_2-\phi_{kj,s}(w_1,z_2)}\frac{\psi(w_1,w_2)}{z_1-\phi_{kj,u}(w_1,z_2)}\,. \tag{2.13}$$

In (2.13), and from now on, $s_{\phi'_{kj,s}}$ is the (well-defined) sign of of $\partial_2\phi_{kj,s}$ on $\mathcal{I}_k^1 \times \mathcal{I}_j^2$. See Remark 3.6 below for a heuristic justification of this choice, and pages 1248–1250 in [Ru1] for a more analytic explanation.

If $(k,j) \in \mathcal{S}_0 \times \mathcal{S}_1$, we use (2.13), replacing $\partial\mathcal{D}_k^2$ by a simple curve Γ_k^2 inside U_k^2 which does not intersect K_k^2 in case (P.a) (this ensures that ψ is well defined on $(\partial\mathcal{D}_k^1, \partial\Gamma_k^2)$ and the integral is holomorphic). If k is of type (P.b), we replace $\partial\mathcal{D}_k^1$ by a curve Γ_k^1 inside U_k^1 such that (see Definition 2.5)

$$\cup_{j\in\mathcal{S}_1}(\phi_{kj,u}(\Gamma_j^1,\mathcal{D}_k^2)) \cap -\mathcal{D}_j^1 = \emptyset\,.$$

If $k \in \mathcal{S}_0$ is of type (P.a), we define $\widehat{\mathcal{L}}_{kk}\psi\,(z_1,z_2)$ for $(z_1,z_2) \in (\bar{\mathbb{C}} \setminus \mathcal{D}_k^1) \times U_k^2$ by

$$\widehat{\mathcal{L}}_{kk}\psi\,(z_1,z_2) = \oint_{\partial\mathcal{D}_k^1}\frac{dw_1}{2i\pi}s_{\phi'_{kk,s}}\partial_2\phi_{kk,s}(w_1,z_2)\frac{\psi(w_1,\phi_{kk,s}(w_1,z_2))}{z_1-\phi_{kk,u}(w_1,z_2)}\,. \tag{2.14}$$

In case (P.b), for $(z_1,z_2) \in U_k^1 \times \mathcal{D}_k^2$, we use

$$\begin{aligned}\widehat{\mathcal{L}}_{kk}\psi\,(z_1,z_2) = s_{\phi'_{kk,s}}\cdot&\\ &\frac{\partial_2\phi_{kk,s}(\phi_{kk,u}^{-1}(z_1,z_2),z_2)}{\partial_1\phi_{kk,u}(\phi_{kk,u}^{-1}(z_1,z_2),z_2)}\psi(\phi_{kk,u}^{-1}(z_1,z_2),\phi_{kk,s}(\phi_{kk,u}^{-1}(z_1,z_2),z_2))\,.\end{aligned} \tag{2.15}$$

Note that

$$\widehat{\mathcal{L}}_{kk}\psi\,(z_1,z_2) = \oint_{\partial\mathcal{D}_k^2}\frac{dw_2}{2i\pi}\frac{s_{\phi'_{kk,s}}\partial_2\phi_{kk,s}(\phi_{kk,u}^{-1}(z_1,z_2),z_2)}{\partial_1\phi_{kk,u}(\phi_{kk,u}^{-1}(z_1,z_2),z_2)}\frac{\psi(\phi_{kk,u}^{-1}(z_1,z_2),w_2)}{w_2-\phi_{kk,s}(\phi_{kk,u}^{-1}(z_1,z_2),z_2)}\,. \tag{2.16}$$

(The two signs cancels in the residue computation because z_1 is outside of $\mathcal{D}_k^1$.)

Remark 2.9.

1. If $j \in \mathcal{S}_1$, then $\widehat{\mathcal{L}}_{kj}$ is bounded from $\mathcal{B}_k$ to $\mathcal{B}_j = \mathcal{B}'_j$.
2. If $k \in \mathcal{S}_1$ and $j \in \mathcal{S}_0$ then $\widehat{\mathcal{L}}_{kj}$ is bounded from $\mathcal{B}_k$ to $\mathcal{B}'_j$, and $\mathcal{B}'_j \subset \mathcal{B}_j$, continuously by our conditions on U_j^1, U_j^2 in cases (P.a), respectively (P.b), so that $\widehat{\mathcal{L}}_{kj}$ is bounded from $\mathcal{B}_k$ to $\mathcal{B}_j$.

Approximate Fatou coordinates, more about Banach spaces

Our next step is to give a precise definition of the Banach spaces $\mathcal{B}_k$ associated to $k \in \mathcal{S}_0$ and to study the corresponding elementary transfer operators.

Let $\hat{f}_{kk}$ be of type (P.a). It is well-known that the injective map $\mathcal{F}(z) = 1/z$ is an approximate Fatou coordinate, i.e.,

$$\begin{aligned} &\mathcal{F}(\phi_{kk,s}(w_1, z_2)) = \mathcal{F}(z_2) + 1 + z_2 \cdot \mathcal{E}_{k,2}(w_1, z_2)\,, \\ &\mathcal{E}_{k,2} : \widetilde{\mathcal{D}}_k^1 \times \mathcal{D}_k^2 \to \mathbb{C}\,, \text{ holomorphic and bounded.} \end{aligned} \tag{2.17}$$

The set $\Omega_k^2 = \mathcal{F}(\operatorname{int} \mathcal{D}_k^2) \subset \mathbb{C}$ is open and simply connected. It is easy to check that $\Omega_k^2 + 1$ is contained in Ω_k^2 and that for some (large) R_k the domain Ω_k^2 contains the closed "right" half plane $\overline{H_{R_k}}$ where

$$H_R = \{z \in \mathbb{C} \mid \Re z > R\}\,.$$

Note that the one-dimensional Fatou coordinates $\mathcal{F}_{w_1}$ associated to each $\phi_{kk,s}(w_1, \cdot)$ (see e.g [Mi] or [Ru3, Lemma 2.1]) solves

$$\mathcal{F}_{w_1}(\phi_{kk,s}(w_1, z_2)) = \mathcal{F}_{w_1}(z_2) + 1\,, \tag{2.18}$$

while we would "like" $\widetilde{\mathcal{F}}_{w_1}(\phi_{kk,s}(w_1, z_2)) = \widetilde{\mathcal{F}}_{\phi_{kk,u}(w_1,z_2)}(z_2) + 1$, which is not immediately available in the attracting petal. Our argument will be perturbative — in Lemma 3.2 we shall compare our elementary operator $\widehat{\mathcal{L}}_{kk}$ to a direct product — it hence is possible to work with the approximate Fatou coordinate $\mathcal{F}$, which also has the important feature of being holomorphic on $\mathbb{C}^*$.

In case (P.b), recall from Definition 2.5 that the map $\phi_{kk,u}^{-1}(\cdot, z_2)$ is the inverse of $w_1 \mapsto \phi_{kk,u}(w_1, z_2)$ for fixed $z_2 \in \widetilde{\mathcal{D}}_k^2$ and $w_1 \in -\mathcal{D}_k^1$. We have

$$\begin{aligned} &\mathcal{F}(\phi_{kk,u}^{-1}(z_1, z_2)) = \mathcal{F}(z_1) - 1 + z_1 \cdot \mathcal{E}_{k,1}(z_1, z_2)\,, \\ &\mathcal{E}_{k,1} : -\mathcal{D}_k^1 \times \widetilde{\mathcal{D}}_k^2 \to \mathbb{C}\,, \text{ holomorphic and bounded}\,. \end{aligned} \tag{2.19}$$

Here $\Omega_k^{1-} = \mathcal{F}(\operatorname{int}(-\mathcal{D}_k^1))$ is open, simply connected, and there is $R_k < 0$ so that Ω_k^{1-} contains the closed left half plane $-\overline{H_{-R_k}}$.

Definition of the Banach space $X(H_R)$. For $R > 0$, let $X(H_R)$ be the isometric image of $L^1(\mathbb{R}_+, \text{Lebesgue})$ under the shifted Laplace transform

$$\tilde{\psi}(\tilde{w}) = \int_0^\infty \psi_L(t) e^{-(\tilde{w}-R)t}\, dt\,,\ \tilde{w} \in H_R\,, \tag{2.20}$$

inside the space of holomorphic functions in H_R, with induced norm.

(We refer to Doetsch [Doe] for the basics of the Laplace transform.) Functions in $X(H_R)$ are in fact bounded in H_R by the L^1 norm of ψ. One can easily check (see e.g. Lemma 2.5 in [Ru3] for similar ideas) that for any closed right-half plane $H_{R'}$ with $R' > R$ there is a constant $C_{R'-R}$ so that for each $\tilde\psi \in X(H_R)$ the derivative $\tilde\psi'$ is bounded in $H_{R'}$ by $C \cdot \int |\psi_L(t)|\, dt$, where ψ_L is an L^1 representant of the inverse Laplace transform of $\tilde\psi$ in H_R (use that $e^{-\delta t} t$ is bounded on $[0,\infty]$). It is not difficult [Ru3, Lemmas 2.2 and 2.3] to prove that the spectrum of the translation operator $S : X(H_R) \to X(H_R)$ defined by $S\tilde\psi(w) = \tilde\psi(w+1)$ is the unit interval $[0,1]$.

Laplace coordinates and $\mathcal{B}_k$ norm in case (P.a)

We next exploit the Fatou coordinates, adapting some definitions from [Ru3]. Let us consider first the case (P.a). Although Ω_k^2 contains the closed right half-plane H_{R_k}, we shall need to work with a slightly larger domain. The set K_k^2 from (2.11) is a compact subset of $\operatorname{int}\mathcal{D}_k^2$. We may thus adapt Lemma 2.4 in [Ru3], together with the arguments presented just after it, finding $R_k > m_k > 0$ and an open connected and simply connected subset N_k^2 of Ω_k^2 so that

$$\overline{H_{R_k}} \subset N_k^2 \subset H_{R_k} - m_k\,, \quad K_k^2 \subset \mathcal{F}^{-1}(N_k^2)\,,$$

$$N_k^2 + 1 \pm \frac{\sup|\mathcal{E}_{k,2}|}{|z|} \subset N_k^2\,, \forall z \in H_{R_k - m_k}\,.$$

Definition ($X(N_k^2)$). For $\widetilde R_k > R_k$, we let $X(N_k^2)$ be the subset of $X(H_{\widetilde R_k})$ consisting in those functions which admit an analytic continuation to N_k^2 with a continuous extension to the boundary. We take as norm the sum of the supremum norm on N_k^2 with the $X(H_{\widetilde R_k})$ norm of the restriction to $H_{\widetilde R_k}$.

Strictly speaking, we should replace our translation operator S on $X(H_{\widetilde R_k})$ by a translation operator T which "lives" in $X(N_k^2)$. Since this does not influence the spectrum (details are to be found in [Ru3, (2.27–2.30)]) we shall instead abuse notation.

Definition of U_k^2 and $\mathcal{B}_k$, $\widetilde{\mathcal{B}}_k$. Define an open subset of $\mathcal{D}_k^2$ by $U_k^2 = \operatorname{int}\mathcal{F}_2^{-1}(N_k^2)$, it has the required properties. We use the notation $\mathcal{A}(\bar{\mathbb{C}} \setminus \mathcal{D}_k^1)$ for the Banach space of holomorphic functions in $\bar{\mathbb{C}} \setminus \mathcal{D}_k^1$ vanishing at infinity and extending continuously to the boundary. Let $\mathcal{B}_k$ be the set of analytic functions in $(\bar{\mathbb{C}} \setminus \mathcal{D}_k^1) \times U_k^2$ given by the isometric image of the Banach space tensor product $\widetilde{\mathcal{B}}_k = \mathcal{A}(\bar{\mathbb{C}} \setminus \mathcal{D}_k^1) \otimes X(N_k^2)$ under J^{-1} defined by

$$\psi = (J^{-1}\tilde\psi)\,, \quad \psi(z_1, z_2) = -\tilde\psi(z_1, \mathcal{F}(z_2)) \cdot \mathcal{F}'(z_2) = \tilde\psi(z_1, \mathcal{F}(z_2)) z_2^{-2}\,, \tag{2.21}$$

with induced norm. In particular, if $\tilde\psi \in \widetilde{\mathcal{B}}_k$ then $\tilde\psi(w_1, \cdot) \in X(N_k^2)$ for every $w_1 \in \bar{\mathbb{C}} \setminus \mathcal{D}_k^1$.

Writing $\mathcal{E}'_{k,2} = \partial_2 \mathcal{E}_{k,2}$, the elementary transfer operator $\widehat{\mathcal{L}}_{kk}$ (2.14) in the Fatou coordinates (i.e., acting on $\widetilde{\mathcal{B}}_k$ via (2.21)) can be written as

$$\widetilde{\mathcal{L}}_{kk}\tilde\psi(z_1, \tilde z_2) = s_{\phi'_{kk,s}} \oint_{\partial D_k^1}$$

$$\frac{dw_1}{2i\pi}\left(1 - \frac{\mathcal{E}_{k,2}(w_1, 1/\tilde z_2) + \tilde z_2^{-1}\mathcal{E}'_{k,2}(w_1, 1/\tilde z_2)}{\tilde z_2^2}\right) \cdot \frac{\tilde\psi(w_1, \tilde z_2 + 1 + \tilde z_2^{-1}\mathcal{E}_{k,2}(w_1, 1/\tilde z_2))}{z_1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))}\,. \tag{2.22}$$

Indeed, differentiating both sides of (2.17) with respect to z_2 yields

$$\mathcal{F}'(\phi_{kk,s}(w_1,z_2))\partial_2\phi_{kk,s}(w_1,z_2) = \mathcal{F}'(z_2)\left(1+\frac{\mathcal{E}_{k,2}(w_1,z_2)+z_2\mathcal{E}'_{k,2}(w_1,z_2)}{\mathcal{F}'(z_2)}\right),$$

so that $\widehat{\mathcal{L}}_{kk}(J^{-1}\tilde{\psi})$ coincides with $J^{-1}(\widetilde{\mathcal{L}}_{kk}\tilde{\psi})$, using (2.17) again.

Laplace coordinates and $\mathcal{B}_k$ norm in case (P.b)

Let us now discuss case (P.b). We shall use left half-planes $-H_{-R}$ for $R<0$ and spaces $X(-H_{-R})$ of "left" Laplace transforms

$$\tilde{\psi}(\tilde{z}) = \int_0^\infty \psi_L(t)e^{(\tilde{z}-R)t}\,dt\,,\ \Re\tilde{z}<R\,,\ \psi_L\in L^1(\mathbb{R}^+,\text{ Lebesgue})\,. \tag{2.23}$$

Since the closed attracting petal $-\mathcal{D}^1_k$ does not intersect G^1_k from (2.12), the open left half plane $N^1_k=-H_{-R_k}$ satisfies $\mathcal{F}^{-1}(N^1_k)\cap G^1_k=\emptyset$ and

$$N^1_k - 1 \pm \frac{\sup|\mathcal{E}_{k,1}|}{|z|}\subset N^1_k\,,\forall z\in -H_{-R_k}\,.$$

Definition of $X(N^1_k)$, U^1_k, $\mathcal{B}_k$, $\widetilde{\mathcal{B}}_k$. Fix $\widetilde{R}_k<R_k<0$. Let $X(N^1_k)$ be the subset of $X(-H_{-\widetilde{R}_k})$ consisting in those functions which extend analytically to N^1_k and continuously to its boundary, with norm the sum of the L^1 norm of the inverse Laplace transform with the supremum in N^1_k. Set $\widetilde{\mathcal{B}}_k=X(N^1_k)\otimes\mathcal{A}(\mathcal{D}^2_k)$ (with $\mathcal{A}(\mathcal{D}^2_k)$ the Banach space of holomorphic functions in $\operatorname{int}\mathcal{D}^2_k$ extending continuously to the boundary).

Defining the following analogue of (2.21)

$$\psi=(J^{-1}\tilde{\psi})\,,\quad \psi(z_1,z_2)=-\tilde{\psi}(\mathcal{F}(z_1),z_2)\mathcal{F}'(z_1)\,, \tag{2.24}$$

we finally let $\mathcal{B}_k$ be the isometric image of $\widetilde{\mathcal{B}}_k$ under J^{-1}.

Thus, elements of $\mathcal{B}_k$ are analytic functions in the interior of $U^1_k\times\mathcal{D}^2_k$ for the open connected set $U^1_k=\mathcal{F}^{-1}(N^1_k)$.

Differentiating (2.19) with respect to z_1 (writing $\mathcal{E}'_{k,1}=\partial_1\mathcal{E}_{k,1}$) yields

$$\mathcal{F}'(\phi^{-1}_{kk,u}(z_1,z_2))\partial_1\phi^{-1}_{kk,u}(z_1,z_2)=\mathcal{F}'(z_1)\left(1+\frac{\mathcal{E}_{k,1}(z_1,z_2)+z_1\mathcal{E}'_{k,1}(z_1,z_2)}{\mathcal{F}'(z_1)}\right).$$

Set $v_1=\mathcal{F}^{-1}(\tilde{z}_1-1+\tilde{z}_1^{-1}\mathcal{E}_{k,1}(\tilde{z}_1^{-1},z_2))$ and note that

$$v_1=\phi^{-1}_{kk,u}(\mathcal{F}^{-1}\tilde{z}_1,z_2)\,. \tag{2.25}$$

The transfer operator (2.15) in the Fatou coordinates can be written for $(\tilde{z}_1,z_2)\in\mathcal{F}(U^1_k)\times\mathcal{D}^2_k$ as

$$\begin{aligned}\widetilde{\mathcal{L}}_{kk}\tilde{\psi}(\tilde{z}_1,z_2)=s_{\phi'_{kk,s}}&\left(1-\frac{\mathcal{E}_{k,1}(1/\tilde{z}_1,z_2)+\tilde{z}_1^{-1}\mathcal{E}'_{k,1}(1/\tilde{z}_1,z_2)}{\tilde{z}_1^2}\right)\\ &\oint_{\partial\mathcal{D}^2_k}\frac{dw_2}{2i\pi}\frac{\partial_2\phi_{kk,s}(v_1,z_2)}{w_2-\phi_{kk,s}(v_1,z_2)}\tilde{\psi}(\mathcal{F}v_1,w_2)\,.\end{aligned} \tag{2.26}$$

(Use $(\partial_1\phi_{kk,u}(\phi^{-1}_{kk,u}(z_1,z_2),z_2))^{-1}=\partial_1\phi^{-1}_{kk,u}(z_1,z_2)$.)

3.A Direct product transfer operators for (P.a)–(P.b).

For $k \in \mathcal{S}_0$ of type (P.a), we introduce a direct tensor product operator, written using the Laplace transforms (2.20) of $\tilde\psi(w_1, \tilde w_2)$ for each fixed w_1 as

$$\widetilde{\mathcal{L}}^{\otimes}_{kk}\tilde\psi(z_1, \tilde z_2) = s_{\phi'_{kk,s}} \oint_{\partial D^1_k} \frac{dw_1}{2i\pi} \frac{1}{(z_1 - \lambda_{k,u} w_1)} \int_0^\infty e^{-(\tilde z_2 - \widetilde R_k + 1)t}\, \psi_L(w_1, t)\, dt\,, \tag{3.1}$$

where $\lambda_{k,u} = \partial_1 \phi_{kk,u}(0,0)$.

If k is of type (P.b), setting $\lambda_{k,s} = \partial_2 \phi_{kk,s}(0,0)$, the corresponding direct tensor product approximation can be written as

$$\widetilde{\mathcal{L}}^{\otimes}_{kk}\tilde\psi(\tilde z_1, z_2) = s_{\phi'_{kk,s}} \lambda_{k,s} \int_0^\infty e^{(\tilde z_1 - \widetilde R_k - 1)t} \psi_L(t, \lambda_{k,s} z_2)\, dt\,. \tag{3.2}$$

We define a direct sum of direct tensor products $\widetilde{\mathcal{L}}^{\otimes}_0$ acting on $\widetilde{\mathcal{B}}_0 = \oplus_{k\in\mathcal{S}_0} \widetilde{\mathcal{B}}_k$ by setting $\widetilde{\mathcal{L}}^{\otimes}_0 = \oplus_{k\in\mathcal{S}_0} \widetilde{\mathcal{L}}^{\otimes}_{kk}$. We use similar notations $\mathcal{B}_0$, $\mathcal{L}^{\otimes}_0$, corresponding to the conjugated operators $\mathcal{L}^{\otimes}_{kk}$ on $\mathcal{B}_k$.

Lemma 3.1 (Spectrum and resolvent of the direct products $\widetilde{\mathcal{L}}^{\otimes}_0$, $\mathcal{L}^{\otimes}_0$).

(1) *Let $\{\lambda_{k,u}\}$, in case (P.a), and $\{\lambda_{k,s}\}$ in case (P.b), be defined above. The operator $\widetilde{\mathcal{L}}^{\otimes}_0$ is bounded on $\widetilde{\mathcal{B}}_0$ and its spectrum is is the following set:*

$$\{[0,1]\,, k \text{ type (P.a)}\,, \lambda_{k,u} > 0\} \cup \{[\lambda_{k,u}, 1]\,, k \text{ type (P.a)}\,, \lambda_{k,u} < 0\}$$
$$\cup\, \{[0, \lambda_{k,s}]\,, k \text{ type (P.b)}\,, \lambda_{k,s} > 0\} \cup \{[-\lambda^2_{k,s}, -\lambda_{k,s}]\,, k \text{ type (P.b)}\,, \lambda_{k,s} < 0\}\,.$$

(2) *Let $\widetilde{\mathcal{B}}_0(\epsilon)$ be the Banach space obtained by replacing discs $\mathcal{D}^{1,2}_k$ of radius r by discs of radius ϵr, and, in case (P.a) $\widetilde R_k$ by $\widetilde R_k + 1/\epsilon$, N^2_k by $N^2_k + 1/\epsilon$, in case (P.b) $\widetilde R_k$ by $\widetilde R_k - 1/\epsilon$, N^1_k by $N^1_k - 1/\epsilon$. Then, for every $1/z \notin \operatorname{sp} \widetilde{\mathcal{L}}^{\otimes}_0$ (on $\widetilde{\mathcal{B}}_0$) there is $C(z)$ so that*

$$\|(1 - z\widetilde{\mathcal{L}}^{\otimes}_0)^{-1}\|_{\widetilde{\mathcal{B}}_0(\epsilon)} \le C(z) \|(1 - z\widetilde{\mathcal{L}}^{\otimes}_0)^{-1}\|_{\widetilde{\mathcal{B}}_0}\,, \quad \forall \epsilon > 0\,.$$

(1–2) *hold for $\mathcal{L}^{\otimes}_0$ acting on $\mathcal{B}_0$ and $\mathcal{B}_0(\epsilon)$.*

Note that in the definition of $\mathcal{B}_0(\epsilon)$ we disregard the condition $K^2_k \subset \mathcal{F}^{-1}(N^2_k)$, but we do use, in case (P.a) e.g., that condition (H.2) also holds for the disc $\epsilon \cdot \mathcal{D}^1_k$ and the restricted domain $\mathcal{D}^2_k \cap \mathcal{F}^{-1}(N^2_k + 1/\epsilon)$. See also Remark 4.7.

Proof of Lemma 3.1. Since $\widetilde{\mathcal{L}}^{\otimes}_0$ is a direct sum, it suffices to consider each term $\widetilde{\mathcal{L}}^{\otimes}_{kk}$ acting on $\widetilde{\mathcal{B}}_k$. So, let us fix $k \in \mathcal{S}_0$, assuming first that we are in case (P.a). Then, since we assumed that the neutral eigenvalue is $+1$, we have $s_{\phi'_{kk,s}} = 1$. The operator $\widetilde{\mathcal{L}}^{\otimes}_{kk}$ (3.1) is thus the direct tensor product $\widetilde{\mathcal{L}}^{\otimes}_{kk} = \mathcal{M}_{k,u} \otimes \mathcal{T}^+$, with

$$\mathcal{M}_{k,u}\psi_1(z_1) = \oint_{\partial D^1_k} \frac{dw_1}{2i\pi} \frac{1}{(z_1 - \lambda_{k,u} w_1)} \psi_1(w_1)\,,$$
$$\mathcal{T}^+ \widetilde{\psi}_2(\tilde{z}_2) = \int_0^\infty e^{-(\tilde{z}_2 - \widetilde{R}_k + 1)t} \psi_L(t)\, dt\,.$$

The results of Ichiniose [Ic] give that its spectrum is just $\{\sigma_1 \cdot \sigma_2\,, \sigma_1 \in \mathrm{sp}(\mathcal{M}_{k,u})\,, \sigma_2 \in \mathrm{sp}(\mathcal{T}^+)\}$. The remarks above and the ideas in [Ru3] easily yield that the spectrum of $\mathcal{T}^+$ on $X(N^2_k)$ is the unit interval $[0,1]$. One also obtains that $\mathcal{M}_{k,u}$ is nuclear on the Banach space $\mathcal{A}(\bar{\mathbb{C}} \setminus \mathcal{D}^1_k)$ of functions holomorphic in $\bar{\mathbb{C}} \setminus \mathcal{D}^1_k$, vanishing at ∞, and extending continuously to $\partial\mathcal{D}^1_k$, and that its spectrum consist of the eigenvalues $\lambda^\ell_{k,u}$ for $\ell \in \mathbb{Z}_+$. (Indeed, the trace of $\mathcal{M}^\ell_{k,u}$ can be shown, following the methods in [Ru1, Ru2, Ru3] to be $1/(1-\lambda^\ell_{k,u})$.)

If we are in case (P.b), we proceed similarly, starting from (3.2) and using

$$\mathcal{T}^- \widetilde{\psi}(\tilde{z}_1) = \int_0^\infty e^{(\tilde{z}_1 - \widetilde{R}_k - 1)t} \psi_L(t)\, dt\,,$$
$$\mathcal{M}_{k,s}\psi_2(z_2) = \oint_{\partial D^2_k} \frac{dw_2}{2i\pi} \frac{s_{\phi'_{kk,s}} \cdot \lambda_{k,s}}{(w_2 - \lambda_{k,s} z_2)} \psi_2(w_2) = |\lambda_{k,s}| \cdot \psi_2(\lambda_{k,s} z_2)\,.$$

For this, just check that the trace of $\mathcal{M}^\ell_{k,s}$ is $|\lambda^\ell_{k,s}|/(1-\lambda^\ell_{k,s})$. Its spectrum is thus $(\mathrm{sgn}\,\lambda_{k,s}) \cdot (\lambda_{k,s})^\ell$ for $\ell \in \mathbb{Z}^*_+$.

For the second claim, observe first that modifying $\widetilde{R}_k$ and N^i_k does not affect the spectrum or the norm of the resolvent, by definition of the Laplace norm and because $\mathcal{F}'(w_1)/\mathcal{F}'(w_1 \pm 1/\epsilon)$ is uniformly bounded. It thus suffices to consider $\widetilde{\mathcal{L}}^{\otimes}_0$ on $\widetilde{\mathcal{B}}_0(\epsilon)$, where only the hyperbolic radii of $\mathcal{D}^{1,2}_k$ have been rescaled.

Then, denoting by $\mathcal{H}_\epsilon : \widetilde{\mathcal{B}}_0(\epsilon) \to \widetilde{\mathcal{B}}_0$ the following isometry on $\widetilde{\mathcal{B}}_k$ for $k \in \mathcal{S}_0$: $\mathcal{H}_{\epsilon,k}\varphi(w_1, \tilde{w}_2) = \varphi(\epsilon w_1, \tilde{w}_2)$, and similarly in case (P.b), we see that $\mathcal{H}^{-1}_\epsilon \widetilde{\mathcal{L}}^{\otimes}_0 \mathcal{H}_\epsilon = \widetilde{\mathcal{L}}^{\otimes}_0$ (this follows from the facts that the "weight" in $\mathcal{M}_{k,(u,s)}$ is constant while the "dynamics" is linear) and thus

$$\mathcal{H}^{-1}_\epsilon (1 - z\widetilde{\mathcal{L}}^{\otimes}_0)^{-1} \mathcal{H}_\epsilon = (1 - z\widetilde{\mathcal{L}}^{\otimes}_0)^{-1}\,.$$

Since $\|\mathcal{H}^{\pm 1}_\epsilon\| = 1$ for all ϵ, we have proved the bounds on $\widetilde{\mathcal{B}}_0(\epsilon)$. $\square$

Lemma 3.2 (Perturbation theory for $\widehat{\mathcal{L}}_0$). *$\widehat{\mathcal{L}}_0$ is bounded on $\mathcal{B}_0$ and on $\mathcal{B}_0(\epsilon)$, and there is a positive function $\mathcal{G}$ with $\mathcal{G}(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$\mathrm{sp}(\widehat{\mathcal{L}}_0|_{\mathcal{B}_0(\epsilon)}) \subset \{z \in \mathbb{C} \mid \exists \tilde{z} \in \mathrm{sp}\,\widetilde{\mathcal{L}}^{\otimes}_0\,, |z - \tilde{z}| = \mathcal{G}(\epsilon)\}\,.$$

We summarize in two sublemmas elementary properties of the Laplace norm which will be used in the proof of Lemma 3.2:

Sublemma 3.3. *Let $\tilde{\psi}_1 \in X(H_R)$ for $R > 0$. Then:*

(1) *For each integer $s \geq 1$, the function $\tilde{\psi}_1(\tilde{z})/\tilde{z}^s$ belongs to $X(H_R)$ and*

$$\|\tilde{\psi}_1(\tilde{z})/\tilde{z}^s\|_{X(H_R)} \leq \|\tilde{\psi}_1\|_{X(H_R)}/R^s \,.$$

(2) *For every w such that $\Re w > 0$, the functions $\tilde{\psi}_1(\tilde{z}+w)$ and $\tilde{\psi}_1'(\tilde{z}+w)$ belong to $X(H_R)$ and*

$$\max\big(\|\tilde{\psi}_1(\tilde{z}+w)\|_{X(H_R)}, \|\tilde{\psi}_1'(\tilde{z}+w)\|_{X(H_R)}\big) \leq \|\tilde{\psi}_1\|_{X(H_R)}/(e \cdot \Re w) \,.$$

(3) *Assume that $\tilde{\psi}_1$ admits a bounded holomorphic extension to $H_{R-2\eta}$ for $\eta > 0$ and $R - \eta > 0$. Let $\Phi : H_{R-2\eta} \to H_{R-2\eta}$ be holomorphic. For each integer $s \geq 2$, the function $\tilde{\psi}_1(\Phi(\tilde{z}))/\tilde{z}^s$ belongs to $X(H_R)$ and*

$$\left\| \frac{\tilde{\psi}_1(\Phi(\tilde{z}))}{\tilde{z}^s} \right\|_{X(H_R)} \leq \sup_{H_{R-2\eta}} |\tilde{\psi}_1| \frac{4}{\eta \cdot (R-\eta)^{s-1}} \,.$$

(4) *If $\mathcal{E}(z)$ is holomorphic and bounded in a neighbourhood $\mathcal{D}$ of zero such that $\mathcal{F}(\mathcal{D})$ contains a closed half-plane $H_{R'}$, $0 < R' < R$, then $\mathcal{E}(1/\tilde{z})\tilde{\psi}_1(\tilde{z}) \in X(H_R)$ and*

$$\|\mathcal{E}(1/\tilde{z})\tilde{\psi}_1(\tilde{z})\|_{X(H_R)} \leq \frac{2\pi}{1 - R'/R} \sup_{\mathcal{D}} |\mathcal{E}| \cdot \|\tilde{\psi}_1\|_{X(H_R)} \,.$$

(There are obvious analogues for $\tilde{\psi}_2 \in X(-H_R^-)$ with $R < 0$.)

Sublemma 3.4. *Assume that $\tilde{\psi}_1 \in X(H_R)$ for $R > 0$, and that $\tilde{\psi}_1$ admits a bounded holomorphic extension to $H_{R-2\eta}$ for some $\eta > 0$. If $\mathcal{E}(z_1, z_2)$ is analytic and bounded in $\mathcal{D}^1 \times \mathcal{D}^2$ such that $\mathcal{F}(\mathcal{D}^1)$ contains a closed half-plane $H_{R'}$ with $0 < R' < R$, and $\mathcal{D}_2$ is a neighbourhood of zero, then for each $\rho < 1$ the product $\mathcal{E}(1/\tilde{z}_1, z_2) \cdot \tilde{\psi}_1(\tilde{z}_1)$ is an element of $X(H_R) \otimes \mathcal{A}(\rho\mathcal{D}^2)$ and*

$$\|\mathcal{E}(1/\tilde{z}_1, z_2) \cdot \tilde{\psi}_1(\tilde{z}_1)\| \leq \sup_{\mathcal{D}^1 \times \mathcal{D}^2} |\mathcal{E}| \frac{\|\tilde{\psi}_1\|_{X(H_R)}}{(1-\rho)(1 - R'/R)} \,.$$

Proof of Sublemma 3.3. (1) By induction, it is enough to consider $s = 1$. We first note that $\tilde{z}^{-1} = \int_0^\infty e^{-(\tilde{z}-R)t} e^{-Rt}\, dt$ belongs to $X(H_R)$. Since e^{-Rt} is also in L^∞, we may obtain an (L^1) inverse Laplace transform of $\tilde{\psi}_1(\tilde{z})/\tilde{z}$ by performing the convolution

$$e^{-Rt} \star \psi_{1,L}(t) = \int_0^t e^{-R\tau} \psi_{1,L}(t-\tau) d\tau \,.$$

Now, $\int_0^\infty |e^{-Rt} \star \psi_{1,L}(t)| dt \leq \int_0^\infty d\tau e^{-R\tau} \int_\tau^\infty dt |\psi_{1,L}(t-\tau)| \leq \|\tilde{\psi}_1\| \int_0^\infty e^{-R\tau} d\tau$ which is equal to $\|\tilde{\psi}_1\|_{X(H_R)}/R$.

(2) By definition $\tilde{\psi}_1'(\tilde{z}+w) = -\int_0^\infty te^{-(\tilde{z}+w-R)t}\psi_L(t)\,dt$. Estimating the L^1 norm of $-te^{-wt}\psi_{1,L}(t)$, we obtain the claim for $\tilde{\psi}_1'(\tilde{z}+w)$ the bound for $\tilde{\psi}_1(\tilde{z}+w)$ is easier.

(3) We can express the inverse Laplace transform of $\tilde{\psi}_1(\Phi(\tilde{z}))/\tilde{z}^s$ as

$$\psi_{1,L,s}(t) = \int_{\partial H_{R-\eta}} \frac{du}{2i\pi} e^{t(u-R)} \frac{\tilde{\psi}_1(\Phi(u))}{u^s} = e^{-t\eta} \int_{\partial H_R} \frac{dv}{2i\pi} e^{t(v-R)} \frac{\tilde{\psi}_1(\Phi(v-\eta))}{(v-\eta)^s}\,.$$

It follows that

$$|\psi_{1,L,s}(t)| \le e^{-\eta t} \sup_{H_{R-\eta}} |\tilde{\psi}_1| \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{1}{((R-\eta)^2+x^2)^{s/2}}\,,$$

which gives the claimed bound.

(4) Let $\mathcal{E}(z) = \sum_{m=0}^\infty a_m z^m$ be a Taylor series for $\mathcal{E}$ at the origin. By the Cauchy formula we have $|a_m| \le 2\pi \sup_{\mathcal{D}} |\mathcal{E}|(R')^m$ for $m \ge 0$. Then

$$\mathcal{E}(1/\tilde{z})\tilde{\psi}_1(\tilde{z}) = \sum_{m=0}^\infty a_m \frac{\tilde{\psi}_1(\tilde{z})}{\tilde{z}^m}\,.$$

Then, bound (1) shows that the sum is $\le \sum_{m\ge 0} |a_m| \|\tilde{\psi}_1\| R^{-m}$. $\square$

Proof of Sublemma 3.4. As in the proof of Sublemma 3.3 (4), we use a Taylor series

$$\mathcal{E}(z_1,z_2) = \sum_{m=0}^\infty a_m(z_1) z_2^m\,,$$

where each $a_m(z_1)$ is holomorphic in $\mathcal{D}^1$ and, if $\delta > 0$ is the radius of $\mathcal{D}^2$,

$$\sup_{\mathcal{D}^1} |a_m| \le \frac{2\pi \sup_{\mathcal{D}^1\times\mathcal{D}^2} |\mathcal{E}|}{\delta^m}\,.$$

By Sublemma 3.3 (4), we know that each $a_m(1/\tilde{z}_1)\tilde{\psi}_1(\tilde{z}_1)$ belongs to $X(H_R)$ with $\|a_m(1/\tilde{z}_1)\tilde{\psi}_1(\tilde{z}_1)\|_{X(H_R)} \le 2\pi \|\tilde{\psi}_1\|_{X(H_R)} \sup_{\mathcal{D}^1} |a_m|/(1-R'/R)$. Finally, we may write

$$\mathcal{E}(1/\tilde{z}_1, \rho z_2)\tilde{\psi}_1(\tilde{z}_1) = \sum_{m=0}^\infty \rho^m z_2^m a_m(1/\tilde{z}_1)\tilde{\psi}_1(\tilde{z}_1)\,. \quad \square$$

Proof of Lemma 3.2. For $k \in \mathcal{S}_0$, recall that $\lambda_{k,u} = \partial_1 \phi_{kk,u}(0,0)$, $\lambda_{k,s} = \partial_2 \phi_{kk,s}(0,0)$ and set

$$\Delta_k = \begin{cases} \sup_{w_1 \in \partial\mathcal{D}_k^1, z_2 \in \mathcal{D}_k^2} |\lambda_{k,u} - \phi_{kk,u}(w_1,z_2)/w_1|\,, & \text{in case (P.a),} \\ \sup_{w_1 \in -\mathcal{D}_k^1, z_2 \in \mathcal{D}_k^2} \max(|\lambda_{k,s} - \phi_{kk,s}(w_1,z_2)/z_2|, & \\ \qquad\qquad |\lambda_{k,s} - \partial_2\phi_{kk,s}(w_1,z_2)|)\,, & \text{in case (P.b).} \end{cases}$$

We can also introduce $\Delta_k(\epsilon)$ replacing the $\mathcal{D}_k^i$ by the modified domains as when defining $\mathcal{B}_0(\epsilon)$. We shall prove that there is $C > 0$ such that (setting $m_k = 0$ in case (P.b))

$$\|\widehat{\mathcal{L}}_0 - \mathcal{L}_0^{\otimes}\|_{\mathcal{B}_0} \le C \max_{k\in\mathcal{S}_0}\Big\{\sup(|\mathcal{E}_{k,i}| + |\mathcal{E}'_{k,i}|)\cdot(|\widetilde{R}_k - m_k|^{-1} + \Delta_k)(1-|\lambda_{k,u/s}|)^{-1}\,.\Big\} \quad (3.4)$$

This will immediately imply that $\widehat{\mathcal{L}}_0$ is bounded on $\mathcal{B}_0$.

The proof of (3.4) also gives the upper bound $C\max_{k\in\mathcal{S}_0}\{(|\widetilde{R}_k - m_k| + 1/\epsilon)^{-1} + \Delta_k(\epsilon)\}$ for $\|\widehat{\mathcal{L}}_0 - \mathcal{L}_0^{\otimes}\|$ on $\mathcal{B}_0(\epsilon)$.

For the spectral claim, we use the fact that $\sup_{k\in\mathcal{S}_0}\Delta_k(\epsilon)$ tends to zero as ϵ goes to zero. We cannot apply ordinary perturbation theory, since the Banach spaces vary. However, we can invoke Lemma 3.1 (2) together with

$$(1 - z\mathcal{L}_0)^{-1} = \sum_{j=0}^{\infty}\big(z(1 - z\mathcal{L}_0^{\otimes})^{-1}\circ(\mathcal{L}_0 - \mathcal{L}_0^{\otimes})\big)^j\cdot(1 - z\mathcal{L}_0^{\otimes})^{-1}\,,\ \text{on } \mathcal{B}_0(\epsilon)\,.$$

Let us prove (3.4), considering first case (P.a). We concentrate on the Laplace component of the norm, the supremum component is easier to handle. Since we may rewrite the direct product (3.1) using $\tilde\psi(w_1, \tilde z_2 + 1)$ in lieu of the Laplace transform, we have from (2.22)

$$\begin{aligned}
s_{\phi'_{kk,s}}&(\widetilde{\mathcal{L}}_{kk} - \widetilde{\mathcal{L}}_{kk}^{\otimes})(\tilde\psi)(z_1, \tilde z_2) = E_A + E_B + E_C\\
&= \oint_{\partial D_k^1}\frac{dw_1}{2i\pi}\left(1 - \frac{\mathcal{E}_{k,2}(w_1, 1/\tilde z_2) + \tilde z_2^{-1}\mathcal{E}'_{k,2}(w_1, 1/\tilde z_2)}{\tilde z_2^2}\right)\\
&\qquad\qquad\cdot\frac{\tilde\psi(w_1, \tilde z_2 + 1 + \tilde z_2^{-1}\mathcal{E}_{k,2}(w_1, 1/\tilde z_2)) - \tilde\psi(w_1, \tilde z_2 + 1)}{z_1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))}\\
&\quad - \oint_{\partial D_k^1}\frac{dw_1}{2i\pi}\frac{\mathcal{E}_{k,2}(w_1, 1/\tilde z_2) + \tilde z_2^{-1}\mathcal{E}'_{k,2}(w_1, 1/\tilde z_2)}{\tilde z_2^2}\left(\frac{\tilde\psi(w_1, \tilde z_2 + 1)}{z_1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))}\right)\\
&\quad + \oint_{\partial D_k^1}\frac{dw_1}{2i\pi}\frac{\tilde\psi(w_1, \tilde z_2 + 1)}{z_1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))}\left(1 - \frac{1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))/z_1}{1 - \lambda_{k,u}w_1/z_1}\right)\,.
\end{aligned}$$

Let us bound the three terms E_A, E_B, E_C, taking (as we may, by linearity and the tensor product topology) $\tilde\psi(w_1, \tilde w_2) = \tilde\psi_1(w_1)\tilde\psi_2(\tilde w_2)$ with $\|\tilde\psi_1\| = \|\tilde\psi_2\| = 1$.

To prove $\|E_C\| \le \Delta_k/(1-|\lambda_{k,u}|)$, use on the one hand that $|1-\lambda_{k,u}w_1/z_1| > 1-|\lambda_{k,u}|$ (since $|z_1| > |w_1|$) and

$$\left(1 - \frac{1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))/z_1}{1 - \lambda_{k,u}w_1/z_1}\right) = \left(\frac{\phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))/z_1 - \lambda_{k,u}w_1/z_1}{1 - \lambda_{k,u}w_1/z_1}\right)\,.$$

On the other hand

$$\psi_1 \mapsto \oint_{\partial D_k^1}\frac{dw_1}{2i\pi}\left(\frac{\psi_1(w_1)}{z_1 - \phi_{kk,u}(w_1, \mathcal{F}^{-1}(\tilde z_2))}\right)$$

is bounded on $\mathcal{A}(\overline{\mathbb{C}} \setminus \mathcal{D}_k^1)$, uniformly as $\epsilon \to 0$, i.e., $\widetilde{R}_k \to \infty$ and the diameter of $\mathcal{D}_k^1$ tends to zero. To show this, note that for small ϵ we may perform the path integral over $\partial\mathcal{D}_k^1$ and use $\phi_{kk,u}^{-1}$ (despite the caveat in Remark 3.6 below). Combining the above two facts with Sublemma 3.4 gives the bound on E_C.

Now, to estimate E_B, we use Sublemma 3.4 again, with $\rho \sim |\lambda_{k,u}|$, and Sublemma 3.3 (1) to see that

$$\psi_2(\tilde{z}_2+1) \cdot \frac{\mathcal{E}_{k,2}(w_1, 1/\tilde{z}_2) + \tilde{z}_2^{-1}\mathcal{E}'_{k,2}(w_1, 1/\tilde{z}_2)}{\tilde{z}_2^2} \cdot \psi_1(\phi_{kk,u}^{-1}(z_1))$$

has norm bounded by $C \sup(|\mathcal{E}_{k,2}| + |\mathcal{E}'_{k,2}|)\widetilde{R}_k^{-2}$.

To see that $\|E_A\| \le \widetilde{R}_k^{-1}$, write

$$\tilde{\psi}_2(\tilde{z}_2+1+O_{w_1}(|\tilde{z}_2|^{-1})) - \tilde{\psi}_2(\tilde{z}_2+1) = \tilde{\psi}'_2(\tilde{z}_2+1) \cdot O_{w_1}(|\tilde{z}_2|^{-1}) + \tilde{\psi}''_2(\Phi_{w_1}(\tilde{z}_2)) \cdot O_{w_1}(|\tilde{z}_2|^{-2}) \,. \tag{3.4}$$

Since $\tilde{z}_2+1$, $\tilde{z}_2+1+O_{w_1}(|\tilde{z}_2|^{-1})$, and thus $\Phi_{w_1}(\tilde{z}_2)$, belong to a smaller half-plane (uniformly in w_1 and $\tilde{z}_2$) we may apply Sublemma 3.3 (2), and, for the term involving the second derivative, Sublemma 3.3 (3).

Case (P.b) is also handled by taking $\tilde{\psi} = \tilde{\psi}_1\tilde{\psi}_2$. For $(\tilde{z}_1, z_2) \in -H_{-\widetilde{R}_k} \times \mathcal{D}_k^2 \subset \mathcal{F}(U_k^1) \times \mathcal{D}_k^2$, it is convenient to avoid as much as possible the simultaneous occurrence of both variables $\tilde{z}_1$ and z_2 in the test functions, and we keep the integration over w_2:

$$\begin{aligned}
&s_{\phi'_{kk,s}}(\widetilde{\mathcal{L}}_{kk} - \widetilde{\mathcal{L}}_{kk}^{\otimes})(\tilde{\psi})(\tilde{z}_1, z_2) = E'_A + E'_B + E'_C \\
&= \left(1 - \frac{\mathcal{E}_{k,1}(1/\tilde{z}_1, z_2) + \tilde{z}_1^{-1}\mathcal{E}'_{k,1}(1/\tilde{z}_1, z_2)}{\tilde{z}_1^2}\right) \\
&\cdot \oint_{\partial\mathcal{D}_k^2} \frac{dw_2}{2i\pi} \frac{\partial_2\phi_{kk,s}(1/\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)}{w_2 - \phi_{kk,s}(\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)} \cdot \big(\tilde{\psi}_1(\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2)) - \tilde{\psi}_1(\tilde{z}_1 - 1)\big)\tilde{\psi}_2(w_2) \\
&- \left(\frac{\mathcal{E}_{k,1}(1/\tilde{z}_1, z_2) + \tilde{z}_1^{-1}\mathcal{E}'_{k,1}(1/\tilde{z}_1, z_2)}{\tilde{z}_1^2}\right) \\
&\qquad\qquad \cdot \oint_{\partial\mathcal{D}_k^2} \frac{dw_2}{2i\pi} \frac{\partial_2\phi_{kk,s}(1/\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)}{w_2 - \phi_{kk,s}(\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)} \cdot \tilde{\psi}_1(\tilde{z}_1 - 1)\tilde{\psi}_2(w_2) \\
&+ \oint_{\partial\mathcal{D}_k^2} \frac{dw_2}{2i\pi} \left(\frac{\partial_2\phi_{kk,s}(1/\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)}{w_2 - \phi_{kk,s}(\phi_{kk,u}^{-1}(1/\tilde{z}_1, z_2), z_2)} - \frac{\lambda_{k,s}}{w_2 - \lambda_{k,s}z_2}\right)\tilde{\psi}_1(\tilde{z}_1 - 1)\tilde{\psi}_2(w_2)\,,
\end{aligned}$$

recalling (3.2) and (2.25–2.26). The norm of E'_C may be bounded by a constant times Δ_k, as in case (P.a). For E'_B, we also adapt the argument above to find a bound $C/|\widetilde{R}_k^2|$. Finally, it is not difficult to see with the help of Sublemma 3.3 (1, 4) and Sublemma 3.4 that the norm of E'_A is of the order of $C/|\widetilde{R}_k|$. $\square$

3.B The full symbolic transfer operator.

Definition 3.5 (Full symbolic operator). The full symbolic transfer operator $\widehat{\mathcal{L}}$ is defined on the direct sum $\mathcal{B} = \bigoplus_{k\in\mathcal{S}_0\cup\mathcal{S}_1} \mathcal{B}_k$ as $\widehat{\mathcal{L}}_0 \oplus \widehat{\mathcal{L}}_1$ where

$$\begin{cases} \left(\widehat{\mathcal{L}}_1(\oplus_k \psi_k)\right)_j = \oplus_{k,j\notin\mathcal{S}_0\times\mathcal{S}_0} t_{kj} \cdot \widehat{\mathcal{L}}_{kj}\psi_k \, , j \in \mathcal{S}_0 \cup \mathcal{S}_1 \, , \\ \left(\widehat{\mathcal{L}}_0(\oplus_k \psi_k)\right)_j = t_{jj} \cdot \widehat{\mathcal{L}}_{jj}\psi_j \, , j \in \mathcal{S}_0 \, , \end{cases} \tag{3.5}$$

for operators $\widehat{\mathcal{L}}_{kj}$ given by (2.13, 2.14, 2.15), respectively.

Remark 3.6 (Reformulating the transfer operator). The jacobian $\det D\hat{f}_{kj}(w_1, w_2)$ may be expressed in terms of the pinning coordinates as $\partial_1\phi_{kj,u}(w_1, z_2)/\partial_2\phi_{kj,s}(w_1, z_2)$ where $\hat{f}_{kj}(w_1, w_2) = (z_1, z_2)$. Performing two successive Cauchy residue computations (formally), we get (note that $\partial_1\phi_{kj,u}(w_1, z_2)$ indeed appears with a + sign, just like in (2.16), because the w_1-pole is outside of the integration curve):

$$\left(\widehat{\mathcal{L}}(\oplus_\ell \psi_\ell)\right)_j(z_1, z_2) = \sum_{k\in S} t_{kj} \frac{s_{\phi'_{kj,s}}}{\det D\hat{f}_{kj}(\hat{f}_{kj}^{-1}(z_1, z_2))} \psi_k(\hat{f}_{kj}^{-1}(z_1, z_2)) \, . \tag{3.6}$$

The reader must beware that formula (3.6) does not make sense in general. Indeed, in the hyperbolic case, if $(z_1, z_2) \in (\mathbb{C} \setminus \mathcal{D}^1_j) \times \mathcal{D}^2_j$ then $\hat{f}_{kj}^{-1}(z_1, z_2)$ *may not* belong to $(\mathbb{C} \setminus \mathcal{D}^1_k) \times \mathcal{D}^2_k$. This is connected to the fact that the pinning coordinates are defined for $(w_1, z_2) \in \mathcal{D}^1_k \times \mathcal{D}^2_j$ and do not necessarily extend to $w_1 \in \mathbb{C} \setminus \mathcal{D}^1_k$. However, we may always perform the dw_2 path integral as in (2.14) and in particular use (2.16) in case (P.b) .

The sign of $\det D\hat{f}_{kj}$ is the product of $s_{\phi'_{kj,s}}$ and the sign $s_{\phi'_{kj,u}}$ of $\partial_1\phi_{kj,u}$. Note however that the real observables $\psi^{\mathbb{R}}(x, y)$ live on $I^1 \times I^2$ and that the "real" transfer operator $\mathcal{L}^{\mathbb{R}}$ is connected to $\widehat{\mathcal{L}}$ via the change of coordinate (see [Ru1, p. 1250] and [Ru4, p. 302])

$$\psi(w_1, w_2) = \int_{I^1_k} dx \frac{\psi^{\mathbb{R}}(x, w_2)}{w_1 - x} \, .$$

The "missing" sign of $\partial_1\phi_{kj,u}$ appears (morally) when replacing z_1 by its inverse image and we get the expected $1/|\det D\hat{f}_{kj}|$ factor.

Although Remark 3.6 means that there is no simple way to express (3.5) as a weighted composition operator in general, we can in some sense pretend it is possible: the kernel expression for the iterates of $\widehat{\mathcal{L}}$ guessed from (3.6) by applying the usual composition and multiplication scheme actually holds:

Lemma 3.7 (Naturality of the transfer operator). *For $n \geq 1$, the difference $\widehat{\mathcal{L}}^n - \widehat{\mathcal{L}}_0^n$ acts on $\mathcal{B}$ according to*

$$\left((\widehat{\mathcal{L}}^n - \widehat{\mathcal{L}}_0^n) \oplus_\ell \psi_\ell\right)_j = \oplus_{\vec{k} \in \mathcal{S}^n \setminus \mathcal{S}_0^n} \prod_{m=1}^{n-1} t_{k_{m-1}k_m} \cdot t_{k_n j} \cdot (\widehat{\mathcal{L}}_{\vec{k}j}^n \psi_{k_1}) \,, \tag{3.7}$$

where, for any admissible $\vec{k}j \in \mathcal{S}^{n+1} \setminus \mathcal{S}_0^{n+1}$,

$$\widehat{\mathcal{L}}_{\vec{k}j}^n \psi(z_1, z_2) = \oint_{\partial\mathcal{D}_{k_1}^1} \oint_{\partial\mathcal{D}_{k_1}^2} \frac{dw_1}{2i\pi} \frac{dw_2}{2i\pi} \frac{s_{(\phi^n)'_{\vec{k}j}} \partial_2 \phi_{\vec{k}j,s}^{(n)}(w_1, z_2)}{w_2 - \phi_{\vec{k}j,s}^n(w_1, z_2)} \frac{\psi(w_1, w_2)}{z_1 - \phi_{\vec{k}j,u}^{(n)}(w_1, z_2)} \,, \tag{3.8}$$

if $k_1 \notin \mathcal{S}_0$ and we replace (3.8) by $\oint_{\partial\Gamma_{k_1}^1} \oint_{\partial\mathcal{D}_{k_1}^2}$ or $\oint_{\partial\mathcal{D}_{k_1}^1} \oint_{\partial\Gamma_{k_1}^2}$ if $k_1 \in \mathcal{S}_0$. Here, $s_{(\phi^n)'_{\vec{k}j}}$ is the sign of $\partial_2(\phi_{\vec{k}j,s}^{(n)})$ on $\mathcal{I}_{k_1}^1 \times \mathcal{I}_j^2$.

Proof of Lemma 3.7. To obtain the formula for the kernel, we shall use Cauchy's theorem again. If $\vec{k} \in \mathcal{S}_1^n$, we can follow exactly Rugh's argument [Ru1], that we repeat for the convenience of the reader (and because it will be adapted to $\vec{k} \notin \mathcal{S}_1^n$). Denote

$$\mathcal{G}_{\vec{k}j}^{(n)}(w, z) = \frac{s_{(\phi^n)'_{\vec{k}j}} \partial_2 \phi_{\vec{k}j,s}^{(n)}(w_1, z_2)}{w_2 - \phi_{\vec{k}j,s}^{(n)}(w_1, z_2)} \frac{1}{z_1 - \phi_{\vec{k}j,u}^{(n)}(w_1, z_2)} \,.$$

By definition, $\mathcal{G}_{kj}^{(1)}(w, z)$ is the kernel of $\widehat{\mathcal{L}}_{kj}$. Lemma 3.7 is clearly true for $n = 1$ and it suffices to prove inductively that for all $n \geq 1$

$$\mathcal{G}_{\vec{k}j}^{(n+1)}(w, z) = \oint_{\partial\mathcal{D}_k^1} \oint_{\partial\mathcal{D}_k^2} \frac{d\xi_1}{2i\pi} \frac{d\xi_2}{2i\pi} \mathcal{G}_{\vec{k}k_n}^{(n)}(w, \xi) \mathcal{G}_{k_n j}^{(1)}(\xi, z) \,. \tag{3.9}$$

To prove the above equality, recall the fixed point $\xi^* = (\xi_1^*, \xi_2^*)$ constructed in the hyperbolic part of the proof of Proposition 2.6. The right-hand side of (3.9) has a single simple pole in each coordinate at $\xi^* = (\xi_1^*, \xi_2^*)$. Thus, writing R_{ξ^*} for the residue at ξ^*

$$\begin{aligned}
&\oint_{\partial\mathcal{D}_k^1} \oint_{\partial\mathcal{D}_k^2} \frac{d\xi_1}{2i\pi} \frac{d\xi_2}{2i\pi} \mathcal{G}_{\vec{k}k_n}^{(n)}(w, \xi) \mathcal{G}_{k_n j}^{(1)}(\xi, z) = \\
&\qquad s_{(\phi)'_{k_n j}} s_{(\phi^n)'_{\vec{k}k_n}} \mathrm{R}_{\xi^*} \left[((\xi_2 - \phi_{k_n j,s}(\xi_1, z_2))(\xi_1 - \phi_{\vec{k}k_n,u}^{(n)}(w_1, \xi_2)))^{-1} \right] \\
&\qquad\qquad \cdot \frac{\partial_2 \phi_{k_n j,s}(\xi_1^*, z_2)}{z_1 - \phi_{k_n j,u}(\xi_1^*, z_2)} \frac{\partial_2 \phi_{\vec{k}k_n,s}^{(n)}(w_1, \xi_2^*)}{w_2 - \phi_{\vec{k}k_n,s}^{(n)}(w_1, \xi_2^*)} \,.
\end{aligned}$$

Now, the two-variable residue is $(1-\partial_1\phi_{k_nj,s}\partial_2\phi^{(n)}_{\vec{k}k_n,u})^{-1}$, and the fixed point property implies $\xi_2^*=\phi_{k_nj,s}(\phi^{(n)}_{\vec{k}k_n,u}(w_1,\xi_2^*),z_2)$, so that

$$\frac{\partial\xi_2^*}{\partial z_2}=\partial_1\phi_{k_nj,s}\partial_2\phi^{(n)}_{\vec{k}k_n,u}\frac{d\xi_2^*}{dz_2}+\partial_2\phi_{k_nj,s}\,.$$

Finally, the definition of $\phi^{(n+1)}_{\vec{k}j,s}$ gives

$$\partial_2\phi^{(n+1)}_{\vec{k}j,s}=\partial_2\phi^{(n)}_{\vec{k}k_n,s}\frac{d\xi_2^*}{dz_2}=\partial_2\phi^{(n)}_{\vec{k}k_n,s}(1-\partial_1\phi_{k_nj,s}\partial_2\phi^{(n)}_{\vec{k}k_n,u})^{-1}\partial_2\phi_{k_nj,s}\,.$$

To finish the proof in the hyperbolic case, use the multiplicative properties of $s'_{(\phi^{n+1})_s}$.

If $\vec{k}\notin\mathcal{S}_1^n$ but there are no consecutive symbols in $\mathcal{S}_0$, the above argument applies, up to replacing $\partial\mathcal{D}^{1,2}_k$ by $\Gamma^{1,2}_k$.

So let us assume that there are at least two consecutive $\mathcal{S}_0$s. We proceed inductively on the number of consecutive $\mathcal{S}_0$ factors, considering the first time when $k_T,k_{T+1}\in\mathcal{S}_0$ and k_{T-1} or $k_{T+2}\in\mathcal{S}_1$. There are four cases to consider, depending on whether (H) is followed or preceded by (P), and on whether we are in case (P.a) or (P.b). In case (P.a), we use formula (2.14) for $\widehat{\mathcal{L}}_{kk}$, and Rugh's proof for the hyperbolic case recalled above gives the claim, using the nonhyperbolic case of Proposition 2.6. In case (P.b), we use (2.15) and the above proof may be adapted again, using Proposition 2.6. $\square$

The following lemma says that if $z^{-1}\notin\mathrm{sp}(\widehat{\mathcal{L}}_0)$ then the "regularised " transfer operator $z\widehat{\mathcal{L}}_1(1-z\widehat{\mathcal{L}}_0)^{-1}$ is nuclear on $\mathcal{B}$ in the sense of Grothendieck, and that its Fredholm determinant $\hat{d}(z)$ is dynamically defined. This will allow us to describe a nontrivial part of the spectrum of $\widehat{\mathcal{L}}=\widehat{\mathcal{L}}_0+\widehat{\mathcal{L}}_1$ via this *regularised determinant* $\hat{d}(z)$, and prove our main theorem. Although we shall mainly refer to Grothendieck's works [Gr1, Gr2], we mention two useful basic references: The recent book [GGK] provides a good introduction to the theory of nuclear operator on Banach spaces, and the survey [Ma] contains useful results for Banach spaces of holomorphic functions.

Lemma 3.8 (Nuclearity of the hyperbolic analytic transfer operator). *If* $z^{-1}\notin\mathrm{sp}(\widehat{\mathcal{L}}_0)$, *the operator* $z\widehat{\mathcal{L}}_1(1-z\widehat{\mathcal{L}}_0)^{-1}$ *acting on* $\mathcal{B}$ *is nuclear of order zero. The Fredholm determinant* $\det(1-z\widehat{\mathcal{L}}_1(1-z\widehat{\mathcal{L}}_0)^{-1})$ *is holomorphic in* $\{z^{-1}\notin\mathrm{sp}(\widehat{\mathcal{L}}_0)\}$ *and*

$$\hat{d}(z)=\det(1-z\widehat{\mathcal{L}}_1(1-z\widehat{\mathcal{L}}_0)^{-1})=\exp-\sum_{m=1}^{\infty}\frac{z^m}{m}\sum_{x\in\mathrm{Fix}_h\hat{f}^{(m)}}\frac{1}{|\det(D\hat{f}^{(m)}_{\vec{\imath}_x}(x)-Id)|}\,. \tag{3.10}$$

(Recall the bijection between hyperbolic fixed points $x_{\vec{\imath}}$ of $\hat{f}^n$ and periodic cycles $\vec{\imath}_x\in\mathcal{S}^n\setminus\mathcal{S}_0^n$ from Corollary 2.8.)

Since $\hat{f}$ is invertible the matrix equality $\det(1+A)\det(1+B)=\det((1+A)(1+B))$ gives

$$\hat{d}(z)=\exp-\sum_{m=1}^{\infty}\frac{z^m}{m}\sum_{x\in \mathrm{Fix}\,{}_h\hat{f}^{(m)}}\frac{1}{|\det D\hat{f}^{(m)}_{\vec{\imath}_x}(x)|}\frac{1}{|\det(\mathrm{Id}\,-D\hat{f}^{(-m)}_{\vec{\imath}_x}(x))|}\,. \tag{3.11}$$

Proof of Lemma 3.8. We adapt the argument in [Ru3, Lemma 2.7].

Using [Gr1], one shows that for any compact sets K, K' of the complex plane such that K is contained in the interior of K', the restriction map from the Banach space $\mathcal{A}(K')$ of analytic functions on $\mathrm{int}\,K'$ extending continuously to K', to the space $\mathcal{A}(K)$ of analytic functions on $\mathrm{int}\,K$ extending continuously to K, is nuclear of order zero. (See e.g. the beginning of the proof of Lemma 2.7 in [Ru3], which was adapted from Ruelle's paper [Rue1].) This argument may be extended to show that if K^1, K^2, $(K^1)'$ and $(K^2)'$ are compact subsets of $\mathbb{C}$ such that

$$K^2\subset \mathrm{int}\,(K^2)'\,,\qquad (K^1)'\subset \mathrm{int}\,K^1\,, \tag{3.12}$$

then the restriction map r_K from the Banach space $\mathcal{A}(\overline{\mathbb{C}}\setminus(K^1)',(K^2)')$ of functions holomorphic in the interior of $(\overline{\mathbb{C}}\setminus(K^1)')\times(K^2)'$, vanishing at infinity, and extending continuously to the boundary, to the Banach space $\mathcal{A}(\overline{\mathbb{C}}\setminus K^1,K^2)$, is nuclear of order zero. (The key step is to observe that the topological vector space of analytic functions on $((\overline{\mathbb{C}}\setminus K^1)\times \mathrm{int}\;K^2)$ is a nuclear space.)

On $\mathcal{B}_1=\oplus_{k\in\mathcal{S}_1}\mathcal{B}_k$, the resolvent $(1-z\widehat{\mathcal{L}}_0)^{-1}$ acts as the identity, while $(1-z\widehat{\mathcal{L}}_0)^{-1}$ is bounded on $\mathcal{B}_0$ by our assumption on z. Thus, $(1-z\widehat{\mathcal{L}}_0)^{-1}$ is bounded on $\mathcal{B}$.

Recall the definitions of U^2_k and U^1_k for $k\in\mathcal{S}_0$ from Section 2. Let $(K^1_k)',(K^2_k)'$ be compact sets such that $(K^2_k)'\subset \mathrm{int}\,\mathcal{D}^2_k$, if $k\in\mathcal{S}_1$ or we are in case (P.b), and $(K^2_k)'\subset U^2_k$ in case (P.a), while $\mathcal{D}^1_k\subset$ int $(K^1_k)'$ if $k\in\mathcal{S}_1$ or we are in case (P.a), and $(K^1_k)'\subset U^1_k$ in case (P.b). Denote

$$\mathcal{B}(K'):=\Big[\oplus_{k\notin(P.b))}\mathcal{A}(\bar{\mathbb{C}}\setminus(K^1_k)',(K^2_k)')\Big]\oplus\Big[\oplus_{k\in(P.b)}\mathcal{A}((K^1_k)',(K^2_k)')\Big]\,.$$

The restriction map $r_{K'}:\mathcal{B}\to\mathcal{B}(K')$ is continuous (see [Ru3, Lemma 2.6] for case (P.a)). Therefore $r_{K'}\circ(1-z\widehat{\mathcal{L}}_0)^{-1}$ is continuous from $\mathcal{B}$ to $\mathcal{B}(K')$.

Recall the definition of the compact set K^2_k for $k\in\mathcal{S}_0$. Extend it to $k\in\mathcal{S}_1$ as follows:

$$K^2_k=\cup_{\ell\in\mathcal{S}}\phi_{k\ell,s}(\mathcal{D}^1_k,\mathcal{D}^2_\ell)\,.$$

For k in case (P.b) we take a compact subset K^1_k of $\mathrm{int}\,(K^1_k)'\subset U^1_k$. In cases (H) or (P.a) we take a compact subset K^1_k of $\widetilde{\mathcal{D}}^1_k$, containing $\mathcal{D}^1_k$ in its interior and so that $\phi_{kj,u}(K^1_k,\mathcal{D}^2_j)\subset \mathrm{int}\,\mathcal{D}^1_j$ for each j. Now, up to slightly changing the $(K^{1,2}_k)'$ introduced above, we may ensure that $K^2_k\subset \mathrm{int}\,(K^2_k)'$ and $(K^1_k)'\subset \mathrm{int}\,K^1_k$ in cases (H) or

(P.a), $K_k^2 \subset \operatorname{int}(K_k^2)'$ and $K_k^1 \subset \operatorname{int}(K_k^1)'$ in case (P.b), while maintaining the other requirements. By the above choices, the restriction r_K is nuclear from $\mathcal{B}(K')$ to $\mathcal{B}(K)$.

Then (use in particular the definition of $\Gamma_k^{1;2}$ if $k \in \mathcal{S}_0$) $\widehat{\mathcal{L}}_1$ is bounded from

$$\mathcal{B}(K) = \left[\oplus_{k \notin (P.b)} \mathcal{A}(\bar{\mathbb{C}} \setminus K_k^1, K_k^2)\right] \oplus \left[\oplus_{k \in (P.b)} \mathcal{A}(K_k^1, K_k^2)\right]$$

to $\mathcal{B}' := \oplus_{k \in \mathcal{S}} \mathcal{B}'_k$. Since the inclusion $j : \mathcal{B}' \subset \mathcal{B}$ is continuous (for $k \in \mathcal{S}_0$ see [Ru3, Lemma 2.5] for a similar result, noting that the constant β there is equal to $+1$ in our case), the composition $j \circ \widehat{\mathcal{L}}_1$, is bounded from $\mathcal{B}(K)$ to $\mathcal{B}$.

As a consequence of the above considerations, the composition

$$\widehat{\mathcal{L}}_1(1 - z\widehat{\mathcal{L}}_0)^{-1} = \left(j \circ \widehat{\mathcal{L}}_1\right) \circ r_K \left(\circ r_{K'} \circ (1 - z\widehat{\mathcal{L}}_0)^{-1}\right) : \mathcal{B} \to \mathcal{B}$$

is nuclear of order zero (just use that a nuclear operator composed with bounded operators is nuclear). By [Gr1, II, pp. 16 and 18] it has a Fredholm determinant $\det(1 - z\widehat{\mathcal{L}}_1(1 - z\widehat{\mathcal{L}}_0)^{-1})$ which is an entire function of $1/z \notin \operatorname{sp}(\widehat{\mathcal{L}}_0)$.

It remains to establish the stated "dynamical" formula for the traces. For this, we combine arguments from [Ru3] and [Ru2]. First, just like on p. 17 of [Ru3], we find that for small enough z

$$\operatorname{tr}\log(1 - z\widehat{\mathcal{L}}_1(1 - z\widehat{\mathcal{L}}_0)^{-1}) = \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{tr}((\widehat{\mathcal{L}}_0 + \widehat{\mathcal{L}}_1)^m - \widehat{\mathcal{L}}_0^m) \, .$$

Defining $d_m = \operatorname{tr}(\widehat{\mathcal{L}}^m - \widehat{\mathcal{L}}_0^m)$, the Fredholm determinant

$$\det(1 - z\widehat{\mathcal{L}}_1(1 - z\widehat{\mathcal{L}}_0)^{-1}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} d_m$$

extends analytically to $\mathbb{C} - \operatorname{sp}(\widehat{\mathcal{L}}_0)$. Using the uniform contraction, we get for the trace

$$d_m = \sum_{\vec{\jmath} \in \mathcal{S}^{m+1} \setminus \mathcal{S}_0^{m+1}, j_1 = j_{m+1}} \prod_{k=1}^{m} t_{j_k j_{k+1}} \cdot \operatorname{tr}(\widehat{\mathcal{L}}_{\vec{\jmath}}^m) \, ,$$

where the iterated operator $\widehat{\mathcal{L}}_{\vec{\jmath}}^m$ may be expressed in kernel form by (3.8). Finally, the trace may be computed by performing a Cauchy integration:

$$\begin{aligned} \operatorname{tr}(\widehat{\mathcal{L}}_{\vec{\jmath}}^m) &= \oint_{\partial \mathcal{D}_{j_1}^1} \oint_{\partial \mathcal{D}_{j_1}^2} \frac{dw_1}{2i\pi} \frac{dw_2}{2i\pi} \frac{s_{(\phi^m)'_{\vec{\jmath}}} \partial_2 \phi_{\vec{\jmath},s}^{(m)}(w_1, w_2)}{w_2 - \phi_{\vec{\jmath},s}^m(w_1, w_2)} \frac{1}{w_1 - \phi_{\vec{\jmath},u}^{(m)}(w_1, w_2)} \\ &= \frac{1}{|\det(D\hat{f}^m(x_{\vec{\jmath}}) - \mathrm{Id}\,)|} \, , \end{aligned}$$

where $x_{\vec{\jmath}}$ is the unique (necessarily hyperbolic, and real) fixed point of $\hat{f}^m$ associated to the admissible periodic sequence $\vec{\jmath} \in \mathcal{S}^{m+1} \setminus \mathcal{S}_0^{m+1}$ (see pp. 1246–1247 of [Ru1] for details). $\square$

To relate the zeroes of the (regularised) Fredholm determinant $\hat{d}(z)$ (to part of) the spectrum of $\widehat{\mathcal{L}}$ we need the following lemma of Rugh, that we state for convenience of the reader:

Lemma [Ru3, Lemma 2.8]. *Let $\mathcal{M}_0 : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator on a Banach space $\mathcal{B}$ and let $\mathcal{M}_1 : \mathcal{B} \to \mathcal{B}$ be nuclear of order zero. Assume that $\mathbb{C} \setminus \mathrm{sp}(\mathcal{M}_0)$ is connected. Then the part of the spectrum of $\mathcal{M}_0 + \mathcal{M}_1$ which does not intersect $\mathrm{sp}(\mathcal{M}_0)$ consists of isolated eigenvalues of finite multiplicity, which can not accumulate in $\mathbb{C} \setminus \mathrm{sp}(\mathcal{M}_0)$. The Fredholm determinant*

$$d(u) = \det(1 - \mathcal{M}_1(u - \mathcal{M}_0)^{-1})$$

is analytic in $u \in \mathbb{C} \setminus \mathrm{sp}(\mathcal{M}_0)$. In this domain, the zero-set of $d(u)$ counted with order is the same as the eigenvalues of $\mathcal{M}_0 + \mathcal{M}_1$ counted with (algebraic) multiplicity.

We do not know a priori that the complement of the spectrum of $\widehat{\mathcal{L}}_0$ is connected. However, the spectra of the direct product operators (3.1) and (3.2) have this property by Lemma 3.1. We saw in Lemma 3.2 how to compare $\widehat{\mathcal{L}}_0$ to $\widehat{\mathcal{L}}_0^{\otimes}$, which is a direct sum of such operators. In our application (see Section 4) we will find a closed set containing the spectrum of $\widehat{\mathcal{L}}_0$, arbitrarily close to the spectrum $\widehat{\mathcal{L}}_0^{\otimes}$, with connected complement in $\mathbb{C}$, and apply the above lemma to this complement.

4. Reducing to (symbolic) analytic almost hyperbolic maps

Let us consider now a real-analytic diffeomorphism $f : M \to M$ for which there exists a dominated splitting $T_\Omega = E \oplus F$ over the nonwandering set. Our starting point will be a decomposition of Ω from [PS2]. The decomposition in [PS2] is stated for the limit set of f. However, if M is a surface and Ω is hyperbolic then Ω coincides with the limit set of f (see [NP]), and this equality also holds when Ω has a dominated splitting.

We recall some notation and results from [PS2]. We say that a compact invariant set $\Lambda \subset \Omega$ admits a *spectral decomposition* if it is a finite disjoint union of transitive compact invariant sets Λ_i (called *basic sets*) which may further be decomposed in a finite union of $n_i \geq 1$ *basic subsets* Λ_{i_j} with $f(\Lambda_{i_j}) = \Lambda_{i_{(j+1 \mod n_i)}}$, and $f^{n_i}|_{\Lambda_{i_j}}$ topologically mixing. We shall assume that the Λ_j are not trivial, i.e., not reduced to a single periodic orbit. It follows from the results in [PS1] and the classical Hirsch-Pugh-Shub [HPS] theory that for each small enough ϵ, there is δ, so that for each $x \in \Omega$, there exist local center stable and unstable manifolds $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(x)$ so that

$$T_x W^{cs}_\epsilon(x) = E \,, \quad T_x W^{cu}_\epsilon(x) = F \,,$$
$$f(W^{cs}_\delta(x)) \subset W^{cs}_\epsilon(f(x)) \,, \quad f^{-1}(W^{cu}_\delta(x)) \subset W^{cu}_\epsilon(f^{-1}(x)) \,.$$

The decomposition proved by Pujals and Sambarino [PS1] (who do not require analyticity, C^2 suffices) says that $\Omega = \Lambda \cup \mathcal{R} \cup \mathcal{I}$. Here, the "quasi-periodic" set $\mathcal{R}$ is a finite union of normally hyperbolic C^2 simple closed curves $\mathcal{C}_i$ on which f^{r_i} is conjugated to an irrational rotation for $r_i \geq 1$. The "periodic" set $\mathcal{I}$ is the union of a finite set of isolated periodic orbits with a set contained in a finite union $\cup_j \mathcal{I}_j$ of normally hyperbolic C^2 arcs or simple closed curves with $f^{m_j}(\mathcal{I}_j) \subset \mathcal{I}_j$ for $m_j \geq 1$. The set $\mathcal{I}$ contains all $\Omega \setminus P$ isolated periodic orbits. Next, f is expansive on the "almost hyperbolic" compact invariant set Λ, which admits a spectral decomposition $\Lambda = \cup_j \Lambda_j$, together with local product structure. (Lemma 4.5.1 in [PS2]: there are γ and $\eta > 0$ so that for any $x, y \in \Lambda_j$ with $d(x, y) < \eta$ then $W^{cs}_\gamma(x) \cap W^{cu}_\gamma(y) \in \Lambda_j$.) Finally, the set $\mathcal{N}$ of nonhyperbolic periodic orbits in Λ is empty or finite. In fact (see e.g. Proposition A.2 below), every basic set Λ_j which does not contain any nonhyperbolic periodic point is uniformly hyperbolic.

It is easy to construct examples where $\cup \mathcal{I}_j$ is not empty: just take a real-analytic flow on the sphere with both poles as sources and the equator as limit set. Our analyticity assumption implies that the arcs and curves $\mathcal{I}_j$ in $\mathcal{I}$ are isolated: Indeed, if they were not isolated, there would be a normally hyperbolic arc $\mathcal{I}_j$ and a basic set Λ_k such that their intersection is a nonhyperbolic periodic point q which is accumulated by periodic points contained in $\mathcal{I}_i$ with the same period than that of q; and this would contradict analyticity. It follows that $\mathcal{I}$ is not only open but also compact in Ω. Note also that the set $\mathcal{H}$ of (isolated) hyperbolic periodic points in $\mathcal{I}$ is finite. (Indeed, if there were infinitely many hyperbolic periodic points in $\cup_j \mathcal{I}_j$, their periods being bounded by $\max m_j$, a subset of constant period would accumulate on a periodic point, contradicting the analyticity assumption.)

Note that $\mathcal{R}$ does not contain any periodic orbits. Consider first the finite set $\mathcal{H} \subset \mathcal{I}$ of isolated hyperbolic periodic orbits. Writing $P \geq 1$ for the period and λ_E, λ_F for the multipliers (eigenvalues of $Df^P(p)$), of a periodic orbit p, each $p \in \mathcal{H}$ contributes to $d_f(z)$ a factor of the following type:

$$\begin{aligned}
d_{f|sink}(z) &= \prod_{j=0}^{\infty} \prod_{k=0}^{\infty} (1 - z^P \lambda_E^j \lambda_F^k)\,, \quad |\lambda_E| < |\lambda_F| < 1\,, \\
d_{f|saddle}(z) &= \prod_{j=0}^{\infty} \prod_{k=1}^{\infty} (1 - z^P \lambda_E^j \lambda_F^{-k})\,, \quad |\lambda_E| < 1\,, |\lambda_F| > 1\,, \\
d_{f|source}(z) &= \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} (1 - z^P \lambda_E^{-j} \lambda_F^{-k})\,, \quad 1 < |\lambda_E| < |\lambda_F|\,.
\end{aligned} \tag{4.1}$$

The infinite products above all converge, and define entire functions with an obvious zero-set. In particular, each $d_{f|sink}(z)$ is zero-free in the open unit disk and admits P simple zeros on the closed disk, at the Pth roots of 1, while each $d_{f|source}(z)$ admits a first zero at $z^P = \lambda_E \lambda_F$, which is outside the open disk, and each $d_{f|saddle}(z)$ admits a first zero at $z^P = \lambda_F$, which is outside the open disk.

We may therefore concentrate on the dynamical determinant $\prod_j d_{f|\Lambda_j}(z)$, where the Λ_j are the basic sets of Λ. Recall the set $\Sigma(p)$ associated to $p \in \mathcal{N}$ by (1.3). To prove Theorem A, we need to see that $d_{f|\Lambda_j}(z)$ is holomorphic in the (possibly) slit plane, or multiply slit plane defined by $\{z \in \mathbb{C} \mid 1/z \notin \cup_{p \in \mathcal{N} \cap \Lambda_j} \Sigma(p)\}$.

In order to do this, we shall associate an almost hyperbolic analytic map $\hat{f}$ to $f|\Lambda_j$ in such a way as to ensure that $d_{\hat{f}}(z)$ is almost $d_{f|\Lambda_j}(z)$ (dealing with the usual over-counting of periodic orbits on the boundaries of the Markov rectangles is postponed until Section 5.A).

Markov partitions.

The starting point in our construction of the symbolic map $\hat{f}$ is the existence of Markov partitions for f. We recall a possible definition in dimension two:

Definition 4.1 (Markov partition). Let $\Lambda \subset \Omega$ be a basic set. A Markov partition $\mathcal{R}$ of Λ is a finite collection $\{R_1, \ldots, R_\ell\}$ of "rectangles," with disjoint interiors, which are diffeomorphic to the square $Q = [-1,1]^2$, through $R_i = \tilde{\psi}_i(Q)$, whose union contains Λ, and such that

$$f(\partial_s R_i) \subset \cup_j \partial_s R_j \,, f^{-1}(\partial_u R_i) \subset \cup_j \partial_u R_j \,,$$

where $\partial_s R_i = \partial_E R_i = \tilde{\psi}_i(\{(x,y) \mid |y| = 1\})$ and $\partial_u R_i = \partial_F R_i = \tilde{\psi}_i(\{(x,y) \mid |x| = 1\})$.

To a Markov partition with ℓ rectangles we may associate an $\ell \times \ell$ transition matrix by setting $t_{ij} = 1$ if the interior of $f(R_i)$ intersects R_j and $t_{ij} = 0$ otherwise. Transitivity of Λ implies that this matrix is irreducible with no wandering states.

Since we are in dimension two, we can adapt the construction of Markov partitions in [PT, Appendix 2] (see [PS2, Lemma 4.5.2], first reduce to a mixing basic subset). The construction shows that a basic set Λ of Ω admits Markov partitions of arbitrarily small diameter (the diameter being the maximum of the diameters of the rectangles R_i). Since we have only a finite number of nonhyperbolic periodic points in Λ, we may assume that each rectangle contains at most one nonhyperbolic periodic point. We may furthermore ensure that if $q \in R_i$ is E-nonhyperbolic (i.e., $\lambda_E = \pm 1$) then $q \in \partial_u R_i = \partial_F R_i$ but $q \notin \partial_s R_i$, while if q is F-nonhyperbolic then $q \in \partial_s R_i$ but $q \notin \partial_u R_i$.

Note that if q is fixed, nonhyperbolic, and the order of $f - \mathrm{Id}$ at q is even (i.e., we have a saddle node situation), then Ω lies entirely on the weakly attracting side of q if $\lambda_E = 1$, while it is on the weakly expanding side of q if $\lambda_F = 1$. If the nonhyperbolic multiplier is $+1$ but the order of $f - \mathrm{Id}$ at a nonhyperbolic fixed point is odd (i.e., we have a saddle), then Ω intersects both sides, so that q will belong to the boundary of two rectangles. If the nonhyperbolic multiplier is -1 then Ω also meets both sides and we need two rectangles, whether the order is even or odd. For periodic points of period larger than one, the above remarks may be applied along the orbit.

We set $i \in \mathcal{S}_0$ if $R_i \cap \mathcal{N} \neq \emptyset$ (i.e., it is a "bad" rectangle) and $i \in \mathcal{S}_1$ otherwise.

Takens (C^∞) local coordinates for nonhyperbolic fixed points.

In the arguments below it will be convenient to use normal forms. We discuss first the C^∞ normal form due to Takens

Let $q \in \mathcal{N} \cap \Lambda_j$ be an F-nonhyperbolic fixed point. In particular, q is not contained in a periodic curve. We assume also that $\lambda_F = +1$. (The other cases in $\mathcal{N} \cap \Lambda_j$, E-nonhyperbolic, period ≥ 2, multiplier -1, are similar, see also §5.B.) By [T], we may express the diffeomorphism f in C^∞ local coordinates at $0 \mapsto q$ as

$$f_T(s,t) = (\lambda(t)s\,, b(t))\,, \tag{4.2}$$

for C^∞ functions λ and b satisfying $0 < |\lambda(0)| < 1$, $b(0) = 0$, and $b'(0) = 1$. (Notice that $\{(s,t) : t = 0\}$ is the strong stable (E) manifold and $\{(s,t) : s = 0\}$ is the central unstable (F) manifold.)

The neutral Takens coordinate $b(t)$ cannot be infinitely flat in our setting:

Lemma 4.2 (Nonflatness of C^∞ normal form). *If f is analytic and q is not contained in a curve of fixed points, setting $\nu + 1 \geq 2$ to be the multiplicity of $f - Id$ at q, then $b^{(\nu+1)}(0) \neq 0$, and $b^{(j)}(0) = 0$ for $2 \leq j \leq \nu$.*

Proof of Lemma 4.2. Since q is not contained in a curve of fixed points, by [H, Proposition 2.3, see also p. 481], we may express the diffeomorphism f in real-analytic local coordinates at 0 as

$$f_H(x,y) = (g(x) + yh(x,y)\,, y + y^{\nu+1} + Ay^{2\nu+1} + a(x)y^{2\nu+2} + \cdots)\,, \tag{4.3}$$

for $\nu + 1 \geq 2$ the multiplicity of $f - \mathrm{Id}$ at 0, with A a complex constant and g, a, and h real-analytic in a neighbourhood of 0 in $\mathbb{C}$ (resp. $\mathbb{C}^2$) and $g(0) = 0$, $h(0,0) = 0$ and $0 < |g'(0)| < 1$. (Hakim deals with holomorphic situation, but real-analytic data gives real A and real-analytic functions g, h, and a.) Notice that in these coordinates $\{(x,y) : y = 0\}$ is the still strong stable manifold. The central manifold, however, does not have an obvious description any more (indeed, it is usually not real-analytic), but it is tangent to $\{(x,y) : x = 0\}$ and can be described as the graph of a C^∞ map $y \mapsto x_F(y)$ with $x_F(0) = 0$. Additionally, it is the image of $\{(s,t) : s = 0\}$ by the conjugacy restricted to this line, which may be encoded in the C^∞ one-dimensional diffeomorphism $t \mapsto y_t$ with inverse $y \mapsto \tau(y)$ and $y_0 = 0$, $y_0' \neq 0$. Now, $b(t)$ can be decomposed as

$$t \mapsto f_H(x_F(y_t), y_t) \mapsto b(t) = \tau(y_t + y_t^{\nu+1} + Ay_t^{2\nu+1} + a(x_F(y_t))y_t^{2\nu+2} + \cdots)\,.$$

Hence, using the mean value theorem, and setting

$$e(t) = y_t^{\nu+1} + Ay_t^{2\nu+1} + a(x_F(y_t))y_t^{2\nu+2} + \cdots\,,$$

we have $b(t) = t + \tau'(\xi(t))e(t)$. Since $e^{(\ell)}(0) = 0$ for $0 \leq \ell < \nu + 1$, it follows that $b^{(\ell)}(0 = 0$ for $\ell < \nu + 1$ and $b^{(\nu+1)}(0) = \tau'(0)e^{(\nu+1)}(0) = \tau'(0)(\nu+1)!\,(y_0')^{\nu+1} \neq 0$. □

Constructing a symbolic model

From now on, we work with a real-analytic atlas of M, $\psi_k : A_k \to M$, where each $A_k \subset \mathbb{R}^2$ is a viewed as a real subset of the complexification $T_x^{\mathbb{C}}M \subset \mathbb{C}^2$ of the tangent space for some chosen $x \in \mathrm{Int}\psi_k(A_k)$ (we refer to pp. 808–809 in [Ru2] for details). We assume that the atlas is compatible with the normal form f_H from (4.3). We let $\widehat{A}_k$ be a complex neighbourhood of A_k. Note that the decomposition $E \oplus F$ extends to $T^{\mathbb{C}}_\Lambda M = E^{\mathbb{C}}_\Lambda + F^{\mathbb{C}}_\Lambda$. For $x \in A_k$ with $\psi_k(x) \in \Lambda$ we denote $E^s_{k,x} = E^{\mathbb{C}}_{k,x}$, $E^u_{k,x} = F^{\mathbb{C}}_x$, let $P^{u,\mathbb{C}}_{k,z} : T^{\mathbb{C}}_z M \to E^{u,\mathbb{C}}_{k,z}$, $P^{s,\mathbb{C}}_{k,z} : T^{\mathbb{C}}_z M \to E^{s,\mathbb{C}}_{k,z}$ denote the complexified projections to the unstable and stable bundles (we sometimes drop the k index). By construction, for j, k with $t_{j,k} \neq 0$, the map $\hat f$ induced by f in the charts extends to a real-analytic map.

We are now almost ready to state Proposition 4.5, which says that a (sequence of) almost hyperbolic symbolic models for f can be constructed with the help of a sequence of Markov partitions of diameters ending to zero. (All Markov partitions involved will be real and compatible with the real-analytic atlas chosen above, in the sense that each rectangle is included in some $\psi_j(A_j)$.) We must introduce further notation:

Definition 4.3 (Admissible complex extension). A subset $\widetilde\Lambda$ of a complex neighbourhood of Λ (in the charts) is an admissible complex extension of Λ if there is a complex neighbourhood V_j of each nonhyperbolic fixed point $q_j \in \mathcal{N}$ such that (in the charts) $\widetilde\Lambda \setminus \cup_j V_j$ is a complex neighbourhood of $\Lambda \setminus \cup_j V_j^{\mathbb{R}}$, and each $\widetilde\Lambda \cap V_j$ in analytic charts (z_E, z_F) (compatible with (4.3)) contains the intersection of a neighbourhood of $q_j = 0$ and a domain $\{\Re(z_E^{\nu_j}) \geq 0\}$, if q_j is E-nonhyperbolic, and $\{\Re(z_F^{\nu_j}) \geq 0\}$ otherwise (as usual, $\nu_j + 1 \geq 2$ is the multiplicity).

Let $|\cdot|_z$ denote the norm on $T^{\mathbb{C}}_z M$ induced by Riemann metric. Adapting the Mather trick and ideas from Crovisier, we show in the Appendix:

Lemma 4.3 (Adapted metrics). *Assume all $q \in \mathcal{N}$ are fixed points with nonhyperbolic multiplier $+1$. There are two semi-norms $\|\cdot\|_{E,z} = \|\cdot\|^s_z$, $\|\cdot\|_{F,z} = \|\cdot\|^u_z$ on the complex tangent bundle $T_{\Lambda^{\mathbb{C}}} M$ over a complex neighbourhood $\Lambda^{\mathbb{C}}$ of Λ, and an admissible complex extension $\widetilde\Lambda$ of Λ, such that:*

(1) *For all w*

$$\|Df^{-1}w\|_{F,f^{-1}z} \leq \|w\|_{F,z}\,, \qquad \|Dfw\|_{E,fz} \leq \|w\|_{E,z}\,, \forall z \in \widetilde\Lambda\,. \tag{4.4}$$

For $w \neq 0$, equality holds in the first bound if and only if z is F-nonhyperbolic, in the second one if and only if z is E-nonhyperbolic. In addition, there are $C > 0$ and $C_\nu > \nu$ so that for each F-nonhyperbolic fixed point $q_j \in \Lambda$ of index $\nu + 1$, letting V_j be the neighbourhood from Definition 4.3:

$$\|Df^{-1}w\|_{F,z} \leq (1 - C|z_E|)(1 - C_\nu|\Re(z_F^\nu)|)\|w\|_{F,f(z)}\,, \forall z = (z_E, z_F) \in \widetilde\Lambda \cap V_j\,, \tag{4.5}$$

and similarly for the E-nonhyperbolic case.

(2) *There is $C < \infty$ so that*

$$\frac{|\cdot|_z}{C} \leq \max(\|\cdot\|_{E,z}, \|\cdot\|_{F,z}) \leq C|\cdot|_z\,, \quad \forall z \in \Lambda^{\mathbb{C}}\,. \tag{4.6}$$

(3) *For any z in $\Lambda^{\mathbb{C}}$, and any z_0 close enough to z so that $(z-z_0)$ can be interpreted as a vector and $(f(z)-f(z_0)-Df_{z_0}(z-z_0))$ is in the chart at $f(z_0)$, we have*

$$\|f(z)-f(z_0)-Df_{z_0}(z-z_0)\|_{E/F,f(z_0)} \le \epsilon_1^{E/F}(|z-z_0|_{z_0})|z-z_0|_{z_0}\,,$$

where the two functions $\epsilon_1^{E/F}:\mathbb{R}^+\to\mathbb{R}^+$ are $\mathcal{C}^\infty$ with $\epsilon_1^{E/F}(0)=0$. Furthermore, there is C so that for any δ, if z and z_0 are in a complex δ-neighbourhood of an E/F-nonhyperbolic fixed point $q\in\Lambda$ of index $\nu+1$, then

$$\epsilon_1^{E/F}(|z-z_0|_{z_0}) \le C\cdot\delta^\nu|z-z_0|_{z_0}\,. \tag{4.7}$$

(4) *Let $\gamma>0$ be the Hoelder smoothness of the stable and unstable foliations. Then:*

$$|\|\cdot\|_{E/F,z}-\|\cdot\|_{E/F,z'}| \le \epsilon_2(d(z,z'))|\cdot|_z\,,\quad \forall z,z'\in\Lambda^{\mathbb{C}}\,, \tag{4.8}$$

for a γ-Hoelder continuous function ϵ_2 on $\mathbb{R}^+$, vanishing at 0. If z and z' are on the same $W^{E/F}$ local stable manifold, we can assume that $\gamma=1$.

We set $\|\cdot\|_z=\max(\|\cdot\|_z^u,\|\cdot\|_z^s)$ We shall work with two types of complex extensions of the rectangles of a Markov partition of Λ (compatible with charts). If $R_k\cap\mathcal{N}=\emptyset$, for $\xi_k\in R_k\subset\psi_j(A_j)$, and small $\delta_k>0$, we consider an ω-*rectangle* (just like in [Ru2])

$$R_k^\omega(\xi_k,\delta_k)=\{z\in\widehat{A}_j\subset\mathbb{C}^2\mid \|z-\psi_j^{-1}(\xi_k)\|_{\psi_j^{-1}(\xi_k)}\le\delta_k\}\,.$$

By definition, R_k^ω factorises as $\mathcal{D}_k^1\times\mathcal{D}_k^2$, with $\mathcal{D}_k^i$ a compact connected subset of $\mathbb{C}$ with smooth boundary (in fact, a disc), and intersecting the real axis on an interval $\mathcal{I}_k^i$.

If $R_k\cap\mathcal{N}=\{q\}$, we shall assume that $q=\psi_j(0)$ in charts $z=(z_E,z_F)$ compatible with the Hakim normal form (4.3), and, if q is F-nonhyperbolic of index ν_k+1, for $\delta_k>0$ and $\pi/(2\nu_k)<\theta_k<\pi/\nu_k$ we consider an ω-*petal*

$$R_k^\omega(\nu_k,\theta_k,\delta_k)=\{z\in\widehat{A}_j\mid z_F\in\mathcal{U}(\theta_k,\delta_k)\text{ and }\|z\|_q^E\le\delta_k\}\,,$$

Again, R_k^ω factorises as $\mathcal{D}_k^1\times\mathcal{D}_k^2$, with $\mathcal{D}_k^i$ a compact connected subset of $\mathbb{C}$ with smooth boundary (except at $0\in\mathcal{D}_k^2$), and intersecting the real axis on an interval $\mathcal{I}_k^i$. If q is E-nonhyperbolic, we proceed in an analogous way. We denote by $\widetilde{R}_k^\omega$ the real projection $\widetilde{R}_k^\omega=\psi_j(R_k^\omega\cap\mathbb{R}^2)$ of an ω-rectangle or an ω-petal.

We may finally state the main result of this section:

Proposition 4.5 (From dominated splitting to almost hyperbolic). *Let f be a real analytic diffeomorphism on a compact real-analytic surface M, with dominated splitting on its nonwandering set Ω. Let Λ be a basic set of f. Assume that all orbits in $\mathcal{N}$ are fixed points with neutral multiplier $+1$ and multiplicity $\nu+1=2$. Then there* *exists a sequence of Markov partitions $\mathcal{R}_n = \{R_{k,n}\}_{k\in\mathcal{S}_n}$ of Λ, with diameters tending to zero and such that, for each fixed n, denoting by $t_{ij} = t_{ij,n}$ the transition matrix:*

(1) *For each k so that $R_k \cap \mathcal{N} = \emptyset$, letting j be such that $R_k \subset \psi_j(A_j)$, there are $\xi_k \in R_k$ and $\delta_k > 0$ so that the projection of the corresponding ω-rectangle satisfies $R_k \subset \widetilde{R}^\omega_k(\xi_k, \delta_k) \subset \psi_j(A_j)$.*

(2) *For each k so that $R_k \cap \mathcal{N} = \{q\}$, of index $\nu_k + 1 = 2$, letting j be such that $R_k \subset \psi_j(A_j)$, there are $\delta_k > 0$ and $\pi/(2\nu_k) < \theta_k < \pi/\nu_k$, so that the projection of the corresponding ω-petal satisfies $R_k \subset \widetilde{R}^\omega_k(\nu_k, \theta_k, \delta_k) \subset \psi_j(A_j)$.*

(3) *The following defines an almost hyperbolic analytic map $\hat f$:*

$$
\begin{aligned}
&\mathcal{S}_0 = \{k \mid R_k \cap \mathcal{N} \neq \emptyset\}\,, \quad \mathcal{S}_1 = \{k \mid R_k \cap \mathcal{N} = \emptyset\}\,, \quad R^\omega_k = \mathcal{D}^1_k \times \mathcal{D}^2_k \subset \widehat{A}_k\,, \\
&\text{for } t_{k\ell} \neq 0: \quad \psi_i \circ \hat f_{k\ell}|_{(\mathcal{I}^1_k \times \mathcal{I}^2_k) \cap \hat f^{-1}_{k\ell}(\mathcal{I}^1_\ell \times \mathcal{I}^2_\ell)} = f \circ \psi_j|_{\psi_j^{-1}(\tilde R^\omega_k \cap f^{-1}(\tilde R^\omega_\ell))}\,.
\end{aligned}
\tag{4.9}
$$

Proof of Proposition 4.5. Taking small enough δ_k and θ_k, the nonhyperbolic requirement of (P.a) or (P.b) is obviously satisfied for the self-transition on an ω-petal by the Hakim normal form (4.3). We therefore concentrate on the hyperbolic condition for the system (4.9). Just like in [Ru2], the key is to reduce to a Schwarz inclusion:

Lemma 4.6 (Schwarz lemma contraction). *Proposition 4.5 holds, replacing condition (H) for $(k,\ell) \in \mathcal{S}^2 \setminus \mathcal{S}^2_0$ by*

$$
P^s_{\psi_j^{-1}(\xi_k)}(\hat f_{k\ell}(R^\omega_k)) \subset int(\mathcal{D}^1_\ell)\,, \quad P^u_{\psi_i^{-1}(\xi_\ell)}(\hat f^{-1}_{k\ell}(R^\omega_\ell)) \subset int(\mathcal{D}^2_k)\,, \tag{4.10}
$$

and the hyperbolic condition for $(k,k) \in \mathcal{S}^2_0$ by the P^u-inclusion above in case (P.a) and the P^s-inclusion in case (P.b).

The apparently weaker condition in Lemma 4.6 implies (H): Indeed, the existence of a partial inverse $\phi_{k\ell,s} : \mathcal{D}^1_k \times \mathcal{D}^2_\ell \to \mathrm{int}\,(\mathcal{D}^2_k)$, which is real-analytic in a neighbourhood of $\mathcal{D}^1_k \times \mathcal{D}^2_\ell$, and is the unique solution of $P^u_{\psi_j^{-1}(\xi_k)} \hat f_{k\ell}(w_1, \phi_{k\ell}(w_1, z_2)) = z_2$, can be obtained as in pp. 812-813 of [Ru2]. $\square$

The hard work consists now in proving Lemma 4.6, the technical but crucial dynamical lemma of this paper:

Proof of Lemma 4.6. We first consider the case of a single F-nonhyperbolic fixed point q of f (which is $(0,0)$ in the charts), of multiplicity $\nu + 1 = 2$.

Let ϵ_0 be small enough so that V, the ϵ_0-neighbourhood of the fixed point $q = 0$, is contained in a chart of the atlas and in a domain of definition of both Takens and Hakim normal forms $f_H(x,y)$ and $f_T(s,t)$ from (4.2–4.3). In particular, we skip the chart index and do no distinguish between ξ and $\psi_j^{-1}(\xi)$ in the notation for this proof. We shall use the fact that for all $t \geq 0$ we have $\Re y(s,t) \geq (1 - O(\epsilon_0))t$ (the local strong stable manifold is the same for both coordinates (s,t) and (x,y)). We also take ϵ_0 small enough so that $f^{\pm m} V \cap V \neq \emptyset$, for $m \neq 0$, is only possible for very large $|m| \geq m(\epsilon_0)$.

Pick a Markov partition $\mathcal{Q}$ like described just after Definition 4.1, ensuring that the rectangle containing q is a subset of V. For each $n \gg 1/\epsilon_0$, we consider $\mathcal{Q}_n$ the nth refinement of $\mathcal{Q}$ under f. We may replace V by the union of rectangles in $\mathcal{Q}_n$ intersecting V, up to slightly changing ϵ_0. We set Q_q to be the rectangle containing $q = 0$ in its (horizontal) boundary and let $Q_{W^s(q)}$ be the set of rectangles of $\mathcal{Q}_n$ along the stable manifold of q, in particular $Q_q \in Q_{W^s(q)}$. (Of course, if $n \gg m(\epsilon_0)$ then $Q_{W^s(q)}$ winds back into V and we get an infinite sequence of homoclinic intersections.) We define $Q_{W^s_{loc}}(q)$ to be the union of those rectangles in $Q_{W^s(q)}$ which are inside V.

Let us examine the rectangles Q of $\mathcal{Q}_n$ in V. Note that in the Takens normal form, the stable boundaries $\partial_s Q$ are horizontal segments, while the weak-unstable boundaries $\partial_u Q$ are curves which are close to vertical segments. We claim that the maximal diameter of $Q \in \mathcal{Q}_n$ with $Q \subset V$ is $O(1/n)$, which is realised only for the vertical length of rectangles of V in $Q_{W^s(q)}$ (rectangles not along the *global* stable manifold have diameter $O(1/n^2)$). This can be seen via the f_T coordinate, since if $b^n(t_0) = \epsilon_0$ then $t_0 = O(1/n)$, using the nonflatness Lemma 4.2 and $\nu + 1 = 2$. In fact $O(1/n)$ is the diameter of $\mathcal{Q}_n$ (also outside V). Since $m(\epsilon_0)$ is large and we have $1 - O(\epsilon_0)$ contraction outside of V, the rectangles of $\mathcal{Q}_n \in Q_{W^s(q)}$ in V which are not along the *local* stable manifold have (vertically realized) diameter at most $\epsilon(\epsilon_0)/n$ with $\epsilon \to 0$ as $\epsilon_0 \to 0$.

Recall that $\gamma \leq 1$ is the Hoelder smoothness of ϵ_2. If $\gamma \geq 1/2$ we may replace it by $0 < \gamma' < 1/2$, keeping the notation γ. Let U be the union of elements of $\mathcal{Q}_n$ which are in an $n^{-\gamma}$ neighbourhood of $q = 0$. We next construct $\mathcal{R}_n$ by modifying $\mathcal{Q}_n$ in U. Our aim is to ensure that we may choose a point ξ_j in every rectangle R_j in $\mathcal{R}_n$, with $R_j \subset U$, and $q \notin R_j$ in such a way as to guarantee that if R_k is another such rectangle, not along the local stable manifold of q, and $t_{jk} = 1$, then $f(\xi_j) = \xi_k$.

Let $\mathcal{Q}_{n,0}$ be the set of rectangles $Q \in \mathcal{Q}_n$ in U such that $f^{-1}(Q) \cap U = \emptyset$. For each Q_i in $\mathcal{Q}_{n,0} \setminus Q_{W^s_{loc}(q)}$, consider all forward iterates which intersect U: $\{f^j(Q_i) \cap U\}$. For Q_i in $\mathcal{Q}_{n,0}$ along the local stable manifold, we perform the same construction, except that we set $R' = f(Q_i) \cap Q_{W^s_{loc}(q)}$ and $R = f(Q_i) \setminus Q_{W^s_{loc}(q)}$, and we continue iterating R, R' until we leave U, decomposing each iterate which meets $Q_{W^s_{loc}(q)}$ into R and R'. The newly created sets R_m are all Markov rectangles, and whenever $R_m \cap Q_i \neq \emptyset$ for some Q_i of the partition $\mathcal{Q}_n$, the complement $\widehat{R} = R_m \setminus (R_m \cap Q_i)$ is also a Markov rectangle. Letting $\mathcal{Q}_{n,1}$ be the set of newly created complements $\widehat{R}$ such that $f^{-1}(\widehat{R}) \cap U = \emptyset$, we proceed as above, considering forward iterates in U and taking appropriate intersections. We repeat this procedure until $\mathcal{Q}_{n,N}$ is empty. Finally, we add to our collection $\{R_m\}$ of rectangles $\widehat{R}_q = Q_q \setminus \cup R_m$, as well as $R_q = \overline{f(\widehat{R}_q) \setminus \widehat{R}_q}$, and all its iterates $f^j(R_q)$ intersecting U.

The rectangles R_m are two by two disjoint and their union is U. We define $\mathcal{R}_n$ to be the union of the R_ms and the rectangles of $\mathcal{Q}_n$ outside of U. It is a Markov partition, which tends to be thinner horizontally and (slightly) fatter vertically than $\mathcal{Q}_n$ in U. Note also (this is the announced feature) that we may choose a point $\xi_m = (s_\xi, t_\xi)$ in each rectangle R_m of $\mathcal{Q}_{n,i}$, and a point ξ_q in R_q, and consider the corresponding iterated points ξ_ℓ in elements R_ℓ of the partition $\mathcal{R}_n$; when there is an R, R' bifurcation we "follow" the orbit in R, and take a new point in $\xi' \in R'$, making sure that $f(\xi) \in R$ and ξ' are in the same W^u leaf. If R_k is not adjacent to $W^s_{loc}(q)$ then $t_{\xi_k} \geq t_0 = O(1/n)$ while if $R_m \neq \widehat{R}_q$ is adjacent to the local stable manifold of q, we take η on the top ∂_s boundary so that $t_{\xi_m} \geq t_0 = O(1/n)$ even in this case.

Note that inside U we have $|\Re x(s,t)| > (1 - O(n^{-\gamma}))|s| - O(n^{-\gamma})|t|$, so that on the boundary of U either $t = Cn^{-\gamma}$ and thus $\Re y \geq cn^{-\gamma}$ or $t < Cn^{-\gamma}$ and $|\Re x| \geq C|s|$ with $|s| \geq cn^{-\gamma}$. Hence, by (4.5), the contraction factor outside of U (and also when entering U from outside of U, or just when leaving U) is $1 - cn^{-\gamma}$ (note that the points involved belong to the rectangles and are thus real points). Replacing U by V, the same argument gives a contraction factor $1 - C\epsilon_0$.

If $t_{jk} = 1$ and both rectangles R_k and R_j are outside of V we can apply the construction in the lemma of [Ru2, p. 811]. Let us recall here the key estimate involved: Fixing $\xi \in R_j$ and $\eta \in R_k$ we first observe (see [Ru2, p. 813]) that $|f^{-1}(\eta) - \xi|_\xi \leq O(1/n)$. Thus, using (4.4) and (4.6, 4.8) (in particular $\epsilon_1^F(a) = O(a)$),

$$\begin{aligned}
\|f^{-1}(v) - \xi\|^u_\xi &\leq \|f^{-1}(\eta) - \xi\|^u_\xi + \epsilon_2(d(\xi, f^{-1}(\eta)) \cdot |f^{-1}(v) - f^{-1}(\eta)|_{f^{-1}(\eta)} \\
&\qquad + \|f^{-1}(v) - f^{-1}(\eta) - Df^{-1}_\eta(v - \eta)\|^u_{f^{-1}(\eta)} + \|Df^{-1}_\eta(v - \eta)\|^u_{f^{-1}(\eta)} \\
&\leq C|f^{-1}(\eta) - \xi|_\xi + C|f^{-1}(\eta) - \xi|^\gamma_{f^{-1}(\eta)}|f^{-1}(v) - f^{-1}(\eta)|_{f^{-1}(\eta)} \\
&\qquad + \epsilon_1^F(|v - \eta|_\eta)|v - \eta|_\eta + (1 - C_\nu\epsilon_0)\|v - \eta\|^u_\eta \\
&\leq \frac{C}{n} + \frac{C}{n^\gamma}|v - \eta|_\eta + \widetilde{C}|v - \eta|^2_\eta + (1 - C_\nu\epsilon_0)\|v - \eta\|^u_\eta \,.
\end{aligned}$$

Taking the size $\delta_k = \delta_j$ of the ω-rectangles $R^\omega_{j,k}$ to be $C\epsilon(\epsilon_0)n^{-\gamma}$ (this choice will turn out to be useful later), we get $\|v - \eta\|_\eta \leq \delta_k \Longrightarrow \|f^{-1}(v) - \xi\|^u_\xi < \delta_j$. The P^s-inclusion is similar (in fact easier), we shall concentrate on the P^u-inclusion.

If $t_{jk} = 1$ and both rectangles R_k and R_j are in V, but outside of $U \cup Q_{W^s_{loc}(q)}$, the $\|\cdot\|^u$- contraction of $f^{-1}|_{R_k \cap f(R_j)}$ is at least $1 - C_\nu n^{-\gamma}$. We can essentially apply the above estimate, using also that $|f^{-1}(\eta) - \xi|_\xi \leq \epsilon/n$ in this case:

$$\begin{aligned}
\|f^{-1}(v) - \xi\|^u_\xi &\leq C|f^{-1}(\eta) - \xi|_\xi + C|f^{-1}(\eta) - \xi|^\gamma_{f^{-1}(\eta)}|f^{-1}(v) - f^{-1}(\eta)|_{f^{-1}(\eta)} \\
&\qquad + C\epsilon_1^F(|v - \eta|_\eta)|v - \eta|_\eta + (1 - C_\nu n^{-\gamma})\|v - \eta\|^u_\eta \\
&\leq \frac{C\epsilon}{n} + \left(\frac{C\epsilon}{n}\right)^\gamma |v - \eta|_\eta + \widetilde{C}|v - \eta|^2_\eta + (1 - \frac{C_\nu}{n^\gamma})\|v - \eta\|^u_\eta \,,
\end{aligned} \tag{4.11}$$

(note that η and $f^{-1}(\eta)$ are in $\Lambda \subset \widetilde{\Lambda}$). Taking the size $\delta_k = \delta_j = \delta$ to be $\delta = C\epsilon(\epsilon_0)n^{-\gamma} = \hat{\epsilon}n^{-\gamma}$, we get $\|v - \eta\|_\eta \leq \delta_k \Longrightarrow \|f^{-1}(v) - \xi\|^u_\xi < \delta_j$.

In order to obtain the P^u-inclusion for R_j or R_k in U, note first that any R_m in $U \setminus Q_{W^s_{loc}(q)}$ lies entirely between two horizontal lines $t \equiv b^\ell(t_0)$ and $t \equiv b^{\ell+1}(t_0)$ for an integer $\ell = \ell(R_m)$ between 0 and $n - 1$ (recall the definition of $t_0 = O(1/n)$). Using Lemma 4.2, it is not very difficult to see that the vertical diameter of such a rectangle

R_m is not larger than $B(n)(b^{\ell(R_m)}(t_0))^2$, for some constant $B(n) \geq 1$ tending to 1 when $n \to \infty$ (for fixed ϵ_0). Indeed, recall $n \gg 1/\epsilon_0$ and notice that if $b^\ell(t_0) \leq n^{-\gamma}$ then $\ell \leq \ell_{max} \leq n - An^\gamma$ for some $A > 0$. For such ℓ we have

$$0 < b^{(\ell+1)}(t_0) - b^{(\ell)}(t_0) = B(n)(b^{(\ell)}(t_0))^2 \leq \left(1 + O\left(\frac{1}{n^\gamma}\right)\right)(b^{(\ell)}(t_0))^2 \,. \tag{4.12}$$

Along the local stable manifold of 0, the diameter is of course $t_0 = O(1/n)$.

Discuss first rectangles R_j, R_k in U with $t_{jk} = 1$, such that R_j is *not* adjacent to the local stable manifold of q. Take as reference points for the (complex) ω-rectangles the chosen points $\xi = \xi_j$ and $\eta = \xi_k$. Note that $\ell(R_j) = \ell(R_k) - 1 \geq 0$ and set $\ell_{k/j} = \ell(R_{k/j})$. Then, since $f^{-1}(\eta) = \xi$, we only have two nonzero terms out of four in (4.11), and if $\|v - \eta\|_\eta \leq \delta_k$,

$$\begin{aligned}\|f^{-1}(v) - \xi\|^u_\xi &= \|f^{-1}(v) - f^{-1}(\eta)\|^u_{f^{-1}(\eta)} \\ &\leq \|f^{-1}(v) - f^{-1}(\eta) - Df^{-1}_\eta(v-\eta)\|^u_{f^{-1}(\eta)} + \|Df^{-1}_\eta(v-\eta)\|^u_{f^{-1}(\eta)} \\ &\leq \widetilde{C}|v-\eta|^2_\eta + (1 - C_\nu b^{\ell_j}(t_0))\|v-\eta\|^u_\eta \\ &\leq (\widetilde{C}'\delta_k + 1 - C_\nu b^{\ell_j}(t_0))\delta_k \,.\end{aligned} \tag{4.13}$$

Since C' does not depend on ϵ_0 or n, and $C_\nu > \nu \geq 1$ while $B(n) \to 1$ when $n \to \infty$, for each ϵ_0, up to taking smaller ϵ_0, we may assume that $\widetilde{C}'\hat{\epsilon}(\epsilon_0)(1+B(n)) - C_\nu < -B(n)$. We take

$$\delta_m = \hat{\epsilon} b^{\ell_m}(t_0)\,, \quad m = k\,, j\,.$$

Also, for $\ell_k = 0, \ldots, \ell_{max}$ we may assume $\delta_k \leq \hat{\epsilon} n^{-\gamma}$. Then, on the one hand the real projection of $R^\omega_{k/j}$ contains $R_{k/j}$, and on the other, if $\|v - \eta\|_\eta \leq \delta_k$, then by (4.13)

$$\begin{aligned}\|f^{-1}(v) - \xi\|^u_\xi &\leq \left(1 + (\widetilde{C}'\hat{\epsilon}(1 + B(n)b^{\ell_j}) - C_\nu)b^{\ell_j}(t_0)\right) \cdot \hat{\epsilon} b^{\ell_k}(t_0) \\ &\leq \left(1 - B(n)b^{\ell_j}(t_0)\right) \cdot \left(1 + B(n)b^{\ell_j}(t_0)\right) \cdot \hat{\epsilon} b^{\ell_j}(t_0) < \delta_j \,,\end{aligned}$$

so that we have the required Schwarz lemma inclusion property. If R_j, R_k in U with $t_{jk} = 1$, are such that R_k is *not* adjacent to the local stable manifold of q, while $R_j \neq \widehat{R}_q$ is adjacent to it, we have $\ell_k = 0$, so that we already set $\delta_k = \hat{\epsilon} t_0$. We take $\delta_j = C_j/n$, where C_j is a constant (independent of n) to be made more precise later on. So, if $\|v - \eta\|_\eta \leq \delta_k$,

$$\|f^{-1}(v) - \xi\|^u_\xi \leq \left(1 + (C'\hat{\epsilon} - C_\nu)t_0\right) \cdot \hat{\epsilon} t_0 \leq C_j/n \,.$$

If $R_j = \widehat{R}_q$, we have $\ell_k = 0$ and $\delta_k = \hat{\epsilon} t_0$ and we need to show that $\|v - \eta\|_\eta \leq \delta_k$ implies that $f^{-1}(v)$ belongs to an ω-petal at q. Taking the size of this ω-petal to be C_q/n for $C_q \geq 1$ but not very large, this follows from the fact that the t- (and thus the real part of the y-) coordinate of $f^{-1}(v)$ is positive, but not much bigger than t_0.

For the transitions t_{jk} from R_k, outside of U, to R_j, inside of U, we have as mentioned above a contraction factor $1-O(n^{-\gamma})$, and we may use the "ordinary" four-term estimate (4.11) since $\delta_j = \hat{\epsilon} b^{\ell_{j_{max}}}(t_0)$ while $\delta_k = \hat{\epsilon} n^{-\gamma}$. For the transitions from U (not along the local stable manifold) to outside of U, we apply (4.5) as in (4.11), using $\delta_k \le \hat{\epsilon} n^{-\gamma}$, and the contraction factor $1 - C_\nu n^{-\gamma}$.

We finally discuss transitions in V along the local stable manifold of q. Let R_j be a rectangle with $f(R_j) \subset \widehat{R}_q$. If $\|v-q\|_q \le 2\delta_q = 2C_q/n$ (this is in fact a weaker condition than the petal condition) we have from the usual four-term estimate, but without any contraction factor,

$$\|f^{-1}(v) - \xi\|^u_\xi \le O(1/n) + O(1/n^\gamma)|v-q|_q + C|v-q|^2_q + \|v-q\|^u_q \le C'_q/n\,. \quad (4.14)$$

We take $\delta^u_j = C'_q/n$, which defines the constant C_j for this rectangle. When we iterate, progressing to the left or to the right along of $W^s_{loc}(q)$, we get (recall that ξ and $f^{-1}\eta$ are on the same unstable leaf, and also that $t_\xi = t_0$ while $b(t_{f^{-1}(\eta)}) = t_\xi$)

$$\|f^{-1}(v)-\xi\|^u_\xi \le O(1/n^2)+O(1/n)|v-\eta|_\eta+C|v-\eta|^2_\eta+(1-c/n)\|v-\eta\|^u_\eta \le (1+O(1/n))\delta_k\,, \quad (4.15)$$

and the diameter $\delta_j = (1 + O(1/n))\delta_k$ grows. However, the number n_0 of iterations of f^{-1} to escape from V to the left, starting from a component of $f^{-1}(\widehat{R}_q) \setminus R_q$ is at most n. So the cumulated factor is smaller than $\zeta_n = (1 + C/n)^n$, which is uniformly bounded as $n \to \infty$, by $\zeta \gg 1$, say. We may choose all the $C_\ell \le \zeta C'_q$. Finally, since $\zeta O(1/n) \le \hat{\epsilon} n^{-\gamma}$, we get the required inclusion when we leave U, and, a fortiori, V.

If there are several F-nonhyperbolic fixed points (of the same multiplicity $\nu+1 = 2$), the construction above works too. The E-nonhyperbolic case is analogous. Dealing with coexistence of E and F-nonhyperbolic points (of the same multiplicity) does not cause any problems, since we never used the exponential smallness in the strong direction in our estimates: $O(1/n)$ was enough outside of V and on $Q_{W_{loc}(q)}$ and $O(\epsilon/n)$ elsewhere. □

Remark 4.7. When $n \to \infty$, the cardinality of $\mathcal{S}_1$ goes to infinity but the cardinality of $\mathcal{S}_0$ is fixed. Also, each map $\hat{f}_{kk,n}$ associated to a nonhyperbolic fixed point (i.e., $k \in \mathcal{S}_0$) for $n \ge 1$ is just the restriction of $\hat{f}_{kk,1}$ to smaller domains $\mathcal{D}^1_k$, $\mathcal{D}^2_k$, which can be constructed in a way compatible with the definition of $\mathcal{B}_0(\epsilon)$ in Lemma 3.1(2).

5. Spectral harvest

5.A Proof of Theorem A in the parabolic case.

In this section, we put together the results of Sections 2–3 and Section 4 to prove Theorem A, under the additional assumptions that the nonhyperbolic periodic points which are not $\Omega \setminus P$ isolated are fixed, never have an eigenvalue equal to -1, and always have multiplicity $\nu + 1 = 2$.

As explained in Section 4, we may restrict to a single basic set Λ_j. Let $\mathcal{N}$ be the set of nonhyperbolic non $\Omega \setminus P$ isolated periodic orbits and recall the definition (1.3) of $\Sigma(p)$

for $p \in \mathcal{N}$. We fix an arbitrary open neighbourhood $\mathcal{W}$ of $\cup_{p\in\mathcal{N}\cap\Lambda_j}\{1/z \mid z \in \Sigma(p)\}$, and we shall show that $d_{f|\Lambda_j}(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{W}$.

Putting together Proposition 4.5 and Lemmas 3.8 and 3.1–3.2, with Remark 4.7, and using Manning's counting trick [M] (see also [Ru2, p. 817]), we obtain that for each Markov partition $\mathcal{R}_n$ of small enough diameter, the determinant may be written as a ratio of two functions which are analytic in $\mathbb{C} \setminus \mathcal{W}$:

$$d_{f|\Lambda_j}(z) = \frac{d_1(z)}{d_2(z)}\,.$$

Indeed, d_1 is the regularised Fredholm determinant of the transfer operator of the hyperbolic analytic map $\hat{f}$ associated by Proposition 4.5 to a Markov partition $\mathcal{R}_n$, while d_2 is the determinant of a second symbolic map $\hat{f}_2$, associated to the auxiliary subshift corresponding to the pairs of adjacent rectangles in $\mathcal{R}_n$. (Note that the nonhyperbolic periodic orbits are not counted at all and therefore not overcounted, while the hyperbolic periodic orbits on boundaries of rectangles of the partitions coinciding with boundaries of our basic set are only counted once.)

It thus only remains to check that all zeroes of $d_2(z)$ are cancelled by zeroes of $d_1(z)$, for each n. The argument for this uses our assumption that M is two-dimensional:

Sublemma 5.1 (Disjoint boundary periodic orbits). *Let Λ be a basic set of a C^2 diffeomorphism f on a compact surface having dominated splitting over its nonwandering set. Then for each $\epsilon > 0$ there are two finite Markov partitions of diameter smaller than ϵ, and such that the sets B_1, B_2 of periodic orbits of f lying on their respective boundaries may only intersect on the boundary of Λ.*

See the Proposition on p. 817 of [Ru2], to which we refer for a proof of Sublemma 5.1, valid in the Anosov case (there, B_1 and B_2 can be taken disjoint), and which may be extended to Axiom A and dominated splitting.

End of the proof of Theorem A (parabolic case). Sublemma 5.1 finishes the proof. Indeed we may take two sequences $\mathcal{R}_{1,n}$, $\mathcal{R}_{2,n}$ of Markov partitions so that, on the one hand, $\mathcal{G}(\epsilon)$ in Lemma 3.2 goes to zero, and on the other, the sets B_1^n and B_2^n satisfy the properties in Sublemma 5.1. The argument on p. 818 of [Ru2] applies for each n. $\square$

5.B The general case.

We discuss briefly the changes needed to handle the general case. First note that the arguments in the appendix apply to neutral eigenvalues -1, working on both sides of the central manifold, and to periods larger than one, by considering the corresponding iterate of f. Let us now explain the changes in Sections 2–3 and 4.

Regarding $\lambda_{E,F} = -1$, we may easily generalise (P.a), (P.b) by allowing normal forms $-z_2 - z_2^{1+\nu} + h.o.t.$ instead of (2.3) and $-w_1 - w_1^{1+\nu} + h.o.t.$ instead of (2.4). We then use two petals, one on each side of 0. The square of the local (P) dynamics has neutral eigenvalue $+1$ and the same multiplicity $\nu + 1$. We may thus adapt the proof of Lemma 3.1, using the fact that the spectrum of $(\mathcal{T}^{\pm})^2$ is $[0, 1]$ so that the spectrum of

$\mathcal{T}^{\pm}$ is contained in $[-1,1]$. We get the sets announced in (1.3) for $P=1$. In Section 4, we have to take into account the fact that the dynamics oscillates between both sides of the strong local manifold, but this does not require any serious changes.

Regarding *fixed points of multiplicities* $\nu+1\geq 3$, using the conformal change of variables $(x,y)\mapsto(x,y^\nu)$ as in [Ru3, §2.6], we may adapt the contents of Sections 2 and 3. (Note that for even ν we need to consider two real petals for each $j\in\mathcal{S}_0$, while for odd ν we have just one petal.) We must also adapt the proof of Lemma 4.6. The main changes are the weaker contraction $1-C_\nu t^\nu$ (note however that ϵ_1^ν has stronger decay, see (4.7)), and the larger diameter of the refined partition: $O(1/n^{1/\nu})$ in general, and $O(\epsilon(\epsilon_0)/n^{1/\nu})$ (except along the local strong manifold) in an $O(\epsilon)$ neighbourhood V of the nonhyperbolic periodic point $p=0$. To deal with this, we replace γ by $\gamma'<1/(\nu(\nu+1))$ if $\gamma\geq 1/(\nu(\nu+1))$. Then, $\nu\gamma+\gamma<1/\nu$. We take U to be an $O(n^{-\gamma})$ neighbourhood of 0 (for the new, possibly smaller γ) and choose $\delta_m=C\epsilon/n^\gamma$ outside of U. Then, (4.11) becomes

$$\|f^{-1}(v)-\xi\|_\xi^u\leq\frac{C\epsilon}{n^{1/\nu}}+\left(\frac{C\epsilon}{n^{1/\nu}}\right)^\gamma|v-\eta|_\eta+\frac{\widetilde{C}}{n^{(\nu-1)\gamma}}|v-\eta|_\eta^2+(1-\frac{C_\nu}{n^{\nu\gamma}})\|v-\eta\|_\eta^u\,, \tag{5.1}$$

so that $\|v-\eta\|_\eta\leq\delta_k$ implies $\|f^{-1}(v)-\xi\|_\xi^u\leq\delta_j$. Then, (4.12) becomes

$$0<b^{(\ell+1)}(t_0)-b^{(\ell)}(t_0)\leq B(n)(b^{(\ell)}(t_0))^{\nu+1}\leq\big(1+O(n^{-\gamma})\big)(b^{(\ell)}(t_0))^{\nu+1}\,. \tag{5.2}$$

For the the two-term transitions from R_k to R_j within U, we set $\delta_m=\hat\epsilon b^{\ell_m}(t_0)$, and invoke (4.7) inside the $b^{(\ell_k+1)}(t_0)$ neighbourhood. Then we may assume that $\hat\epsilon\widetilde{C}'(1+B(n))^{2\nu-1}-C_\nu<-B(n)$ and (4.13) becomes

$$\begin{aligned}\|f^{-1}(v)-\xi\|_\xi^u&\leq\widetilde{C}(b^{\ell_k+1}(t_0))^{\nu-1}|v-\eta|_\eta^2+(1-C_\nu(b^{\ell_j}(t_0))^\nu)\|v-\eta\|_\eta^u\\&\leq(1+(\widetilde{C}'(b^{\ell_k+1}(t_0))^{\nu-1}\delta_k-C_\nu(b^{\ell_j}(t_0))^\nu)\delta_k\\&\leq[1+(\hat\epsilon\widetilde{C}'(1+B(n))^{2\nu-1}-C_\nu)(b^{\ell_j}(t_0))^\nu](1+B(n)(b^{\ell_j}(t_0))^\nu)\delta_j<\delta_j\,.\end{aligned} \tag{5.3}$$

Along the local strong manifold, we take $\delta_m=C_m/n^{1/\nu}$, and the diameter grows

$$\begin{aligned}&\|f^{-1}(v)-\xi\|_\xi^u\\&\quad\leq O(1/n^{1+1/\nu})+O(1/n^{1/\nu})|v-\eta|_\eta+C|v-\eta|_\eta^2/n^{1-1/\nu}+(1-C_\nu/n)\|v-\eta\|_\eta^u\\&\quad\leq(1+O(1/n^{1/\nu}))\delta_k\,,\end{aligned} \tag{5.4}$$

but (up to considering as a single Markov rectangle at p all the rectangles to the left and to the right in a horizontal neighbourhood of horizontal size ϵ/n^ν, which does not interfere with the other computations) we may assume that the number of iterations n_0 within U is smaller than $C\log n\ll n^{\gamma/\nu}$. Since $\zeta/n^{1/\nu}\ll 1/n^\gamma$, we are done.

Finally, if there are nonhyperbolic fixed points of different indices, we set $\nu=\max\nu_j$ and use the same $\gamma'\leq\gamma$ with $\gamma'<1/(\nu(\nu+1))$ for all neighbourhoods V_j and U_j.

If we have periodic points of *periods* $P \geq 2$ *in* $\mathcal{N}$, they will be associated to a periodic cycle in the symbolic set $\mathcal{S}_0$ (work first with f^P). The corresponding symbolic transfer operator $\widehat{\mathcal{L}}_0$ will have associated periodic blocks of periods P_i. Its spectrum is thus contained in the union over i of the P_ith roots of the spectrum of the block $\widehat{\mathcal{L}}_{0,i}^{P_i}$. This gives the P-th roots announced in (1.3) and thus the slits in Theorem A. $\square$

Appendix: Constructing an adapted metric

Let f be a real-analytic diffeomorphism of a compact surface having a dominated splitting $T_\Omega M = E \oplus F$. Let Λ be a basic set from the decomposition [PS2] of the nonwandering set Ω. We consider an analytic atlas for f, as in Section 4 and denote by $|\cdot|_z$ the riemannian norm induced on complex charts. (We systematically drop the chart index in this appendix.) Let $\mathcal{N}$ be the finite set of nonhyperbolic periodic points of f in Λ. For each $q_j \in \mathcal{N}$, we denote by $\nu_j + 1 \geq 2$ its index, i.e., the order of the zero $f - \mathrm{Id}$ at q_j in the charts (in other words, the multiplicity of $f - \mathrm{Id}$, recall from Section 4 that this multiplicity is finite). In this appendix, we allow all $\nu_j \geq 1$, however, for simplicity, we assume that all points in $\mathcal{N}$ are *fixed points* and that their nonhyperbolic multiplier is equal to $+1$ (not -1). See Section 5.B for the general case.

Our aim in this appendix is to prove Lemma 4.3, i.e., to construct two adapted semi-norms (see [Ru2 pp. 809–810] for a brief account of the hyperbolic case) $\|\cdot\|_{E,z} = \|\cdot\|_z^s$, $\|\cdot\|_{F,z} = \|\cdot\|_z^u$ on the complex tangent bundle $T_{\Lambda^{\mathbb{C}}} M$ over a complex neighbourhood $\Lambda^{\mathbb{C}}$ of Λ, and an admissible complex extension $\widetilde{\Lambda}$ of Λ, such that (4.4–4.8) hold.

We shall recycle some ideas of Crovisier [Cr, §5.3] (introducing simplifications arising from [PS1, PS2]), but we must modify his construction which uses smooth (not analytic) coordinates (see [Cr, §4.2]), since we need to control the complex extensions. Another difference is that Crovisier constructs an adapted metric in the sense that $\|Df^{-1}_{/F(z)}\|_z^C < 1$ and $\|Df_{/E(z)}\|_z^C < 1$ if $z \notin \mathcal{N}$, while we need the quantitative estimate (4.5).

Adapted metrics close to $\mathcal{N}$ — The first global norm $\|\cdot\|'_z$

Our first step is to construct local semi-norms $\|\cdot\|'_{F/E,z}$ in a neighbourhood of each nonhyperbolic fixed point of $f|_\Lambda$.

Recall (see e.g. [PS1, Lemma 3.2.1]) that for suitable $\alpha < 1$ the (complex) cone

$$\mathcal{C}^F_{\alpha,z} = \{w \in T^{\mathbb{C}}_z \Lambda \mid w = u + v\,, u \in E(z)\,, v \in F(z)\,, |u| \leq \alpha |v|\} \tag{A.1}$$

is invariant under Df, for z in a complex neighbourhood of Λ.

Lemma A.1 (Local semi-norm). *Let f be a real-analytic surface diffeomorphism with dominated splitting over Ω, and Λ a basic set. Let $q \in \Lambda$ be an F-nonhyperbolic fixed point of index $\nu + 1 \geq 2$. Then there exist a semi-norm $\|\cdot\|'_{F,z}$ over a complex neighbourhood $B^{\mathbb{C}}$ of a (real) neighbourhood B of $q = 0$, and an admissible complex extension $\widetilde{B}$ of B, so that* (4.5) *holds for all $w \in \mathcal{C}^F_{\alpha,z}$, i.e., for some $C_\nu > \nu$*

$$\|Df^{-1} w\|'_{F,z} \leq (1 - C|z_E|)(1 - C_\nu |\Re(z_F^\nu)|)) \|w\|'_{F,f(z)}\,, \ \forall z \in \widetilde{B}\,, \forall w \in \mathcal{C}^F_{\alpha,z}\,. \tag{A.2}$$

Furthermore $z \mapsto \|\cdot\|'_{F,z}$ is C^∞, (4.7) holds on $B^{\mathbb{C}}$ if $(z-z_0) \in \mathcal{C}_{\alpha,z_0}$, and (4.8) holds. We have analogous statements if q is E-nonhyperbolic.

Proof of Lemma A.1. As mentioned in the proof of Lemma 4.2, by [H], we may express f in real-analytic coordinates in a neighbourhood of 0 as

$$f_H(x,y) = (g(x) + yh(x,y)\,, y + y^{\nu+1} + Ay^{2\nu+1} + \sum_{j=2\nu+2}^{\infty} a_j(x)y^j)\,,$$

where A is a constant, and g, a_j, and h, respectively, are real-analytic in a neighbourhood of 0 in $\mathbb{C}$, respectively $\mathbb{C}^2$, with $g(0) = 0$, $h(0,0) = 0$, and $0 < |g'(0)| < 1$. Writing $h_x = \partial_x h$, $h_y = \partial_y h$, we get

$$\begin{aligned} Df_{H,(x,y)}(u,v) = ((g'(x) + yh_x(x,y))u + (h(x,y) + yh_y(x,y))v,\\ \sum_{j=2\nu+2}^{\infty} (a_j'(x)y^j)u + (1 + (\nu+1)y^\nu + \cdots)v)\,. \end{aligned} \tag{A.3}$$

Note that if $\Re(y^\nu) \geq 0$

$$|(1 + (\nu+1)y^\nu + O(y^{2\nu}))v| \geq (1 + C_\nu|\Re(y^\nu)|)|v|\,. \tag{A.4}$$

We may view 0 as a partially hyperbolic fixed point of a $\mathcal{C}^\infty$ map on a neighbourhood of 0 in $\mathbb{R}^4$ and use the Takens [T] standard coordinates $(\vec{s},\vec{t}) = (s_1,s_2,t_1,t_2)$. Here, $f_T(\vec{s},\vec{t}) = (G_{\vec{t}}(\vec{s}), H(\vec{t}))$ where $G_{\vec{t}}$ is a contraction, i.e., both its eigenvalues have absolute values < 1, uniformly in t, while the two eigenvalues of H have moduli 1. In particular (up to taking a smaller neighbourhood) there is $\gamma < 1$ so that $\sup_{\vec{t}} |G_{\vec{t}}(\vec{s})| \leq \gamma|\vec{s}|$. For small $\beta > 0$, define a semi-norm for a complex vector (u,v) over $z = (x,y)$ by

$$\|(u,v)\|'_{F,(x,y)} = (1 - \beta|\vec{s}(x,y)|)|v|\,. \tag{A.5}$$

Take $c < \beta(1-\gamma)$. By definition, and by (A.3–A.4), we have for $\Re y^\nu \geq 0$ and $(u,v) \in \mathcal{C}^F_\alpha$

$$\begin{aligned} &\|Df_{H,(x,y)}(u,v)\|'_{F,f(x,y)}\\ &\qquad = (1 - \beta|\vec{s}(f_H(x,y))|) \cdot |\sum_{j\geq 2\nu+2} (a_j'(x)y^j)u + (1 + (\nu+1)y^\nu + \cdots)v|\\ &\qquad \geq (1 - \gamma\beta|\vec{s}(x,y)|)(1 + C_\nu|\Re(y^\nu)|)|v|\\ &\qquad \geq (1 + c|\vec{s}(x,y)|)(1 + C_\nu|\Re(y^\nu)|)\|(u,v)\|'_{F,(x,y)}\\ &\qquad \geq (1 + \tilde{c}|x|)(1 + C_\nu|\Re(y^\nu)|)\|(u,v)\|'_{F,(x,y)}\,. \quad \square \end{aligned} \tag{A.6}$$

Define a local norm in a neighbourhood of an F-nonhyperbolic fixed point $q \in \Lambda$:

$$\|(u,v)\|'_z := \begin{cases} \|(u,v)\|'_{F,z} & \text{if } (u,v) \in \mathcal{C}^F_\alpha\,,\\ |u| =: \|(u,v)\|'_{E,z} & \text{otherwise}\,. \end{cases}$$

It is not difficult to check, using Lemma A.1 and the cone property, that the new norm is adapted in $\widetilde{B}$ in the sense that

$$\begin{cases} \|Df_{/E(z)}\|'_z < 1\,, \forall z \in B^{\mathbb{C}}\,, \\ \|Df^{-1}_{/F(z)}\|'_z \leq 1\,, \forall z \in \widetilde{B}\,, \text{ with equality if and only if } z = q\,. \end{cases} \tag{A.7}$$

We may extend the norm $\|\cdot\|'_z$ continuously to $\widetilde{\Lambda}$ by glueing it with the riemannian norm $|\cdot|_z$. We replace the riemannian norm by this new equivalent norm $\|\cdot\|'_z$. Of course, we still have a dominated splitting.

Controlling hyperbolicity away from $\mathcal{N}$ — The second global norm $\|\cdot\|''_z$

We will use the following results proved in [PS1] and [PS2]:

Proposition A.2 [PS1, Theorem B]. *Let f be a C^2-diffeomorphism on a compact surface, and Λ a compact f-invariant set having a dominated splitting. Assume that all the periodic points in Λ are hyperbolic of saddle-type and that Λ does not contain normally invariant curves. Then, Λ is hyperbolic.*

Lemma A.3 [PS2, see Proposition 3.1]. *Let f be a C^2-diffeomorphism on a compact surface, and Λ a compact f-invariant set having a dominated splitting. Let q be a periodic point of Λ and let $z_0 \in \Lambda$ be a point in the local unstable manifold of q. Then, there are a neighbourhood V of z_0 and an integer k_0 such that if $z \in V \cap \Lambda$ and $k > k_0$ then $\|Df^{-k}_{/F(z)}\|' < \frac{1}{2}$.*

We will also use the following two lemmas (and their versions exchanging E and F):

Lemma A.4. *Let f and Λ be as in Lemma A.3 and assume that $q \in \Lambda$ is an F-nonhyperbolic fixed point, and is the only nonhyperbolic periodic point of f in Λ. Then, there is $m_0 \geq 1$ such that for any $z \in \Lambda \setminus \{q\}$ there is $1 \leq n(z) \leq m_0$ such that $\|Df^{-n(z)}_{/F(z)}\|' < 1$. We shall take $n(z)$ minimal with this property.*

Proof of Lemma A.4. If z is in some (real) neighbourhood B of q, we may take $n(z) = 1$ by the construction after Lemma A.1.

Assume for a contradiction that the conclusion of the lemma does not hold (outside of B). Then there exists a sequence $z_n \notin B$ such that $\|Df^{-j}_{/F(z_n)}\|' \geq 1$, for all $1 \leq j \leq n$. Taking z to be an accumulation point of the z_n, we have $z \notin B$ and $\|Df^{-j}_{/F(z)}\|' \geq 1$ for all $j \geq 1$. Now, if $q \notin \alpha(z)$ then $\alpha(z)$ is a compact set with all periodic points hyperbolic, and from Proposition A.2 this set must be hyperbolic, a contradiction.

Consider now the case $q \in \alpha(z)$. We analyse two situations: $\{q\} = \alpha(z)$ or $\{q\} \subsetneq \alpha(z)$. In the first case, $z \in W^u(q)$ and thus there is k_0 such that $f^{-k_0}(z) \in W^u_{loc}(q)$. Therefore $\|Df^{-j}_{/F(f^{-k_0}(z))}\|' \to 0$, and so also $\|Df^{-j}_{/F(z)}\|' \to 0$. Then by continuity, taking z_n close enough to z, we also get a contradiction. Finally, if $\{q\} \subsetneq \alpha(z)$, we may take a neighbourhood V in the local unstable manifold of q and a subsequence $f^{-j_n}(z_n) \in V$ with j_n uniformly bounded. From Lemma A.3, we also get a contradiction. $\square$

Lemma A.5. *Let f, Λ be as in Lemma A.4. For each neighbourhood B_1 of q and every $0 < \lambda_1 < 1$ there exists m_1 and a complex neighbourhood W_1 of $\Lambda \setminus B_1$ such that for each $z \in W_1$ then $\|Df^{-m_1}_{/F(z)}\|' < \lambda_1$.*

Proof of Lemma A.5. For arbitrary $z \in \Lambda \setminus \{q\}$ let $n(z) \le m_0$ be given by Lemma A.4. It is not difficult to see that $\sup_{z\in\Lambda\setminus B_1} \|Df^{-n(z)}_{/F(z)}\|' < \mu < 1$ (for the real norm). Let $C = \sup_{z\in\Lambda}\{\|Df^{-j}_{/F(z)}\|' \ 1 \le j \le m_0\}$. Take $k > m_0$ such that $\mu^k C < \lambda_2 < \lambda_1$. Also notice that there exists $r > m_0$ such that if $z \in \Lambda \setminus B_1$ and $f^{-j}(z) \in B_1$ for all $n(z) \le j \le n$ with $n \ge r$ then $\|Df^n_{/F(z)}\|' < \lambda_2/C$.

We will show the lemma for $m_1 = (k+1)r$: For $z \in \Lambda \setminus B_1$, set $n_0 = 0$, $n_1 = n(z)$, $n_2 = n(f^{n_1}(z)), \ldots$. It follows that for some $i \ge 2$ we have $m_1 = n_1 + n_2 + ... + n_i + s$ with $0 \le s \le m_0$. If $s = 0$ there is nothing to do. Consider $j_0 = 0, j_1, j_2, ..., j_\ell$ such that $f^{n_0+n_1+...+n_{j_s}}(z) \notin B_1$. If $\ell > k$ the result follows. If $\ell \le k$ then there is j_s such that $n_{j_{s+1}} - n_{j_s} > r$, and we also conclude. Since we proved the real claim for $\lambda_2 < \lambda_1$, there is a small complex neighbourhood W_1 so that the lemma holds for W_1 and λ_1. $\square$

Assume that Λ contains only one, say F-nonhyperbolic fixed point q. (The general case follows in a very similar way, in particular Lemmas A.4–5 can be adapted to the situation where $\mathcal{N}$ contains more than one point.) Using m_1 from Lemma A.5, we define a second global norm by:

$$\|w\|''_z = \sum_{j=0}^{m_1-1} \|Df^{-j}w\|'_{f^{-j}z}, \quad z \in \Lambda^{\mathbb{C}}. \tag{A.8}$$

More generally, we shall use auxiliary norms $\|w\|''_{i,z} = \sum_{j=0}^{i-1} \|Df^{-j}w\|'_{f^{-j}z}$ for $1 \le i \le m_1$. (In particular $\|w\|''_{1,z} = \|w\|'_z$.) If $z \in W_1$ then by Lemma A.5, $\|Df^{-1}_{/F(z)}\|''_z < 1$.

The global adapted norm and the global adapted semi-norms

We have two global complex norms $\|\cdot\|'$ and $\|\cdot\|''$: one which is adapted in an admissible neighbourhood $\widetilde{B}$ of q; the other one adapted outside a neighbourhood B_1 of q, and we may assume that B_1 is much smaller than B. We must glue these two norms in order to get a global adapted complex norm $\|\cdot\|_z$ over $\widetilde{\Lambda} \subset \Lambda^{\mathbb{C}}$.

This can be done as in [Cr, pp. 1112–1114], using the auxiliary norms $\|w\|'_{-i,z}$, and $\|w\|'^{,\ell}_{-i,z}$ defined there, because we have the equivalent of (19) and (17)& (20) in Crovisier's paper:

$$\begin{gathered}\|w\|'_z < \|Dfw\|'_{f(z)} = \|Dfw\|''_{1,f(z)} < \|Df^2w\|''_{2,f^2(z)} < \cdots < \|Df^{m_1}w\|''_{f^{m_1}z}, \\ \forall z \in \widetilde{B}, 0 \ne w \in F(z);\end{gathered} \tag{A.9}$$

if $0 \ne w \in F(z)$, and $n-1 \ge m_1$, we get from Lemma A.5 and (A.7):

$$\|Df^n w\|'_z > 4\|w\|'_z, \forall z \text{ with } f^{n-1}z \notin B_1; \tag{A.10}$$

finally, using Lemma A.5 again, there is $m_2 \gg m_1$ so that if $0 \neq w \in F(z)$

$$\|Df^{m_2} w\|'_z > Cm_1 \|w\|'_z \geq 4\|w\|''_z \,, \forall z \text{ with } f^{m_2-1}z \notin B_1 \,. \tag{A.11}$$

(Note that Lemma A.5 gives us better control than what is available in [Cr] so that his construction can in fact be simplified.)

Finally, the two adapted semi-norms are just

$$\|w\|_{E,z} = \|P^E_z w\|_z \,, \quad \|w\|_{F,z} = \|P^F_z w\|_z \,, \quad z \in \Lambda^{\mathbb{C}} \,. \tag{A.12}$$

The smoothness property (4.8) follows from the fact that $z \mapsto \|\cdot\|_z$ is C^∞, combined with the Hoelder smoothness of the foliations and C^∞ property of leaves. $\square$

References

[BoVi] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math. **115** (2000), 157–193.

[Cas] A. Castro, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors*, Preprint (2000).

[Cr] S. Crovisier, *Saddle-node bifurcations for hyperbolic sets*, Ergodic Theory Dynam. Systems **22** (2002), 1079–1115.

[Doe] G. Doetsch, *Introduction to the theory and application of the Laplace transformation*, Springer-Verlag, Berlin, 1974.

[Do] D. Dolgopyat, *On dynamics of mostly contracting diffeomorphisms.*, Comm. Math. Phys. **213** (2000), 181–201.

[Fr] D. Fried, *Meromorphic zeta functions for analytic flows*, Comm. Math. Phys. **174** (1995), 161–190.

[GGK] I. Gohberg, S. Goldberg, and N. Krupnik, *Traces and Determinants of Linear Operators*, Birkhäuser, Basel, 2000.

[Gr1] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[Gr2] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France **84** (1956), 319–384.

[H] M. Hakim, *Attracting domains for semi-attractive transformations of $\mathbb{C}^p$*, Publicacions Matemàtiques **38** (1994), 479–499.

[HPS] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics Vol. 583, Springer, Berlin, New York, 1977.

[Ic] T. Ichinose, *On the spectra of tensor products of linear operators in Banach spaces*, Journal für Mathematik **244** (1970), 119-153.

[Ke] G. Keller, *Dynamical zeta functions for S-unimodal maps*, Colloquium Mathematicum **84/85** (2000), 229-233.

[M] A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215–220.

[Ma] D. Mayer, *Continued fractions and related transformations*, Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces (T. Bedford, M. Keane, and C. Series, ed.), Oxford University Press, Oxford, 1991.

[Mi] J. Milnor, *Dynamics in one complex variable: Introductory lectures*, Friedr. Vieweg & Sohn, Braunschweig, 1999.

[NP] S. Newhouse and J. Palis, *Hyperbolic nonwandering sets on two-dimensional manifolds*, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, 1973, pp. 293–301.

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors*, Cambridge University Press, 1993.

[PY] J. Palis and J.-C. Yoccoz, *Implicit formalism for affine-like maps and parabolic composition*, Global Analysis, Festschrift dedicated to Floris Takens on the occasion of his 60th birthday (H. Broer, B. Krauskopf, and G. Vegter, ed.), Institute of Physics Publishing, Bristol, 2001, pp. 67–87.

[PeS] Ya.B. Pesin and Ya.G. Sinai, *Gibbs measures for partially hyperbolic attractors*, Ergodic Theory Dynam. Systems **2** (1982), 417–438.

[PS1] E.R. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, Ann. of Math. (2) **151** (2000), 961–1023.

[PS2] E.R. Pujals and M. Sambarino, *On the dynamics of the dominated splitting*, Preprint, 2003.

[Rue1] D. Ruelle, *Zeta functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), 231–242.

[Rue2] D. Ruelle, *One-dimensional Gibbs states and Axiom A diffeomorphisms*, J. Diff. Geom. **25** (1987), 117–137.

[Ru1] H.H. Rugh, *The correlation spectrum for hyperbolic analytic maps*, Nonlinearity **5** (1992), 1237–1263.

[Ru2] H.H. Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), 805–819.

[Ru3] H.H. Rugh, *Intermittency and regularized Fredholm determinants*, Invent. Math. **135** (1999), 1–25.

[Ru4] H.H. Rugh, *Fredholm determinants for real-analytic hyperbolic diffeomorphisms of surfaces.*, Proceedings XIth International Congress of Mathematical Physics (Paris, 1994), Internat. Press, Cambridge, MA, 1995, pp. 297–303.

[Ta] F. Takens, *Partially hyperbolic fixed points*, Topology **10** (1971), 133–147.

V. Baladi: CNRS UMR 7586, Institut Mathématique de Jussieu, 75251 Paris, FRANCE
E-mail address: baladi@math.jussieu.fr

E.R. Pujals: Instituto de Matemática, Universidade Federal do Rio de Janeiro, CEP 21.945-970, Rio de Janeiro, BRAZIL
E-mail address: enrique@impa.br

M. Sambarino: IMERL, Fac. Ingeniería, Montevideo C.C. 30, URUGUAY
E-mail address: samba@fing.edu.uy