SLAB PERCOLATION FOR THE ISING MODEL

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Abstract. For the FK representation of the Ising model, we prove that the slab percolation threshold coincides with the critical temperature in any dimension $d \geqslant 3$.

1. Introduction

Renormalization arguments are at the core of the description of microscopic systems. In particular, they are necessary close to the critical point where perturbative techniques no longer apply. The rigorous implementation of renormalization techniques is model dependent and in this paper we shall focus on the q-Potts model. In this case, the critical temperature is characterized by a breaking of symmetry (spontaneous magnetization) or by the occurrence of percolation (in the FK representation). In principle, this characterization should suffice to obtain further information about the sub-critical or super-critical phases, like the control of the susceptibility, the classification of the phases and so on. Unfortunately, some of these issues can only be settled at very high or very low temperatures and otherwise they remain open. In fact, in the intermediate regime of temperatures, the knowledge of the absence/occurrence of percolation might not be enough to implement a mathematical argument. Nevertheless, progress can be made under stronger assumptions which sometimes can be proven afterward.

The art of statistical mechanics is to propose the right criteria from which concrete results can be deduced and which should be valid in the whole sub-critical/super-critical regime. As an example, the Dobrushin and Shlosman strong mixing property [DS] implies that the sub-critical phase is well behaved (complete analyticity ...), as well as the dynamics. In some instances this property can be checked: for example Schonmann and Shlosman [SS] have shown that it is valid in the whole uniqueness regime of the two-dimensional Ising model with nearest neighbor interaction.

In this paper we address another type of hypothesis, namely the percolation in slabs. This concept was introduced in the context of Bernoulli percolation by Aizenman, Chayes, Chayes, Fröhlich, Russo [ACCFR] and turned out to be a crucial tool to derive many facts about the geometry of open clusters (see e.g. [G1]). Let us be more specific. If p_c denotes the critical intensity for the Bernoulli percolation, then for any $p > p_c$, the origin has a positive probability to be connected to infinity by an

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open path in $\mathbb{Z}^d$. On the other hand, one can consider the stronger constraint that the origin is connected to infinity in a two-dimensional slice of $\mathbb{Z}^d$ for $d \geqslant 3$. This happens with positive probability for any p larger than a critical value $\hat{p}_c$. Above $\hat{p}_c$ several procedures have been developed to analyze many physical phenomena. Thus, all that is needed to generalize these results to the entire super-critical regime, is to prove that $\hat{p}_c$ and p_c coincide. This conjecture was solved in the breakthrough work of Grimmett and Marstrand [GM] also using important ideas introduced by Barsky, Grimmett, Newman [BGN]. The concept of slab percolation was generalized successfully to the random cluster measure by Pisztora [Pi], who proposed an extremely powerful renormalization scheme under the hypothesis that slab percolation occurs. Furthermore Pisztora conjectured that for FK percolation, slab percolation should also be valid in the whole super-critical phase. This coarse graining provides a very good description of the super-critical regime and in particular, it is a crucial tool in the derivation of the Wulff construction [C, CP, B, BIV1].

The main object of this paper is to prove that for the random cluster measure associated to the Ising model, the slab percolation threshold coincides with the critical temperature. The proof uses heavily the strategy of dynamic renormalization introduced by Barsky, Grimmett, Newman [BGN] in the context of Bernoulli percolation. The starting point of their method was the assumption of percolation in a half space, from which a coarse graining could be implemented. For the random cluster measure, the basic characterization of the super-critical regime is a positive probability of percolation in any finite box with wired boundary conditions. The main difficulty with implementing this information is created by the dependence on the boundary conditions.

Our approach relies on the surface tension from which accurate estimates on percolation in finite size volumes can be obtained uniformly over the boundary conditions. This requires precise controls on the surface tension which are known for the Ising model, thanks to inequalities. The most important property being the positivity of the surface tension derived by Lebowitz and Pfister [LP] in the whole phase transition regime. The consequences of the surface tension estimates are summarized in Corollary 3.1 from which the coarse graining will be constructed. Besides Subsection 3.1, the renormalization procedure developed in the rest of the paper is valid for general random cluster measures $q \geqslant 1$. Further heuristics as well as the scheme of the proof will be presented in Subsection 2.3.

2. Notation and Result

2.1. **The random cluster measure.** Let $\mathbb{E}^d$ be the set of bonds, i.e. of pairs of nearest neighbor vertices (i,j). The set $\Omega = \{0,1\}^{\mathbb{E}^d}$ is the state space for the dependent percolation measures. Given $\omega \in \Omega$ and a bond $b = (i,j) \in \mathbb{E}^d$, we say that b is open if $\omega(b) = 1$. Two sites of $\mathbb{Z}^d$ are said to be connected if one can be reached from another via a chain of open bonds. Thus, each $\omega \in \Omega$ splits $\mathbb{Z}^d$ into the disjoint union of maximal connected components, which are called the open clusters of Ω. Given a finite subset $B \subset \mathbb{Z}^d$ we use $c_B(\omega)$ to denote the number of different open finite clusters of ω which have a non-empty intersection with B.

For any $\Lambda \subset \mathbb{Z}^d$ we define the random cluster measure on the bond configurations $\omega \in \Omega_\Lambda = \{0,1\}^{\mathbb{E}_\Lambda}$, where $\mathbb{E}_\Lambda$ is the set of bonds $b \in \mathbb{E}^d$ intersecting Λ. The boundary conditions are specified by a frozen percolation configuration $\pi \in \Omega_\Lambda^c = \Omega \backslash \Omega_\Lambda$. Using the shortcut $c_\Lambda^\pi(\omega) = c_\Lambda(\omega \vee \pi)$ for the joint configuration $\omega \vee \pi \in \mathbb{E}^d$, we define the finite volume random cluster measure $\Phi_\Lambda^{p,\pi}$ on Ω_Λ with the boundary conditions π as:

$$\Phi_\Lambda^{p,\pi}(\omega) \;=\; \frac{1}{Z_\Lambda^{p,\pi}} \left(\prod_{b \in \mathbb{E}_\Lambda} \big(1-p\big)^{1-\omega_b}\, p^{\omega_b} \right) q^{c_\Lambda^\pi(\omega)}, \tag{2.1}$$

for $q \geqslant 1$. In this paper, we will sometimes use bond dependent intensities $p(b)$. When there is no risk of confusion, we drop the upper-script p and write Φ_Λ^π.

The random cluster measure associated to the box $\Lambda_N = \{-N, \dots, N\}^d$ will be denoted by Φ_N^π. The measures Φ_N^π are FKG partially ordered with respect to the lexicographical order of the boundary condition π. Thus, the extremal ones correspond to the free ($\pi \equiv 0$) and wired ($\pi \equiv 1$) boundary conditions and are denoted as Φ_N^{f} and Φ_N^{w} respectively. The corresponding infinite volume ($N \to \infty$) limits Φ^{f} and Φ^{w} always exist. In the following, the same set of bonds $\mathbb{E}_\Lambda$ will be used for random cluster measures in Λ with free or with wired boundary conditions.

A correspondence between the q-Potts model and the random cluster measure was established by Fortuin and Kasteleyn [FK] (see also [ES, G2]). This representation of the Potts model will be referred as FK representation. Of particular interest for us is the Ising model at inverse temperature β which can be related to the previous model by setting $q = 2$ and choosing the bond intensity $p = 1 - \exp(-2\beta)$. More precisely, the Ising model on $\mathbb{Z}^d$ with nearest neighbor interaction is defined in terms of spins $\{\sigma_i\}_{i \in \mathbb{Z}^d}$ taking values ± 1. Let $\sigma_{\Lambda_N} \in \{\pm 1\}^{\Lambda_N}$ be the spin configuration restricted to Λ_N. The Hamiltonian associated to σ_{Λ_N} with boundary conditions $\sigma_{\partial\Lambda_N}$ is defined by

$$H(\sigma_{\Lambda_N} \,|\, \sigma_{\partial\Lambda_N}) = -\frac{1}{2} \sum_{\substack{i \sim j \\ i,j \in \Lambda_N}} \sigma_i \sigma_j - \sum_{\substack{i \sim j \\ i \in \Lambda_N, j \in \partial\Lambda_N}} \sigma_i \sigma_j .$$

The Gibbs measure in Λ_N at inverse temperature $\beta > 0$ is defined by

$$\mu_{\beta,\Lambda_N}^{\sigma_{\partial\Lambda_N}}(\sigma_{\Lambda_N}) = \frac{1}{Z_{\Lambda_N}^{\sigma_{\partial\Lambda_N}}} \exp\big(-\beta H(\sigma_{\Lambda_N} \,|\, \sigma_{\partial\Lambda_N})\big),$$

where the partition function $Z_{\Lambda_N}^{\sigma_{\partial\Lambda_N}}$ is the normalizing factor. The boundary conditions act as boundary fields, therefore more general values of the boundary conditions can be used.

2.2. **The slab percolation threshold.** The phase transition of the random cluster model is characterized by the occurrence of percolation above the critical intensity p_c

$$\forall p > p_c, \qquad \lim_{N \to \infty} \Phi_N^{p,\mathrm{w}}(0 \leftrightarrow \Lambda_N^c) \;=\; \Phi^{p,\mathrm{w}}(0 \leftrightarrow \infty) > 0. \tag{2.2}$$

If β_c denotes the critical inverse temperature of the Ising model, the FK representation implies that $p_c = 1 - \exp(-2\beta_c)$ for $q = 2$.

For $d \geqslant 3$, one may wonder if the stronger property of percolation in a slab also holds up to the critical value. For any integers (L, N) we define the slabs of thickness L as

$$\mathcal{S}_{L,N} = \{-L, \dots, L\}^{d-2} \times \{-N, \dots, N\}^2 , \qquad \mathcal{S}_L = \{-L, \dots, L\}^{d-2} \times \mathbb{Z}^2 .$$

A critical value can be associated to any slab thickness L

$$\hat{p}_c(L) = \inf \left\{ p \geqslant 0, \qquad \liminf_N \inf_{x \in \mathcal{S}_{L,N}} \Phi^{p,\mathrm{f}}_{\mathcal{S}_{L,N}} (0 \leftrightarrow x) > 0 \right\} . \tag{2.3}$$

As the function $L \to \hat{p}_c(L)$ is non-increasing, it admits a limit $\hat{p}_c$ as L tends to infinity. The critical value $\hat{p}_c$ is the **slab percolation threshold** and satisfies $\hat{p}_c \geqslant p_c$. The main result of this paper is to prove that both critical values coincide if $q = 2$.

Theorem 2.1.
- *If $d \geqslant 3$ and $q = 2$, then $\hat{p}_c = p_c$.*
- *More generally for $q \geqslant 1$, if Θ_q is the set defined before Corollary 3.1 then $]\hat{p}_c, 1] \subset \Theta_q$.*

To avoid many technicalities, the Theorem has been derived for random cluster measures with nearest neighbor interactions. Nevertheless, the proof can be adapted in a straightforward manner to the case of finite range interactions. Unbounded range interactions would require a more delicate treatment (see e.g. [MS, GH]).

As a by-product of the proof, we get

Corollary 2.1. *If $d \geqslant 3$ and $q = 2$, then the surface tension of the Ising model (see Definition 3.1) is equal to 0 at p_c.*

This follows from an argument similar to the one used in [BGN] to prove the continuity of phase transition for half-space Bernoulli percolation.

2.3. **Heuristics and scheme of the proof.** In the super-critical regime, the percolation cluster can be seen as a backbone on which small open clusters are attached (the leaves). When the bond density approaches p_c, the backbone becomes thinner and the structure of the leaves becomes more chaotic as the correlation length $\zeta(p)$ diverges. Nevertheless, for any $p > p_c$, patterns with similar features are repeated on a scale of the order of the correlation length, and the bond configurations which are distant from each other by at least $\zeta(p)$ behave essentially independently. Thus the backbone of the percolation cluster has enough space to spread in any slab of thickness much larger than $\zeta(p)$. This heuristic justifies the slab percolation conjecture.

Some information about the backbone structure is encoded in the surface tension. If the surface tension is positive then the probability that one face of a cube of

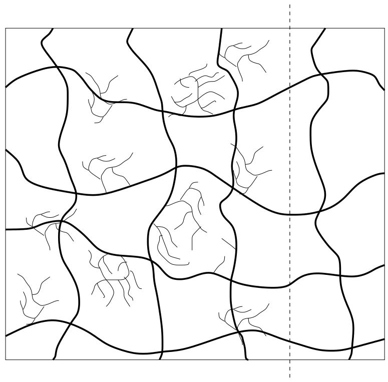

FIGURE 1. A backbone with some leaves is depicted. The dashed line shows which bonds have to be closed to disconnect the two faces of the box.

side length N is not connected to the opposite face decays like $\exp(-\tau N^{d-1})$. This means that to disconnect one face from the other a number of bonds of the order of N^{d-1} have to be closed (see Figure 2). Thus one deduces that the intersection of the backbone with any hyperplan contains a density of bonds.

The implication of the surface tension on the percolation are derived in Subsection 3.1. For the Ising model, the positivity of the surface tension has been established in the whole super-critical regime [LP]. Furthermore, in the framework of the Ising model, several estimates on the surface tension can be derived uniformly wrt the boundary conditions. This independence wrt the boundary conditions will be crucial to implement the renormalization scheme. With the exception of Subsection 3.1, the rest of the paper does not rely on the properties of the Ising model. Thus for any $q \geqslant 1$, the coarse graining also applies under the assumption that p belongs to the range of intensities Θ_q for which Corollary 3.1 holds. We believe that the estimates on the surface tension derived for the Ising model should be valid for any random cluster model if $p > p_c$.

This information about the backbone can be used to implement a dynamic renormalization procedure in the spirit of the one introduced by Barsky, Grimmett, Newman [BGN]. Before going into the details, let us comment on the results obtained in the case of Bernoulli percolation. Under the assumption of half-space percolation, Barsky et al. proposed a renormalization scheme in two steps. First they construct large blocks, so that with high probability the faces of these blocks are interconnected in such a way that further connections can be launched from any faces of these blocks. Using the independence between disjoint blocks, they were able in a second step of renormalization to create an infinite open cluster by piling up these blocks. The simplifying feature of the half-space percolation is to decouple the different blocks, the drawback being that the threshold of half-space percolation may not coincide with p_c. Grimmett and Marstrand overcame this problem by using the sprinkling technique. At the cost of a small increase of bond intensity, they were able to control the block connections without assuming half-space percolation.

When considering the random cluster measure, the dependence on the boundary conditions adds up and it seems difficult to generalize their proof directly from the knowledge that percolation occurs in any finite box with wired boundary conditions.

The surface tension estimates imply precise controls on the probability of percolation in a half box uniformly wrt the boundary conditions (see Subsection 3.3). This will be used to adapt the strategy of Barsky et al. : the main effort is devoted to performing the first renormalization step, i.e. to constructing coarse grained blocks such that with high probability they satisfy good properties uniformly wrt the boundary conditions. As in [BGN], the blocks are such that a small region on the underside is connected to seeds lying in every facet of the block. Nevertheless, the lack of independence wrt the boundary conditions has prevented us from directly applying the ideas of [BGN] and several detours are necessary. This task is performed in Section 4. Once this is done the coarse grained blocks can be piled up as if they were independent and the second renormalization step applies as in [BGN]. For the sake of completeness, the main ideas of the geometric construction, including the steering and branching rules, are recalled in Section 5.

Finally, we would like to comment on two related works based on very different strategies.

The convergence of the critical temperatures of three-dimensional slabs has been derived by Aizenman in the context of Ising model [A1, A2]. The framework is different from ours and in particular the slabs are of the form $\mathbb{Z}^3 \times \{-L, L\}$ and the boundary conditions are periodic. As the proof relies on the random current representation of the Ising model and on reflection positivity, it seems difficult to relate Aizenman's results to the slab percolation threshold defined in the FK setting (2.3) and therefore to Pisztora's coarse graining.

For very large q, the FK measure can be analyzed by means of Pirogov Sinai theory (see e.g. [BKM, KLMR]). At p_c the transition is first order and above p_c, one can show that the only stable phase is the one with high density of open bonds. Thus for large q, the slab percolation threshold becomes a trivial matter and renormalization techniques do not seem necessary to describe the super-critical phase[1].

3. Crossing clusters

3.1. **Surface tension estimates.** In this Subsection we study the dependence on the boundary conditions of the surface tension.

Let us first recall the definition of the surface tension along the coordinate axis $\vec{e}_d$. Let δ be a positive parameter (typically chosen very small) and N, L be two integers such that $N \gg L$. In the following these parameters are chosen of the form $N = 2^n, L = 2^\ell, \delta = 2^{-p}$, where n, ℓ, p are integers. We consider the increasing family of rectangles

$$\mathcal{R}^L(N, \delta) = \{-N, \dots, N\}^{d-1} \times \{-\delta N - L, \dots, \delta N + L\},$$

[1]Private communication by R. Kotecky.

and simply write $\mathcal{R}(N,\delta)$ when $L=0$. Finally, we introduce the set $\mathfrak{J}(N,\delta)$ of bond configurations for which there is no connection from the top face to the bottom face of $\mathcal{R}(N,\delta)$ (i.e. the two faces orthogonal to $\vec{e}_d$)

$$\mathfrak{J}(N,\delta) = \left\{\partial^{\text{bot}}\mathcal{R}(N,\delta) \ \not\leftrightarrow \ \partial^{\text{top}}\mathcal{R}(N,\delta)\right\} . \tag{3.1}$$

Definition 3.1. *The surface tension in the direction $\vec{e}_d$ is defined by*

$$\tau_p = \lim_{\delta\to 0}\ \lim_{N\to\infty} -\frac{1}{N^{d-1}} \log \Phi^{p,\text{w}}_{\mathcal{R}(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) .$$

For $q=2$ and $p = 1-\exp(-2\beta)$, the Ising counterpart is

$$\tau_p = \lim_{\delta\to 0}\ \lim_{N\to\infty} -\frac{1}{N^{d-1}} \log \frac{Z^{\pm}_{\mathcal{R}(N,\delta)}}{Z^{+}_{\mathcal{R}(N,\delta)}} ,$$

where $Z^{\pm}_{\mathcal{R}(N,\delta)}$ denotes the partition function with mixed boundary conditions.

The convergence of the thermodynamic limit has been derived in [MMR]. Notice also that the lateral sides have a vanishing perimeter and therefore they have no influence on the value of the surface tension.

For our purposes, it will be useful to consider the equivalent formulation

$$\tau_p = \lim_{\delta\to 0}\ \lim_{N\to\infty} -\frac{1}{N^{d-1}} \log \Phi^{p,\text{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) . \tag{3.2}$$

The dependence on L can be estimated by using FKG inequality and the fact that $\mathfrak{J}(N,\delta)$ is a non increasing event

$$\Phi^{\text{w}}_{\mathcal{R}(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) \leqslant \Phi^{\text{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) \leqslant \Phi^{\text{w}}_{\mathcal{R}^L(N,\delta)}\Big(\left\{\partial^{\text{top}}\mathcal{R}^L(N,\delta) \ \not\leftrightarrow \ \partial^{\text{bot}}\mathcal{R}^L(N,\delta)\right\} \Big) .$$

As the LHS and the RHS (properly renormalized) converge to the surface tension, the identity (3.2) holds.

The following result enables us to compare the surface tension for different boundary conditions. The parameter L should be taken large enough in order to screen the boundary conditions.

Theorem 3.1. *For $q=2$, then for any $p\in[0,1]$,*

$$\left|\log \Phi^{p,\text{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) - \log \Phi^{p,\text{f}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big)\right| \leqslant \left(\varepsilon_L + c_p\delta + c_d\frac{L}{N}\right) N^{d-1} , \tag{3.3}$$

where ε_L vanishes as L tends to infinity.

The specificity of the Ising model will be used only for the derivation of (3.4). Thus for any $q\geqslant 1$, we introduce the set of bond intensities Θ_q for which (3.4) holds. The rest of the paper will apply for any random cluster measure with $p\in\Theta_q$.

We can now state the main result of this Subsection.

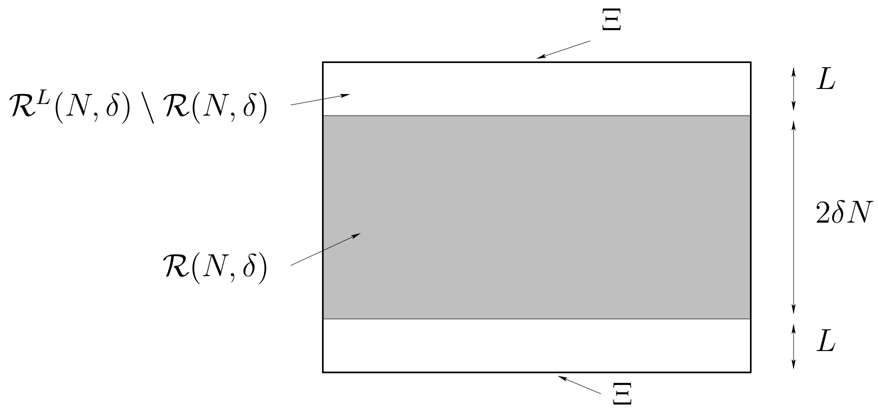

FIGURE 2. The event $\mathfrak{J}(N,\delta)$ is supported by the shaded region which is decoupled from Ξ.

Corollary 3.1. *Let $q=2$, then for any $p>p_c$, there exists $\delta=\delta(p)$ and L large such that for any N large enough*

$$\Phi^{p,\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big(\big\{\partial^{\mathrm{bot}}\mathcal{R}(N,\delta)\ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\}\Big)\geqslant 1-\exp(-C_pN^{d-1})\,, \tag{3.4}$$

where C_p is a positive constant.

This follows from Theorem 3.1 and the positivity of surface tension which was proven for the Ising model by Lebowitz and Pfister [LP] when β is larger than β_c.

Proof of Theorem 3.1.

The estimate (3.3) will be obtained by interpolating the boundary conditions wired and free. Define the set of boundary bonds at the top and bottom faces of $\mathcal{R}^L(N,\delta)$ as

$$\Xi=\big\{(i,i+\vec e_d),(j,j-\vec e_d)\in\mathbb{E}_{\mathcal{R}^L(N,\delta)},\qquad i,j\in\mathcal{R}^L(N,\delta);\ i_d=-j_d=L+\delta N\big\}\ .$$

Let $\Phi^{s,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}$ be the wired FK measure for which the bonds in Ξ have intensity s instead of p. The parameter s acts as a boundary magnetic field (on the faces orthogonal to $\vec e_d$) and interpolates between $\Phi^{\mathrm{w}}_{\mathcal{R}^L(N,\delta)}$ (for $s=p$) and $\Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}$ (for $s=0$). The latter measure is the FK measure with free boundary conditions on the top and bottom faces of $\mathcal{R}^L(N,\delta)$ and wired otherwise.

Remark 3.1. *In Subsection 2.1, the FK measure $\Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}$ was defined on the set of bonds intersecting $\mathcal{R}^L(N,\delta)$. When $s=0$, the measure $\Phi^{s=0,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}$ is equal to the free measure conditionally to the fact that the bonds in Ξ are closed. Nevertheless this has no impact on the probability of events which are not supported by Ξ and thus the probability of $\mathfrak{J}(N,\delta)$ is the same under $\Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}$ or $\Phi^{s=0,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}$.*

$$\log \Phi^{\mathrm{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) - \log \Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) \tag{3.5}$$
$$= \sum_{b\in\Xi} \int_0^p \frac{ds}{s(1-s)} \left(\frac{\Phi^{s,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\,\omega_b\Big)}{\Phi^{s,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big)} - \Phi^{s,\mathrm{w}}_{\mathcal{R}^L(N,\delta)}(\omega_b) \right).$$

At this stage, a mixing property is required. We introduce now the FK measures on the box $\mathbb{B}_K = \{-K,\dots,K\}^{d-1}\times\{0,\dots,K\}$ with intensity s for the bonds below the bottom face of $\mathbb{B}_K$ (i.e. the bonds in $\{(i, i-\vec{e}_d), \quad i_d = 0, i \in \mathbb{B}_K\}$) and wired boundary conditions at the bottom face. When the boundary conditions on the remaining faces are free the corresponding FK measure is denoted by $\Phi^{s,\mathrm{f}}_{\mathbb{B}_K}$ and $\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}$ for wired.

Proposition 3.1. *Let $\mathfrak{b}_0$ be the bond $(0,-\vec{e}_d)$. For any $p\in[0,1]$ and $s\in(0,p]$,*
$$\Big|\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0}) - \Phi^{s,\mathrm{f}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0})\Big| \leqslant \varepsilon_K(s)\,, \tag{3.6}$$
where $\varepsilon_K(s)$ vanishes as K diverges.

We postpone the proof of this Lemma and first estimate (3.5). As $\mathfrak{J}(N,\delta)$ is supported by $\mathcal{R}(N,\delta)$, it is decoupled from the bonds in Ξ. Thus using FKG property one can reduce the estimates to the set $\mathcal{R}^L(N,\delta)\setminus\mathcal{R}(N,\delta)$
$$\Big|\log \Phi^{\mathrm{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) - \log \Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big)\Big|$$
$$\leqslant \sum_{b\in\Xi}\int_0^p \frac{ds}{s(1-s)} \Big(\Phi^{s,\mathrm{w}}_{\mathcal{R}^L(N,\delta)\setminus\mathcal{R}(N,\delta)}(\omega_b) - \Phi^{s,\mathrm{f}}_{\mathcal{R}^L(N,\delta)\setminus\mathcal{R}(N,\delta)}(\omega_b)\Big)\,,$$
$$\leqslant 2N^{d-1}\int_0^p \frac{ds}{s(1-s)}\Big(\Phi^{s,\mathrm{w}}_{\mathbb{B}_L}(\omega_{\mathfrak{b}_0}) - \Phi^{s,\mathrm{f}}_{\mathbb{B}_L}(\omega_{\mathfrak{b}_0})\Big) + c_d L N^{d-2}\,.$$

The final bound has been obtained by applying again FKG inequality to reduce further the domain of the FK measure. The rest is an upper bound of the expectation of the terms lying at a distance smaller than L from the lateral sides of $\mathcal{R}^L(N,\delta)$.

We remark that for any $s\in(0,p]$
$$\frac{1}{s(1-s)}\Big(\Phi^{s,\mathrm{w}}_{\mathbb{B}_L}(\omega_{\mathfrak{b}_0}) - \Phi^{s,\mathrm{f}}_{\mathbb{B}_L}(\omega_{\mathfrak{b}_0})\Big) \leqslant \frac{1}{s(1-s)}\Big(\Phi^{s,\mathrm{w}}_{\{\mathfrak{b}_0\}}(\omega_{\mathfrak{b}_0}) - \Phi^{s,\mathrm{f}}_{\{\mathfrak{b}_0\}}(\omega_{\mathfrak{b}_0})\Big)\,,$$
$$\leqslant \frac{1}{s(1-s)}\left(s - \frac{s}{s+(1-s)q}\right) \leqslant \frac{1}{1-p}\,.$$

Thus the dominated convergence theorem and Proposition 3.1 imply that there exists a sequence ε_L vanishing to 0 as L diverges such that
$$\Big|\log \Phi^{\mathrm{w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big) - \log \Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}\Big(\mathfrak{J}(N,\delta)\Big)\Big| \leqslant 2N^{d-1}\varepsilon_L + c_d L N^{d-2}\,.$$

It remains to compare the FK measures $\Phi^{\mathrm{f,w}}_{\mathcal{R}^L(N,\delta)}$ and $\Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}$ by modifying the boundary conditions on the lateral sides. This can be achieved at a finite cost proportional to the perimeter of the sides parallel to $\vec{e}_d$. The corresponding error is bounded by the term δN^{d-1} in the LHS of (3.3). This completes Theorem 3.1. $\square$

Proof of Proposition 3.1.

The strategy is to use the Gibbssian counterpart of (3.6) for which a mixing property is known.

Step 1.

We denote by $\partial \mathbb{B}_K$ the set of boundary vertices of $\mathbb{B}_K$ and define the set of vertices lying in the bottom face of $\partial \mathbb{B}_K$ as

$$\partial^s \mathbb{B}_K \quad = \quad \{i, \qquad i_d = -1, \; -K \leqslant i_\ell \leqslant K \;\; \text{if} \;\; \ell < d\} \; .$$

The bond $\mathfrak{b}_0$ links the site 0 to the site $\mathfrak{g} = (0, \dots, 0, -1)$. The event that 0 is connected to the boundary without using the bond $\mathfrak{b}_0$ is in the wired and free case

$$\mathcal{C}^{\mathrm{w}} = \{0 \leftrightarrow \partial \mathbb{B}_K \setminus \{\mathfrak{g}\}\} \quad \text{and} \quad \mathcal{C}^{\mathrm{f}} = \{0 \leftrightarrow \partial^s \mathbb{B}_K \setminus \{\mathfrak{g}\}\} \; .$$

Let $h = h(s) = -\frac{1}{2}\log(1-s)$ be the positive magnetic field associated to the bond intensity s.

$$\mu_{\mathbb{B}_K}^{h,+}(\sigma_0) = \Phi_{\mathbb{B}_K}^{s,\mathrm{w}}(0 \leftrightarrow \partial \mathbb{B}_K) = \Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big(1_{\mathcal{C}^{\mathrm{w}}}\big) + \Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big((1 - 1_{\mathcal{C}^{\mathrm{w}}})\, \omega_{\mathfrak{b}_0}\big) \, .$$

Conditioning outside $\{\mathfrak{b}_0\}$, we have for any boundary condition π which does not belong to $\mathcal{C}^{\mathrm{w}}$ (see e.g. equation (3.10) in [G2])

$$\Phi_{\{0\}}^{s,\pi}\big(\omega_{\mathfrak{b}_0}\big) = \frac{s}{s + (1-s)q} \, .$$

Thus

$$\mu_{\mathbb{B}_K}^{h,+}(\sigma_0) = \Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big(1_{\mathcal{C}^{\mathrm{w}}}\big) + \big(1 - \Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big(1_{\mathcal{C}^{\mathrm{w}}}\big)\big) \, \frac{s}{s + (1-s)q} \, .$$

In the same way, with free boundary conditions (i.e. with zero magnetic field)

$$\mu_{\mathbb{B}_K}^{h,0}(\sigma_0) = \Phi_{\mathbb{B}_K}^{s,\mathrm{f}}\left(1_{\mathcal{C}^{\mathrm{f}}}\right) + \Big(1 - \Phi_{\mathbb{B}_K}^{s,\mathrm{f}}\left(1_{\mathcal{C}^{\mathrm{f}}}\right)\Big) \, \frac{s}{s + (1-s)q} \, .$$

This leads to

$$\Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big(1_{\mathcal{C}^{\mathrm{w}}}\big) - \Phi_{\mathbb{B}_K}^{s,\mathrm{f}}\left(1_{\mathcal{C}^{\mathrm{f}}}\right) = \frac{s + (1-s)q}{(1-s)q} \Big(\mu_{\mathbb{B}_K}^{h,+}(\sigma_0) - \mu_{\mathbb{B}_K}^{h,0}(\sigma_0)\Big) \, .$$

Thus we have established the following equivalence

Lemma 3.1. *For $h = -\frac{1}{2}\log(1-s)$, if the difference $\big(\mu_{\mathbb{B}_K}^{h,+}(\sigma_0) - \mu_{\mathbb{B}_K}^{h,0}(\sigma_0)\big)$ vanishes to 0 as K diverges then there exists a sequence $\varepsilon_K(s)$ such that*

$$\Big|\Phi_{\mathbb{B}_K}^{s,\mathrm{w}}\big(\mathcal{C}^{\mathrm{w}}\big) - \Phi_{\mathbb{B}_K}^{s,\mathrm{f}}\left(\mathcal{C}^{\mathrm{f}}\right)\Big| \leqslant \varepsilon_K(s) \, ,$$

and $\lim_K \; \varepsilon_K(s) = 0$.

Step 2.

Let us prove that for any $h > 0$

$$\lim_{K\to\infty} \mu^{h,+}_{\mathbb{B}_K}(\sigma_0) - \mu^{h,0}_{\mathbb{B}_K}(\sigma_0) = 0\,. \tag{3.7}$$

The above thermodynamic limits are taken in the upper half space. We denote by $\mu^{h,+}(\sigma_0)$ and $\mu^{h,0}(\sigma_0)$ the corresponding limits.

From [MMP], we know that the functions $\mu^{h,+}_{\mathbb{B}_K}(\sigma_0)$, $\mu^{h,0}_{\mathbb{B}_K}(\sigma_0)$ as well as $\mu^{h,+}(\sigma_0)$ and $\mu^{h,0}(\sigma_0)$ are analytic in the complex domain $\{\mathrm{Re}(h) > |\mathrm{Im}(h)|\}$. Therefore the functions below enjoy the same properties

$$F_K(h) = \mu^{h,+}_{\mathbb{B}_K}(\sigma_0) - \mu^{h,0}_{\mathbb{B}_K}(\sigma_0) \quad \text{and} \quad F(h) = \mu^{h,+}(\sigma_0) - \mu^{h,0}(\sigma_0)\,.$$

When $h > 1$, then $F(h) = 0$. This follows from an estimate derived by Fröhlich and Pfister (see (2.13) and (2.21) of [FP])

$$\forall h > 1, \qquad \lim_{K\to\infty} \mu^{h,+}_{\mathbb{B}_K}(\sigma_0) - \mu^{h,-}_{\mathbb{B}_K}(\sigma_0) = 0\,.$$

From the FKG inequality, the LHS dominates $F_K(h)$ and one obtains that $F(h) = 0$ for any $h > 1$. As F is analytic it must be constant for any $h > 0$. Thus (3.7) holds.

Step 3.

Finally, it remains to control the expectation of $\omega_{\mathfrak{b}_0}$

$$\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0}) = \Phi^{s,\mathrm{w}}_{\mathbb{B}_K}\big(1_{\mathcal{C}^{\mathrm{w}}}\omega_{\mathfrak{b}_0}\big) + \Phi^{s,\mathrm{w}}_{\mathbb{B}_K}\big((1-1_{\mathcal{C}^{\mathrm{w}}})\omega_{\mathfrak{b}_0}\big)$$

Conditionally to $\mathcal{C}^{\mathrm{w}}$ or its complement, the distribution of $\omega_{\mathfrak{b}_0}$ is determined thus

$$\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0}) = s\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}\big(\mathcal{C}^{\mathrm{w}}\big) + \big(1 - \Phi^{s,\mathrm{w}}_{\mathbb{B}_K}\big(\mathcal{C}^{\mathrm{w}}\big)\big)\,\frac{s}{s+(1-s)q}\,.$$

A similar relation holds in the case of free boundary conditions, this leads to

$$\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0}) - \Phi^{s,\mathrm{f}}_{\mathbb{B}_K}(\omega_{\mathfrak{b}_0}) = s\left(1 - \frac{1}{s+(1-s)q}\right)\left(\Phi^{s,\mathrm{w}}_{\mathbb{B}_K}\big(\mathcal{C}^{\mathrm{w}}\big) - \Phi^{s,\mathrm{f}}_{\mathbb{B}_K}\big(\mathcal{C}^{\mathrm{f}}\big)\right)\,.$$

Combining Lemma 3.1 and (3.7) we conclude Proposition 3.1. □

3.2. **Crossing clusters.** Corollary 3.1 can be improved by showing that a connection from the top to the bottom of $\mathcal{R}^L(N,\delta)$ occurs with high probability.

Proposition 3.2. *For any $p \in \Theta_q$, there exists δ and L such that for any N large enough*

$$\Phi^{p,\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big(\,\big\{\partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta) \;\leftrightarrow\; \partial^{\mathrm{top}}\mathcal{R}^L(N,\delta)\big\}\,\Big) \geqslant 1 - \exp(-CN^{d-1})\,, \tag{3.8}$$

where $C = C(p,L,\delta)$ is a positive constant.

Proof.

The parameters δ and L are fixed according to Corollary 3.1. We first analyze the connections from $\partial^{\mathrm{top}}\mathcal{R}(N,\delta)$ to the bottom face of $\mathcal{R}^L(N,\delta)$ and show that there exists $C>0$ such that

$$\Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big(\big\{\partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta)\ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\}\Big)\geqslant 1-\exp(-CN^{d-1})\,. \tag{3.9}$$

Given $\alpha>0$, we introduce $\mathcal{A}_N^\alpha$ the set of bond configurations for which there are at least αN^{d-1} sites at the level $\{x_d=-\delta N+1\}$ connected to $\partial^{\mathrm{top}}\mathcal{R}(N,\delta)$ by open paths contained within $\{x_d\geqslant -\delta N+1\}$. There exists $\alpha>0$ and $C_1>0$ such that

$$\Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big(\mathcal{A}_N^\alpha\Big)\geqslant 1-\exp(-C_1N^{d-1})\,. \tag{3.10}$$

To see this, we first notice that the set $(\mathcal{A}_N^\alpha)^c$ is supported by the bonds lying in $\{x_d\geqslant -\delta N+1\}$. Given any configuration ω in $(\mathcal{A}_N^\alpha)^c$, it is enough to close at most αN^{d-1} bonds in order to cut all the connections from $\partial^{\mathrm{top}}\mathcal{R}(N,\delta)$ to $\partial^{\mathrm{bot}}\mathcal{R}(N,\delta)$. Therefore, there exists a constant C_2 such that

$$\Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big((\mathcal{A}_N^\alpha)^c\Big)\leqslant\ \exp(C_2\alpha N^{d-1})\ \Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\big(\mathfrak{J}(N,\delta)\big)\,. \tag{3.11}$$

Using the inequality (3.4), one can choose α small enough such that (3.10) holds.

The next step is to extend the connections below the level $\{x_d=-\delta N+1\}$. For any bond configuration π in $\mathcal{A}_N^\alpha$, we denote by $\{x^{(i)}\}_i$ the first αN^{d-1} sites (wrt the lexicographic order) in $\{x_d=-\delta N+1\}$ which are connected to $\partial^{\mathrm{top}}\mathcal{R}(N,\delta)$. Let $\mathbf{C}(x^{(i)})$ be the event that $x^{(i)}$ is connected to $\partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta)$ by a straight vertical path, i.e. using only the edges in

$$\mathcal{P}(x^{(i)})=\big(\langle x^{(i)}-(k-1)\vec{e}_d,\ x^{(i)}-k\vec{e}_d\rangle\big)_{k\,\leqslant\,L}\ .$$

Uniformly over the boundary conditions ω outside $\mathcal{P}(x^{(i)})$, we have

$$\Phi^{\omega}_{\mathcal{P}(x^{(i)})}\Big(\mathbf{C}(x^{(i)})\Big)\geqslant\Phi^{\mathrm{f}}_{\mathcal{P}(x^{(i)})}\Big(\mathbf{C}(x^{(i)})\Big)\geqslant c_L>0\,. \tag{3.12}$$

To simplify the notation, we set $\Delta=\mathcal{R}^L(N,\delta)\cap\{x_d\leqslant -\delta N\}$. For any bond configuration $\pi\in\mathcal{A}_N^\alpha$ supported by $\{x_d\geqslant -\delta N+1\}$, we get

$$\begin{aligned}\Phi^\pi_\Delta\Big(\big\{\partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta)\ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\}\Big)\quad&\geqslant\quad\Phi^\pi_\Delta\Big(\bigcup_i\mathbf{C}(x^{(i)})\Big)\\ &\geqslant\quad 1-\Phi^\pi_\Delta\left(\bigcap_i\big(\mathbf{C}(x^{(i)})\big)^c\right)\,.\end{aligned}$$

The supports of the events $\mathbf{C}(x^{(i)})$ are disjoint. Using repeated conditionings and the lower bound (3.12), we get uniformly over π in $\mathcal{A}_N^\alpha$

$$\Phi^\pi_\Delta\Big(\big\{\partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta)\ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\}\Big)\geqslant 1-(1-c_L)^{\alpha N^{d-1}}\geqslant 1-\exp(-C_L\,\alpha N^{d-1})\,,$$

for some $C_L>0$.

Finally, combining the previous estimate and (3.9), there is $C > 0$ such that

$$\begin{aligned}
&\Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)}\Big(\big\{ \partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta) \ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\} \Big) \\
&\quad \geqslant \Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)} \Big[\Phi^{\pi}_{\Delta}\Big(\big\{ \partial^{\mathrm{bot}}\mathcal{R}^L(N,\delta) \ \leftrightarrow\ \partial^{\mathrm{top}}\mathcal{R}(N,\delta)\big\} \Big) 1_{\mathcal{A}^{\alpha}_N}(\pi) \Big] \\
&\quad \geqslant \Phi^{\mathrm{f}}_{\mathcal{R}^L(N,\delta)} \Big(\mathcal{A}^{\alpha}_N \Big) \left(1 - \exp(-C_L\, \alpha N^{d-1}) \right) \geqslant 1 - \exp(-C\, \alpha N^{d-1}) \,.
\end{aligned}$$

Thus (3.9) is satisfied. The Theorem can be completed by following the same strategy to extend the connections to the top face of $\mathcal{R}^L(N,\delta)$.

□

3.3. **Half box percolation.** The surface tension estimates imply not only the occurrence of half space percolation, but also uniform bounds wrt the boundary conditions for the corresponding finite size events. At this stage, it will be easier to consider sets in the half space $\{x_d \geqslant 0\}$ instead of the rectangles $\mathcal{R}^L(N,\delta)$. We define below the corresponding notation which will be used throughout the paper.

Definition 3.2.

(1) *For any integers* (ℓ, h), *a block is defined as*

$$\mathbb{B}(\ell,h) = \{-\ell,\dots,\ell\}^{d-1} \times \{0,\dots,h\} \,.$$

The interior of $\mathbb{B}(\ell,h)$ *is*

$$\mathbb{B}^*(\ell,h) = \big\{ -\ell+1,\dots,\ell-1 \big\}^{d-1} \times \big\{1,\dots,h-1\big\} \,.$$

(2) *The top face of the block* $\mathbb{B}(\ell,h)$ *will be denoted by*

$$T(\ell,h) = \{-\ell,\dots,\ell\}^{d-1} \times \{h\} \,.$$

The top face is split into 2^{d-1} *sub-regions*

$$\begin{aligned}
T_1(\ell,h) &= \{0,\dots,\ell\}^{d-1} \times \{h\} \,, \\
T_2(\ell,h) &= \{0,\dots,\ell\}^{d-2} \times \{-\ell,\dots,0\} \times \{h\} \,, \\
&\dots \\
T_{2^{d-1}}(\ell,h) &= \{-\ell,\dots,0\}^{d-1} \times \{h\} \,.
\end{aligned}$$

(3) *Denote by* $S(\ell,h)$ *the union of the sides of* $\mathbb{B}(\ell,h)$. *It is divided into* $2(d-1)2^{d-2}$ *sub-regions*

$$\begin{aligned}
S_1(\ell,h) &= \{\ell\} \times \{0,\dots,\ell\}^{d-2} \times \{0,\dots,h\} \,, \\
S_2(\ell,h) &= \{\ell\} \times \{0,\dots,\ell\}^{d-3} \times \{-\ell,\dots,0\} \times \{0,\dots,h\} \,, \\
&\dots \\
S_{2(d-1)2^{d-2}}(\ell,h) &= \{-\ell,\dots,0\}^{d-2} \times \{\ell\} \times \{0,\dots,h\} \,.
\end{aligned}$$

For any $i \leqslant d$ and $x \in \mathbb{Z}^d$, let $b^i_K(x)$ be the $(d-1)$-dimensional hypercube centered around x and orthogonal to $\vec{e}_i$

$$b^i_K(x) = \big\{ y \in \mathbb{Z}^d, \quad y_i = x_i, \ \forall j \neq i, \ |y_j - x_j| \leqslant K \big\} \,.$$

In the following we will be interested to the connections from $b_K^i(0)$ to the faces of $\mathbb{B}(\ell,h)$ by open paths strictly within $\mathbb{B}(\ell,h)$, i.e. paths lying in $\mathbb{E}_{\mathbb{B}^*(\ell,h)}$. Recall that $\mathbb{E}_{\mathbb{B}^*(\ell,h)}$ denotes the set of bonds intersecting $\mathbb{B}^*(\ell,h)$ (see Subsection 2.1).

Proposition 3.3. *For any $p \in \Theta_q$, there exists $\delta > 0$ and L such that the following holds uniformly in N and $K \leqslant N$*

$$\Phi^{p,\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\big\{ b_K^d(0) \;\leftrightarrow\; T(N,H_N) \big\} \Big) \geqslant 1 - \exp(-CK^{d-1})\,, \tag{3.13}$$

where $H_N = \delta N + L$ and C is a positive constant.

Proof.
Let $B_N = \big\{ -N,\dots,N \big\}^{d-1} \times \big\{0\big\}$ be the bottom face of $\mathbb{B}^*(N,H)$. According to Proposition 3.2 the following holds for appropriate choices of $\delta > 0$ and L

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H'_N)}\Big(\{ B_N \;\leftrightarrow\; T(N,H'_N) \} \Big) \geqslant 1 - \exp(-CN^{d-1})\,, \tag{3.14}$$

where $H'_N = 2\delta N + 2L$.

We consider also

$$B_N^{(K)} = \big\{ x \in B_N, \qquad x = 2Kj \quad \text{for } j \in \mathbb{Z}^d \big\}\,.$$

Then B_N is covered by $\bigcup_{x\in B_N^{(K)}} b_K^d(x)$ and

$$\big\{ B_N \not\leftrightarrow T(N,H'_N) \big\} = \bigcap_{x \in B_N^{(K)}} \big\{ b_K^d(x) \not\leftrightarrow T(N,H'_N) \big\}\;.$$

These events are decreasing, therefore (3.14) and FKG inequality imply

$$\exp(-CN^{d-1}) \geqslant \prod_{x\in B_N^{(K)}} \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H'_N)} \left(\big\{ b_K^d(x) \not\leftrightarrow T(N,H'_N) \big\} \right)\;. \tag{3.15}$$

Applying again the FKG property, we see that for any x in $B_N^{(K)}$

$$\begin{aligned} \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H'_N)} \left(\big\{ b_K^d(x) \not\leftrightarrow T(N,H'_N) \big\} \right) &\geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,H'_N)} \left(\big\{ b_K^d(0) \not\leftrightarrow T(N,H'_N) - x \big\} \right) \\ &\geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,H'_N)} \left(\big\{ b_K^d(0) \not\leftrightarrow T(2N,H'_N) \big\} \right)\;. \end{aligned}$$

Plugging this inequality into (3.15) leads to

$$-C \frac{N^{d-1}}{|B_N^{(K)}|} \geqslant \log \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,H'_N)} \left(\big\{ b_K^d(0) \not\leftrightarrow T(2N,H'_N) \big\} \right)\;,$$

where $|B_N^{(K)}| = (N/2K)^{d-1}$ is the cardinal of $B_N^{(K)}$. This completes the Proposition with $H_N = H'_{N/2}$.

□

3.4. **The seeds.** We introduce now the notion of seed which will be essential to iterate (3.13).

Definition 3.3.

(1) *We say that* $b_K^d(x)$ *is a* **seed** *centered at site* $x \in \mathbb{Z}^d$ *if all the bonds lying in* $b_K^d(x)$ *are open.*
(2) *For any* (ℓ, h)*, let* $\mathcal{C}_K(\ell, h)$ *be the event that* $b_K^d(0)$ *is connected by a path of open bonds strictly within* $\mathbb{B}(\ell, h)$ *to a seed* $b_K^d(x)$ *included in* $T(\ell, h)$*.*
(3) *For any* (ℓ, h)*,* $\mathcal{C}_K^i(\ell, h)$ *denotes the event that* $b_K^d(0)$ *is connected strictly within* $\mathbb{B}(\ell, h)$ *to a seed* $b_K^d(x)$ *included in* $T(\ell, h)$ *and centered at site* x *in* $T_i(\ell, h)$*, with* $i \leqslant 2^{d-1}$*.*

In the following, we choose $p \in \Theta_q$ and H_N according to Proposition 3.3.

Proposition 3.4. *For any integer* K *there exists an integer* $M = M(K)$ *and a height* $h = h(K, N) \in [H_N - M, H_N]$ *such that uniformly over* N *large enough*

$$\Phi^{p,\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{C}_K(N, h)\Big) \geqslant 1 - 4\exp\big(-CK^{d-1}\big),$$

for some positive constant C*.*

Proof.
For a given height h, define $Y(\ell, h)$ as the number of sites in $T(\ell, h)$ which are connected strictly within $\mathbb{B}(\ell, h)$ to $b_K^d(0)$ by paths of open bonds.

Let m, M be two integers which are going to be chosen later. The sequence of random heights $\mathcal{H}_i$ is defined recursively. We set $\mathcal{H}_0 = H_N - M$ and

$$\mathcal{H}_{i+1} = \inf\big\{h > \mathcal{H}_i, \quad 1 \leqslant Y(N, h) \leqslant m\big\} \ \wedge\ H_N\,.$$

We first check that there exists $r_p = r_p(m) < 1$ such that uniformly in N and M

$$\forall n \in [2, M], \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{H}_n < H_N\Big) \leqslant r_p^{n-1}\,. \tag{3.16}$$

To see this, we write for $n \geqslant 2$

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{H}_n < H_N\Big) = \sum_{h < H_N - 1} \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{H}_{n-1} = h, \mathcal{H}_n < H_N\Big)\,.$$

Conditionally to the event $\{\mathcal{H}_{n-1} = h\}$ any bond configuration in $\{\mathcal{H}_n < H_N\}$ is such that $Y(N, h+1) \geqslant 1$. On the other hand, there are at most m sites of $T(N, h)$ connected to $b_K^d(0)$. Thus

$$\begin{aligned}
\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{H}_n < H_N \,\Big|\, \mathcal{H}_{n-1} = h\Big) &\leqslant 1 - \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(Y(N, h+1) = 0 \,\Big|\, \mathcal{H}_{n-1} = h\Big)\,,\\
&\leqslant 1 - c_p^m = r_p < 1\,,
\end{aligned}$$

where $c_p^m > 0$ is a lower bond of the energetic cost for closing the bonds which could connect $b_K^d(0)$ to the height $h+1$. By iterating the procedure, we obtain (3.16).

For any K, there exists $n = n(K,m)$ such that for any M large enough and uniformly over N,

$$\Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\left(\sum_{h=H_N-M}^{H_N} 1_{\{Y(N,h)>m\}} \geqslant M-n\right) \geqslant 1-2\exp(-CK^{d-1})\,, \qquad (3.17)$$

where C is the constant of Proposition 3.3.

To see this, we write

$$\Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\left(\sum_{h=H_N-M}^{H_N} 1_{\{Y(N,h)\,\leqslant\, m\}} \geqslant n\right) \quad\leqslant\quad \Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\left(Y(N,H_N)=0\right)$$
$$+\Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\left(\mathcal{H}_n < H_N\right)\,.$$

The first term in the LHS means that there is no connection from $b_K^d(0)$ to the top face of $\mathbb{B}(N,H_N)$ thus it is bounded by Proposition 3.3. Let $n = n(K,m)$ be such that $r_p^{n-1} \leqslant \exp(-CK^{d-1})$. Then the second term is bounded by the estimate (3.16). Thus (3.17) is satisfied.

We need to check that there exists $h \in [H_N - M, H_N]$ such that

$$\Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\Big(Y(N,h) \geqslant m\Big) \geqslant 1-3\exp\left(-CK^{d-1}\right)\,. \qquad (3.18)$$

Applying Tchebyshev inequality to the estimate (3.17), we get

$$\big(1-2\exp(-CK^{d-1})\big)(M-n) \leqslant \sum_{h=H_N-M}^{H_N} \Phi^{\rm f}_{\mathbb{B}^*(N,H_N)}\big(Y(N,h)\geqslant m\big)\,.$$

We choose $M = M(K,m)$ such that $n(K,m) < \exp(-CK^{d-1})M$ then the above inequality ensures that there exists a height h in $[H_N - M, H_N]$ such that (3.18) holds. The height h depends on K, N and m, nevertheless the parameter m will be fixed according to K (see (3.19)). Thus, ultimately, only the dependency on K, N will remain.

The estimate (3.18) implies that with high probability, there are at least m sites in $T(N,h)$ connected to $b_K^d(0)$. Thus it is enough to find at least a seed lying on top of one of these sites. The probability that uniformly wrt the boundary conditions there exists a seed of side length K is bounded from below by $c_p^{K^{d-1}}$, with $c_p > 0$. Let us choose $m = m(K)$ such that

$$\big(1-c_p^{K^{d-1}}\big)^{m/(4K)^{d-1}} < \exp(-CK^{d-1})\,. \qquad (3.19)$$

Conditioning wrt the bond configurations below h which belong to the event $\{Y(N,h) \geqslant m\}$, we can find at least $m/(4K)^{d-1}$ sites at distance larger than $4K$ from each other. Then the probability that a seed (lying in $T(N,h)$) is attached on top of one of these sites is dominated by independent Bernoulli variables. As m has been chosen large enough, there must be at least a seed on one of the attachment site with probability larger than $1-\exp(-CK^{d-1})$.

This concludes the proof of Proposition 3.4. $\square$

Proposition 3.4 has to be enhanced in order to control the position of the seeds.

Proposition 3.5. *For any large K there exists an integer $M = M(K)$ and a height $h = h(K,N) \in [H_N - M, H_N]$ such that uniformly over N large enough*

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{p,\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\mathcal{C}^i_K(N,h)\Big) \geqslant 1 - \exp\big(-CK^{d-1}\big)\,,$$

for some positive constant C. The event $\mathcal{C}^i_K(N,h)$ was introduced in Definition 3.3.

Proof.
Following [BGN], we use the symmetry of the model to write

$$\begin{aligned}\Big(\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big((\mathcal{C}^i_K(N,h))^c\Big)\Big)^{2^{d-1}} &= \prod_j \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big((\mathcal{C}^j_K(N,h))^c\Big)\\ &\leqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big(\bigcap_j (\mathcal{C}^j_K(N,h))^c\Big) = \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big((\mathcal{C}_K(N,h))^c\Big)\,,\end{aligned}$$

where we used the FKG inequality and the fact that the events $\mathcal{C}^j_K(N,h)$ are increasing.

Using the lower bound derived in Proposition 3.4, we obtain that

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,H_N)}\Big((\mathcal{C}^i_K(N,h))^c\Big) \leqslant 4^{1/2^{d-1}} \exp\left(-\frac{C}{2^{d-1}}K^{d-1}\right)\,.$$

This completes the proof. □

4. The coarse graining: first renormalization step

Recall that p is chosen in the set Θ_q introduced before Corollary 3.1.

4.1. **Top connections.** The next step is to iterate Proposition 3.5 in order to obtain a connection in a box $\mathbb{B}(N, L_N)$ for some $L_N \geqslant 3N$.

Theorem 4.1. *For any $\eta > 0$, one can find $K = K(\eta)$ such that uniformly in N (large enough), there exists an integer $M = M(K)$ and a height $\varphi = \varphi(K,N) \in [L_N - M, L_N]$ such that*

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{p,\mathrm{f}}_{\mathbb{B}^*(N,L_N)}\Big(\mathcal{C}^i_K(N,\varphi)\Big) \geqslant 1 - \eta\,,$$

where $L_N = L_N(K)$ is such that $3N \leqslant L_N - M(K) \leqslant \varphi(K,N)$ (see (4.1)).

Proof.
Let $H_N = \delta N + L$, where δ and L are chosen according to Proposition 3.3. Fix K large enough such that

$$\big(1 - \exp\big(-CK^{d-1}\big)\big)^{1+6/\delta} \geqslant 1 - \eta\,,$$

where C is the constant obtained in Proposition 3.5. Then choose N large enough and set $h = h(K,N), M = M(K)$ according to Proposition 3.5.

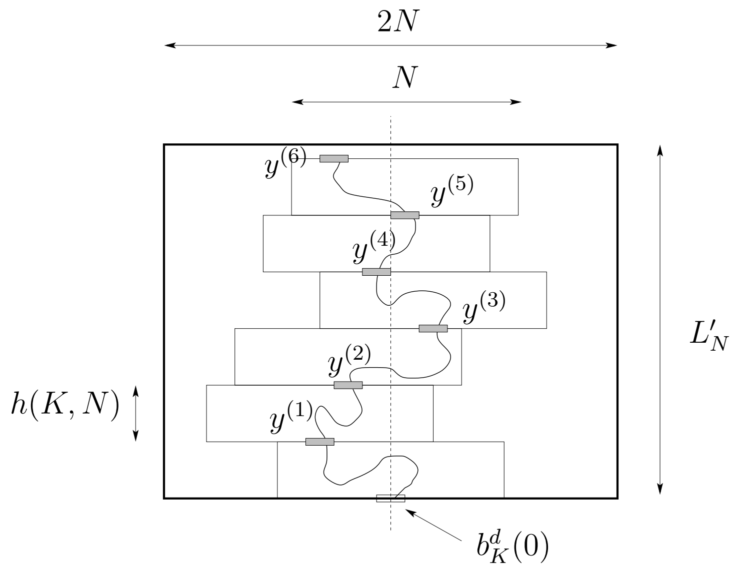

FIGURE 3. The seeds are connected according to the steering rule.

We set

$$n = 6/\delta, \qquad L'_N = H_N + nh(K,N), \qquad L_N = L'_{N/2}\,. \tag{4.1}$$

Let $\mathcal{A}_i$ be the event that $b_K^d(0)$ is connected to a seed in $T\big(N, ih\big)$ by an open path strictly contained within $\mathbb{B}(2N, i\,h)$

$$\forall i \leqslant n, \qquad \mathcal{A}_i = \left\{ b_K^d(0) \leftrightarrow \text{seed} \subset T\big(N, i\,h\big) \right\}\,.$$

We stress the fact that the event $\mathcal{A}_i$ differs from $\mathcal{C}_K(2N, ih)$ since the attachment site of a seed must be at distance less than N from the axis $\{x_1 = 0, \dots, x_{d-1} = 0\}$, nevertheless the open path leading to this seed is allowed to use any bonds within $\mathbb{B}(2N, i\,h)$. If $\varphi'(K,N) = (n+1)h(K,N)$ then $\mathcal{A}_{n+1} \subset \mathcal{C}_K(2N, \varphi')$.

Conditioning wrt π_n the bond configuration below the height $nh(K,N)$, we get

$$\begin{aligned}
\Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,L'_N)}\left(\mathcal{A}_{n+1}\right) &\geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,L'_N)}\left(\mathcal{A}_n \cap \mathcal{A}_{n+1}\right)\\
&\geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,L'_N)}\Big(\mathcal{A}_n\ \Phi^{\pi_n,\mathrm{f}}_{\mathbb{B}^*(2N,L'_N)}(\mathcal{A}_{n+1})\Big)\\
&\geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(2N,L'_N)}\Big(\mathcal{A}_n\ \Phi^{\pi_n,\mathrm{f}}_{\mathbb{B}^*(2N,H_N)}\left(\left\{b_K^d(y^{(n)}) \leftrightarrow \text{seed} \subset T(N,(n+1)h)\right\}\right)\Big)\,,
\end{aligned}$$

where $y^{(n)}$ is defined for each π_n as the attachment site of a seed in $T(N, nh)$ (if there are many such sites choose the smallest one wrt the lexicographic order). The box $y^{(n)} + \mathbb{B}(N, H_N)$ is included in $\mathbb{B}(2N, H_N)$ and with high probability $b_K^d(y^{(n)})$ is connected to a seed in $y^{(n)} + T(N,h)$. Nevertheless, piling up the boxes in an arbitrary way would give little control on the location of the boxes and therefore the connections might not remain in the box $\mathbb{B}(2N, L'_N)$. In order to ensure that this seed is in $T(N,(n+1)h)$ one needs to choose a proper subfacet according to the **steering rule** described below.

We write $y^{(n)} = \{y_i^{(n)}\}_{i \leqslant d}$. Choose $j(\pi_n)$ such that the site $\{-y_1^{(n)}, \dots, -y_{d-1}^{(n)}, h\}$ belongs to $T_{j(\pi_n)}(N, h)$. By construction $y^{(n)} + T_{j(\pi_n)}(N, h) \subset T(N, (n+1)h)$.

$$\Phi^{\pi_n, \mathrm{f}}_{\mathbb{B}^*(2N, H_N)} \left(\left\{ b^d_K(y^{(n)}) \leftrightarrow \text{seed} \subset T(N, (n+1)h) \right\} \right) \geqslant \Phi^{\pi_n, \mathrm{f}}_{\mathbb{B}^*(2N, H_N)} \left(\mathcal{C}_K^{j(\pi_n)}(N, h) \right) ,$$

where the event $\mathcal{C}_K^{j(\pi_n)}(N, h)$ is a translate of the event introduced in Definition 3.3 and is supported by the domain $y^{(n)} + \mathbb{B}^*(N, H_N)$. As $\mathcal{C}_K^j(N, h)$ is an increasing event, we derive a lower bound uniformly over the configurations π_n in $\mathcal{A}_n$

$$\Phi^{\pi_n, \mathrm{f}}_{\mathbb{B}^*(2N, H_N)} \left(\mathcal{C}_K^{j(\pi_n)}(2N, h) \right) \geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(N, H_N)} \left(\mathcal{C}^1_K(N, h) \right) \geqslant 1 - \exp\left(- C K^{d-1} \right) ,$$

where the last estimate follows from Proposition 3.5.

Proceeding recursively, and applying the steering rule at each step we obtain

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N, L_N)} \Big(\mathcal{C}_K(N, \varphi) \Big) \geqslant 1 - \eta \, ,$$

with $L_N = L'_{N/2}$ and $\varphi(K, N) = \varphi'(K, N/2)$. Using the same strategy as in Proposition 3.5, the position of the seeds can be controlled. This completes the proof.

□

The steering rule was introduced in [BGN]. It will be also extremely useful to perform the dynamic renormalization in a two-dimensional slab (see Section 5).

Remark 4.1. *The previous Theorem enables us to generalize inequality* (3.18) *to the domain* $\mathbb{B}(N, L_N)$. *For a given* $\eta > 0$, *choose* K *and* $\varphi(K, N)$ *according to Theorem 4.1. Then, for any* $m > 0$, *there exists* $M' = M'(\eta, m)$ *such that for any* N *large enough, one can find a height* $\hat{\varphi} = \hat{\varphi}(\eta, m, K, N) \in [\varphi(K, N) - M', \varphi(K, N)]$ *so that*

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N, L_N)} \big(Y(N, \hat{\varphi}) > m \big) \geqslant 1 - 2\eta \, . \tag{4.2}$$

The derivation of this inequality follows the scheme of the proof of (3.18).

4.2. **Occupied blocks.** Theorem 4.1 implies that a connection occurs with high probability from a small region around the origin to a height almost at the top of a box $\mathbb{B}(N, L_N)$. To perform the dynamic renormalization, connections from the bottom to the lateral sides of a box will be necessary. This will require different arguments.

We state now the counterpart of Definition 3.3 for the lateral connections.

Definition 4.1.

(1) *For any integers* (ℓ, h), *denote by* $\hat{\mathcal{C}}_K(\ell, h)$ *the event that* $b^d_K(0)$ *is connected by open bonds strictly within* $\mathbb{B}(\ell, h)$ *to a seed lying entirely in* $S(\ell, h)$ *(the union of the sides). Depending on the orientation of the side, the corresponding seed will be of the type* $b^j_K(\cdot)$ *for some appropriate* j.

(2) *We also define the collection of events* $\hat{\mathcal{C}}^i_K(\ell, h)$ *for all* $i \leqslant (d-1)2^{d-1}$. *Any bond configuration in* $\hat{\mathcal{C}}^i_K(\ell, h)$ *is such that* $b^d_K(0)$ *is connected strictly within* $\mathbb{B}(\ell, h)$ *to a seed included in* $S(\ell, h)$ *and centered at a site in* $S_i(\ell, h)$.

(3) *A block $\mathbb{B}(\ell,h)$ is said* **occupied** *if the bond configuration within this block belongs to the intersection of all the sets $\mathcal{C}_K^i(\ell,h)$ and $\hat{\mathcal{C}}_K^j(\ell,h)$.*

The main step will be to prove that there is a class of blocks which are **occupied** with high probability.

Theorem 4.2. *Fix $\eta > 0$, then there exists $K = K(\eta)$ such that for N large enough one can find $M(K)$, $h = h(K,N) \in [L'_N - M(K), L'_N]$ and $\ell = \ell(K,N) \leqslant h(K,N)$ so that*

$$\Phi^{p,\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\bigcap_i \mathcal{C}_K^i(\ell,h)\ \bigcap_j \hat{\mathcal{C}}_K^j(\ell,h)\right) \geqslant 1-\eta\,,$$

with $L'_N = L'_N(K) \geqslant 3N$.

The Theorem is a consequence of Propositions 4.1 and 4.2 which are derived in the following Subsections.

4.2.1. *Lateral connections.* Recall that $Y(\ell,h)$ denotes the number of sites in $T(\ell,h)$ which are connected strictly within $\mathbb{B}(\ell,h)$ to $b_K^d(0)$ by open bonds. We also define $X(\ell,h)$ as the number of sites in the sides $S(\ell,h)$ which are connected strictly within $\mathbb{B}(\ell,h)$ to $b_K^d(0)$ by a path of open bonds.

Proposition 4.1. *Fix $\eta > 0$, then there exists $K = K(\eta)$ such that for N large enough one can find $\varphi^1 = \varphi^1(K,N)$, $\ell^1 = \ell^1(K,N)$ and $L'_N \geqslant 3N$ so that*

$$\forall j \leqslant (d-1)2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\big(\hat{\mathcal{C}}_K^j(\ell^1,\varphi^1)\big) \geqslant 1-\varepsilon_1(\eta)\,,$$

where $\varepsilon_1(\cdot)$ converges to 0 as η tends to 0.

Proof.
We start by fixing the parameters according to η. Remark 4.1 implies that there exists $K = K(\eta)$ and a sequence L_n, such that for any $m > 0$ one can find $M = M(\eta,m,K)$ and $\varphi_n = \varphi_n(\eta,m,K) \in [L_n - M, L_n]$ such that for any n large enough,

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(n,\varphi_n) \geqslant m\big) \geqslant 1 - \frac{1}{2}(\eta^2)^{2^{d-1}} \gg 1-\eta^2\,. \tag{4.3}$$

Finally, let $m = m(\eta,K)$ be such that (see (3.19))

$$\big(1 - c_p^{K^{d-1}}\big)^{m/(4K)^{d-1}} < \frac{1}{2}(\eta^2)^{2^{d-1}}\,.$$

As m has been chosen large enough, (4.3) implies that uniformly over n large enough (see Propositions 3.4 and 3.5)

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(\mathcal{C}_K^i(n,\varphi_n)\big) \geqslant 1-\eta^2\,. \tag{4.4}$$

For any n large enough, there exists ℓ_n such that

$$\begin{cases} \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n - 1,\varphi_n) \leqslant m-1\big) > \eta\,, \\ \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n,\varphi_n) \geqslant m\big) \geqslant 1-\eta\,. \end{cases} \tag{4.5}$$

To see that (4.5) is well defined, first notice that

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(Y(\ell-1,\varphi_n)\leqslant m-1\Big)=1-\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(Y(\ell-1,\varphi_n)\geqslant m\Big)\,.$$

Then it is enough to observe that the function $\ell\to\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell,\varphi_n)\geqslant m\big)$ is non-decreasing and

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(Y(1,\varphi_n)\geqslant m\Big)=0\quad\text{and}\quad\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(Y(n,\varphi_n)\geqslant m\Big)\geqslant 1-\eta\,.$$

Thus one defines ℓ_n as the smallest value for which (4.5) holds.

The proof will be decomposed into 3 steps.

Step 1.

In this first step, we show that lateral connections occur in a domain smaller than $\mathbb{B}(\ell_n,\varphi_n)$.

Lemma 4.1. *There exists an integer $U=U(\eta)$ such that for any n large enough, one can find $\ell_n'\in[\ell_n-1-U,\ell_n-1]$ and*

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(\hat{\mathcal{C}}_K(\ell_n',\varphi_n)\Big)\geqslant 1-2\eta\,,$$

where K and φ_n were chosen in (4.3) *and ℓ_n was determined in* (4.5)*.*

Proof.

We first remark that for ℓ_n defined as above then

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\Big(X(\ell_n-1,\varphi_n)=0\Big)\leqslant\eta\,. \tag{4.6}$$

To see this, we apply (4.3) and obtain

$$\begin{aligned}\eta^2&\geqslant\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(n,\varphi_n)<m\big)\\&\geqslant\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n-1,\varphi_n)\leqslant m-1,\ X(\ell_n-1,\varphi_n)=0\big)\\&\geqslant\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n-1,\varphi_n)\leqslant m-1\big)\ \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(X(\ell_n-1,\varphi_n)=0\big)\,,\end{aligned}$$

where the last inequality follows from the fact that both events in the RHS are decreasing. Thus inequality (4.5) implies (4.6).

Let us also check that ℓ_n diverges with n. First notice that there exists $c_p>0$ such that

$$\forall k\leqslant\varphi_n,\qquad\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n,k)=0\,\big|\,Y(\ell_n,k-1)\geqslant 1\big)\geqslant c_p^{\ell_n}\,, \tag{4.7}$$

where $c_p^{\ell_n}$ is simply the cost of closing all the bonds linking the level $k-1$ to the level k.

$$1-\eta\leqslant\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(Y(\ell_n,\varphi_n)\geqslant m\big)\leqslant\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(\forall k\leqslant\varphi_n,\quad Y(\ell_n,k)\geqslant 1\big)\,.$$

In order to estimate the RHS, we follow the strategy used in Proposition 3.4. We proceed recursively by conditioning at each level and using (4.7).

$$1-\eta\leqslant(1-c_p^{\ell_n})^{\varphi_n}\leqslant\exp(-c_p^{\ell_n}\,n)\,,$$

where we used that $\varphi_n \geqslant 3n$. This implies that for n large enough

$$\ell_n \geqslant \frac{1}{\log(1/c_p)} \log n \,. \tag{4.8}$$

The derivation of Lemma 4.1 follows closely the one of Proposition 3.4. Let u, U be two integers which are going to be fixed later. Define recursively the sequence of random lengths by $\mathcal{L}_0 = \ell_n - 1 - U$ and

$$\mathcal{L}_{i+1} = \inf \left\{ l > \mathcal{L}_i, \quad 1 \leqslant X(l, \varphi_n) \leqslant u \right\} \ \wedge \ (\ell_n - 1) \,.$$

Notice that for n large enough $\mathcal{L}_0$ is positive (see (4.8)).

As in (3.16), we get

$$\forall i \leqslant U, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \Big(\mathcal{L}_i < \ell_n - 1 \Big) \leqslant r_p(u)^{i-1} \,,$$

where $r_p(u) < 1$ is related to the probabilistic cost of closing at most u open bonds (see (3.16)).

Fix $u = u(\eta)$ large enough such that $\big(1 - c_p^{K^{d-1}}\big)^{u/(4K)^{d-1}} < \eta/3$ (see (3.19)). Then choose $v = v(\eta, u)$ such that $r_p(u)^{v-1} < \eta/3$. Following the derivation of (3.17), we get

$$\begin{aligned} \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \left(\sum_{l=\ell_n-1-U}^{\ell_n-1} 1_{\{X(l,\varphi_n) \leqslant u\}} \geqslant v \right) \quad &\leqslant \quad \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \left(X(\ell_n - 1, \varphi_n) = 0 \right) \\ &\qquad + \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \left(\mathcal{L}_v < \ell_n - 1 \right) \leqslant \frac{4}{3}\eta \,, \end{aligned}$$

where we used (4.6).

Finally as in the derivation of (3.18), one can find $U(\eta, u)$ large enough such that uniformly in n there exists $\ell'_n \in [\ell_n - 1 - U, \ell_n - 1]$

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \Big(X(\ell'_n, \varphi_n) \geqslant u \Big) \geqslant 1 - \frac{5}{3}\eta \,.$$

Therefore the probability that a seed lies on top of one of the u attachment sites is large. This concludes Lemma 4.1.

□

Lemma 4.2. *There exists an integer $U = U(\eta)$ such that for any n large enough, one can find $\ell'_n \in [\ell_n - 1 - U, \ell_n - 1]$ and*

$$\forall i \leqslant (d-1)2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)} \Big(\hat{\mathcal{C}}^i_K(\ell'_n, \varphi_n) \Big) \geqslant 1 - (2\eta)^{\frac{1}{(d-1)2^{d-1}}} \,, \tag{4.9}$$

where K and φ_n were chosen in (4.3) and ℓ_n was determined in (4.5).

The derivation is similar to the one of Proposition 3.5.

Step 2. At this stage, we know that with high probability $b^d_K(0)$ is connected to a seed in each side of the box $\mathbb{B}(\ell'_n, \varphi_n)$ with $\ell'_n \in [\ell_n - 1 - U, \ell_n - 1]$. As $\mathbb{B}(\ell'_n, \varphi_n)$ is contained in $\mathbb{B}(\ell_n, \varphi_n)$, we do not have any information on the connections from

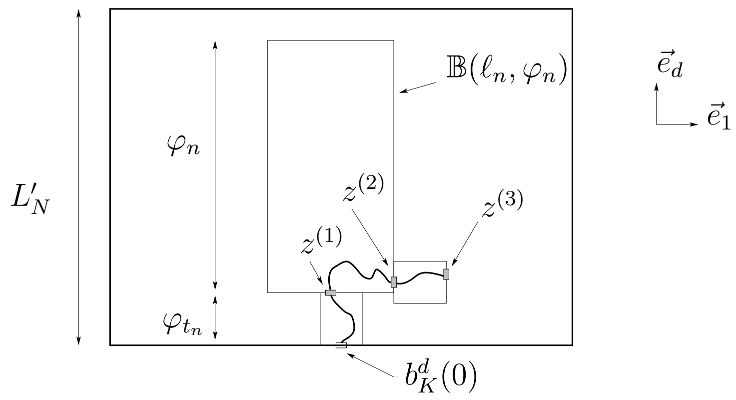

FIGURE 4. The events $\mathcal{E}^1, \mathcal{E}^2$ and $\mathcal{E}^3$ are depicted.

$b^d_K(0)$ to the top face $T(\ell'_n, \varphi_n)$, thus we need to prove that the lateral connections occur as well in a larger box.

We will use n as the reference scale and consider blocks defined in terms of the new parameters (see figure 4)

$$\begin{aligned} t_n &= [\sqrt{\log n}], \quad \varphi^1_n = t_n + \varphi_n + \varphi_{t_n}, \\ N &= n + t_n + L_{t_n}, \quad L'_N = \varphi_n + 4\varphi_{t_n} + L_{t_n}. \end{aligned} \tag{4.10}$$

As $\varphi_n \geqslant 3n$ (see Theorem 4.1), one has $L'_N \geqslant 3N$.

We are going to show that $b^d_K(0)$ is connected with high probability strictly within $\mathbb{B}(\ell'_n + 3t_n, \varphi^1_n)$ to the sides of this block.

Lemma 4.3. *Uniformly over n large enough,*

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\big(X(\ell'_n + 3t_n, \varphi^1_n) \geqslant 1\big) \geqslant 1 - \delta(\eta)\,, \tag{4.11}$$

where $\lim_{\eta\to 0} \delta(\eta) = 0$.

Proof. Let $\mathcal{E}^1 = \mathcal{C}^1_K(t_n, \varphi_{t_n})$ be the event that $b^d_K(0)$ is connected to a seed attached in $T^1(t_n, \varphi_{t_n})$ by a path strictly within $\mathbb{B}(t_n, \varphi_{t_n})$. From (4.4) and the FKG inequality, we have

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\mathcal{E}^1\right) \geqslant 1 - \eta\,. \tag{4.12}$$

For any bond configuration in $\mathcal{E}^1$, we denote by $z^{(1)}$ the center of the seed in $T^1(t_n, \varphi_{t_n})$ with the largest coordinate in the lexicographic order. By construction

$$\forall i \leqslant d-1, \quad 0 \leqslant z^{(1)}_i \leqslant t_n, \quad \text{and} \quad z^{(1)}_d = \varphi_{t_n}\,.$$

The set $\mathcal{E}^1$ can be partitioned into

$$\mathcal{E}^1 = \bigcup_{z^{(1)} \in T_1(t_n, \varphi_{t_n})} \mathcal{E}^1(z^{(1)})\,,$$

where $\mathcal{E}^1(z^{(1)})$ is the set of bond configurations in $\mathcal{E}^1$ for which the seed connected to $b^d_K(0)$ with the largest coordinate is centered in $z^{(1)}$.

Given $z^{(1)}$ in $T^1(t_n, \varphi_{t_n})$, we introduce the event $\mathcal{E}^2(z^{(1)})$ such that $b_K^d(z^{(1)})$ is connected strictly within $z^{(1)} + \mathbb{B}(\ell'_n, \varphi_n)$ to a seed $b_K^1(z^{(2)})$ lying in $z^{(1)} + S_1(\ell'_n, \varphi_n)$ with attachment site $z^{(2)} = (z_i^{(2)})_{i \leqslant d}$,

$$z_1^{(2)} = \ell'_n + z_1^{(1)}, \quad \varphi_{t_n} \leqslant z_d^{(2)} \leqslant \varphi_{t_n} + \varphi_n, \qquad \forall i \in \{2, d-1\}, \quad 0 \leqslant z_i^{(2)} \leqslant t_n + \ell'_n\,.$$

Once again, $z^{(2)}$ is chosen wrt some arbitrary order and therefore is uniquely determined.

Let $\mathcal{F}^{(1)}$ be the σ-algebra generated by the bond configurations in $\mathbb{B}(t_n, \varphi_{t_n})$. Thanks to Lemma 4.2, we have for any $z^{(1)}$,

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\mathcal{E}^2(z^{(1)}) \,\middle|\, \mathcal{F}^{(1)}\right) \geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\left(\hat{\mathcal{C}}^1_K(\ell'_n, \varphi_n)\right) \geqslant 1 - (2\eta)^{\frac{1}{(d-1)2^{d-1}}}\,, \tag{4.13}$$

where we used the FKG property and the fact that for any $z^{(1)} \in T^1(t_n, \varphi_{t_n})$ then $z^{(1)} + \mathbb{B}(n, L_n)$ remains in $\mathbb{B}(N, L'_N)$.

Similarly, $\mathcal{E}^2(z^{(1)})$ can be partitioned into

$$\mathcal{E}^2(z^{(1)}) = \bigcup_{z^{(2)} \in \{z^{(1)} + S_1(\ell'_n, \varphi_n)\}} \mathcal{E}^2(z^{(1)}, z^{(2)})\,.$$

For a given $z^{(2)}$, let $\mathcal{F}^{z^{(2)}}$ be the σ-algebra generated by the bond configurations in the half plane $\{x, \quad x_1 \leqslant z_1^{(2)}\}$. Define the event $\mathcal{E}_3(z^{(2)})$ that $b_K^1(z^{(2)})$ is connected to a site $z^{(3)}$ which satisfies

$$\begin{cases} z_1^{(3)} = z_1^{(2)} + \varphi_{t_n} \geqslant \ell'_n + 3t_n\,, \\ z_i^{(2)} - t_n \leqslant z_i^{(3)} \leqslant z_i^{(2)} + t_n, \qquad \forall i \in \{2, d-1\}\,, \\ 0 \leqslant z_d^{(3)} \leqslant \varphi_{t_n} + \varphi_n + t_n\,, \end{cases}$$

furthermore the connection must occur strictly within the set $\{z_1^{(2)}, z_1^{(2)} + \varphi_{t_n}\} \times \{-t_n, \ell'_n + 2t_n\}^{d-2} \times \{0, \varphi_n + \varphi_{t_n} + t_n\}$. In the previous computation we used the fact that $\varphi_n \geqslant 3n$ (see Theorem 4.1).

We are going to check that (4.11) is implied by the following inequality

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\mathcal{E}^3(z^{(2)}) \,\middle|\, \mathcal{F}^{z^{(2)}}\right) \geqslant 1 - \eta\,. \tag{4.14}$$

By construction

$$\begin{aligned} &\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\big(X(\ell'_n + 3t_n, \varphi_n^1) \geqslant 1\big) \\ &\quad \geqslant \sum_{z^{(1)} \in T_1(t_n, \varphi_{t_n})} \sum_{z^{(2)} \in \{z^{(1)} + S_1(\ell'_n, \varphi_n)\}} \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\mathcal{E}^1(z^{(1)}) \cap \mathcal{E}^2(z^{(1)}, z^{(2)}) \cap \mathcal{E}^3(z^{(2)})\right)\,, \end{aligned}$$

where the events in the RHS are disjoint by construction. Applying (4.14) we obtain

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\big(X(\ell'_n + 3t_n, \varphi_n^1) \geqslant 1\big) \geqslant \sum_{z^{(1)} \in T_1(t_n, \varphi_{t_n})} \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\left(\mathcal{E}^1(z^{(1)}) \cap \mathcal{E}^2(z^{(1)})\right)(1 - \eta)\,.$$

Using (4.13) and then (4.12), Lemma 4.3 is satisfied with

$$\delta(\eta) = 1 - \left(1 - (2\eta)^{\frac{1}{(d-1)2^{d-1}}}\right)(1 - \eta)^2 \leqslant (2\eta)^{\frac{1}{(d-1)2^{d-1}}} + 2\eta\,.$$

It remains to derive (4.14) by piling a block oriented in the direction $\vec{e}_1$. We introduce $\mathbb{B}^{(1)}(L,H) = \{0,H\} \times \{-L,L\}^{d-1}$ and its top face is $T^{(1)}(L,H) = \{H\} \times \{-L,L\}^{d-1}$. According to (4.8), $\ell'_n \gg t_n$ for n large enough. Thus $z^{(2)} + \mathbb{B}^{(1)}(t_n, L_{t_n})$ is included in $\mathbb{B}(N, L'_N)$ but does not intersect $\mathbb{B}(t_n, \varphi_{t_n})$. Inequality (4.4) implies that with probability larger than $1-\eta$, the seed $b^1_K(z^{(2)})$ is connected within $z^{(2)} + \mathbb{B}^{(1)}(t_n, \varphi_{t_n})$ to a seed in $z^{(2)} + T^{(1)}(t_n, \varphi_{t_n})$ attached at the site $z^{(3)}$.

□

Step 3.
Lemma 4.3 implies that with high probability, $b^d_K(0)$ is connected to $S(\ell'_n + 3t_n, \varphi^1_n)$ within $\mathbb{B}(\ell'_n + 3t_n, \varphi^1_n)$. Thus we can proceed as in the first step and derive on a larger domain a result similar to (4.9). There exists an integer $U = U(\eta)$ such that for any n large enough, one can find ℓ^1_n in $[\ell'_n + 3t_n - U, \ell'_n + 3t_n]$ such that

$$\forall j \leqslant (d-1)2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\Big(\hat{\mathcal{C}}^j_K(\ell^1_n, \varphi^1_n)\Big) \geqslant 1 - \big(2\delta(\eta)\big)^{\frac{1}{(d-1)2^{d-1}}}. \tag{4.15}$$

This completes the proof of Proposition 4.1 with $\varepsilon_1(\eta)$ defined by the RHS of the previous inequality.

□

4.2.2. *Top connections.* For any η, the parameters are fixed according to Proposition 4.1. In particular, K was introduced in (4.3), φ^1_n, N in (4.10) and ℓ^1_n in (4.15).

Proposition 4.2. *There exists $\ell^2 = \ell^2(K,N)$ and $\varphi^2 = \varphi^2(K,N) \in [L'_N - M(K), L'_N]$ such that*

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(N,L'_N)}\big(\mathcal{C}^i_K(\ell^2, \varphi^2)\big) \geqslant 1 - \varepsilon_2(\eta),$$

where $\varepsilon_2(\cdot)$ converges to 0 as η tends to 0. By construction, $\ell^2(K,N) \leqslant \ell^1(K,N)$ and $\varphi^2(K,N) \geqslant \varphi^1(K,N)$.

Proof. Let ℓ_n be defined according to (4.5). Proceeding as in the derivation of (4.3) and since m has been chosen large enough, we have

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(n,L_n)}\big(\mathcal{C}_K(\ell_n, \varphi_n)\big) \geqslant 1 - 2\eta. \tag{4.16}$$

We recall that (4.3) also implies for n large enough that

$$\forall i \leqslant 2^{d-1}, \qquad \Phi^{\mathrm{f}}_{\mathbb{B}^*(t_n,\varphi_{t_n})}\big(\mathcal{C}^i_K(t_n, \varphi_{t_n})\big) \geqslant 1 - \eta^2. \tag{4.17}$$

As in (4.10), the scaling parameter is n

$$\ell^2(K,N) = \ell^2_n = \ell_n + t_n, \qquad \varphi^2(K,N) = \varphi^2_n = 5\varphi_{t_n} + \varphi_n. \tag{4.18}$$

According to Theorem 4.1, $\varphi_{t_n} \geqslant 3t_n$, thus for n large enough

$$\ell^1_n = \ell_n - 1 - 2U + 3t_n \geqslant \ell^2_n, \qquad \varphi^1_n = t_n + \varphi_n + \varphi_{t_n} \leqslant \varphi^2_n.$$

Finally, we recall that $L'_N = \varphi_n + 4\varphi_{t_n} + L_{t_n}$ (see (4.10)) so that $0 \leqslant L'_N - \varphi^2_n \leqslant M(K)$.

In order to connect $b^d_K(0)$ to the height φ^2_n, it is enough to link $b^d_K(0)$ to a seed in $T(\ell_n, \varphi_n)$ and then to join it to the height φ^2_n by piling up 5 blocks of type $\mathbb{B}(t_n, \varphi_{t_n})$.

As in the derivation of Theorem 4.1, the steering rule will be necessary to control the position of the seeds. This ensures that the connection from $b_K^d(0)$ to $T(\ell_n^2, \varphi_n^2)$ occurs strictly within $\mathbb{B}(\ell_n^2, \varphi_n^2)$.

Combining inequalities (4.16) and (4.17), we derive the Proposition with

$$\varepsilon_2(\eta) = 1 - (1 - 2\eta)\big(1 - \eta^2\big)^5 \leqslant 2\eta + 5\eta^2 .$$

□

4.2.3. *Proof of Theorem 4.2.* Setting $\ell(K,N) = \ell^1(K,N)$ and $h(K,N) = \varphi^2(K,N)$, the Theorem 4.2 follows by combining the FKG inequality and Propositions 4.1 and 4.2.

5. Slab percolation: second renormalization step

Using the renormalized blocks, we will prove that percolation occurs in a slab with positive probability as soon as $p \in \Theta_q$. For this, we follow the cluster-growth algorithm introduced in [BGN] and prove that on a coarse grained scale large enough, the renormalized process dominates a supercritical two-dimensional Bernoulli process. This will be enough to ensure percolation in a slab.

Most of the work was done in the previous Sections to establish uniform estimates on the occupied blocks. As a consequence, this final step is very similar to the one devised for independent percolation [BGN]. With some respect, it is even simpler since the shape of the blocks (ratio height/width) is under control.

Let us fix $p \in \Theta_q$ and summarize the result of Theorem 4.2. For any $\eta > 0$, there exists K, (L,H) and (ℓ,h) such that

$$\Phi^{\mathrm{f}}_{\mathbb{B}^*(L,H)}\big(\mathbb{B}^*(\ell,h) \;\text{ is occupied}\big) \geqslant 1 - \eta , \tag{5.1}$$

where K characterizes the size of the seeds and the parameters are chosen such that

$$H \geqslant 3L, \qquad 0 \leqslant H - h \leqslant H/100 .$$

At this stage the dependency on N is no longer relevant.

We sketch below the main steps of the proof and detail only the new features.

5.1. **Reduction to two dimensions.** The strategy will be to create new connections by stacking rotated translates of the block $\mathbb{B}(\ell,h)$. In the derivation of Theorem 4.1, we already encountered two basic features: **top stacking** and **steering**. We will also present a third one: **branching**. This will enables us to define a new dependent percolation process restricted to the slab $\mathcal{S}_L = \{-2L, \dots, 2L\}^{d-2} \times \mathbb{Z}^2$.

Steering is necessary to control the deviation of the renormalized paths. By choosing carefully on which particular subfacet of a block another block is attached, we may localize a trajectory. In particular, the first $(d-2)$ coordinates happen to be irrelevant because any stacking in a direction $\vec{e_i}$ (for $i \leqslant d-2$) can be centered along the i-axis (we refer to [BGN] for a complete explanation). We stress the fact that by steering the sequence of boxes $\mathbb{B}(\ell,h)$ will remain inside $\mathcal{S}_{2\ell} = \{-2\ell, \dots, 2\ell\}^{d-2} \times \mathbb{Z}^2$, nevertheless to evaluate the probability of an occupied block one needs to average over a bigger block $\mathbb{B}(L, H)$ (see (5.1)). This explain why we have to consider the renormalized process in the thicker slab $\mathcal{S}_{L+\ell} = \{-L-\ell, \dots, L+\ell\}^{d-2} \times \mathbb{Z}^2$.

Let us now introduce some notation to emphasize the role of the plane $(\vec{e}_{d-1}, \vec{e}_d)$ which we shall refer later on as the (x, y)-plane. In order to describe the geometrical construction in the (x, y)-plane, it is convenient to rename the subfacets of the blocks. The direction $\vec{e}_d$ will be referred as **North**, $-\vec{e}_d$ as **South**, $\vec{e}_{d-1}$ as **East** and $-\vec{e}_{d-1}$ as **West**. In particular $\mathbb{B}(\ell, h)$ will be dubbed north block and denoted by $\mathbb{B}_{\text{North}}(\ell, h)$. Its north face $T_{\text{North}}(\ell, h) = T(\ell, h)$ is split into

$$\begin{aligned} T_{\text{North,East}} &= \{-\ell, \ell\}^{d-2} \times \{0, \ell\} \times \{H\}\,, \\ T_{\text{North,West}} &= \{-\ell, \ell\}^{d-2} \times \{-\ell, 0\} \times \{H\}\,. \end{aligned}$$

We also distinguish the western and eastern sides

$$\begin{aligned} S_{\text{North,East}} &= \{-\ell, \ell\}^{d-2} \times \{\ell\} \times \{0, H\}\,, \\ S_{\text{North,West}} &= \{-\ell, \ell\}^{d-2} \times \{-\ell\} \times \{0, H\}\,. \end{aligned}$$

Rotating $\mathbb{B}_{\text{North}}$ leads to define blocks oriented in the other 3 directions $\mathbb{B}_{\text{East}}, \mathbb{B}_{\text{South}}$ and $\mathbb{B}_{\text{West}}$. Their faces will be named by transposing the previous notation.

We can now describe the main ingredients to concatenate paths from different occupied blocks. Starting with an occupied north block, **top stacking** enables to pile up n occupied blocks in the north direction with a probability of success at least $(1-\eta)^n$. As this procedure is doomed to fail sooner or later, a branching procedure will be necessary to allow percolation in the other directions.

An occupied block contains attachment sites on each of its faces, so that several blocks can be stacked on it. A **branching** occurs when two blocks are stacked simultaneously on the top and on one side of an occupied block. These blocks should be positioned in a careful way to remain essentially independent. For example, starting with the occupied block $\mathbb{B}_{\text{North}}(\ell, h)$, a block $y + \mathbb{B}_{\text{North}}(\ell, h)$ can be stacked on a seed centered in $y \in T_{\text{North,East}}(\ell, h)$ and a west block $z + \mathbb{B}_{\text{West}}(\ell, h)$ on a seed in $S_{\text{North,West}}$ (see figure 5). Alternatively, exchanging East and West would lead to a branching in the North/East directions. By construction, the event that these new blocks are occupied, is supported by the disjoint set of bonds $y + \mathbb{B}^*_{\text{North}}(\ell, h)$ and $z + \mathbb{B}^*_{\text{West}}(\ell, h)$. Nevertheless to evaluate the probability of a block to be occupied requires averaging over a larger domain $\mathbb{B}_\bullet(L, H)$ (see (5.1)). This raises some measurability issues which will be detailed in the example below.

We are going to evaluate the probability that the following sequence of occupied blocks occurs (see figure 5). The derivation will be similar to the one of Lemma 4.3. In the procedure below, the steering in the first $(d-2)$ coordinates is implicit.

First $b^d_K(0)$ is connected strictly within $\mathbb{B}_{\text{North}}(\ell, h)$ to a seed centered at the site $y^{(1)}$ which belongs to $T_{\text{North,West}}(\ell, h)$. The site $y^{(1)}$ is uniquely determined and is chosen as the first site (wrt to the lexicographic order) for which the previous event holds. This event is denoted by $\mathcal{E}^1(y^{(1)})$. Thanks to the key estimate (5.1), we have

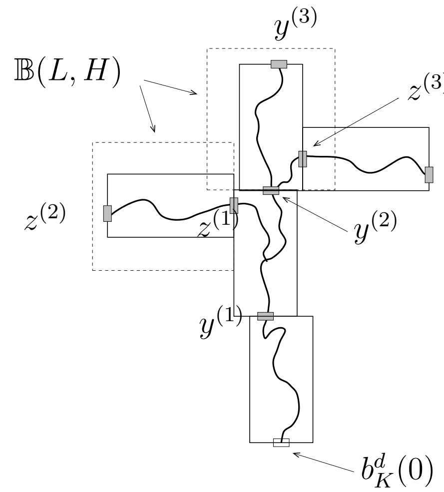

FIGURE 5. A branching sequence in the North/West directions. The domains $y^{(2)} + \mathbb{B}_{\text{North}}(L,H)$ and $z^{(1)} + \mathbb{B}_{\text{West}}(L,H)$ are depicted by the dashed blocks.

for any domain Λ containing $\mathbb{B}(L,H)$

$$\sum_{y^{(1)}} \Phi_\Lambda^{\mathrm{f}}\left(\mathcal{E}^1(y^{(1)})\right) = \Phi_\Lambda^{\mathrm{f}}\left(\mathcal{C}_K^1(\ell,h)\right) \geqslant \Phi^{\mathrm{f}}_{\mathbb{B}^*(L,H)}\big(\mathbb{B}^*(\ell,h) \;\text{ is occupied}\big) \geqslant 1-\eta\,, \quad (5.2)$$

where the sum over $y^{(1)}$ is restricted to $T_{\text{North,West}}(\ell,h)$.

Then a branching occurs within $y^{(1)} + \mathbb{B}_{\text{North}}(\ell,h)$, i.e. $b_K^d(y^{(1)})$ is connected to a seed $b_K^d(y^{(2)})$ centered in $y^{(2)} \in y^{(1)} + T_{\text{North,East}}(\ell,h)$ and $b_K^d(y^{(1)})$ is also connected to a seed $b_K^{d-1}(z^{(1)})$ centered in $z^{(1)} \in y^{(1)} + S_{\text{North,West}}(\ell,h)$. This event is denoted by $\mathcal{E}^2(y^{(1)},y^{(2)},z^{(1)})$. As the attachment sites $y^{(2)}, z^{(1)}$ are uniquely determined, the previous events are disjoint for different attachment sites.

Finally, the connections evolve in the West and North directions: $b_K^{d-1}(z^{(1)})$ is connected to a seed $b_K^{d-1}(z^{(2)})$ in $z^{(1)} + T_{\text{West}}(\ell,h)$ and $b_K^d(y^{(2)})$ is connected to a seed in $y^{(2)} + T_{\text{North,East}}(\ell,h)$ or possibly to a seed in $y^{(2)} + S_{\text{North,East}}(\ell,h)$ (on figure 5, a side stacking from $z^{(3)}$ oriented in the East direction is also depicted). We also define

$$\begin{cases} \mathcal{E}^3(y^{(2)}) = \big\{y^{(2)} + \mathbb{B}_{\text{North}}(\ell,h) \text{ is occupied}\big\}\,, \\ \mathcal{E}^4(z^{(1)}) = \big\{z^{(1)} + \mathbb{B}_{\text{West}}(\ell,h) \text{ is occupied}\big\}\,. \end{cases}$$

We are going to check that a branching occurs with high probability

$$\sum_{y^{(1)},y^{(2)},z^{(1)}} \Phi_\Lambda^{\mathrm{f}}\Big(\mathcal{E}^1(y^{(1)}) \cap \mathcal{E}^2(y^{(1)},y^{(2)},z^{(1)}) \cap \mathcal{E}^3(y^{(2)}) \cap \mathcal{E}^4(z^{(1)})\Big) \geqslant (1-\eta)^4\,. \quad (5.3)$$

It is enough to iterate in the right order the key estimate (5.1). Let us fix a triplet $\{y^{(1)},y^{(2)},z^{(1)}\}$ and drop temporarily the dependency on the sites in the events $\mathcal{E}^\bullet$. The event $\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3$ is supported by bond configurations in the hyperplane $\{x_{d-1} \geqslant y^{(1)}_{d-1} - \ell\}$ and the support of $\mathcal{E}^4$ lies in $\{x_{d-1} < y^{(1)}_{d-1} - \ell\}$. Thus conditioning outside $z^{(1)} + \mathbb{B}_{\text{West}}(L,H)$ and using the fact that $\mathcal{E}^4$ is non decreasing, we get

$$\Phi^{\mathrm{f}}_{\Lambda}\left(\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3 \; \Phi^{\pi}_{z^{(1)}+\mathbb{B}_{\text{West}}(L,H)}(\mathcal{E}^4)\right)$$
$$\geqslant \Phi^{\mathrm{f}}_{\Lambda}\left(\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3\right) \Phi^{\mathrm{f}}_{z^{(1)}+\mathbb{B}_{\text{West}}(L,H)}(\mathcal{E}^4) \geqslant \Phi^{\mathrm{f}}_{\Lambda}\left(\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3\right)(1-\eta)\,.$$

At this stage there are no more ambiguities and we estimate the remaining events one after the other (starting from the top) thanks to (5.1). This completes (5.3).

5.2. **Cluster-growth algorithm.** We describe now the second level of renormalization.

As explained in the previous Subsection, it is enough to consider a two-dimensional projection of the system. The projection of the block $\mathbb{B}_{\bullet}(\ell,h)$ onto the plane (x,y) will be called a **brick** and denoted by $\tilde{\mathbb{B}}_{\bullet}(\ell,h)$. The orientation convention applies as well for the bricks. To an occupied block is associated a **successful** brick with four **connection sites** one in each set $\tilde{T}_{\text{North,East}}, \tilde{T}_{\text{North,West}}, \tilde{S}_{\text{North,East}}, \tilde{S}_{\text{North,West}}$ which are the projections of the centers of the seeds. A sequence of bricks build by concatenating occupied blocks according to the stacking procedures previously described will be called a **successful brick sequence**.

Thus there is an immediate correspondence between blocks and bricks and it is enough to build the second level of renormalization in terms of bricks in $\mathbb{Z}^2$. The second renormalization level lives on the coarse grained scale $N = 10L + 10H$. The lattice $\mathbb{Z}^2$ is partitioned into translates of the square $\mathbb{S}_0 = \{-N+1,\dots,N\}^2$ which are indexed wrt an arbitrary order $\{\mathbb{S}_i\}_{i\,\geqslant\,0}$. The algorithm is performed by inspecting the squares one after the other and checking iteratively some properties which will be detailed later on. If these properties are satisfied then the square contains a crossing cluster made of successful bricks. A random variable $Z_i(\omega)$ depending on the bond configuration ω is associated to the square $\mathbb{S}_i$. The square $\mathbb{S}_i$ is declared to be **good** if these properties hold and we set $Z_i = 1$, otherwise $Z_i = 0$. As it will be clear later on, any bond configuration must contain an open cluster intersecting all the good squares.

We explain now the construction of the renormalized process $\{Z_i\}_i$ indexed by the squares in the domain $\{-M,\dots,M\}^2$ for some $M = 2mN$. Let us suppose that k squares $\mathbb{S}_0, \mathbb{S}_{i_1}, \dots, \mathbb{S}_{i_{k-1}}$ have been examined. Choose the next square $\mathbb{S}_{i_k}$ as the earliest square (in the fixed ordering) which has a face in common with one of the previous good squares. If no such square exists then the algorithm stops, otherwise we are going to prove that for a suitable tuning of the coarse grained scales

$$\Phi^{\mathrm{f}}_{\mathcal{S}_{L+\ell,M}}\left(Z_{i_k} = 1 \,\middle|\, Z_{i_{k-1}},\dots,Z_0\right) \geqslant \alpha\,, \tag{5.4}$$

where $\alpha = \alpha(\eta)$ can be chosen arbitrarily close to 1 and in particular much larger than the critical value of the site percolation in $\mathbb{Z}^2$. We stress the fact that α depends only on η, so that all the parameters are determined thanks to (5.1). Inequality (5.4) implies that the cluster formed by the good squares dominates stochastically the open cluster growing from the origin of a supercritical site percolation process. We refer to Grimmett, Marstrand [GM] for a detailed account of the stochastic domination.

Before completing the construction of the renormalized process and deriving (5.4), we pause to look at the proof of Theorem 2.1.

Proof of Theorem 2.1.

Fix $M = 2mN$ and choose a site x in $\mathcal{S}_{L+\ell,M}$. Let $\mathbb{S}$ be the square such that the tube $T_x = \{-L-\ell, L+\ell\}^{d-2} \times \mathbb{S}$ contains x. If $x = 0$, then $T_0 = \{-L-\ell, L+\ell\}^{d-2} \times \mathbb{S}_0$. Define $\mathcal{Z}(0,x)$ as the set of bond configurations such that there exists a renormalized connected path of good squares from $\mathbb{S}_0$ to $\mathbb{S}$. By construction, one has

$$\Phi^{\mathrm{f}}_{\mathcal{S}_{L+\ell,M}}\big(T_0 \leftrightarrow T_x\big) \geqslant \Phi^{\mathrm{f}}_{\mathcal{S}_{L+\ell,M}}\big(\mathcal{Z}(0,x)\big)\,. \tag{5.5}$$

This means that the tubes are connected by an open path if a connection occurs on the coarse grained level. Thanks to the stochastic domination (see (5.4)), one can choose $\alpha = \alpha(\eta)$ such that the RHS is bounded from below uniformly wrt to x and M. Therefore $p \geqslant \hat{p}_c(L+\ell) \geqslant \hat{p}_c$ and Theorem 2.1 holds.

□

We turn now to the derivation of (5.4). As explained previously, the cluster-growth algorithm examines in turn each square: a square $\mathbb{S}$ will be inspected only if there exists a square $\mathbb{S}'$ which shares a common facet with $\mathbb{S}$ and has been previously declared to be good. In this case the state of $\mathbb{S}$ will be determined independently of the bonds lying outside $\mathbb{S} \cup \mathbb{S}'$ (more precisely, only a small portion of $\mathbb{S}'$ will be relevant). The procedure we are going to apply is translational and rotational invariant, thus it is sufficient to describe it in a special case: conditionally to the fact that $\mathbb{S}_0$ is a good square, we would like to determine the state of its northern neighbor $\mathbb{S}_1 = \mathbb{S}_0 + (0, N)$.

We define the **target regions** of $\mathbb{S}_0$ as

$$\mathcal{T}^0_{\mathrm{North}} = \{-N/2, N/2\} \times \{N - 2H\}$$

and the other target regions $\mathcal{T}^0_{\mathrm{East}}, \mathcal{T}^0_{\mathrm{South}}, \mathcal{T}^0_{\mathrm{West}}$ are deduced by rotation. In the same way, the target regions of $\mathbb{S}_1$ are obtained by translation $\mathcal{T}^1_\bullet = \mathcal{T}^0_\bullet + (0, N)$. We will see that a necessary condition for a square to be good is to contain a successful brick $\tilde{\mathbb{B}}_\bullet(\ell, h)$ intersecting the corresponding target region $\mathcal{T}_\bullet$, where $\bullet$ ranges over the 4 directions.

Conditionally to the fact that $\mathbb{S}_0$ is a good square, there exists by definition a successful north brick $\tilde{\mathbb{B}}$ intersecting the target region $\mathcal{T}^0_{\mathrm{North}}$. This brick will be used as a starting point to launch connections to the target regions of $\mathbb{S}_1$. The main ingredients are the centering and the bifurcation rules (see Figure 6).

Centering rules:

First, bricks are piled up on top of $\tilde{\mathbb{B}}$ in the north direction using the steering rule in order to center the brick sequence along the axis $x = 0$. At some point a brick, denoted by $\tilde{\mathbb{B}}_0$ will intersect the level $\{y = N + N/2\}$, this triggers the bifurcation.

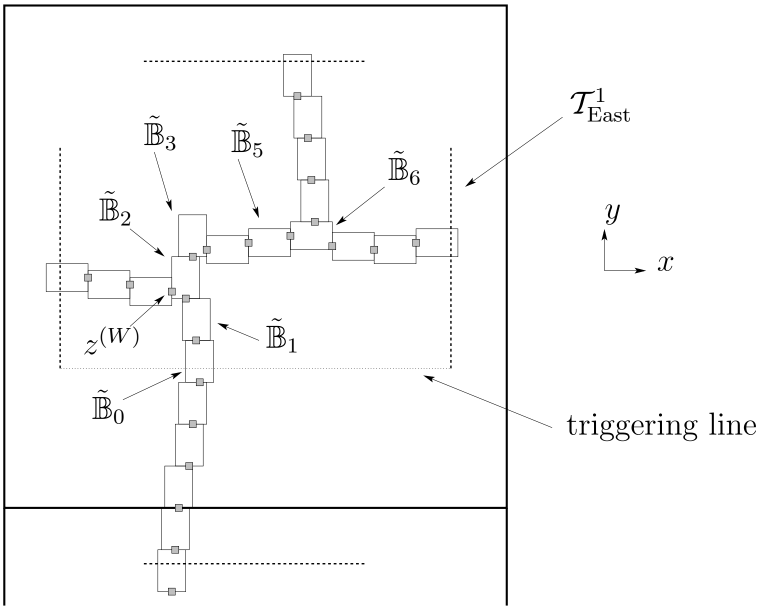

FIGURE 6. A successful sequence of bricks is depicted starting from the northern part of $\mathbb{S}_0$ and invading $\mathbb{S}_1$. The dashed lines are the target regions.

Bifurcation:

On top of $\tilde{\mathbb{B}}_0$ lies a connection site $z^{(1)} = (x_1, y_1)$ (recall that $z^{(1)}$ is simply the image of the center of the seed in the top face of the occupied block associated to $\tilde{\mathbb{B}}_0$). By construction x_1 belongs to $\{-5H - 5L, 5H + 5L\}$ and we also suppose that $x_1 < 0$, the other case can be treated by symmetry.

First a branching occurs in the North/West directions, to do so three bricks are piled up in the north direction.

$$\begin{cases} \tilde{\mathbb{B}}_1 = z^{(1)} + \tilde{\mathbb{B}}_{\text{North}}(\ell, h)\,, \\ \tilde{\mathbb{B}}_2 = z^{(2)} + \tilde{\mathbb{B}}_{\text{North}}(\ell, h) \quad \text{with} \quad z^{(2)} \in z^{(1)} + \tilde{T}_{\text{North,West}}(\ell, h)\,, \\ \tilde{\mathbb{B}}_3 = z^{(3)} + \tilde{\mathbb{B}}_{\text{North}}(\ell, h) \quad \text{with} \quad z^{(3)} \in z^{(2)} + \tilde{T}_{\text{North,East}}(\ell, h)\,. \end{cases}$$

From $\tilde{\mathbb{B}}_2$ a brick sequence branches in the West direction towards the target region $\mathcal{T}^1_{\text{West}}$. We will come back to this later.

Then an east brick $\tilde{\mathbb{B}}_4$ is attached to the east side of $\tilde{\mathbb{B}}_3$ and another branching occurs in the East/North directions: three bricks are piled up in the East direction on top of $\tilde{\mathbb{B}}_4$

$$\begin{cases} \tilde{\mathbb{B}}_4 = z^{(4)} + \tilde{\mathbb{B}}_{\text{East}}(\ell, h) \quad \text{with} \quad z^{(4)} \in S_{\text{North,East}} + z^{(3)}\,, \\ \tilde{\mathbb{B}}_5 = z^{(5)} + \tilde{\mathbb{B}}_{\text{East}}(\ell, h) \quad \text{with} \quad z^{(5)} \in T_{\text{East,North}} + z^{(4)}\,, \\ \tilde{\mathbb{B}}_6 = z^{(6)} + \tilde{\mathbb{B}}_{\text{East}}(\ell, h) \quad \text{with} \quad z^{(6)} \in T_{\text{East,North}} + z^{(5)}\,, \\ \tilde{\mathbb{B}}_7 = z^{(7)} + \tilde{\mathbb{B}}_{\text{East}}(\ell, h) \quad \text{with} \quad z^{(7)} \in T_{\text{East,South}} + z^{(6)}\,. \end{cases}$$

From $\tilde{\mathbb{B}}_6$ a brick sequence branches in the North direction towards the target region $\mathcal{T}^1_{\text{North}}$ and another sequence goes on in the East direction towards $\mathcal{T}^1_{\text{East}}$.

Final connections:

After the bifurcation, the steering rules are applied once again to center along the axis of the target region $\mathcal{T}^1_{\text{West}}$ (resp. North, East) the brick sequence piled up in the West direction (resp. North, East directions). It remains to check that using this construction the sequences reach the target regions and that they do not overlap.

The starting point of the western sequence $z^{(W)} = (x_W, y_W)$ belongs to the western side of $\tilde{\mathbb{B}}_3$. By construction

$$x_W \in [-2\ell, \ell] + x_0, \qquad y_W \in [N + N/2 + H, N + N/2 + 3H]$$

By piling up bricks in the western direction, the brick sequence is localized in the strip $\{-N, x_W\} \times \{y_W - L, L\}$ and therefore it encounters the target region $\mathcal{T}^1_{\text{West}}$ by using less than 20 bricks.

We proceed in the same way for the other directions and see that the sequences evolve in non overlapping regions.

The inspection of a square requires less than 100 bricks, thus the probability for $\mathbb{S}_1$ to be good conditionally to the fact that $\mathbb{S}_0$ is good, is larger than $\alpha(\eta) = (1-\eta)^{100}$. Furthermore, the procedure described above does not depend on the bond configurations outside $\mathbb{S}_0 \cap \mathbb{S}_1$. Thus the stochastic domination inequality (5.4) holds at any step of the cluster algorithm.

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