On the smoothness of gap boundaries for generalized Harper operators

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Abstract

Results concerning set theoretic continuity properties of the spectrum of the Harper operator are extended to a large class (generalized Harper operators (GHO)) of operators in $L^2(\mathbb{Z}^2)$.

1 Introduction: the setting and the main results.

Consider in $L^2(\mathbb{Z}^2)$ the following class of operators

$$(h_\epsilon)\psi(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{y})} h(\mathbf{x},\mathbf{y})\psi(\mathbf{y}) \tag{1.1}$$

where $\epsilon \in \mathbb{R}$ and $h(\mathbf{x},\mathbf{y}),\ \phi(\mathbf{x},\mathbf{y})$ satisfy the conditions:

$$h(\mathbf{x},\mathbf{y}) = \overline{h(\mathbf{x},\mathbf{y})} \tag{1.2}$$

$$|h(\mathbf{x},\mathbf{y})| \le Ce^{-\beta|\mathbf{x}-\mathbf{y}|};\ \ C<\infty,\ \ 0<\beta\le 1 \tag{1.3}$$

$$\phi(\mathbf{x},\mathbf{y}) = -\phi(\mathbf{y},\mathbf{x}) \tag{1.4}$$

$$|F(\mathbf{x},\mathbf{y},\mathbf{z})| \le \text{area}\ \ \Delta(\mathbf{x},\mathbf{y},\mathbf{z}) \tag{1.5}$$

where

$$F(\mathbf{x},\mathbf{y},\mathbf{z}) = \phi(\mathbf{x},\mathbf{y}) + \phi(\mathbf{y},\mathbf{z}) + \phi(\mathbf{z},\mathbf{x}) \tag{1.6}$$

and $\Delta(\mathbf{x},\mathbf{y},\mathbf{z})$ is the triangle in $\mathbb{R}^2$ determined by the points $\mathbf{x},\mathbf{y},\mathbf{z}$.

Under the conditions (1.2-1.6), h_ϵ is a uniformly bounded family of self-adjoint operators in $L^2(\mathbb{Z}^2)$. While the family h_ϵ is (by a simple argument based on Lebesgue dominated convergence theorem) strongly continuous, in general it is not normic continuous and this makes the problem of set theoretic continuity properties of the spectrum, $\sigma(h_\epsilon)$, a highly nontrivial one.

Some particular cases of the above class (called for the reasons below) generalized Harper operators (GHO)) are well known. If

$$h(\mathbf{x},\mathbf{y}) = h(\mathbf{x}-\mathbf{y}), \tag{1.7}$$

$$\phi(\mathbf{x},\mathbf{y}) = -x_1y_2 + x_2y_1; \;\; \mathbf{x} = (x_1,x_2), \; \mathbf{y} = (y_1,y_2), \tag{1.8}$$

then h_ϵ are nothing but the "magnetic matrices" appearing in so called Peierls-Onsager substitution for 2D-electrons moving in a periodic potential and subjected to a constant magnetic field [15],[16],[12]. Also, they are the discrete version of twisted convolutions [19]. If, in addition

$$h(\mathbf{c}) = \begin{cases} h, & \text{if } \; |\mathbf{x}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

one recovers the famous Harper operator which, as well known, has a fascinating Hofstatder butterfly like spectrum (see e.g. [9], [6] and references therein). In particular it has been conjectured that for all irrational ϵ, the spectrum has a Cantor set structure. The proof of this conjecture for some classes for irrational ϵ [8] used in an essential way set continuity properties of $\sigma(h_\epsilon)$.

The aim of this note is to prove that most of the set theoretic continuity properties of $\sigma(h_\epsilon)$ known for the particular case given by (1.7),(1.8) hold true in the general case. More precisely:

Theorem 1. *For h_ϵ given by (1.1)-(1.6):*

i. Let $E \in \sigma(h_\epsilon)$. Then there exists an absolute constant $K < \infty$, such that for $|\epsilon - \eta| \leq 1/2$

$$dist(E, \sigma(h_\eta)) \leq K \frac{C}{\beta^6} |\epsilon - \eta|^{1/2}. \tag{1.9}$$

ii. Let

$$E_+(\epsilon) = sup\, \sigma(h_\epsilon). \tag{1.10}$$

Then there exists an absolute constant $K < \infty$, such that for $|\epsilon - \eta| \leq 1/2$:

$$|E_+(\epsilon) - E_+(\eta)| \leq K \frac{C}{\beta^4} |\epsilon - \eta| |\ln|\epsilon - \eta||. \tag{1.11}$$

Suppose now that for some ϵ_0 there is a gap $\Delta = (a,b),\ \ b-a = 4d > 0$ in the spectrum of h_{ϵ_0}, i.e. $\sigma(h_{\epsilon_0}) = \sigma_1(\epsilon_0) \cup \sigma_2(\epsilon_0)$, $sup\sigma_1(\epsilon_0) = a,\ inf\sigma_2(\epsilon_0) = b$. Then from Theorem 1i. one has that for $|\,\epsilon - \epsilon_0\,| \leq nd^2$, where $n > 0$ is a constant depending upon C and β, h_ϵ still has a gap, $\Delta(\epsilon)$ of length larger than $2d$, i.e. $\sigma(h_\epsilon) = \sigma_1(\epsilon) \cup \sigma_2(\epsilon)$, $\inf_{E\in\sigma_1(\epsilon),E'\in\sigma_2(\epsilon)} |E - E'| \geq 2d$ and $\sigma_1(\epsilon)$ coincide with $\sigma_1(\epsilon_0)$ in the limit $\epsilon \to \epsilon_0$. So for $|\,\epsilon - \epsilon_0\,| \leq nd^2$ one can define

$$E_1(\epsilon) = sup\sigma_1(\epsilon),\ \ E_2(\epsilon) = inf\sigma_2(\epsilon). \tag{1.12}$$

Theorem 2. *There exist constants $m > 0,\ \ M < \infty$, depending upon C and β such that for*

$$|\,\epsilon - \epsilon_0\,| \leq md^4, \tag{1.13}$$

$$|E_j(\epsilon) - E_j(\epsilon_0)| \leq M|\,\epsilon - \epsilon_0\,|(d^{-7} + d^{-5}|\ln|\,\epsilon - \epsilon_0\,||). \tag{1.14}$$

Theorem 1i. implies that for $|\,\epsilon - \eta| \leq 1/2$

$$dist_H(\sigma(h_\epsilon), \sigma(h_\eta)) \leq K\frac{C}{\beta^6}|\,\epsilon - \eta|^{1/2} \tag{1.15}$$

where $dist_H(A,B)$ is the Hausdorff distance between two compact sets in $\mathbb{R}$. For the Harper operator (1.15) has been proved in [4] (improving an earlier result in [8] giving the exponent 1/3 in (1.15)). For the more general case of magnetic matrices (see (1.7), (1.8))(1.15) follows from the existence of a 1/2-Hölder continuous field of rotation algebras [10],[14]. It seems [5], [11] that the result in Theorem 1i. is optimal in the sense that the exponent 1/2 in (1.9) cannot be improved *uniformly* in the gap's length. As concerning Theorem 1ii. and Theorem 2, for the case of magnetic matrices there exists a better result due to Bellissard [5], namely that the gap boundaries are actually Lipschitz continuous, i.e. the logarithmic factor in the r.h.s. of (1.11), (1.14) can be removed. We believe this is true also in the general case but we were not able (at least up to now) to prove it. Concerning the dependence of constants upon gap's length, both the results in [5] and (1.14) are far from optimal; one has to use Theorem 1i. for small gaps and Theorem 2 for large ones.

The proofs in [10], [14], [5] (the proof in [4] uses the specific form of the Harper operator) rest heavily on the fact that under the conditions (1.7), (1.8), h_ϵ belongs to a rotation algebra and then one can use the powerful techniques of C^*-algebras theory. In other words the translation invariance, (1.7), as well as the "homogeneity of the magnetic field" ,(1.8), seems to be essential for these proofs.

The basic idea of our proof is the one already used in [15] (see also [16]) for the first general proof (see [3] for a particular case) of set theoretic continuity of the spectra of magnetic Schrödinger operators against variations of the magnetic field. It is based on exploiting, at the technical level, the gauge symmetry. Accordingly, it is expected to work under very general assumptions on $h(\mathbf{x},\mathbf{y})$, and indeed while we restricted ourselves to the discrete two-dimensional case, the results in Theorems 1 and 2 can be generalized to cover higher dimensions, matrix (or even operator) valued $h(\mathbf{x},\mathbf{y})$ and more important, the continuous case i.e. the case of "twisted" integral operators in $L^2(\mathbb{R}^n, d\mu)$ [18].

Twisted integral operators in $L^2(\mathbb{R}^n, d\mu)$ are intimately related to magnetic Schrödinger (and Dirac) operators. Let us outline, at the heuristic level, the main idea of this connection (see [17] for details). Let

$$H_{\epsilon,\mathbf{a}} = (\mathbf{P} - \mathbf{A}_0(\mathbf{x}) - \epsilon\,\mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x}) \tag{1.16}$$

with $\mathbf{b}(\mathbf{x}) = curl\mathbf{a}(\mathbf{x})$ uniformly bounded together with its first order derivatives and $\mathbf{A}_0(\mathbf{x})$, $V(\mathbf{x})$ satisfying the appropriate conditions as to assure that H_ϵ is a family of semi-bounded self-adjoint operators in $L^2(\mathbb{R}^n, d\mathbf{x})$, $\ n = 2, 3$. Take $-E_0$ sufficiently large so that $(-\infty, E_0 + 1) \subset \rho(H_\epsilon)$. Then it turns out that [17]

$$(H_\epsilon - E_0)^{-1} = S_{\epsilon,E_0} + \epsilon\, V(\epsilon, E_0) \tag{1.17}$$

where $V(\epsilon, E_0)$ is uniformly bounded as $\epsilon \to 0$ and S_{ϵ,E_0} is the integral operator:

$$(S_{\epsilon,E_0} f)(\mathbf{x}) = \int_{\mathbf{R}^3} e^{i\,\epsilon\,\phi_{\mathbf{a}}(\mathbf{x},\mathbf{y})} G_0(\mathbf{x},\mathbf{y}; E_0) f(\mathbf{y}) d\mathbf{y} \tag{1.18}$$

where

$$\phi_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = \int_{\mathbf{y}}^{\mathbf{x}} \mathbf{a}(\mathbf{u}) \cdot d\mathbf{u} \tag{1.19}$$

and $G_0(\mathbf{x},\mathbf{y}; E_0)$ is the integral kernel of $((\mathbf{P} - \mathbf{A}_0(\mathbf{x}))^2 + V(\mathbf{x}) - E_0)^{-1}$. Notice that in this case (see(1.6)) $F(\mathbf{x},\mathbf{y},\mathbf{z})$ is nothing but the flux of $\mathbf{b}(\mathbf{x})$ through the triangle $\Delta(\mathbf{x},\mathbf{y},\mathbf{z})$. Now the spectral properties of H_ϵ can be read from the spectral properties of $(H_\epsilon - E_0)^{-1}$ and since the second term in the r.h.s. of (1.17) can be controlled by regular perturbation theory one is led to the study of S_{ϵ,E_0}. Along this way one obtains the analog of Theorems 1 and 2 for magnetic Schrödinger and (with an easy extension) Dirac operators. To our knowledge the best result to date about set theoretic continuity of the spectra of Schrödinger and Dirac operators is the $2/3 - \delta$-Hölder continuity result in [7](the 1/2-Hölder continuity result is contained implicitly in [15] and [12]).

On the way of proving Theorem 2ii. we obtain the following result about smoothness of "almost" convex functions, which might be interesting in itself. We give the result only in the one-dimensional case but it can extended (as the similar results for the mid-point convex functions [2]) to a more general context.

Proposition 3. *Let $F : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\sup_x F(x) - \inf_x F(x) \leq 2P < \infty \tag{1.20}$$

$$F(x) - \frac{F(x+\eta) + F(x-\eta)}{2} \leq N|\eta|^\alpha; \quad |\eta| \leq 1/2, \ x \in \mathbb{R}, \ N < \infty, \ \alpha > 0. \tag{1.21}$$

Then for $x, y \in \mathbb{R}, \quad |x - y| \leq 1/2$:

$$|F(x) - F(y)| \leq \begin{cases} (4P + 3N\frac{1}{1-2^{1-\alpha}})|x-y| & if \quad \alpha > 1 \\ (4P + 6N)|x-y||\ln|x-y|| & if \quad \alpha = 1 \\ (4P + 2N\frac{1}{1-2^{\alpha-1}})|x-y|^\alpha & if \quad \alpha < 1 \end{cases} \tag{1.22}$$

2 The proofs.

Proof of Theorem 1

It is sufficient to consider only the case $\eta = 0:$ write $h_\epsilon(\mathbf{x}, \mathbf{y}) = h_\eta(\mathbf{x}, \mathbf{y}) e^{i(\epsilon - \eta)\phi(\mathbf{x}, \mathbf{y})}$ and observe that $h_\eta(\mathbf{x}, \mathbf{y})$ satisfies (1.2), (1.3). Before entering the technicalities let us point out the main idea of the proof (borrowed from [15]). For $\mathbf{c} \in \mathbb{Z}^2$, consider the "gauge transformation"

$$(U_{\mathbf{c},\epsilon} f)(\mathbf{x}) = e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{c})} f(\mathbf{x}).$$

By direct computation for $E \in \mathbb{R}$:

$$U^*_{\mathbf{c},\epsilon}(h_\epsilon - E) = (h_0 - E)U^*_{\mathbf{c},\epsilon} + (U^*_{\mathbf{c},\epsilon} h_\epsilon U_{\mathbf{c},\epsilon} - h_0)U^*_{\mathbf{c},\epsilon}.$$

The main point is that (see the proof of (2.25) below) on functions, ψ, supported on a ball centered at $\mathbf{c}$ and of radius L, $\parallel (U^*_{\mathbf{c},\epsilon} h_\epsilon U_{\mathbf{c},\epsilon} - h_0)U^*_{\mathbf{c},\epsilon}\psi \parallel$ is at most of order $L|\,\epsilon\,| \parallel \psi \parallel$. Suppose now that $E \in \sigma(h_\epsilon)$. Then one can find ψ with $\parallel \psi \parallel = 1$ such that $\parallel (h_\epsilon - E)\psi \parallel$ is small. If, in addition ψ is localized somewhere, then one can find $\mathbf{c}$ such that by the above argument $\parallel (h_0 - E)U^*_{\mathbf{c},\epsilon}\psi \parallel$ is also small and then E must be close to $\sigma(h_0)$. The trouble with this argument is that the functions for which $\parallel (h_\epsilon - E)\psi \parallel$ is small might not be localized. So one has either to localize them and estimate the "localization error"(as done in [4] for the almost Mathieu operator) or to design an appropriate, ψ dependent "partition of unity" as done in [15]) We shall follow the second route.

A finite number of strictly positive, absolute constants will appear during the proof; all of them will be denoted by $k > 0$. Also a finite number of finite, positive, absolute constants will appear and will be denoted by $K < \infty$. We begin with a preliminary lemma containing a technical result. It is patterned after a similar result in [15]. Let $\mathbf{a} \in \mathbb{Z}^2$, $N \in \mathbb{N}^+$,

$$C(\mathbf{a}, N) = \{\mathbf{x} = (x_1, x_2) | |x_\mu - a_\mu| \leq N, \;\; \mu = 1, 2\}, \tag{2.1}$$

$\chi_{\mathbf{a},N}$ the characteristic function of $C(\mathbf{a}, N)$, $f_N(\mathbf{x})$ satisfying

$$0 \leq f_N(\mathbf{x}) \leq 1,$$

$$f_N(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in C(\mathbf{0}, N) \\ 0 & \text{if} \quad \mathbf{x} \notin C(\mathbf{0}, 2N) \end{cases} \tag{2.2}$$

$$|f_N(\mathbf{x}) - f_N(\mathbf{y})| \leq \frac{|\mathbf{x} - \mathbf{y}|}{N}$$

and $f_{N,\mathbf{a}}$ defined by:

$$f_{N,\mathbf{a}}(\mathbf{x}) = f_N(\mathbf{x} - \mathbf{a}). \tag{2.3}$$

Lemma 4. *Let $\Phi \in L^2(\mathbb{Z}^2)$ supported on a finite set, $N \in \mathbb{N}^+$. Then there exist $\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_p$, $p < \infty$ depending upon Φ such that:*

i.

$$\min_{j \neq l} |\mathbf{a}_j - \mathbf{a}_l| \geq 8N. \tag{2.4}$$

For $j \neq l$

$$\rho_{j,l;N} = \min_{\mathbf{x} \in C(\mathbf{a}_j, 2N), \mathbf{y} \in C(\mathbf{a}_l, 2N)} |\mathbf{x} - \mathbf{y}| \geq 2N. \tag{2.5}$$

There exist $K < \infty$ such that

$$\max_j \sum_{l \neq j} e^{-\beta \rho_{j,l;N}} \leq \frac{K}{\beta^2} e^{-\beta N}. \tag{2.6}$$

ii. If

$$f_{N,\Phi} = \sum_{j=0}^{p} f_{N,\mathbf{a}_j} \tag{2.7}$$

then

$$0 \leq f_{N,\Phi} \leq 1; \;\; |f_{N,\Phi}(\mathbf{x}) - f_{N,\Phi}(\mathbf{y})| \leq \frac{|\mathbf{x} - \mathbf{y}|}{N}, \tag{2.8}$$

$$\| \Phi f_{N,\Phi} \| \geq \frac{\| \Phi \|}{9}. \tag{2.9}$$

Proof. Let $\mathbf{a}_0$ be a point of maximum of $\| \Phi\chi_{\mathbf{a},N} \|$ (as a function of $\mathbf{a}$). Define

$$\Phi_1 = \Phi(1 - \chi_{\mathbf{a}_0,9N}) \tag{2.10}$$

and repeat the procedure by taking $\mathbf{a}_1$ to be a point of maximum for $\| \Phi_1\chi_{\mathbf{a},N} \|$, etc. Since Φ is supported on a finite set, at some $p < \infty$, (depending upon Φ) $\Phi_p(1 - \chi_{\mathbf{a}_p,9N}) \equiv 0$ and the procedure stops. Now (2.4) holds true by construction and (2.5) follows at once from (2.4). For (2.6), set a lattice centered at $\mathbf{a}_j$ with the lattice spacing equal to $5N$. Due to (2.4) each lattice cell contains at most one $\mathbf{a}_l$. Move each $\mathbf{a}_l$ to the corner of the cell (containing it) which is nearest to $\mathbf{a}_j$. By this operation the sum in (2.6) is increased. The resulting sum is dominated by

$$2 \sum_{\mathbf{q}\in\mathbb{Z}^2,\mathbf{q}\neq\mathbf{0}} e^{-\beta N|\mathbf{q}|}$$

which gives (2.6). Further, (2.8) is almost obvious. Indeed by (2.5), $f_{N,\mathbf{a}_j}$ have disjoint supports so $0 \leq f_{N,\Phi} \leq 1$. Suppose $f_{N,\Phi}(\mathbf{x}) \neq 0,\ \ f_{N,\Phi}(\mathbf{y}) \neq 0$. Since $f_{N,\mathbf{a}_j}$ have disjoint supports, $f_{N,\Phi}(\mathbf{x}) - f_{N,\Phi}(\mathbf{y}) = f_{N,\mathbf{a}_j}(\mathbf{x}) - f_{N,\mathbf{a}_l}(\mathbf{y})$ for some j and l. If $j = l$ (2.8) follows from (2.2) and if $j \neq l$ then $|\mathbf{x} - \mathbf{y}| \geq 2N$ and then $|f_{N,\Phi}(\mathbf{x}) - f_{N,\Phi}(\mathbf{y})| \leq \frac{|\mathbf{x}-\mathbf{y}|}{2N}$. Suppose now $f_{N,\Phi}(\mathbf{x}) \neq 0,\ \ f_{N,\Phi}(\mathbf{y}) = 0$ Then for some j, $f_{N,\Phi}(\mathbf{x}) = f_{N,\mathbf{a}_j}(\mathbf{y})$ and $\mathbf{y} \notin supp f_{N,\mathbf{a}_j}$. It follows that $f_{N,\Phi}(\mathbf{x}) - f_{N,\Phi}(\mathbf{y}) = f_{N,\mathbf{a}_j}(\mathbf{x}) = f_{N,\mathbf{a}_j}(\mathbf{x}) - f_{N,\mathbf{a}_j}(\mathbf{y})$ and again (2.8) follows from (2.2). The case $f_{N,\Phi}(\mathbf{x}) = 0,\ \ f_{N,\Phi}(\mathbf{y}) \neq 0$ is similar and the case $f_{N,\Phi}(\mathbf{x}) = f_{N,\Phi}(\mathbf{y}) = 0$ is trivial.

Consider now

$$\tilde{\Phi}_j = \Phi f_{N,\mathbf{a}_j}. \tag{2.11}$$

Since by construction $\Phi_j = \Phi \prod_{l=0}^{j-1}(1 - \chi_{\mathbf{a}_l,9N})$

$$|\tilde{\Phi}_j(\mathbf{x})| \geq |\Phi_j(\mathbf{x}) f_{N,\mathbf{a}_j}(\mathbf{x})| \tag{2.12}$$

Now by definition (see also (2.12) and remember that $f_{N,\mathbf{a}_j}$ have disjoint supports)

$$\| \Phi_j \|^2 = \| \Phi_j\chi_{\mathbf{a}_j,9N} \|^2 + \| \Phi_{j+1} \|^2$$

so that

$$\| \Phi \|^2 = \sum_{j=0}^{p} \| \Phi_j\chi_{\mathbf{a}_j,9N} \|^2 \leq 81 \sum_{j=0}^{p} \| \Phi_j\chi_{\mathbf{a}_j,N} \|^2$$

$$\leq 81 \sum_{j=0}^{p} \| \Phi_j f_{N,\mathbf{a}_j} \|^2 \leq 81 \sum_{j=0}^{p} \| \tilde{\Phi}_j \|^2 = 81 \| \Phi \sum_{j=0}^{p} f_{N,\mathbf{a}_j} \|^2 = 81 \| f_{N,\Phi}\Phi \|^2$$

and the proof of Lemma 4 is finished.

For the sake of easy quotation we collect some simple facts in:

Lemma 5. *i. Let A be the operator given by*

$$(Ag)(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-\beta|\mathbf{x}-\mathbf{y}|/2} g(\mathbf{y}) \tag{2.13}$$

Then

$$a \equiv \parallel A \parallel \leq \frac{K}{\beta^2} \tag{2.14}$$

ii.

$$\sup_{x>0} x^m e^{-\alpha x} = (\frac{m}{\alpha})^m e^{-m}; \quad m, \alpha > 0 \tag{2.15}$$

iii.

$$H \equiv \sup_{\epsilon\in\mathbb{R}} \parallel h_\epsilon \parallel \leq \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-\beta|\mathbf{y}|} \leq K\frac{C}{\beta^2} \tag{2.16}$$

End of proof of Theorem 1i.

Let $0 < \delta \leq 3 \parallel h_\epsilon \parallel \leq 3H$ and suppose $E \in \sigma(h_\epsilon)$. Then we can find Ψ_δ, $\parallel \Psi_\delta \parallel = 1$ and $\parallel (h_\epsilon - E)\Psi_\delta \parallel \leq \delta/4$. Further, there exists Φ_δ supported on a finite set such that $\parallel \Psi_\delta - \Phi_\delta \parallel \leq \frac{\delta}{8H}$. Then

$$\parallel \Phi_\delta \parallel \geq 5/8, \quad \parallel (h_\epsilon - E)\Phi_\delta \parallel \leq \delta. \tag{2.17}$$

Let (we omit to write some indices) f, $\tilde{\Phi}_j$, $\rho_{j,l}$ as given by Lemma 4 applied to Φ_δ. In what follows we shall use the same letter for a function on $\mathbb{Z}^2$ and for the corresponding multiplication operator. From (2.8) and (2.17) ($[\cdot,\cdot]$ means the commutator)

$$\delta \parallel \Phi_\delta \parallel \geq \parallel (h_\epsilon - E)\Phi_\delta \parallel \geq \parallel f(h_\epsilon - E)\Phi_\delta \parallel \geq \parallel (h_\epsilon - E) f\Phi_\delta \parallel - \parallel [f, h_\epsilon]\Phi_\delta \parallel$$

which implies

$$\parallel (h_\epsilon - E) f\Phi_\delta \parallel \leq \delta \parallel \Phi_\delta \parallel + \parallel [f, h_\epsilon]\Phi_\delta \parallel . \tag{2.18}$$

We first estimate from below the l.h.s. of (2.18). The first observation is that for $j \neq l$, $< (h_\epsilon - E)\tilde{\Phi}_j, (h_\epsilon - E)\tilde{\Phi}_l >$ is small for large N. Indeed from Lemma 4, Lemma 5 and $|\mathbf{y} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{x} - \mathbf{y}|$:

$$| < (h_\epsilon - E)\tilde{\Phi}_j, (h_\epsilon - E)\tilde{\Phi}_j > | = | \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \overline{e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{y})}(h(\mathbf{x},\mathbf{y}) - E\delta_{\mathbf{x},\mathbf{y}})\tilde{\Phi}_j(\mathbf{y})}$$

$$e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{z})}(h(\mathbf{x},\mathbf{z}) - E\delta_{\mathbf{x},\mathbf{z}})\tilde{\Phi}_l(\mathbf{z})| \leq (C+H)^2 a^2 e^{-\frac{\beta}{2}\rho_{j,l}} \parallel \tilde{\Phi}_j \parallel \parallel \tilde{\Phi}_l \parallel . \tag{2.19}$$

Further

$$\parallel (h_\epsilon - E) f\Phi_\delta \parallel^2 = \sum_j \parallel (h_\epsilon - E)\tilde{\Phi}_j \parallel^2 + \sum_{j\neq l} < (h_\epsilon - E)\tilde{\Phi}_j, (h_\epsilon - E)\tilde{\Phi}_j >$$

$$\geq \sum_j \| (h_\epsilon - E)\tilde{\Phi}_j \|^2 - (C+H)^2 a^2 \sum_{j\neq l} e^{-\frac{\beta}{2}\rho_{j,l}} \| \tilde{\Phi}_j \| \| \tilde{\Phi}_l \| . \qquad (2.20)$$

Viewing the last sum in (2.20) as a scalar product in $l^2(\mathbb{N})$ of $\| \tilde{\Phi}_j \|$ with $(B \| \tilde{\Phi} \|)_j \equiv \sum_{j\neq l} e^{-\frac{\beta}{2}\rho_{j,l}} \| \tilde{\Phi}_l \|$ and estimating the norm of B by the Schur test using (2.6) one obtains:

$$\| (h_\epsilon - E) f \Phi_\delta \|^2 \geq$$

$$\geq \sum_j \| (h_\epsilon - E)\tilde{\Phi}_j \|^2 - K a^2 (C+H)^2 e^{-\frac{\beta}{2}N} \frac{1}{\beta^2} \| f\Phi_\delta \|^2 . \qquad (2.21)$$

As said at the beginning of the proof the main point is that, due to the fact that $\tilde{\Phi}_j$ are localized around a_j, one can estimate from below the sum in the r.h.s. of (2.21) in terms of h_0. Indeed, let $U_{j,\epsilon}$ be the unitary operator (gauge transformation) defined by:

$$(U_{j,\epsilon} g)(\mathbf{x}) = e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{a}_j)} g(\mathbf{x}). \qquad (2.22)$$

then

$$\| (h_\epsilon - E)\tilde{\Phi}_j \| = \| U^*_{j,\epsilon}(h_\epsilon - E)\tilde{\Phi}_j \| = \| (U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon}\tilde{\Phi}_j + (h_0 - E) U^*_{j,\epsilon}\tilde{\Phi}_j \|$$

which (by the triangle inequality) gives:

$$\| (h_0 - E) U^*_{j,\epsilon}\tilde{\Phi}_j \| \leq \| h_\epsilon - E)\tilde{\Phi}_j \| + \| (U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon}\tilde{\Phi}_j \|$$

and then

$$\sum_j \| (h_0 - E) U^*_{j,\epsilon}\tilde{\Phi}_j \|^2 \leq$$

$$\leq 2\sum_j \| (h_\epsilon - E)\tilde{\Phi}_j \|^2 + 2\sum_j \| (U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon}\tilde{\Phi}_j \|^2 . \qquad (2.23)$$

We estimate now from above the last sum in the r.h.s. of (2.23). The crucial computation is:

$$(U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon}\tilde{\Phi}_j = \sum_{\mathbf{y}} (e^{i\,\epsilon\,F(\mathbf{x},\mathbf{y},\mathbf{a}_j)} - 1) h(\mathbf{x},\mathbf{y}) (U^*_{j,\epsilon}\tilde{\Phi}_j)(\mathbf{y}). \qquad (2.24)$$

Then using (2.15), (1.5) and the fact that $\mathbf{y} \in supp\tilde{\Phi}_j$ implies $|\mathbf{y} - \mathbf{a}_j| \leq 2\sqrt{2}N$ one obtains:

$$|(U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon}\tilde{\Phi}_j(\mathbf{x})| \leq K|\,\epsilon\,| N \frac{C}{\beta} (A|\tilde{\Phi}_j|)(\mathbf{x})$$

whereof by Lemma 5:

$$\| (U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - h_0) U^*_{j,\epsilon} \tilde{\Phi}_j \|^2 \leq K(|\,\epsilon\,| N \frac{C}{\beta^3})^2 \| \tilde{\Phi}_j \|^2 . \tag{2.25}$$

Putting together (2.9), (2.18), (2.21), (2.23) and (2.25) (remark in addition that $\sum_j \| \tilde{\Phi}_j \|^2 = \| f\Phi_\delta \|^2$) one has:

$$\sum_j \| (h_0 - E) U^*_{j,\epsilon} \tilde{\Phi}_j \|^2 \leq 2(\delta \| \Phi_\delta \| + \| [f, h_\epsilon]\Phi_\delta \|)^2 +$$

$$+K(\frac{(C+H)^2}{\beta^2} e^{-\beta N/2} + \epsilon^2 N^2 \frac{c^2}{\beta^6}) \| \Phi_\delta \|^2 . \tag{2.26}$$

The control of $\| [f, h_\epsilon]\Phi_\delta \|$ is easy: from (2.8), (1.3) and Lemma 5

$$|[f, h_\epsilon](\mathbf{x}, \mathbf{y})| \leq |f(\mathbf{x}) - f(\mathbf{y})||h(\mathbf{x}, \mathbf{y})| \leq K \frac{1}{N\beta} e^{-\frac{\beta}{2}|\mathbf{x}-\mathbf{y}|}$$

and then by Lemma 5

$$\| [f, h_\epsilon]\Phi_\delta \| \leq K \frac{C}{N\beta^3} \| \Phi_\delta \| . \tag{2.27}$$

Let

$$\hat{\Phi}_\delta = \sum_j U^*_{j,\epsilon} \tilde{\Phi}_j. \tag{2.28}$$

Observe that

$$\| \hat{\Phi}_\delta \|^2 = \sum_j \| \tilde{\Phi}_j \|^2 = \| f\Phi_\delta \| \tag{2.29}$$

By the same estimation as in the proof of (2.21):

$$\| (h_0 - E) f \hat{\Phi}_\delta \|^2 \leq$$

$$\leq \sum_j \| (h_0 - E) U^*_{j,\epsilon} \tilde{\Phi}_j \|^2 + K a^2 (C+H)^2 \frac{1}{\beta^2} e^{-\frac{\beta}{2}N} \| \hat{\Phi}_\delta \|^2 . \tag{2.30}$$

Putting the things together (see (2.26), (2.27), (2.29), (2.30)) and taking into account that $e^{-\frac{\beta}{2}N} \leq K \frac{1}{\beta^2 N^2}$ one obtains:

$$\| (h_0 - E) f \hat{\Phi}_\delta \|^2 \leq$$

$$\{2(\delta + K\frac{C}{\beta^3 N})^2 + K\frac{C^2}{\beta^{12} N^2} + K\,\epsilon^2 N^2 \frac{C^2}{\beta^6}\} \| \hat{\Phi}_\delta \|^2 . \tag{2.31}$$

Choosing $N = [\frac{1}{|\,\epsilon\,|^{1/2}}]$ (here $[\cdot]$ means the integer part) (remember that $0 < \beta \leq 1$) one has from (2.31)

$$\| \, (h_0 - E) f \hat{\Phi}_\delta \, \|^2 \leq (4\delta^2 + K|\,\epsilon\,| \frac{C^2}{\beta^{12}}) \, \| \, \hat{\Phi}_\delta \, \|^2 \; .$$

which finishes the proof since δ can be taken arbitrarily small and $\| \, \hat{\Phi}_\delta \, \| \geq 5/72$.

Proof of Theorem 1.ii. Theorem 1.ii follows from Proposition 3 and the following Lemma

Lemma 6. *There exists $K < \infty$ such that for $|\,\epsilon\,| \leq 1/2$:*

$$E_+(\epsilon_0) - \frac{E_+(\epsilon_0 + \epsilon) + E_+(\epsilon_0 - \epsilon)}{2} \leq K \frac{C}{\beta^4} |\,\epsilon\,| \tag{2.32}$$

Remark. The method of proof of Lemma 6 also gives for $|\,\epsilon - \epsilon_0\,| \leq 1/2$:

$$|E_+(\epsilon_0) - E_+(\epsilon)| \leq K \frac{C}{\beta^4} |\,\epsilon - \epsilon_0\,|^{2/3} \tag{2.33}$$

Proof of Lemma 6. As before it is sufficient to consider the case $\epsilon_0 = 0$. We shall use the fact that for a self-adjoint operator, A,

$$\sup \sigma(A) = \sup_{\|g\|=1} < g, Ag > \; .$$

The main point there is that this will allow to replace the $1/N$ dependence of the "localization error" in (2.27) by a better one namely $1/N^2$. Suppose $E \in \sigma(h_0)$, and let ,$\mathbf{a}_j$, $f_{N,\mathbf{a}_j}$, Φ_δ, f, $\tilde{\Phi}_j$ as in the proof of Theorem 1.i. The following localization identity goes back at least to Agmon [1] (see also [13],[17])

$$\begin{aligned} &< \tilde{\Phi}_j, (h_0 - E)\tilde{\Phi}_j >= \\ &= Re < f_{N,\mathbf{a}_j}\Phi_\delta, f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta > -1/2 < \Phi_\delta, [f_{N,\mathbf{a}_j}, [f_{N,\mathbf{a}_j}, h_0]]\Phi_\delta > . \end{aligned} \tag{2.34}$$

On the other hand (see (2.22) for $U_{j,\epsilon}$) from the definition of $E_+(\epsilon)$:

$$< \tilde{\Phi}_j, (h_0 - E)\tilde{\Phi}_j >=< \tilde{\Phi}_j, \{(h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon}) + (U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - E_+(\epsilon)) + (E_+(\epsilon) - E)\}\tilde{\Phi}_j > \leq$$

$$\leq (E - E_+(\epsilon)) \, \| \, \tilde{\Phi}_j \, \|^2 + < \tilde{\Phi}_j, ((h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon})\tilde{\Phi}_j > \tag{2.35}$$

From (2.34) and (2.35) (remember that $\| \, f\Phi_\delta \, \|^2 = \sum_j \| \, \tilde{\Phi}_j \, \|^2$):

$$(E_+(\epsilon) - E) \, \| \, f\Phi_\delta \, \|^2 \leq -\sum_j Re < f_{N,\mathbf{a}_j}\Phi_\delta, f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta > +$$

$$+\frac{1}{2}\sum_j < \Phi_\delta, [f_{N,\mathbf{a}_j}, [f_{N,\mathbf{a}_j}, h_0]]\Phi_\delta > + \sum_j < \tilde{\Phi}_j, ((h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon})\tilde{\Phi}_j > . \tag{2.36}$$

Writing (2.36) also for $-\epsilon$ and summing up one obtains:

$$E - \frac{E_+(\epsilon) + E_+(-\epsilon)}{2}) \parallel f\Phi_\delta \parallel^2 \leq -\sum_j Re < f_{N,\mathbf{a}_j}\Phi_\delta, f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta > +$$

$$+\frac{1}{2}\sum_j < \Phi_\delta, [f_{N,\mathbf{a}_j}, [f_{N,\mathbf{a}_j}, h_0]]\Phi_\delta > + \frac{1}{2}\sum_j < \tilde{\Phi}_j, (2h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - U^*_{j,-\epsilon} h_{-\epsilon} U_{j,-\epsilon})\tilde{\Phi}_j > . \tag{2.37}$$

We are left with the problem of estimating the r.h.s. of (2.37). Using twice the Cauchy-Schwartz inequality:

$$-\sum_j Re < f_{N,\mathbf{a}_j}\Phi_\delta, f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta > \leq \sum_j \parallel f_{N,\mathbf{a}_j}\Phi_\delta \parallel \parallel f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta \parallel$$

$$(\sum_j \parallel \tilde{\Phi}_j \parallel^2)^{1/2} (\sum_j \parallel f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta \parallel^2)^{1/2} \tag{2.38}$$

Taking into account that $f_{N,\mathbf{a}_j}$ have disjoint supports,

$$\sum_j f_{N,\mathbf{a}_j}(\mathbf{x})^2 = f(\mathbf{x})^2,$$

which together with $0 \leq f(\mathbf{x}) \leq 1$, (2.17) and (2.9) gives

$$\sum_j \parallel f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta \parallel^2 = \parallel f(h_0 - E)\Phi_\delta \parallel^2 \leq 81\delta^2 \parallel f\Phi_\delta \parallel^2 . \tag{2.39}$$

Combining (2.38) with (2.39) one has

$$-\sum_j Re < f_{N,\mathbf{a}_j}\Phi_\delta, f_{N,\mathbf{a}_j}(h_0 - E)\Phi_\delta > \leq 81\delta^2 \parallel f\Phi_\delta \parallel^2 . \tag{2.40}$$

Consider now the second term in the r.h.s. of (2.37). Observe that since $\max_j(f_{N,\mathbf{a}_j}(\mathbf{x}) - f_{N,\mathbf{a}_j}(\mathbf{y}))^2 \leq \frac{\mathbf{x}-\mathbf{y}|^2}{N^2}$ and that, for fixed $\mathbf{x}$ and $\mathbf{y}$ at most two terms in the sum $\sum_j (f_{N,\mathbf{a}_j}(\mathbf{x}) - f_{N,\mathbf{a}_j}(\mathbf{y}))^2$ are nonzero one has that

$$\sum_j (f_{N,\mathbf{a}_j}(\mathbf{x}) - f_{N,\mathbf{a}_j}(\mathbf{y}))^2 \leq 2\frac{\mathbf{x}-\mathbf{y}|^2}{N^2}. \tag{2.41}$$

Now from Lemma 4, Lemma 5 and (2.41)

$$\sum_j < \Phi_\delta, [f_{N,\mathbf{a}_j}, [f_{N,\mathbf{a}_j}, h_0]]\Phi_\delta >= \sum_{j,\mathbf{x},\mathbf{y}} \overline{\Phi_\delta(\mathbf{x})}((f_{N,\mathbf{a}_j}(\mathbf{x})-f_{N,\mathbf{a}_j}(\mathbf{y}))^2 h(\mathbf{x},\mathbf{y})\Phi_\delta(\mathbf{y}) =$$

$$\sum_{\mathbf{x},\mathbf{y}} \overline{\Phi_\delta(\mathbf{x})}(\sum_j (f_{N,\mathbf{a}_j}(\mathbf{x})-f_{N,\mathbf{a}_j}(\mathbf{y}))^2) h(\mathbf{x},\mathbf{y})\Phi_\delta(\mathbf{y}) \leq \frac{2}{N^2} C \sum_{\mathbf{x},\mathbf{y}} |\overline{\Phi_\delta(\mathbf{x})}||\mathbf{x}-\mathbf{y}|^2 e^{-\beta|\mathbf{x}-\mathbf{y}|}|\Phi_\delta(\mathbf{y})| \leq$$

$$\leq K\frac{C}{N^2\beta^2} < |\Phi_\delta|, A|\Phi_\delta| > \leq K\frac{C}{N^2\beta^2} \parallel f\Phi_\delta \parallel^2 . \tag{2.42}$$

A straightforward computation gives:

$$\frac{1}{2}\sum_j < \tilde{\Phi}_j, (2h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - U^*_{j,-\,\epsilon} h_{-\,\epsilon} U_{j,-\,\epsilon})\tilde{\Phi}_j >=$$

$$= \sum_{j,\mathbf{x},\mathbf{y}} \overline{\Phi_\delta(\mathbf{x})}(1 - cos\,\epsilon\, F(\mathbf{x},\mathbf{y},\mathbf{a}_j))h(\mathbf{x},\mathbf{y})\Phi_\delta(\mathbf{y}) \tag{2.43}$$

Since for $\mathbf{y} \in supp\tilde{\Phi}_j$, $|\mathbf{y}-\mathbf{a}_j| \leq 2\sqrt{2}N$ and using Lemma 5 (remember also that $|F(\mathbf{x},\mathbf{y},\mathbf{a}_j)| \leq area\Delta(\mathbf{x},\mathbf{y},\mathbf{a}_j) \leq |\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{a}_j|/2$ one obtains from (2.43)

$$\sum_j < \tilde{\Phi}_j, (2h_0 - U^*_{j,\epsilon} h_\epsilon U_{j,\epsilon} - U^*_{j,-\,\epsilon} h_{-\,\epsilon} U_{j,-\,\epsilon})\tilde{\Phi}_j > \leq K\,\epsilon^2\, N^2 \frac{C}{\beta^4} \parallel f\Phi_\delta \parallel^2 . \tag{2.44}$$

Summing up (2.40), 2.42) and (2.44) one gets:

$$E - \frac{E_+(\epsilon) + E_+(-\,\epsilon)}{2}) \parallel f\Phi_\delta \parallel^2 \leq K(\delta + \frac{C}{N^2\beta^4} + \epsilon^2\, N^2 \frac{C}{\beta^4}) \parallel f\Phi_\delta \parallel^2 . \tag{2.45}$$

Choosing again $N = [\epsilon^{-1/2}]$ and taking into account that (2.45) holds true for all $E \in \sigma(h_0)$ and δ can be arbitrarily small, one has

$$E_+(0) - \frac{E_+(\epsilon) + E_+(-\,\epsilon)}{2}) \leq K\frac{C}{\beta^4}|\,\epsilon\,| \tag{2.46}$$

and the proof of Lemma 6 is complete.

Remark. The above estimations applied to (2.36) leads to $E_+(0) - E_+(\epsilon) \leq K\frac{C}{\beta^4}|\,\epsilon\,|^{2/3}$. By interchanging 0 and ϵ one obtains also $E_+(\epsilon) - E_+(0) \leq K\frac{C}{\beta^4}|\,\epsilon\,|^{2/3}$ and then $|E_+(\epsilon) - E_+(0)| \leq K\frac{C}{\beta^4}|\,\epsilon\,|^{2/3}$.

Proof of Proposition 3. Without restricting the generality one can replace (1.20) by

$$|F(x)| \leq P. \tag{2.47}$$

We give the proof for $\alpha = 1$ (the case we need) and leave the details to the reader the details for $\alpha \neq 1$.

For $x \in \mathbb{R}$ consider the function

$$g_x(u) = F(x+u) - F(x),\ u \in \mathbb{R}. \tag{2.48}$$

One has (uniformly in x):

$$|g_x(u)| \leq 2P, \tag{2.49}$$

$$g_x(0) = 0, \tag{2.50}$$

$$g_x(u) - \frac{g_x(u+\eta) + g_x(u-\eta)}{2} \leq N|\eta|,\ |\eta| \leq 1/2. \tag{2.51}$$

Every $u \in (0, 1/2]$ can be written (uniquely) as

$$u = \frac{a}{2^n};\ a \in (1/2, 1],\ n \in \mathbb{N}^+. \tag{2.52}$$

Let $a \in (1/2, 1]$ be fixed. Then from (2.50), (2.51) with $u = \eta = a/2$ one has

$$g_x(\frac{a}{2}) \leq \frac{g_x(a)}{2} + N\frac{a}{2}. \tag{2.53}$$

Then by induction over n:

$$g_x(\frac{a}{2^n}) \leq \frac{g_x(a)}{2^n} + Nn\frac{a}{2^n}. \tag{2.54}$$

Indeed taking $u = \eta = \frac{a}{2^n}$ in (2.51):

$$g_x(\frac{a}{2^n}) \leq \frac{1}{2}g(\frac{a}{2^{n-1}}) + N\frac{a}{2^n} \leq$$

$$\leq \frac{1}{2}[\frac{g_x(a)}{2^{n-1}} + N(n-1)\frac{a}{2^{n-1}}] + N\frac{a}{2^n} = \frac{g_x(a)}{2^n} + Nn\frac{a}{2^n}. \tag{2.55}$$

Take now $u \in (0, 1/2]$ and write it as in (2.52). Since $\ln 2 \geq 1/2$ one has

$$n \leq 2|\ln\ u|. \tag{2.56}$$

Then from (2.49) and $|\ln\ u| \geq 1/2$:

$$g_x(u) = g_x(u) - g_x(0) \leq \frac{g_x(a)}{a}u + 2Nu|\ln\ u| \leq 2(2P+N)u|\ln\ u|. \tag{2.57}$$

In the same way for $u \in [-1/2, 0)$:

$$g_x(u) - g_x(0) \leq 2(2P+N)|u||\ln\ u|. \tag{2.58}$$

Let $v \in (0, 1/2]$. From (2.51) with $u = 0,\ \eta = v$:

$$g_x(-v) \geq -g_x(v) - 2Nv \tag{2.59}$$

and using (2.57)

$$g_x(-v) \geq -2(2P+N)v|\ln\ v| - \frac{2N}{|\ln\ v|}v|\ln\ v| \geq -(4P+6N)v|\ln\ v|. \tag{2.60}$$

In the same way for $u \in (0, 1/2]$

$$g_x(u) \geq -(4P + 6N)u|\ln\ u|$$

which together with (2.60), (2.57) and(2.58)

$$|g_x(u) - g_x(0)| = |g_x(u)| \leq (4P + 6N)u|\ln\ u|,\ \ |u \leq 1/2. \tag{2.61}$$

Writing (2.61) in terms of $F(x)$ one obtains (1.22) for $\alpha = 1$.

Proof of Theorem 2. Again it is sufficient to consider $\epsilon_0 = 0$. We shall prove (1.14) for $E_1(\epsilon)$; the proof for $E_2(\epsilon)$ is similar. In what follows a finite number of constants depending upon C and β will appear; they are all denoted $m > 0$ (when we want to stress that they are strictly positive) of $M < \infty$ (when we want to stress that they are positive and finite). Let Γ_j be contours of finite length enclosing $\sigma_j(\epsilon)$ such that $\sup_{z\in\Gamma_1\cup\Gamma_2, |\,\epsilon\,| \leq md^2} dist(z, \sigma(h_\epsilon) \geq d$. Let $\lambda < \inf_\epsilon \sigma(h_\epsilon)$ and consider for $|\,\epsilon\,| \leq md$:

$$h_1(\epsilon) = \frac{i}{2\pi}\int_{\Gamma_1} z(h_\epsilon - z)^{-1}dz + (\lambda - 1)\frac{i}{2\pi}\int_{\Gamma_2}(h_\epsilon - z)^{-1}dz. \tag{2.62}$$

By construction (for $|\,\epsilon\,| \leq md^2$)

$$E_1(\epsilon) = \sup \sigma(h_1(\epsilon)). \tag{2.63}$$

We shall prove that up to errors which are Lipschitz (in norm), $h_1(\epsilon)$ has the same form as h_ϵ (with a d dependent β!). We begin by estimating $\sup_{z\in\Gamma_1\cup\Gamma_2} |G_0(\mathbf{x}, \mathbf{y}; z)|$ where $G_0(\mathbf{x}, \mathbf{y}; z) \equiv (h_0 - z)^{-1}(\mathbf{x}, \mathbf{y})$. For that we use (like in [18]) some elementary facts from Agmon-Combes-Thomas theory (see e.g. [13]). Consider for $\mu \in \mathbb{R}^+,\ \mathbf{x}_0 \in \mathbb{Z}^2$ the rotated operator

$$h_{\mu,\mathbf{x}_0} = e^{\mu|\cdot-\mathbf{x}_0|}h_0 e^{-\mu|\cdot-\mathbf{x}_0|}. \tag{2.64}$$

given by the kernel

$$h_{\mu,\mathbf{x}_0}(\mathbf{x}, \mathbf{y}) = e^{\mu|\mathbf{x}-\mathbf{x}_0|}h(\mathbf{x}, \mathbf{y})e^{-\mu|\mathbf{y}-\mathbf{x}_0|}. \tag{2.65}$$

Using $||\mathbf{x}-\mathbf{x}_0|-|\mathbf{y}-\mathbf{x}_0|| \leq |\mathbf{x}-\mathbf{y}|$; $|e^x - 1| \leq |x| e^{|x|}$ one has

$$|e^{\mu(|\mathbf{x}-\mathbf{x}_0|-|\mathbf{y}-\mathbf{x}_0|)} - 1| \leq \mu|\mathbf{x}-\mathbf{y}|e^{\mu|\mathbf{x}-\mathbf{y}|}. \tag{2.66}$$

If one writes

$$h_{\mu,\mathbf{x}_0} = h_0 + \mu B_{\mu,\mathbf{x}_0}$$

then from (2.65) and (2.66)

$$\sup_{\mathbf{x}_0 \in \mathbb{Z}^2, 0 \leq \mu \leq \beta/2} \parallel B_{\mu,\mathbf{x}_0} \parallel \equiv b < \infty. \tag{2.67}$$

Since $\sup_{z\in\Gamma_1\cup\Gamma_2} \parallel (h_0 - z)^{-1} \parallel \leq \frac{1}{d}$, for

$$\mu \leq min\{\frac{\beta}{2}, \frac{d}{2b}\} \tag{2.68}$$

and $z \in \Gamma_1 \cup \Gamma_2$:

$$\mu \parallel B_{\mu,\mathbf{x}_0}(h_0 - z)^{-1} \parallel \leq 1/2$$

so that by perturbation theory, on $\Gamma_1 \cup \Gamma_2$

$$(h_{\mu,\mathbf{x}_0} - z)^{-1} = (h_0 - z)^{-1}[1 + \mu B_{\mu,\mathbf{x}_0}(h_0 - z)^{-1}]^{-1}$$

and

$$\sup_{z\in\Gamma_1\cup\Gamma_2} \parallel (h_{\mu,\mathbf{x}_0} - z)^{-1}. \parallel \leq \frac{2}{d} \tag{2.69}$$

Due to the fact that $e^{-\mu|\cdot-\mathbf{x}_0|}$ is injective, $(h_0 - z)\mathrm{Ran}(e^{-\mu|\cdot-\mathbf{x}_0|}) \subset \mathrm{Ran}(e^{-\mu|\cdot-\mathbf{x}_0|})$ and that $h_{\mu,\mathbf{x}_0} - z$ has a bounded inverse one has that $(h_0 - z)^{-1}\mathrm{Ran}(e^{-\mu|\cdot-\mathbf{x}_0|}) \subset \mathrm{Ran}(e^{-\mu|\cdot-\mathbf{x}_0|})$ and

$$e^{\mu|\cdot-\mathbf{x}_0|}(h_0 - z)^{-1}e^{-\mu|\cdot-\mathbf{x}_0|} = (h_{\mu,\mathbf{x}_0} - z)^{-1}. \tag{2.70}$$

Let now for $\mathbf{c} \in \mathbb{Z}^2$

$$\Psi_{\mathbf{c}}(\mathbf{x}) = \delta_{\mathbf{x},\mathbf{c}}$$

where $\delta_{\mathbf{x},\mathbf{c}}$ is the usual Kronecker symbol. Then by (2.69) and (2.70)

$$\sup_{z\in\Gamma_1\cup\Gamma_2} |G_0(\mathbf{x},\mathbf{y};z)| = \sup_{z\in\Gamma_1\cup\Gamma_2} | < \Psi_{\mathbf{x}}, (h_0 - z)^{-1}\Psi_{\mathbf{y}} > | =$$

$$= \sup_{z\in\Gamma_1\cup\Gamma_2} | < e^{-\mu|\cdot-\mathbf{y}|}\Psi_{\mathbf{x}}, (h_{\mu,\mathbf{x}_0} - z)^{-1}e^{\mu|\cdot-\mathbf{y}|}\Psi_{\mathbf{y}} > | \leq \frac{2}{d}e^{-\mu|\mathbf{x}-\mathbf{y}|}. \tag{2.71}$$

Let now $S_{\epsilon,z}$ be the operator given by

$$(S_{\epsilon,z}g)(\mathbf{x}) = \sum_{\mathbf{y}} e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{y})}G_0(\mathbf{x},\mathbf{y};z)g(\mathbf{y}). \tag{2.72}$$

By direct computation

$$(h_\epsilon - z)S_{\epsilon,z} = 1 + \epsilon\, T_{\epsilon,z} \tag{2.73}$$

with

$$T_{\epsilon,z}(\mathbf{x},\mathbf{y}) = e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{y})}\sum_{\mathbf{u}}\frac{e^{i\,\epsilon\,F(\mathbf{x},\mathbf{u},\mathbf{y})}-1}{\epsilon}(h(\mathbf{x},\mathbf{u} - z\delta_{\mathbf{x},\mathbf{u}})G_0(\mathbf{u},\mathbf{y};z). \tag{2.74}$$

From (1.3), (1.5) and (2.71), for $z \in \Gamma_1 \cup \Gamma_2$:

$$|T_{\epsilon,z}(\mathbf{x},\mathbf{y})| \leq \frac{M}{d}\sum_{\mathbf{u}}|\mathbf{x}-\mathbf{u}||\mathbf{y}-\mathbf{u}|e^{-\beta|\mathbf{x}-\mathbf{u}|}e^{-\mu|\mathbf{y}-\mathbf{u}|}$$

and then from the Young inequality

$$\sup_{\mathbf{y}}\sum_{\mathbf{x}}|T_{\epsilon,z}(\mathbf{x},\mathbf{y})| \leq \frac{M}{\mu^3 d}. \tag{2.75}$$

For

$$|\,\epsilon\,| \leq m\mu^3 d \tag{2.76}$$

$$\epsilon \parallel T_{\epsilon,z} \parallel \leq 1/2 \tag{2.77}$$

and then for $z \in \Gamma_1 \cup \Gamma_2$:

$$(h_\epsilon - z)^{-1} = S_{\epsilon,z}(1+\epsilon\, T_{\epsilon,z})^{-1} = S_{\epsilon,z} - \epsilon\, S_{\epsilon,z}T_{\epsilon,z}(1+\epsilon\, T_{\epsilon,z})^{-1} \equiv S_{\epsilon,z} + \epsilon\, V_{\epsilon,z} \tag{2.78}$$

with

$$\parallel V_{\epsilon,z} \parallel \leq 2 \parallel S_{\epsilon,z} \parallel\parallel T_{\epsilon,z} \parallel \leq \frac{M}{d^2\mu^5}. \tag{2.79}$$

By the definition of $h_1(\epsilon)$ (see (2.62) and (2.78)

$$h_1(\epsilon) = \tilde{h}_\epsilon + \epsilon\, W_\epsilon \tag{2.80}$$

where

$$\tilde{h}_\epsilon = \frac{i}{2\pi}\int_{\Gamma_1} zS_{\epsilon,z}dz + (\lambda - 1)\frac{i}{2\pi}\int_{\Gamma_2} S_{\epsilon,z}dz. \tag{2.81}$$

From (2.72)

$$\tilde{h}_\epsilon(\mathbf{x},\mathbf{y}) = e^{i\,\epsilon\,\phi(\mathbf{x},\mathbf{y})}\tilde{h}(\mathbf{x},\mathbf{y}); |\tilde{h}(\mathbf{x},\mathbf{y})| \leq \frac{M}{d}e^{-\mu|\mathbf{x}-\mathbf{y}|}. \tag{2.82}$$

Take now (see (2.68)) $\mu = md$. Since W_ϵ is uniformly bounded as $\epsilon \to 0$ by perturbation theory $|\sup\sigma(h_1(\epsilon) - \sup\sigma(\tilde{h}_\epsilon)| \leq Md^{-7}$, and the application of Theorem 1.ii to $\tilde{h}_\epsilon$ finishes the proof of Theorem 2.

Acknowledgements.

The paper has been partly written at Mittag-Leffler Institute during a one month participation in the programme "*Partial differential equations and spectral theory*", Fall 2002. I thank the scientific steering committee for the invitation and Mittag-Leffler Institute for financial support.

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