A RANDOM NECKLACE MODEL

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Dedicated to Elliott Lieb on occasion of his 70-th birthday

ABSTRACT. We consider a Laplace operator on a random graph consisting of infinitely many loops joined symmetrically by intervals of unit length. The arc lengths of the loops are considered to be independent, identically distributed random variables. The integrated density of states of this Laplace operator is shown to have discontinuities provided that the distribution of arc lengths of the loops has a nontrivial pure point part. Some numerical illustrations are also presented.

1. INTRODUCTION

In this paper we study a model which perhaps provides the simplest example of a differential operator on a nontrivial infinite random metric graph. The graph consists of infinitely many loops joined symmetrically by intervals of unit length. The arc lengths of the loops are independent, identically distributed random variables. A similar model where the arc lengths of the loops are kept fixed, was considered by Avron, Exner, and Last in [2]. They called this model a Necklace of Rings. Mimicking this terminology we will use the name Random Necklace for the model considered here.

The main objective of the present work is to study the integrated density of states and the Lyapunov exponent for the Random Necklace Model. It is well known that the integrated density of states for metrically transitive Schrödinger operators on the line is continuous at all energies since the multiplicity of their spectrum is not greater than two. In contrast, the Laplacian of the Random Necklace Model can have eigenvalues of infinite multiplicity. Therefore, the integrated density of states may have discontinuities at those energies. We will show that this is indeed the case provided that the distribution of arc lengths of the loops has a nontrivial pure point part. An explicit description of the set of all energies where the integrated density of states is discontinuous will also be given. Moreover, we will show that the perturbation of the Laplacian by a magnetic field in general smoothes out the integrated density of states such that some of its discontinuities disappear.

Discontinuities of the integrated density of states for some discrete random Laplacians on Delone sets have been studied recently by Klassert, Lenz, and Stollmann in [9]. Earlier such discontinuities had been observed in [1], [7], [10], [18] for discrete Laplacians associated with Penrose tilings. The appearance of discontinuities is again related to the existence of infinitely degenerated eigenvalues.

The plan of the present work is as follows. The model is defined in Section 2. In Section 3 we decompose the integrated density of states $N(E)$ into the integrated density of loop states $N^{\mathrm{loop}}(E)$ (i.e., the states supported on loops of the graph) and the remainder $\widetilde{N}(E)$.

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Next we prove that $N^{\mathrm{loop}}(E)$ has discontinuities if and only if the distribution measure has a nontrivial pure point part. In Section 4 we show the smoothing effect of a constant magnetic field. In Sections 5 and 6 we accommodate the scattering theoretic method proposed in [13] (see also [12]) to calculate the integrated density of states and the Lyapunov exponent for the Random Necklace Model. Using this approach we prove the positivity of the Lyapunov exponent for almost all energies and study the set of its zeroes. In Section 7 we prove the continuity of $\widetilde{N}(E)$ and discuss its further regularity properties.

We present also some numerical illustrations performed by the method developed in [13]. Part of the material presented here has previously appeared in [11].

2. Random Necklace

In this section we give a precise formulation of the Random Necklace Model and discuss some of its basic properties. Consider an infinite graph Γ consisting of loops L_j with arc lengths $2\omega_j$ joined symmetrically by unit intervals $I_j = [0, 1]$ (see Fig. 1). Any loop will be realized as a union of its upper $L_j^{(+)} \cong [0, \omega_j]$ and lower $L_j^{(-)} \cong [0, \omega_j]$ parts. We always assume that the left vertex of any loop L_j corresponds to the point $x = 0$ for all three bonds $L_j^{(+)}$, $L_j^{(-)}$, and I_j adjacent to this vertex. Thus, the right vertex of the loop L_j corresponds to the point $x = 1$ of the interval I_{j+1} and to the point $x = \omega_j$ of the intervals $L_j^{(\pm)}$. Further, we suppose that $\omega = \{\omega_j\}_{j\in\mathbb{Z}}$ forms an i.i.d. sequence of random variables with distribution measure $\varkappa$ and satisfying $0 < \omega_j \le K$. The underlying probability space is, therefore, $\Omega = [0, K]^{\mathbb{Z}}$ with the product probability measure $\mathbb{P} = \times_{j\in\mathbb{Z}}\varkappa$.

With the graph Γ we associate the Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_{j\in\mathbb{Z}} L^2(I_j) \oplus L^2(L_j) \qquad \text{with} \qquad L^2(L_j) = L^2(L_j^{(+)}) \oplus L^2(L_j^{(-)}). \tag{1}$$

According to the decomposition (1) we will write the elements ψ of $\mathcal{H}$ as follows

$$\psi = \bigoplus_{j\in\mathbb{Z}} \psi_j^{(0)} \oplus \psi_j^{(+)} \oplus \psi_j^{(-)},$$

where $\psi_j^{(0)} \in L^2(I_j)$ and $\psi_j^{(\pm)} \in L^2(L_j^{(\pm)})$.

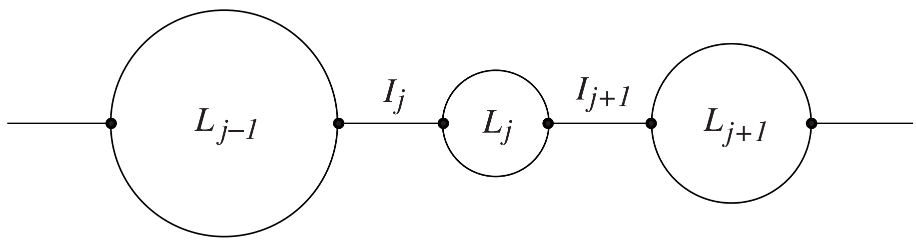

FIG. 1. Random necklace.

Note that the Hilbert space $\mathcal{H}$ depends on the random variable $\omega = \{\omega_j\}_{j\in\mathbb{Z}}$. By proper scaling it is possible to formulate the problem equivalently on a Hilbert space independent of ω and work with a random operator on a deterministic graph. However, we will not use this formulation.

On $\mathcal{H}$ we consider the negative Laplacian $-\Delta(\omega)$ as the operator of second derivative with local boundary conditions of the form

$$A\begin{pmatrix}\psi_j^{(0)}(0)\\ \psi_j^{(+)}(0)\\ \psi_j^{(-)}(0)\end{pmatrix}+B\begin{pmatrix}\psi_j^{(0)\prime}(0)\\ \psi_j^{(+)\prime}(0)\\ \psi_j^{(-)\prime}(0)\end{pmatrix}=0,\qquad j\in\mathbb{Z}\tag{2}$$

at the left vertex of the loop L_j and

$$A\begin{pmatrix}\psi_j^{(0)}(1)\\ \psi_j^{(+)}(\omega_j)\\ \psi_j^{(-)}(\omega_j)\end{pmatrix}+B\begin{pmatrix}-\psi_j^{(0)\prime}(1)\\ -\psi_j^{(+)\prime}(\omega_j)\\ -\psi_j^{(-)\prime}(\omega_j)\end{pmatrix}=0,\qquad j\in\mathbb{Z}\tag{3}$$

at the right vertex. Here A and B are such that they define so called standard boundary conditions,

$$A=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&0\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}.\tag{4}$$

By a general result in [14] the operator $-\Delta(\omega)$ is self-adjoint and nonnegative. By the general theory of metrically transitive operators (see, e.g., [22]) the spectrum of $-\Delta(\omega)$ as well as its components are deterministic.

It is easy to see that the numbers $E_{jl}=(\pi l/\omega_j)^2$, $j,l\in\mathbb{Z}$ are eigenvalues of $-\Delta(\omega)$ with the eigenfunctions

$$\psi_j^{(\pm)}(x)=\pm\sin(\sqrt{E}x)\qquad\text{and}\qquad\psi_k^{(0)}(x)=0\quad\text{for all}\quad k\in\mathbb{Z}.\tag{5}$$

Each of these eigenfunctions is compactly supported on the j-th loop. Therefore, we will call these eigenvalues E_{jl} the *loop eigenvalues*. In fact, $-\Delta(\omega)$ has no other eigenvalues with compactly supported eigenfunctions. The fact that there are no other eigenvalues with eigenfunctions supported on a single loop can be verified directly. Moreover, in Section 5 below we will prove the following proposition.

Proposition 2.1. *Let ψ be an arbitrary solution of the Schrödinger equation $H\psi=E\psi$. Assume that $\psi|_{I_j}\neq 0$ on an open subset of some interval I_j. Then $\psi\neq 0$ in almost all internal points of every interval I_k, $k\in\mathbb{Z}$.*

3. Discontinuities of the Integrated Density of States

Let $\Gamma^{m,n}$ with $m,n\in\mathbb{Z}$, $-m\le n$ be the graph consisting of $n+m+1$ loops L_j of arc length ω_j, $j=-m,\dots,n$ joined symmetrically by intervals of unit length. Let $-\Delta^{m,n}$ be minus the Laplacian for the graph on $\Gamma^{m,n}$ with the boundary condition (4) at all vertices except those vertices of the loops L_{-m} and L_n, where no intervals are attached. At those vertices we impose Dirichlet boundary conditions.

Obviously, the operator $-\Delta^{m,n}$ has a discrete spectrum. Therefore the finite volume integrated density of states is well-defined:

$$N^{m,n}(E):=\frac{\operatorname{tr}\mathsf{E}_{-\Delta^{m,n}}((-\infty,E))}{m+n+1},$$

where $\mathsf{E}_{-\Delta^{m,n}}(\delta)$ denotes the spectral projection of the operator $-\Delta^{m,n}$ corresponding to a Borel subset $\delta \subset \mathbb{R}$. By standard arguments (see, e.g., [8]) one can prove that the limit

$$\lim_{m,n\to\infty} N^{m,n}(E) =: N(E)$$

exists almost surely for all points of continuity of $N(E)$. This limit is deterministic and is called the integrated density of states. At possible (at most countably many) discontinuity points of $N(E)$ we make the convention $N(E) = N(E-0)$.

Let P denote the orthogonal projection in $\mathcal{H}$ onto the subspace generated by the eigenfunctions corresponding to loop eigenvalues. Then we can decompose the integrated density of states into two parts

$$N^{\text{loop}}(E) = \lim_{m,n\to\infty} \frac{\operatorname{tr} P\mathsf{E}_{-\Delta^{m,n}(\omega)}((-\infty, E))}{m+n+1} \tag{6}$$

and

$$\widetilde{N}(E) = \lim_{m,n\to\infty} \frac{\operatorname{tr}(I-P)\mathsf{E}_{-\Delta^{m,n}(\omega)}((-\infty, E))}{m+n+1} \tag{7}$$

such that $N(E) = \widetilde{N}(E) + N^{\text{loop}}(E)$. It is quite easy to calculate $N^{\text{loop}}(E)$ explicitly. Indeed, for $E > 0$ we have

$$\begin{aligned} N^{\text{loop}}(E) &= \lim_{m,n\to\infty} (m+n+1)^{-1} \sum_{j=-m}^{n} \#\{E_{jl} \mid E_{jl} < E\} \\ &= \lim_{m,n\to\infty} (m+n+1)^{-1} \sum_{j=-m}^{n} \left\lceil \frac{\omega_j\sqrt{E}}{\pi} \right\rceil, \end{aligned}$$

where $\lceil t \rceil$ denotes the largest integer strictly smaller than t, $\lceil t \rceil < t$. By the Birkhoff ergodic theorem the limit exists almost surely and is equal to

$$N^{\text{loop}}(E) = \int_0^\infty \left\lceil \frac{\omega_0\sqrt{E}}{\pi} \right\rceil d\varkappa(\omega_0). \tag{8}$$

From (8) it follows that $N^{\text{loop}}(E) = 0$ for all $E \le \pi^2/K^2$, where K is the supremum of the topological support of the measure $\varkappa$.

We have the following elementary result on the integrated density of loop states.

Proposition 3.1. *Let $E \geq \pi^2/K^2$. For all sufficiently small $\varepsilon > 0$*

$$N^{\text{loop}}(E+\varepsilon) - N^{\text{loop}}(E) = \varkappa(S_\varepsilon),$$

where

$$S_\varepsilon = \bigcup_{k=1}^{\lfloor K\sqrt{E}/\pi \rfloor} \left(\frac{k\pi}{\sqrt{E+\epsilon}}, \frac{k\pi}{\sqrt{E}} \right]$$

and $\lfloor t \rfloor$ denotes the largest integer not exceeding t, $\lfloor t \rfloor \le t$.

The Lebesgue measure of the set S_ε is obviously bounded by

$$\varepsilon \sum_{k=1}^{\lfloor K\sqrt{E}/\pi \rfloor} \frac{k\pi}{2E^{3/2}} \le \frac{\varepsilon}{4E}(1 + K\sqrt{E}/\pi).$$

Therefore, Proposition 3.1 implies that if the measure $\varkappa$ is purely continuous, then $N^{\mathrm{loop}}(E)$ is continuous. Moreover, if $\varkappa$ is purely absolutely continuous with bounded density, then $N^{\mathrm{loop}}(E)$ is Lipschitz continuous. On the other hand, one can construct purely absolutely continuous $\varkappa$'s with *unbounded* density such that the integrated density of loop states is not Hölder continuous of any prescribed order. For instance, fix some $\alpha \in (0,1)$. If $d\varkappa(\omega_0) = \sum_i c_i|\omega_0 - t_i|^{-\alpha} d\omega_0$ with t_i being dense in $(0,K)$ and $c_i > 0$ satisfying $\sum_i c_i < \infty$, then $N^{\mathrm{loop}}(E)$ is Hölder continuous of order $1-\alpha$ and not of order $\beta > 1-\alpha$. Also, there are singular continuous $\varkappa$'s such that the integrated density of loop states is not Hölder continuous of any order $\alpha > 0$.

Proof of Proposition 3.1. For given $E > 0$ choose $\varepsilon > 0$ so small that

$$\varepsilon < r(E) := \frac{\pi}{K}\left(\frac{\pi}{K} + 2\sqrt{E}\right). \tag{9}$$

Then

$$\frac{K}{\pi}(\sqrt{E+\varepsilon} - \sqrt{E}) < 1$$

and therefore $\left\lceil \frac{\omega_0\sqrt{E+\varepsilon}}{\pi} \right\rceil - \left\lceil \frac{\omega_0\sqrt{E}}{\pi} \right\rceil \leq 1$ for any $0 < \omega_0 \leq K$. The equality sign occurs if and only if

$$\frac{\omega_0\sqrt{E+\varepsilon}}{\pi} > k \geq \frac{\omega_0\sqrt{E}}{\pi}$$

for some integer k. Thus,

$$\left\lceil \frac{\omega_0\sqrt{E+\varepsilon}}{\pi} \right\rceil - \left\lceil \frac{\omega_0\sqrt{E}}{\pi} \right\rceil = \chi_{S_\varepsilon}(\omega_0),$$

where χ_{S_ε} is the characteristic function of the set S_ε. The claim follows from equation (8). □

Theorem 3.2. *Assume that the probability measure $\varkappa$ has a nontrivial pure point part $\varkappa_{\mathrm{pp}}$,*

$$\varkappa_{\mathrm{pp}}(\cdot) = \sum_{i=1}^{\infty} p_i \delta_{s_i}(\cdot) \qquad \text{with} \qquad p_i \geq 0, \qquad 0 < \sum_{i=1}^{\infty} p_i \leq 1, \tag{10}$$

where δ_{s_i} denotes the Dirac measure concentrated at s_i. Consider the nonempty set

$$D_\varkappa := \{E \in \mathbb{R}_+ |\; E = (\pi k/s_i)^2, \quad s_i \neq 0, \quad p_i \neq 0, \quad k \in \mathbb{N}\}. \tag{11}$$

Then $D_\varkappa$ is the set of all points of discontinuity of the integrated density of states $N(E)$ and for any $E \in D_\varkappa$

$$N(E+0) - N(E) = \sum_{i=1}^{\infty} \alpha_i p_i > 0 \tag{12}$$

with $\alpha_i = 1$ if $s_i\sqrt{E}/\pi$ is integer and $\alpha_i = 0$ otherwise.

Proof. From Proposition 3.1 it follows that

$$N^{\mathrm{loop}}(E+0) - N^{\mathrm{loop}}(E) = \varkappa(M),$$

where $M = \{\omega_0 |\ \omega_0\sqrt{E}/\pi$ is integer$\}$. Obviously, for any E the set M is at most discrete. Therefore, we have

$$N^{\mathrm{loop}}(E+0) - N^{\mathrm{loop}}(E) = \varkappa_{\mathrm{pp}}(M) = \sum_{i=1}^{\infty} \alpha_i p_i \neq 0 \tag{13}$$

if $E \in D_{\varkappa}$ and zero otherwise.

Since $\widetilde{N}(E)$ is non-decreasing it follows that $N(E)$ is discontinuous on $D_{\varkappa}$. Moreover, we obtain (12) with "$=$" being replaced by "$\geq$". Thus, to complete the proof it suffices to show that $\widetilde{N}(E)$ defined by (7) is continuous. We will prove this fact in Section 7. □

Since the set M is discrete the sum in equation (12) involves only a finite number of nonvanishing terms.

If the distribution measure $\varkappa$ has a nontrivial singular continuous component, then the canonical decomposition of $N^{\mathrm{loop}}(E)$ contains a nontrivial singular continuous part. This is established in the following proposition.

Proposition 3.3. *If $\varkappa$ is purely singular continuous, then $N^{\mathrm{loop}}(E)$ is a singular continuous function.*

Proof. Since $N^{\mathrm{loop}}(E)$ is non-decreasing, it is differentiable for Lebesgue almost all $E \in \mathbb{R}$. By Theorem 3.2 it is continuous. Therefore, to prove the claim it suffices to show that $dN^{\mathrm{loop}}(E)/dE = 0$ for a.e. E.

Fix an arbitrary $E_0 \geq \pi^2/K^2$ and consider

$$F_k(\varepsilon) = \varkappa\left(\left(\frac{k\pi}{\sqrt{E_0+\varepsilon}}, \frac{k\pi}{\sqrt{E_0}}\right]\right), \qquad k \in \mathbb{N}$$

as a function of $\varepsilon \in (0, r(E_0))$ with $r(E_0)$ being defined in (9). Since $\varkappa$ is singular continuous, we have $F_k'(\varepsilon) = 0$ for a.e. ε. From Proposition 3.1 it follows that

$$N^{\mathrm{loop}}(E_0+\varepsilon) = N^{\mathrm{loop}}(E_0) + \sum_{k=1}^{\lfloor K\sqrt{E_0}/\pi \rfloor} F_k(\varepsilon)$$

for all $\varepsilon \in (0, r(E_0))$. Thus, for almost all $E \in (E_0, E_0 + r(E_0))$ we have

$$\frac{dN^{\mathrm{loop}}(E)}{dE} = \frac{dN^{\mathrm{loop}}(E_0+\varepsilon)}{d\varepsilon} = \sum_{k=1}^{\lfloor K\sqrt{E_0}/\pi \rfloor} F_k'(\varepsilon) = 0.$$

This proves the claim since E_0 is arbitrary. □

4. Magnetic Field Smoothes the Integrated Density of States

Without loss of generality we can assume that the graph Γ is imbedded in a plane in $\mathbb{R}^3$ and the loops L_j are circles. In this section we consider a magnetic perturbation $-\Delta(\omega; \mathcal{B})$ of the Laplacian described in Section 2. Assume there is a constant magnetic field $\mathcal{B}$ perpendicular to the plane containing the graph. Since the j-th loop encloses an area ω_j^2/π the magnetic flux through this loop is given by $\Phi_j = \omega_j^2\mathcal{B}/\pi$. As shown in [17] prescribing magnetic fluxes through all loops of the graph defines a magnetic Laplacian uniquely up to a gauge transformation. The resulting magnetic Laplacian is again defined to be the operator of second derivative but now with different boundary conditions. The boundary condition

(2) at the left vertex of the loop L_j remains unchanged and the boundary condition (3) at the right vertex takes the form

$$A_j \begin{pmatrix} \psi_j^{(0)}(1) \\ \psi_j^{(+)}(\omega_j) \\ \psi_j^{(-)}(\omega_j) \end{pmatrix} + B_j \begin{pmatrix} -\psi_j^{(0)\prime}(1) \\ -\psi_j^{(+)\prime}(\omega_j) \\ -\psi_j^{(-)\prime}(\omega_j) \end{pmatrix} = 0, \qquad j \in \mathbb{Z}$$

with

$$A_j = \begin{pmatrix} 1 & -1 & 0 \\ 0 & e^{i\Phi_j/2} & -e^{-i\Phi_j/2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & e^{i\Phi_j/2} & e^{-i\Phi_j/2} \end{pmatrix}. \tag{14}$$

It is easy to see that the numbers $E_{jl} = (\pi l/\omega_j)^2$ are eigenvalues of $-\Delta(\omega; \mathcal{B})$ if and only if $\mathcal{B}\omega_j^2/\pi^2$ is integer. The corresponding eigenfunctions are given by (5). Also, there are no other eigenvalues with compactly supported eigenfunctions.

Assume again that the probability measure for the i.i.d. distributed arc lengths ω_j has a nontrivial pure point part given by (10) and consider the set

$$D_\varkappa(\mathcal{B}) := \{E \in \mathbb{R}_+ \,|\, E = (\pi k/s_i)^2, \quad s_i \neq 0, \quad \mathcal{B}s_j^2/\pi^2 \in \mathbb{N}_0, \quad p_i \neq 0, \quad k \in \mathbb{N}\}.$$

Similarly to the analysis of Section 3 one can show that $N^{\mathrm{loop}}(E)$ is discontinuous on the set $D_\varkappa(\mathcal{B}) \subseteq D_\varkappa$ and nowhere else. Note that for $\mathcal{B} \neq 0$ this set is in general strictly smaller than $D_\varkappa$ defined by (11). Since all discontinuities of the integrated density of states are contained in $N^{\mathrm{loop}}(E)$ this implies that generically (if $\mathcal{B}s_j^2/\pi$ is not integer) the discontinuities disappear under the perturbation by a magnetic field.

5. The Integrated Density of States and Scattering Amplitudes

To proceed further with the analysis of the Laplacian $-\Delta(\omega)$ we will use some results from scattering theory. Let $\Gamma_{m,n}$ with $m, n \in \mathbb{Z}$, $-m \leq n$ be the graph consisting of $n + m + 1$ loops L_j of arc length ω_j, $j = -m, \ldots, n$ joined symmetrically by the intervals of unit length and of two semi-lines attached to the loops L_{-m} and L_n. With this graph we associate the Hilbert space $\mathcal{H}_{m,n} = \mathcal{H}^{\mathrm{ext}} \oplus \mathcal{H}^{\mathrm{int}}_{m,n}$, where $\mathcal{H}^{\mathrm{ext}} = L^2(0, \infty) \oplus L^2(0, \infty)$ and

$$\mathcal{H}^{\mathrm{int}}_{m,n} = \begin{cases} L^2(L_{-m}) \oplus \displaystyle\bigoplus_{j=-m+1}^{n} \left[L^2(I_j) \oplus L^2(L_j)\right], & m, n \neq 0, \\ L^2(L_0), & m = n = 0. \end{cases}$$

By $-\Delta_{m,n}(\omega)$ we denote minus the Laplacian acting on $\mathcal{H}_{m,n}$ with the boundary conditions (4). We consider the scattering matrix and the spectral shift function for the pair of operators $(-\Delta_{m,n}(\omega), -\Delta)$ where Δ is a usual Laplacian on $L^2(\mathbb{R})$. Although these operators act in different Hilbert spaces, the scattering matrix as well as the spectral shift function can be constructed in this case (see, e.g., [23] and the A).

Identifying in a natural way $L^2(\mathbb{R})$ and $\mathcal{H}^{\mathrm{ext}}$ we define the isometric identification operator $\mathcal{J} : L^2(\mathbb{R}) \to \mathcal{H}$ such that $\mathrm{Ran}\mathcal{J} = \mathcal{H}^{\mathrm{ext}}$. Obviously, $I - \mathcal{J}^*\mathcal{J} = 0$ and $I - \mathcal{J}\mathcal{J}^* = P_{\mathcal{H}^{\mathrm{ext}}}$ with $P_{\mathcal{H}^{\mathrm{ext}}}$ being the projection in $\mathcal{H}_{m,n}$ onto the subspace $\mathcal{H}^{\mathrm{ext}}$. It is easy to check that the conditions (30) in the A are fulfilled for, e.g., $k = 1$. Thus the spectral shift function $\xi_{m,n}(E; \omega) := \xi(E; -\Delta_{m,n}(\omega), -\Delta; \mathcal{J})$ exists and satisfies the trace formula (31). The condition $\xi_{m,n}(E; \omega) = 0$ for $E < 0$ fixes the spectral shift function uniquely. The scattering matrix

$$S_{m,n}(E;\omega) := S(-\Delta_{m,n}(\omega), -\Delta; \mathfrak{J}) = \begin{pmatrix} T_{m,n}(E;\omega) & R_{m,n}(E;\omega) \\ L_{m,n}(E;\omega) & T_{m,n}(E;\omega) \end{pmatrix}$$

is defined as in Section 2 of [16].

The operator $-\Delta_{0,0}(\omega)$ on the graph $\Gamma_{0,0}$ consisting of a single loop and two half-lines was considered in Example 3.2 in [14], where it was shown that the transmission and reflection amplitudes are given by

$$T_{0,0}(E;\omega) = -\frac{8e^{i\omega_0\sqrt{E}}}{e^{2i\omega_0\sqrt{E}} - 9}, \quad R_{0,0}(E;\omega) = L_{0,0}(E;\omega) = -\frac{3(e^{2i\omega_0\sqrt{E}} - 1)}{e^{2i\omega_0\sqrt{E}} - 9}. \tag{15}$$

The operator $-\Delta_{0,0}(\omega)$ has infinitely many eigenvalues $\{\pi^2 n^2/\omega_0^2,\ n \in \mathbb{N}\}$ imbedded in the absolutely continuous spectrum. At those energies the reflection coefficients vanish and $T_{0,0}(\pi^2 n^2/a^2;\omega) = (-1)^{n+1}$.

The spectral shift function $\xi_{0,0}(E;\omega)$ can be calculated explicitly. Indeed, by the chain rule for the spectral shift function we have

$$\xi_{0,0}(E;\omega) = \xi(E; -\Delta_{0,0}(\omega), (-\Delta)\oplus(-\Delta^{\mathrm{loop}})) + \xi(E; (-\Delta)\oplus(-\Delta^{\mathrm{loop}}), -\Delta; \mathfrak{J}),$$

where Δ^{loop} denotes the (self-adjoint) Laplace operator on $\mathcal{H}_{0,0}^{\mathrm{int}} = L^2([0,\omega_0])\oplus L^2([0,\omega_0])$ with the boundary conditions

$$\begin{aligned} \psi^{(+)}(0) &= \psi^{(-)}(0), & \psi^{(+)\prime}(0) + \psi^{(-)\prime}(0) &= 0, \\ \psi^{(+)}(\omega_0) &= \psi^{(-)}(\omega_0), & \psi^{(+)\prime}(\omega_0) + \psi^{(-)\prime}(\omega_0) &= 0. \end{aligned}$$

Combining the trace formula (31) with Birman-Krein Theorem (32) (both with with $\mathfrak{J} = I$) we obtain

$$\xi(E; -\Delta_{0,0}(\omega), (-\Delta)\oplus(-\Delta^{\mathrm{loop}})) = -\frac{1}{\pi}\phi_{0,0}(E;\omega)$$

with

$$\begin{aligned} \phi_{0,0}(E;\omega) &:= \frac{1}{2i}\log\det S(E; -\Delta_{0,0}(\omega), (-\Delta)\oplus(-\Delta^{\mathrm{loop}}); I) \\ &= \mathrm{Arctan}\left(\frac{5}{4}\tan(\sqrt{E}\omega_0)\right), \end{aligned}$$

where Arctan is chosen such that $x \mapsto \mathrm{Arctan}(\tan x)$ is continuous for all $x \in \mathbb{R}$ and $\mathrm{Arctan}(0) = 0$. In particular, $\phi_{0,0}(\pi^2 k^2/a^2;\omega) = \pi k$ and $\phi_{0,0}(\pi^2(k+1/2)^2/a^2;\omega) = \pi(k+1/2)$ for all $k \in \mathbb{N}_0$ and all $\omega \in \Omega$. Furthermore,

$$\xi(E; (-\Delta)\oplus(-\Delta^{\mathrm{loop}}), -\Delta; \mathfrak{J}) = \xi(E; (-\Delta)\oplus(-\Delta^{\mathrm{loop}}), -\Delta\oplus 0; I)$$

and, thus, by Lemma 3.1 in [13] equals minus the eigenvalue counting function for the operator $-\Delta^{\mathrm{loop}}$,

$$\xi(E; (-\Delta)\oplus(-\Delta^{\mathrm{loop}}), -\Delta; \mathfrak{J}) = -\lceil\sqrt{E}\omega_0/\pi\rceil.$$

Therefore, we obtain

$$\xi_{0,0}(E;\omega) = -\frac{1}{\pi}\mathrm{Arctan}\left(\frac{5}{4}\tan(\sqrt{E}\omega_0)\right) - \lceil\sqrt{E}\omega_0/\pi\rceil.$$

In a similar way one can calculate the spectral shift function $\xi_{m,n}(E;\omega)$ for arbitrary integers m and n such that $-m \le n$,

$$\xi_{m,n}(E;\omega) = -\frac{1}{\pi}\phi_{m,n}(E;\omega) - \sum_{j=-m}^{n} \lceil \sqrt{E}\omega_j/\pi \rceil,$$

where $\phi_{m,n}(E;\omega)$ is the scattering phase for the pair of operators $(-\Delta_{m,n}(\omega), -\Delta)$.

Denoting by $N_0(E) = \sqrt{E}/\pi$ the integrated density of states for the Laplacian $-\Delta$ on $L^2(\mathbb{R})$, by results of [13] (Theorem 4.1, equation (4.4), and Theorem 4.4) we obtain that

$$N(E) = N_0(E) - \lim_{m,n\to\infty} \frac{\xi_{m,n}(E;\omega)}{n+m+1} \tag{16}$$

almost surely. Although the results of [13] are formulated and proven for Schrödinger operators on the line, all proofs extend verbatim to the present context.

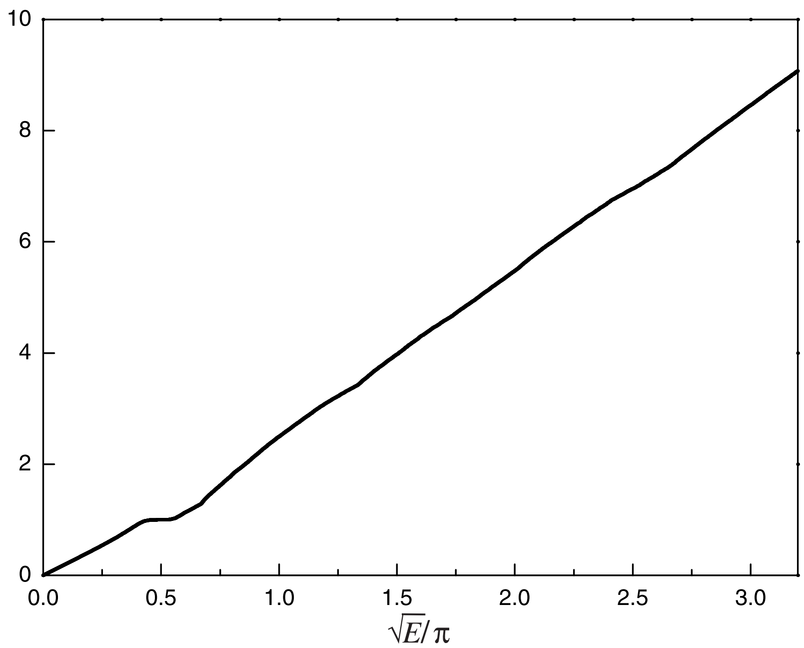

FIG. 2. The integrated density of states $N(E)$ for the case of uniformly continuously distributed $\omega_0 \in [1/2, 3/2]$. The horizontal plateau corresponds to a spectral gap.

Let

$$\Lambda_{\omega_0}(E) = \begin{pmatrix} \frac{e^{-i\sqrt{E}}}{T_{0,0}(E;\omega)} & -\frac{R_{0,0}(E;\omega)}{T_{0,0}(E;\omega)} \\ \frac{L_{0,0}(E;\omega)}{T_{0,0}(E;\omega)} & \frac{e^{i\sqrt{E}}}{T_{0,0}(E;\omega)^*} \end{pmatrix} \tag{17}$$

and

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By the arguments presented in [13] from equation (16) we obtain the following theorem.

Theorem 5.1. *For $E>0$ the integrated density of states $N(E)$ is given by*

$$N(E)=\mp\frac{1}{\pi}\lim_{m,n\to\infty}\frac{\varphi^{\pm}_{m,n}(E;\omega)}{m+n+1}+N^{\mathrm{loop}}(E), \tag{18}$$

where

$$\varphi^{\pm}_{mn}(E;\omega):=\arg\left\langle e_{\pm},\ \prod_{j=-m}^{n}\Lambda_{\omega_j}(E)e_{\pm}\right\rangle, \tag{19}$$

$\langle\cdot,\cdot\rangle$ *denotes the inner product in $\mathbb{C}^2$ and N^{loop} is given by* (8). *The choice of the argument is uniquely fixed by the condition*

$$\varphi^{\pm}_{m,k}+\varphi^{\pm}_{-k,n}-\frac{\pi}{2}\le\varphi^{\pm}_{mn}\le\varphi^{\pm}_{m,k}+\varphi^{\pm}_{-k,n}+\frac{\pi}{2}$$

for arbitrary $k\in\mathbb{Z}$ satisfying $-m\le k$ and $-k\le n$.

In equation (19) and below the product $\prod$ is to be understood in the ordered sense. Theorem 5.1 provides an algorithm for numerical calculations of the integrated density of states. Figures 2 and 3 show examples of such calculations. Some other numerical results for Schrödinger operators on the line are presented in [12].

By means of Theorem 5.1 one can obtain (see [15]) a two-sided estimate on the integrated density of states $\widetilde{N}(E)$

$$\left|\widetilde{N}(E)-N_0(E)-\frac{1}{\pi}\int_0^\infty \operatorname{Arctan}\left(\frac{5}{4}\tan(\omega_0\sqrt{E})\right)d\varkappa(\omega_0)\right|\le\frac{1}{2}. \tag{20}$$

Comparing (18) with equations (6) and (7) we obtain that

$$\widetilde{N}(E)=\mp\frac{1}{\pi}\lim_{m,n\to\infty}\frac{\varphi^{\pm}_{m,n}(E)}{m+n+1}$$

for almost all $\omega\in\Omega$. A simple calculation now leads to the following representation for $\widetilde{N}(E)$:

$$\widetilde{N}(E)=N_0(E)-\frac{1}{\pi}\lim_{m,n\to\infty}\frac{\phi_{m,n}(E;\omega)}{m+n+1}, \tag{21}$$

where $\phi_{m,n}(E;\omega)$ is the scattering phase for the pair of operators $(-\Delta_{m,n}(E;\omega),-\Delta)$. In the next section we use equation (21) to prove the Thouless formula in the present context.

Now we are in position to prove Proposition 2.1. Let $\Lambda_{m,n}(E;\omega)$ denote the transfer matrix,

$$\Lambda_{m,n}(E;\omega)=\begin{pmatrix}\frac{1}{T_{m,n}(E;\omega)} & -\frac{R_{m,n}(E;\omega)}{T_{m,n}(E;\omega)}\\ \frac{L_{m,n}(E;\omega)}{T_{m,n}(E;\omega)} & \frac{1}{T_{m,n}(E;\omega)^*}\end{pmatrix}. \tag{22}$$

Proof of Proposition 2.1. Note that $\psi|_{I_k}$ for any $k\in\mathbb{Z}$ is necessarily of the form $a_ke^{i\sqrt{E}x}+b_ke^{-i\sqrt{E}x}$. Therefore, if $\psi|_{I_j}\neq 0$ on an open subset of some interval I_j, then $|a_j|+|b_j|\neq 0$. For $k\neq j$ the coefficients a_k, b_k are determined by the equation (see, e.g., [16])

$$\begin{pmatrix}a_k\\ b_k\end{pmatrix}=\Lambda_{-k,j}(E;\omega)\begin{pmatrix}a_j\\ b_j\end{pmatrix}, \tag{23}$$

where $\det\Lambda_{-k,j}(E,\omega)=1$. Assume that $a_k=b_k=0$ for some $k\in\mathbb{Z}$. Then from (23) it follows that $a_j=b_j=0$, which is a contradiction. $\square$

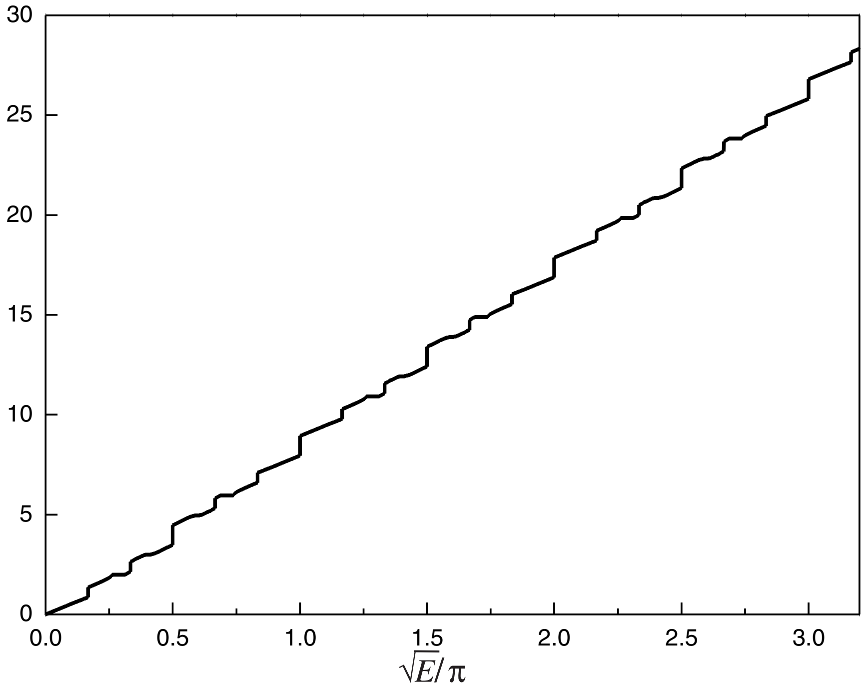

FIG. 3. The integrated density of states $N(E)$ for the Bernoulli distribution $\varkappa = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_6$, i.e., $\omega_j \in \{2, 6\}$ with equal probability. Horizontal plateaus correspond to spectral gaps, vertical strokes represent discontinuities.

6. THE LYAPUNOV EXPONENT

Following convention we define the Lyapunov exponent for the Random Necklace Model as the exponential growth rate of the norm of the transfer matrix,

$$\gamma(E) = \lim_{m,n\to\infty} \frac{\log \|\Lambda_{m,n}(E,\omega)\|}{n+m+1}. \tag{24}$$

It is a general fact that for every $E > 0$ this limit exists almost surely and is nonnegative. Moreover, (see Theorems 5.1 and 5.3 in [13], also cf. [19], [20], [21])

$$\gamma(E) = -\lim_{m,n\to\infty} \frac{\log \|T_{m,n}(E;\omega)\|}{n+m+1}, \tag{25}$$

where $T_{m,n}(E;\omega)$ is the transmission coefficient corresponding to the Laplacian $-\Delta_{m,n}(\omega)$. Also, (24) can be rewritten as

$$\begin{aligned} \gamma(E) &= \lim_{m,n\to\infty} \frac{1}{m+n+1} \log \left| \left\langle e_\pm, \prod_{j=-m}^{n} \Lambda_{\omega_j}(E) e_\pm \right\rangle \right| \\ &= \lim_{m,n\to\infty} \frac{1}{m+n+1} \log \left\| \prod_{j=-m}^{n} \Lambda_{\omega_j}(E) \right\|, \end{aligned} \tag{26}$$

where the matrices $\Lambda_{\omega_j}(E)$ are defined by (17).

Kotani's Theorem, which states that the Lyapunov exponent of every Schrödinger operator on the line with non-deterministic potential is almost everywhere positive (see, e.g., [3]), does not apply directly to the model considered here. Therefore, to prove that $\gamma(E)$ is positive for almost all $E > 0$ in the case when supp $\varkappa$ contains at least one non-isolated point we refer to Theorem 5.6 of [13]. By this result for $E > 0$ the Lyapunov exponent vanishes on the set $\{E = (\pi n)^2 |\ n \in \mathbb{N}\}$ and nowhere else. Also, for $E = 0$ the matrix $\Lambda_{\omega_0}(E)$ equals the unit matrix and hence $\gamma(0) = 0$. As an illustration we have computed the the Lyapunov exponent for uniformly continuously distributed ω_j's on the interval $[1/2, 3/2]$, i.e., for $d\varkappa(E) = \chi_{[1/2,3/2]}(E)$ (see Figure 4).

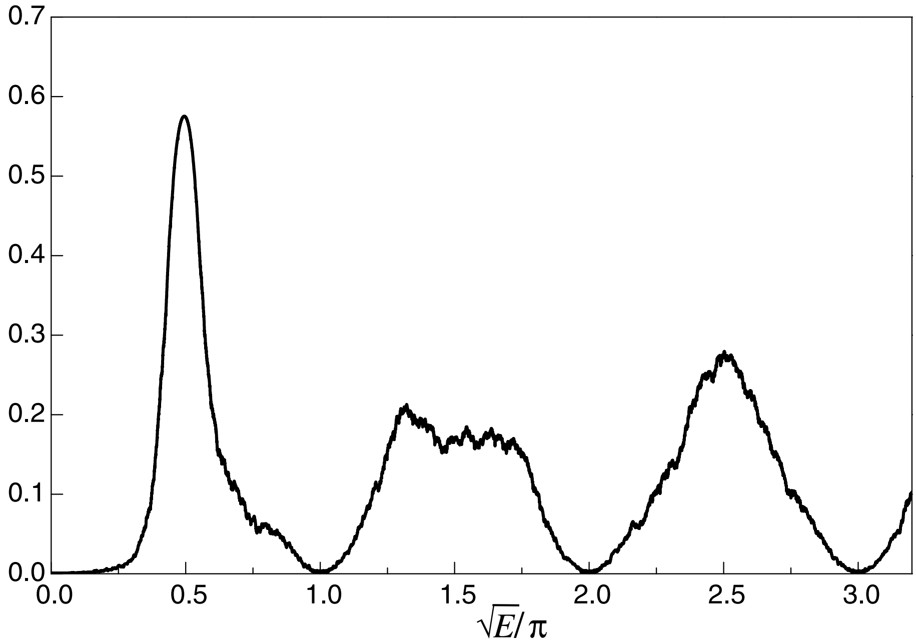

FIG. 4. The Lyapunov exponent for the case of uniformly continuously distributed $\omega_0 \in [1/2, 3/2]$.

The case when the measure $\varkappa$ is purely discrete (and thus is a finite convex combination of Dirac measures) is not covered by Theorem 5.6 in [13]. First, consider the Bernoulli distribution $\varkappa = p\delta_{s_1} + (1-p)\delta_{s_2}$, $0 < p < 1$ since in this case we may invoke the recent results of Damanik, Sims, and Stolz [5] (see also [6]). To apply this result we need to introduce the scattering amplitudes $T^{(s_1,s_0)}$, $R^{(s_1,s_0)}$, $L^{(s_1,s_0)}$ for $-\Delta_{0,0}(s_1)$ relative to the "background" operator $-\Delta_{0,0}(s_0)$. (As discussed above the fact that these operators act in different Hilbert spaces is not relevant). A simple calculation shows that they can be determined from the relation

$$\begin{pmatrix} \frac{1}{T^{(s_1,s_0)}(E)} & -\frac{R^{(s_1,s_0)}(E)}{T^{(s_1,s_0)}(E)} \\ \frac{L^{(s_1,s_0)}(E)}{T^{(s_1,s_0)}(E)} & \frac{1}{T^{(s_1,s_0)}(E)^*} \end{pmatrix} = \Lambda_{0,0}(E; s_1)\ \Lambda_{0,0}(E; s_0)^{-1},$$

where $\Lambda_{0,0}$ is defined in (22). In particular, we obtain

$$R^{(s_1,s_0)}(E) = L^{(s_1,s_0)}(E) = -\frac{3(e^{2is_1\sqrt{E}} - e^{2is_0\sqrt{E}})}{e^{2is_1\sqrt{E}} - 9e^{2is_0\sqrt{E}}}$$

such that the reflection coefficients vanish if and only if $\sqrt{E}(s_1 - s_0)/\pi$ is an integer. Therefore, by applying Theorem 1 of [5] we conclude that the Lyapunov exponent vanishes on the set

$$S(s_0, s_1) := \left\{ E = \left(\frac{\pi k}{2}\right)^2,\ k \in \mathbb{N} \right\} \cup \left\{ E = \left(\frac{\pi k}{s_1 - s_0}\right)^2,\ k \in \mathbb{N} \right\} \tag{27}$$

and nowhere else.

Let now $\varkappa$ be an arbitrary discrete measure given by (10) with a finite number of non-trivial terms. As noted in [5] the set of zeroes of the Lyapunov exponent is contained in the union

$$\bigcup_{\substack{s \neq s' \\ s,s' \in \operatorname{supp} \varkappa}} S(s, s')$$

of the sets (27). The Lyapunov exponent is strictly positive for all E away from this discrete set.

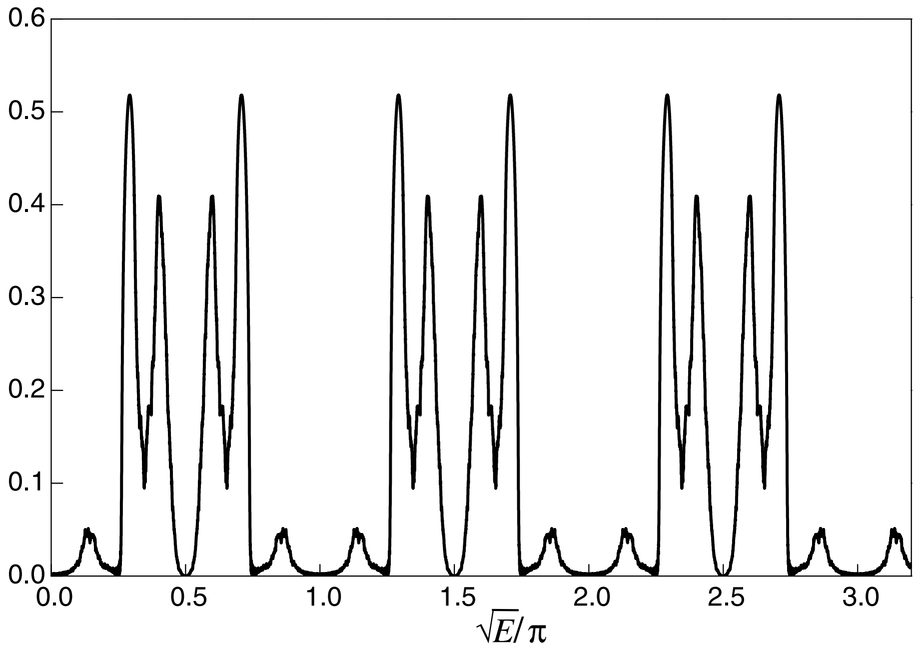

FIG. 5. The Lyapunov exponent for the Bernoulli distribution $\varkappa = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_6$, i.e., $\omega_j \in \{2, 6\}$ with equal probability.

Results of numerical computations of the Lyapunov exponent using (26) for the Bernoulli distribution with $\varkappa = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_6$, i.e., $\omega_j \in \{2, 6\}$ with equal probability, are presented in Figure 5. We mention two properties of $\gamma(E)$:

1. The Lyapunov exponent is periodic in $\sqrt{E}$ with period π, i.e.

$$\gamma((\sqrt{E}+\pi k)^2)=\gamma(E),\quad E>0,\quad k\in\mathbb{N}.$$

This follows immediately from the fact that for even s

$$\Lambda_s((\sqrt{E+\pi k})^2)=\Lambda_s(E).$$

2. The Lyapunov exponent is reflection symmetric,

$$\gamma((k\pi-\sqrt{E})^2)=\gamma(E),\quad E>0,\quad k\in\mathbb{N},\quad k>\sqrt{E}/\pi, \tag{28}$$

i.e., the points $k/2$, $k\in\mathbb{N}$ are the axes of reflection symmetry (on the scale of $\sqrt{E}/\pi$). To prove this we note that

$$\begin{aligned}\Lambda_s((k\pi-\sqrt{E})^2)&=\begin{pmatrix}-\frac{e^{-2is\sqrt{E}}-9}{8e^{-is\sqrt{E}}}e^{i\sqrt{E}}e^{-ik\pi} & \frac{3i}{4}\sin(s\sqrt{E})\\ -\frac{3i}{4}\sin(s\sqrt{E}) & -\frac{e^{2is\sqrt{E}}-9}{8e^{is\sqrt{E}}}e^{-i\sqrt{E}}e^{ik\pi}\end{pmatrix}\\ &=\begin{pmatrix}0 & e^{ik\pi}\\ 1 & 0\end{pmatrix}\Lambda_s(E)\begin{pmatrix}0 & e^{ik\pi}\\ 1 & 0\end{pmatrix}.\end{aligned}$$

Since

$$\begin{pmatrix}0 & e^{ik\pi}\\ 1 & 0\end{pmatrix}^2=e^{ik\pi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

up to a sign the product $\prod_{j=-n}^{m}\Lambda_{\omega_j}((k\pi-\sqrt{E})^2)$ equals

$$\begin{pmatrix}0 & e^{ik\pi}\\ 1 & 0\end{pmatrix}\prod_{j=-n}^{m}\Lambda_{\omega_j}(E)\begin{pmatrix}0 & e^{ik\pi}\\ 1 & 0\end{pmatrix}.$$

Thus, equality (28) follows from (26).

Using equation (26) one can also analyze the periodic case $\omega_j=\omega_0$ for all $j\in\mathbb{Z}$. In this case the spectrum of $-\Delta(\omega)$ consists of the absolutely continuous part and the eigenvalues $E_k=\pi^2k^2/\omega_0^2$, $k\in\mathbb{N}$ of infinite multiplicity. The absolutely continuous spectrum has a band structure such that $E\in\operatorname{spec}_{\mathrm{ac}}(-\Delta(\omega))$ if and only if Hill's discriminant (cf. equation (8) in [2])

$$H(E)=\frac{2\cos(\sqrt{E}+\phi_{0,0}(E;\omega))}{|T_{0,0}(E;\omega)|}=\frac{9}{4}\cos(\sqrt{E}(\omega_0+1))-\frac{1}{4}\cos(\sqrt{E}(\omega_0-1))$$

satisfies the inequality $|H(E)|\leq 2$. From this and the fact that $|T_{0,0}(\pi^2n^2/a^2;\omega)|=1$ it follows immediately that all eigenvalues are imbedded in the absolutely continuous spectrum or lie at the edges of the spectral bands.

7. CONTINUITY OF $\widetilde{N}(E)$

For almost all $E>0$ the Lyapunov exponent satisfies the Thouless formula

$$\gamma(E)=\alpha+\int_{\mathbb{R}}\log\left|\frac{\lambda-E}{\lambda-i}\right|d\widetilde{N}(\lambda), \tag{29}$$

where α is some positive number. The existence of the integral on the r.h.s. is guaranteed by the estimate (20) and Lemma 11.7 in [22]. We emphasize that $N^{\mathrm{loop}}(E)$ does not enter this formula.

Before we proceed, we discuss the implications of (29) to the continuity properties of $\widetilde{N}(E)$. By a modification of an argument due to Craig and Simon [4] (see Theorem 11.9 in

[22]) the positivity of the Lyapunov exponent implies the log-Hölder continuity of $\widetilde{N}(E)$, that is, the inequality

$$\widetilde{N}(E_2) - \widetilde{N}(E_1) \leq C\,|\log|E_2 - E_1||^{-1}$$

for arbitrary sufficiently small intervals $[E_1, E_2]$. Moreover, using the arguments (in a slightly modified form) of Damanik, Sims, and Stolz from [6] one proves that $\widetilde{N}(E)$ is actually Hölder continuous, i.e., there is a number $0 < \mu < 1$ such that

$$\widetilde{N}(E_2) - \widetilde{N}(E_1) \leq C|E_2 - E_1|^{\mu}$$

for arbitrary sufficiently small intervals $[E_1, E_2]$ not containing the points where the Lyapunov exponent vanishes. Both these properties hold for general distribution measures $\varkappa$.

We only sketch the proof of (29), but the interested reader can easily fill in the details. We closely follow the line of arguments given in [13].

Let $f(z)$ be an analytic function in open cut plane $\mathbb{C}_0 = \mathbb{C} \setminus [0, \infty)$, continuous in the closure of $\mathbb{C}_0$ and satisfying $|f(z)| \leq c\sqrt{|z|}$ for all $z \in \mathbb{C}_0$. Moreover, we assume that $f(E + i0) = \overline{f(E - i0)}$ is continuously differentiable for all $E > 0$. Using the Cauchy integral formula it is easy to show that

$$\operatorname{Re} f(z) = \operatorname{Re} f(i) + \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{\lambda - z}{\lambda - i} \right| d \operatorname{Im} f(\lambda + i0).$$

From equations (15) and by the factorization rule for the matrices (22) (see [16]) it follows that one can take $f(z) = \log T_{m,n}(z; \omega)$ for arbitrary m, n, and $\omega \in \Omega$, thus, obtaining

$$\log|T_{m,n}(z;\omega)| = \log|T_{m,n}(i;\omega)| + \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{\lambda - z}{\lambda - i} \right| d\,\phi_{m,n}(\lambda;\omega).$$

Using Lemma 11.7 in [22] we conclude that for almost all $E > 0$, all $\omega \in \Omega$, and arbitrary integers m, n

$$\log|T_{m,n}(E;\omega)| = \log|T_{m,n}(i;\omega)| + \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{\lambda - E}{\lambda - i} \right| d\,\phi_{m,n}(\lambda;\omega).$$

Now divide both sides of this equation by $n + m + 1$ and consider the limit $m, n \to \infty$. By (25) the l.h.s. converges almost surely to $-\gamma(E)$. The signed measures $(n + m + 1)^{-1} d\,\phi_{m,n}(E;\omega)$ converge vaguely to $\pi(dN_0(E) - d\widetilde{N}(E))$. Thus, again by Theorem 11.7 in [22] there are subsequences m_k, n_k such that

$$\lim_{k\to\infty} \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{\lambda - E}{\lambda - i} \right| \frac{d\,\phi_{m_k,n_k}(\lambda;\omega)}{m_k + n_k + 1} = \int_{\mathbb{R}} \log \left| \frac{\lambda - E}{\lambda - i} \right| (dN_0(\lambda) - dN(\lambda))$$

for almost all E. Noting that

$$\int_{\mathbb{R}} \log \left| \frac{\lambda - E}{\lambda - i} \right| dN_0(\lambda) = \gamma_0(E) - \gamma_0(i) = -\frac{\sqrt{2}}{2},$$

where $\gamma_0(z) = |\operatorname{Re}\sqrt{-z}|$ is the Lyapunov exponent of the Laplacian $-\Delta$ on $L^2(\mathbb{R})$, we obtain (29).

APPENDIX A. THE SPECTRAL SHIFT FUNCTION

Here we briefly collect some facts from the theory of the spectral shift function in the case where the operators involved act in different Hilbert spaces. For a comprehensive presentation we refer the reader to the book [23]. Consider two (possibly unbounded) self-adjoint operators $T_0 \geq I$ and $T \geq I$ acting in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, and a bounded operator $\mathcal{J}: \mathcal{H}_0 \to \mathcal{H}$. Suppose that the operators

$$\left.\begin{array}{l} T^{-1}\mathcal{J} - \mathcal{J}T_0^{-1} \\ (\mathcal{J}^*\mathcal{J} - I)T_0^{-1} \\ T^{-1}(\mathcal{J}\mathcal{J}^* - I) \end{array}\right\} \qquad \text{are trace class.} \tag{30}$$

Under these conditions there exists a spectral shift function $\xi(E; T, T_0; \mathcal{J})$ for which the trace formula

$$\operatorname{tr}\,[\phi(T) - \mathcal{J}\phi(T_0)\mathcal{J}^*] + \operatorname{tr}\,[(\mathcal{J}^*\mathcal{J} - I)\phi(T_0)] = \int_{\mathbb{R}} \xi(E; T, T_0; \mathcal{J})\phi'(E)dE \tag{31}$$

holds, where ϕ is an arbitrary bounded continuously differentiable complex-valued function. The relation to the scattering matrix is given by the Birman-Krein theorem

$$\det S(E; T, T_0; \mathcal{J}) = \exp\{-2\pi i \xi(E; T, T_0; \mathcal{J})\}, \tag{32}$$

where $S(E; T, T_0; \mathcal{J})$ is the scattering matrix for the operators (T, T_0).

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