Phase turbulence in the Complex Ginzburg–Landau equation via Kuramoto–Sivashinsky phase dynamics.

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Abstract

We study the Complex Ginzburg–Landau initial value problem

$$\partial_t u = (1 + i\alpha)\, \partial_x^2 u + u - (1 + i\beta)\, u\, |u|^2\,, \quad u(x, 0) = u_0(x)\,, \tag{CGL}$$

for a complex field $u \in \mathbf{C}$, with $\alpha, \beta \in \mathbf{R}$. We consider the Benjamin–Feir linear instability region $1 + \alpha\beta = -\varepsilon^2$ with $\varepsilon \ll 1$ and $\alpha^2 < 1/2$. We show that for all $\varepsilon \leq \mathcal{O}(\sqrt{1 - 2\alpha^2}\, L_0^{-32/37})$, and for all initial data u_0 sufficiently close to 1 (up to a global phase factor $e^{i\,\phi_0}$, $\phi_0 \in \mathbf{R}$) in the appropriate space, there exists a unique (spatially) periodic solution of space period L_0. These solutions are small *even* perturbations of the traveling wave solution, $u = (1 + \alpha^2\, s)\, e^{i\,\phi_0 - i\beta\, t}\, e^{i\alpha\, \eta}$, and s, η have bounded norms in various L^p and Sobolev spaces. We prove that $s \approx -\frac{1}{2}\, \eta''$ apart from $\mathcal{O}(\varepsilon^2)$ corrections whenever the initial data satisfy this condition, and that in the linear instability range $L_0^{-1} \leq \varepsilon \leq \mathcal{O}(L_0^{-32/37})$, the dynamics is essentially determined by the motion of the phase alone, and so exhibits 'phase turbulence'. Indeed, we prove that the phase η satisfies the Kuramoto–Sivashinsky equation

$$\partial_t \eta = -(\tfrac{1+\alpha^2}{2})\, \triangle^2 \eta - \varepsilon^2 \triangle \eta - (1 + \alpha^2)\, (\eta')^2 \tag{KS}$$

for times $t_0 \leq \mathcal{O}(\varepsilon^{-52/5}\, L_0^{-32/5})$, while the amplitude $1 + \alpha^2\, s$ is essentially constant.

1 Introduction

1.1 Generalities about the Ginzburg–Landau equation

The Complex Ginzburg–Landau equation (CGL) admits explicit traveling wave solutions of the form

$$u(x, t) = c(p)\, \exp\big(i(\phi_0 + p\, x - \omega(p)\, t)\big)\,, \tag{1.1}$$

with $\phi_0 \in \mathbf{R}$, $p \in [-1, 1]$, $c(p) = \sqrt{1 - p^2}$ and $\omega(p) = \alpha\, p^2 + \beta\, (1 - p^2)$. For all α, β with $1 + \alpha\, \beta > 0$, there exists a parameter $p_E = p_E(\alpha, \beta)$, with $p_E \to 0$ as $1 + \alpha\, \beta \to 0^+$ such that traveling wave solutions (1.1) with $|p| \geq p_E(\alpha, \beta)$ are linearly unstable, a phenomenon called 'sideband' or 'Eckhaus' instability, while those with $|p| \leq p_E$ are linearly stable (see e.g. [CH93] and the references therein). When $1 + \alpha\beta < 0$, all traveling wave solutions are linearly unstable, a phenomenon called 'Benjamin–Feir' or 'Benjamin–Feir–Newell' instability (see e.g. [BF67] and [New74]).

In this paper, we consider the case $1 + \alpha\, \beta = -\varepsilon^2$. When ε is small enough, numerical simulations on finite domains (see e.g. [MHAM97] and the references therein) indicate that the dynamics of the phase is turbulent, the phase evolving irregularly, (with fluctuations of order ε^2 around the global phase ϕ_0), while the amplitude of u is constant up to $\mathcal{O}(\varepsilon^4)$ corrections. This type of behavior is called 'phase turbulence'. The persistence of phase turbulence on infinite

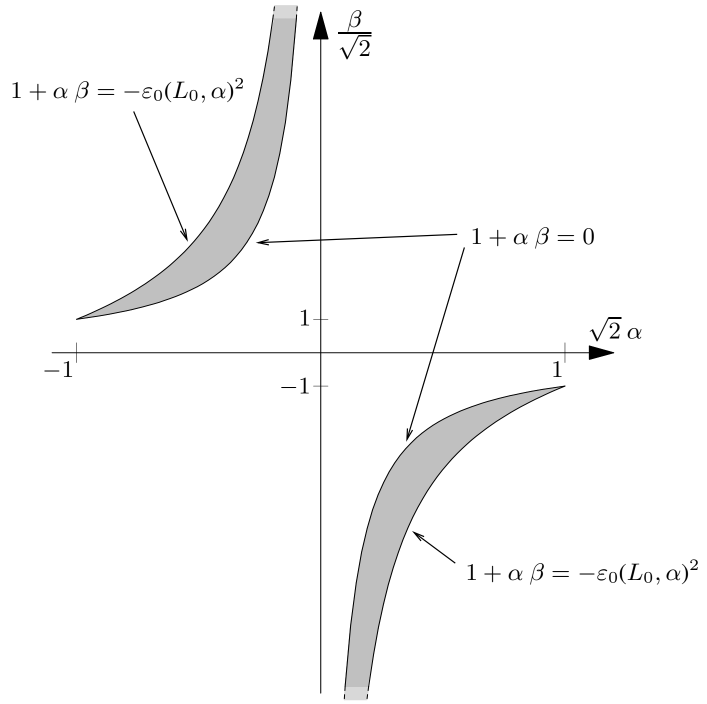

Figure 1: Parameter space for (CGL). Linear instability occurs for $1 + \alpha\,\beta < 0$, and phase turbulence is shown in this paper to occur in shaded region.

domains is not known, while its existence on finite domains is, to our knowledge, not proven rigorously.

As ε increases (or the domain is larger), 'amplitude' or 'defect' turbulence occurs, the amplitude of u vanishing at some instants and places, called 'defects' or 'phase slips' (see also [EGW95]). Note that 'phase' and 'amplitude' turbulence may coexist at the same time in the α, β parameter space, depending on initial conditions, in which case one speaks of 'bichaos'.

The 'amplitude' turbulence regime is technically difficult because the phase is not well defined when the amplitude vanishes. In this paper, we concentrate on the easier phase turbulence regime and prove that for the particular case[1] $p = 0$, phase turbulence occurs for small initial perturbations of the traveling wave $e^{i\,\phi_0 - i\beta\,t}$ on domains of size L_0 for all $\alpha^2 < 1/2$ and for all $\varepsilon \le \varepsilon_0(L_0,\alpha)$ with $\varepsilon_0(L_0,\alpha) \to 0$ as $L_0 \to \infty$ or $\alpha^2 \to 1/2$, see figure 1. We restrict ourselves to *even* perturbations for concision, though general perturbations could be treated as well (see Remark 2.4 below). We believe the restriction $\alpha^2 < 1/2$ to be an artifact of our technical treatment (see the discussion at the end of Section 1.4), though we expect some restriction on the size of α to be necessary anyway, because the large α limit of (CGL) is the so–called Non–linear Schrödinger equation, whose dynamics is completely different from the above picture.

[1]The case $p \neq 0$ should give a similar result but is more challenging.

1.2 Setting

We consider perturbations of $e^{i\,\phi_0 - i\beta\, t}$, (this is a solution of (CGL)) which are of the form[2]

$$u(x,t) = (1 + \alpha^2\; s(x,t))\; e^{i\;\phi_0 - i\beta t}\; e^{i\alpha\;\eta(x,t)}\;, \tag{1.2}$$

for (small) $s, \eta \in \mathbf{R}$. To state our results, we introduce the following scalings[3]

$$\eta(x,t) = \tfrac{1}{4}\,\hat{\varepsilon}^2\;\hat{\eta}(\hat{x},\hat{t})\;, \tag{1.3}$$
$$s(x,t) = \;\hat{\varepsilon}^4\;\hat{s}(\hat{x},\hat{t})\;, \tag{1.4}$$

with $\chi = \frac{4}{1+\alpha^2}$, $\hat{\varepsilon} = \sqrt{\frac{\chi}{2}}\,\varepsilon$, $\hat{x} = \hat{\varepsilon}\;x$ and $\hat{t} = \frac{2}{\chi}\,\hat{\varepsilon}^4\;t$.

We consider the initial value problem (CGL) with $\eta(x,0) = \eta_0(x)$ and $s(x,0) = s_0(x)$, where η_0 and s_0 are *even* periodic functions of period L_0, or equivalently, in terms of the 'hat' variables, $\hat{\eta}_0$ and $\hat{s}_0$ are *even* periodic functions of period $L = \hat{\varepsilon}\;L_0$. To state our conditions on the initial data $\hat{s}_0$ and $\hat{\eta}_0$, we introduce the Banach space $\mathcal{W}_{0,\sigma}$ obtained by completing $\mathcal{C}^\infty_{\text{per}}([-L/2, L/2], \mathbf{R})$ under the norm $\|\cdot\|_\sigma = \|\cdot\|_{\mathrm{L}^2([-L/2,L/2))} + \|\cdot\|_{\mathcal{W},\sigma}$, where $\|\cdot\|_{\mathcal{W},\sigma}$ is a sup norm with algebraic weight (going like $|k|^\sigma$ at infinity) on the Fourier transform, see section 2.3 for details. Essentially $\mathcal{W}_{0,\sigma}$ consists of functions in $\mathrm{L}^2([-L/2, L/2])$, whose Fourier transform decays (at least) like $|k|^{-\sigma}$ as $|k| \to \infty$ (this is a regularity assumption). In the sequel we will often use the shorthand notation L^2 instead of $\mathrm{L}^2([-L/2, L/2])$, while we will always write $\mathrm{L}^2([-L_0/2, L_0/2])$ to avoid confusion.

Definition 1.1 *We say that $\hat{\eta}_0$ and $\hat{s}_0$ are in the class $\mathcal{C}(K, L, \alpha, \hat{\varepsilon}, \hat{\varepsilon}_0, c_{s_0}, c_{\eta_0}, \sigma) \subset \mathcal{W}_{0,\sigma} \times \mathcal{W}_{0,\sigma-1}$ if*

$$\hat{\eta}_0(0) = 0\;, \quad \|\hat{\eta}_0'\|_\sigma \le c_{\eta_0}\;\rho\;, \quad \left\|\hat{s}_0 - \frac{\hat{\varepsilon}^2\;\hat{s}_0''}{2}\right\|_{\sigma-1} \le c_{s_0}\;\rho^3\;, \tag{1.5}$$

for $\rho = K\;L^{8/5}$, and if

$$\left\|\hat{s}_0 - \frac{\hat{\varepsilon}^2\;\hat{s}_0''}{2} + \frac{\hat{\eta}_0''}{8} + \frac{\hat{\varepsilon}^2\;(\hat{\eta}_0')^2}{32}\right\|_{\sigma-1} < 2^{-8}\;\min\Big(\frac{1}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2}\Big)\Big(\frac{\hat{\varepsilon}}{\hat{\varepsilon}_0}\Big)^2\;\hat{\varepsilon}^2\;c_{\eta_0}\;\rho\;. \tag{1.6}$$

The parameter L is the (space) period (in the scaled variables) of the solution. The constant K is essentially the same as that of [CEE93] in their discussion of the Kuramoto–Sivashinsky equation

$$\partial_{\hat{t}}\hat{\eta}_c = -\triangle^2\hat{\eta}_c - \triangle\hat{\eta}_c - \frac{1}{2}(\hat{\eta}_c')^2\;, \tag{1.7}$$

where it appears in the bound $\lim\limits_{t\to\infty}\|\hat{\eta}_c'(\cdot,t)\|_{\mathrm{L}^2} \le K\;L^{8/5}$ for symmetric periodic solutions. Therefore, K is independent of α, ε and L. The parameters α and $\hat{\varepsilon}$ are those of (CGL), with $\hat{\varepsilon}^2 = -2\,\frac{1+\alpha\,\beta}{1+\alpha^2}$, while $\hat{\varepsilon}_0$ is the maximal value of $\hat{\varepsilon}$ for which our results hold. The parameters c_{η_0} and c_{s_0} measure the size of the initial perturbation. Note that only $\hat{\eta}_0'$ and $\hat{\eta}_0(0)$ appear in the

[2]The α factors in front of s and η are only a convenient normalization.

[3]They will be justified in the next subsection.

conditions. We can motivate this by noting that (CGL) has a $U(1)$ symmetry (the global phase factor $e^{i\,\phi_0}$). Expressing all constraints in terms of $\hat{\eta}_0'$ and $\hat{\eta}_0(0)$ is a convenient way to take this invariance into account. The condition $\eta_0(0) = 0$ can always be satisfied, up to a redefinition of the global phase ϕ_0. Furthermore, this condition is preserved by the evolution (see e.g. (1.18)). We will prove that if $\hat{\eta}_0$ and $\hat{s}_0$ are in the class $\mathcal{C}(K, L, \alpha, \hat{\varepsilon}, \hat{\varepsilon}_0, c_{s_0}, c_{\eta_0}, \sigma)$, the (CGL) dynamics (which has a complex function as initial condition) is increasingly well approximated as $\hat{\varepsilon} \to 0$ by the Kuramoto–Sivashinsky dynamics (1.7), which has a real function as initial condition. For this to hold, $\hat{s}_0$ and $\hat{\eta}_0$ have to be tightly related as $\hat{\varepsilon} \to 0$. This relation is quantified by (1.6), which says that, up to $\mathcal{O}(\varepsilon^4)$ corrections, $\hat{s}_0$ and $\hat{\eta}_0$ are related by

$$\hat{s}_0 = -\frac{1}{8}\,\hat{G}\,\hat{\eta}_0'' - \frac{\hat{\varepsilon}^2}{32}\,\hat{G}\,(\hat{\eta}_0')^2\ ,$$

where $\hat{G}$ is the operator with symbol

$$\hat{G}(k) = \frac{1}{1 + \frac{\hat{\varepsilon}^2}{2}\,k^2}\ . \tag{1.8}$$

Note that $\hat{G}$ is the inverse of the (positive) operator $1 - \frac{\hat{\varepsilon}^2}{2}\,\partial_{\hat{x}}^2$.

1.3 Main results and their physical discussion

Our main results are twofold. We first have an existence and unicity result for the solutions of (CGL), see Theorem 1.2 below, and then an approximation result in Theorem 1.3.

From now on, we will denote generic constants by the letters C and c. We will use the letter c with different labels to recall the quantity on which the bound is. By constants, we mean quantities which do not depend on $\alpha, \hat{\varepsilon}, L$ and σ in the ranges

$$0 \le \hat{\varepsilon} \le 1\ ,\quad \alpha^2 < 1/2\ ,\quad L > 2\pi \quad \text{and } \sigma \le \sigma_0$$

for some finite $\sigma_0 > \frac{11}{2}$.

Theorem 1.2 *Let $\alpha^2 < 1/2$, $\sigma > \frac{11}{2}$, $c_{s_0} > 0$, $c_\eta > 0$ and $L > 2\,\pi$. There exist constants K and c_ε such that for all $m_\varepsilon \ge 4$, for any $\hat{\varepsilon} \le \hat{\varepsilon}_0 = c_\varepsilon\,\sqrt{1-2\alpha^2}\,\rho^{-m_\varepsilon}$ and for all $\hat{\eta}_0$ and $\hat{s}_0$ in the class $\mathcal{C}(K, L, \alpha, \hat{\varepsilon}, \hat{\varepsilon}_0, c_{s_0}, c_{\eta_0}, \sigma)$, the solution of (CGL) with parameters α and $\beta = -\frac{2+(1+\alpha^2)\,\hat{\varepsilon}^2}{2\,\alpha}$ exists for all times, is of the form (1.2) and satisfies*

$$\sup_{\hat{t}\ge 0} \|\hat{\eta}(\cdot,\hat{t})'\|_\sigma \le c_\eta\,\rho\ ,\quad \sup_{\hat{t}\ge 0} \|\hat{s}(\cdot,\hat{t})\|_{\sigma-1} \le c_s\,\rho^3, \tag{1.9}$$

$$\sup_{\hat{t}\ge 0} \left\| \hat{s}(\cdot,\hat{t}) + \frac{1}{8}\,\hat{G}\,\hat{\eta}''(\cdot,\hat{t}) + \frac{\hat{\varepsilon}^2}{32}\,\hat{G}\,(\hat{\eta}')^2(\cdot,\hat{t}) \right\|_{\mathrm{L}^2} \le \left(\frac{\hat{\varepsilon}}{\hat{\varepsilon}_0}\right)^2 c_\eta\,\rho\ , \tag{1.10}$$

with $\rho = K\,L^{8/5}$, $c_\eta > 1 + c_{\eta_0}$ and $c_s > c_{s_0}$. This solution is unique among functions satisfying (1.9).

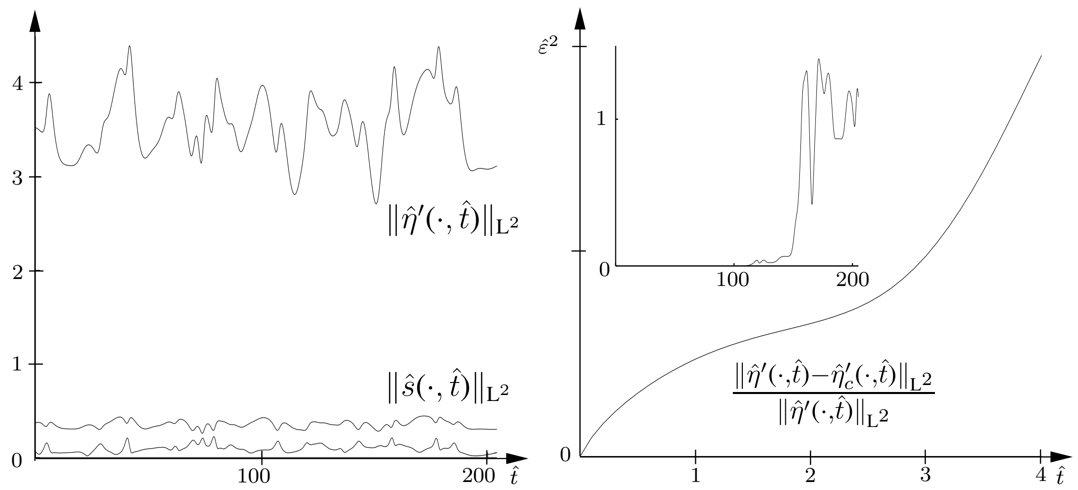

Figure 2: Numerical results for $\hat{\varepsilon} = 10^{-3}$, $\alpha = 10^{-2}$ and $L_0 = 10^4 \cdot 2\pi$.

Our results are valid for any $\hat{\varepsilon} \leq \hat{\varepsilon}_0 = c_\varepsilon \sqrt{1-2\alpha^2}\ \rho^{-4}$ and for any $L > 2\ \pi$. Since $L = \hat{\varepsilon}\ L_0$ and $\rho = K\ L^{8/5}$, we see that the applicability range is

$$C\ L_0^{-1} \leq \varepsilon \leq C' \sqrt{1-2\alpha^2}\ L_0^{-32/37}\ .$$

The lower bound is the linear instability condition.

In terms of the original variables, Theorem 1.2 shows that solutions of (CGL) of the form (1.2) exist, and that (see Appendix G for details)

$$\sup_{t\geq 0} \|\eta(\cdot,t)'\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \leq C\ \varepsilon^{5/2-1/m_\varepsilon}\ , \tag{1.11}$$

$$\sup_{t\geq 0} \|s(\cdot,t)\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \leq C\ \varepsilon^{7/2-3/m_\varepsilon}\ , \tag{1.12}$$

$$\sup_{t\geq 0} \sup_{x\in[-L_0/2,L_0/2]} |\eta(x,t)| \leq C\ \varepsilon^{2-13/(8\ m_\varepsilon)}\ , \tag{1.13}$$

$$\sup_{t\geq 0} \sup_{x\in[-L_0/2,L_0/2]} |s(x,t)| \leq C\ \varepsilon^{4-4/m_\varepsilon}\ . \tag{1.14}$$

The inequalities (1.13) and (1.14) quantify the 'physical intuition' $\eta = \mathcal{O}(\varepsilon^2)$ and $s = \mathcal{O}(\varepsilon^4)$, see section 1.1.

Inequalities (1.9) or (1.11)–(1.14) also show that the solutions belongs to a (local) attractor, while (1.10) shows that on that attractor, the 'amplitude' s satisfies $s = -\frac{1}{2}\ \eta'' + \mathcal{O}(\varepsilon^2)$. The attractor is thus well approximated by the graph $s = -\frac{1}{2}\ \eta''$ in the s, η space. This result was discovered at a heuristic level by Kuramoto and Tsuzuki in [KT76].

We do not expect the bounds (1.9) and (1.10) to be optimal. Numerical simulations show that $\hat{\eta}'$ and $\hat{s}$ are uniformly bounded in space and time, at least for a large range of $L = \varepsilon\ L_0$. This suggests that $\|\hat{\eta}'\|_{\mathrm{L}^2}$ and $\|\hat{s}\|_{\mathrm{L}^2}$ should both scale with L like $\sqrt{L}$ and not like $L^{8/5}$ and $L^{24/5}$, hence we should have $\rho \sim \sqrt{L}$. In the left panel of Figure 2, we display as a function of $\hat{t} \in [0, 200]$ (by decreasing size) the typical behavior of $\|\hat{\eta}'(\cdot,\hat{t})\|_{\mathrm{L}^2}$, $\|\hat{s}(\cdot,\hat{t})\|_{\mathrm{L}^2}$ and $\hat{\varepsilon}^{-4}\|\hat{s}(\cdot,\hat{t})+$

$\frac{1}{8}\,\hat{G}\,\hat{\eta}''(\cdot,\hat{t}) + \frac{\hat{\varepsilon}^2}{32}\,\hat{G}\,(\hat{\eta}')^2(\cdot,\hat{t})\|_{\mathrm{L}^2}$ in units proportional to $\sqrt{\varepsilon L_0}$. We also see that $\hat{s} + \frac{1}{8}\,\hat{G}\,\hat{\eta}'' + \frac{\hat{\varepsilon}^2}{32}\,\hat{G}\,(\hat{\eta}')^2$ is of order $\hat{\varepsilon}^4$ and not $\hat{\varepsilon}^2$ as in (1.10).

We now show that the dynamics of the phase on the attractor is well approximated by the Kuramoto–Sivashinsky equation.

Theorem 1.3 *Under the assumptions of Theorem 1.2, there exists a constant c_t such that if $\hat{t}_1 \le c_t\,\rho^{-4}$, then for all $\hat{t}_0 \ge 0$,*

$$\sup_{0\le\hat{t}\le\hat{t}_1} \|\hat{\eta}(\cdot,\hat{t}_0+\hat{t})' - \hat{\eta}_c(\cdot,\hat{t})'\|_{\mathrm{L}^2} \le \left(\frac{\hat{\varepsilon}}{\hat{\varepsilon}_1}\right) c_\eta\,\rho\,, \tag{1.15}$$

where $\hat{\eta}_c$ satisfies the Kuramoto–Sivashinsky equation (1.7) with initial condition $\hat{\eta}_c(\hat{x},0) = \hat{\eta}(\hat{x},\hat{t}_0)$.

In physical terms, Theorem 1.3 says that on each time interval $[t_0, t_0+t_1]$, the distance between η and the solution of the Kuramoto–Sivashinsky equation with initial condition $\eta(t_0)$ is small compared to the size of the attractor (see (1.15)), at least for time intervals of length t_1 of order $\varepsilon^{-4}\,\rho^{-4} = \varepsilon^{-52/5}\,L_0^{-32/5}$. This result gives a rigorous foundation to the heuristic derivation in [KT76]) of the Kuramoto–Sivashinsky equation as a phase equation for the Complex Ginzburg–Landau equation near the Benjamin–Feir line (see also [Man90]). Furthermore, if ε is sufficiently small, the amplitude $1+\alpha^2\,s$ does not vanish by (1.14). This proves that the solution exhibits phase turbulence for all times, the solutions of the Kuramoto–Sivashinsky equation being believed to be chaotic.

The bound (1.15) for $\hat{t}_1 \le c_t\,\rho^{-4}$ is again certainly not optimal. Numerical simulations show that $\hat{t}_1$ scales like L^{-2} (this is in agreement with $\rho \sim \sqrt{L}$) with a bound of order $\hat{\varepsilon}^2$ and not $\hat{\varepsilon}$ as in (1.15). In the right panel of Figure 2, we show in the large plot $\frac{\|\hat{\eta}'(\cdot,\hat{t})-\hat{\eta}_c'(\cdot,\hat{t})\|_{\mathrm{L}^2}}{\|\hat{\eta}'(\cdot,\hat{t})\|_{\mathrm{L}^2}}$ in units of $\hat{\varepsilon}^2$ for short times (large times are displayed in small inserted plot in absolute units).

In the remainder of this section, we derive the dynamical equations for $\hat{s}$ and $\hat{\eta}$, then we discuss informally these equations to motivate the analytical treatment that we will present in the next sections. In particular, we will explain the particular choice of the scalings (1.3) and (1.4). We will treat the phase dynamical equation in section 2, while the treatment of the dynamical equation for s is postponed to section 3, s being 'slaved' to η by that equation.

1.4 Derivation of the amplitude and phase equations

The ansatz (1.2) leads, after separation of the real and imaginary parts of equation (CGL), to

$$\partial_t s = s'' - 2\,s - \eta'' - (\eta')^2 - \alpha^2\,\left(3\,s^2 + 2\,s'\,\eta' + s\,\eta'' + s\,(\eta')^2 + \alpha^2\,s^3\right)\,, \tag{1.16}$$

$$\partial_t\eta = \eta'' + \alpha^2\,s'' - 2\,\alpha\,\beta\,s - \alpha^2\,\left((\eta')^2 + \alpha\,\beta\,s^2 - \frac{2s'\,\eta'}{1+\alpha^2\,s} + \frac{\alpha^2\,s\,s''}{1+\alpha^2\,s}\right)\,. \tag{1.17}$$

Since these equations preserve the subspace of functions that are *even* in the space variable, we restrict ourselves to that particular case. We also use $\alpha, -\frac{1+\varepsilon^2}{\alpha}$ as parameters instead of α, β as it allows to emphasize the dependence on the small parameter ε. Finally, as the right hand sides of (1.16) and (1.17) contain only (space) derivatives of the function η, we introduce the odd function μ (the phase derivative) by

$$\eta(x,t) = \int_0^x \mathrm{d}y \; \mu(y,t) \;, \tag{1.18}$$

and obtain

$$\begin{aligned} \partial_t s &= s'' - 2\,s - \mu' - \mu^2 \\ &\quad - \alpha^2 \left(3\,s^2 + 2\,s'\,\mu + s\,\mu' + s\,\mu^2 + \alpha^2\,s^3\right) \;, \end{aligned} \tag{1.19}$$

$$\begin{aligned} \partial_t \mu &= \mu'' + \alpha^2\,s''' + 2\,(1+\varepsilon^2)\,s' - \alpha^2\,(\mu^2)' \\ &\quad + \alpha^2 \left((1+\varepsilon^2)\,s^2 + \frac{2s'\,\mu}{1+\alpha^2\,s} - \frac{\alpha^2\,s\,s''}{1+\alpha^2\,s}\right)' \;. \end{aligned} \tag{1.20}$$

We expect $\partial_t s, s' \ll s, \mu' \ll \mu \ll 1$ when $\varepsilon \ll 1$. We then have

$$\partial_t s = s'' - 2\,s - \mu' - \mu^2 + f_s(s,\mu) \;, \tag{1.21}$$

$$\begin{aligned} \partial_t \mu &= -\left(s'' - 2\,s - \mu' - \mu^2\right)' + (1+\alpha^2)\,s''' + 2\,\varepsilon^2\,s' \\ &\quad - 2\,(1+\alpha^2)\,\mu\,\mu' + f_\mu(s,\mu)' \;, \end{aligned} \tag{1.22}$$

where $f_s(s,\mu)$, respectively $f_\mu(s,\mu)$, is defined as the function appearing in the second line of (1.19) resp. (1.20). The $-2s$ term in (1.21) strongly damps s, which therefore is 'slaved' to μ. Indeed, as we will show in Section 3, for given μ satisfying appropriate bounds, the map $\mu \mapsto s(\mu)$ defined by the (global and strong) solution of (1.21) is well defined and Lipschitz in μ. Furthermore, to third order in ε, the map is given by the solution s_1 of $s_1'' - 2\,s_1 - \mu' - \mu^2 = 0$, which can be represented as

$$s_1(\mu) = -\frac{1}{2}\,G\,\left(\mu' + \mu^2\right) \;, \tag{1.23}$$

where G is the convolution operator with the fundamental solution $\mathcal{G}$ of $\mathcal{G}(x) - \frac{1}{2}\mathcal{G}''(x) = \delta(x)$. Note that G acts multiplicatively in Fourier space, with symbol $(1+\frac{k^2}{2})^{-1}$, in particular, $G\,f$ has two more derivatives than f. As we will also show in Section 3, $s(\mu)$ will have the same structure as $s_1(\mu)$, that is, the $\mathcal{G}$–convolution of another map with the same regularity as μ'. As such, $s(\mu)$ is once more differentiable than μ, due to the regularizing properties of G, and $s(\mu) = s(\eta')$ is as regular as η. This is reasonable, since from $u = (1+\alpha^2\,s)\,e^{-i\beta\,t + i\alpha\eta}$ we see that s and η should have both the same degree of regularity as u.

Inserting (1.23) into (1.22) and neglecting f_μ leads to the (modified) Kuramoto–Siva-shinsky equation for the phase

$$\partial_t \mu = -\tfrac{1+\alpha^2}{2}\,G\,\mu'''' - \varepsilon^2\,G\,\mu'' - 2(1+\alpha^2)\,\mu\mu' - \varepsilon^2\,G\,(\mu^2)' - \tfrac{1+\alpha^2}{2}\,G\,(\mu^2)''' \;, \tag{1.24}$$

from which we recover the Benjamin–Feir linear instability criterion $1+\alpha\beta < 0$. Namely, linear stability analysis in Fourier space (set $\mu = \varepsilon_0\,e^{ikx+\lambda(k)t}$ with $\varepsilon_0 \ll 1$) gives the dispersion relation

$$\lambda(k) = \frac{\varepsilon^2\,k^2 - k^4\,(\frac{1+\alpha^2}{2})}{1+\frac{k^2}{2}} = \frac{-(1+\alpha\beta)\,k^2 - k^4\,(\frac{1+\alpha^2}{2})}{1+\frac{k^2}{2}} \;.$$

This shows that there are linearly unstable modes for $|k| \leq \varepsilon \ll 1$, growing at most like $e^{\varepsilon^4 t}$. This suggests that the dynamics of (1.24) should be dominated by the dynamics of the Fourier modes in the small $|k|$ region, the high $|k|$ modes being slaved to them. For $|k| \ll 1$, we have $G \approx 1$, and neglecting the last two terms of (1.24), we get the Kuramoto–Sivashinsky equation in derivative form

$$\partial_t \mu \approx -\tfrac{1+\alpha^2}{2}\, \mu'''' - \varepsilon^2 \mu'' - 2(1+\alpha^2)\, \mu\mu' \ . \tag{1.25}$$

Defining

$$\mu(x,t) = \tfrac{1}{4}\, \hat{\varepsilon}^3\, \hat{\mu}(\hat{x},\hat{t}) \ , \tag{1.26}$$

with

$$\chi = \frac{4}{1+\alpha^2} \ , \ \ \hat{\varepsilon} = \sqrt{\tfrac{\chi}{2}}\, \varepsilon \ , \ \ \hat{x} = \hat{\varepsilon}\, x \ , \ \ \hat{t} = \tfrac{2}{\chi}\, \hat{\varepsilon}^4\, t \ ,$$

we get from (1.25)

$$\partial_t \hat{\mu} = -\hat{\mu}'''' - \hat{\mu}'' - \hat{\mu}\, \hat{\mu}' \ , \tag{1.27}$$

which is the original Kuramoto–Sivashinsky equation in derivative form. This justifies the scalings (1.3). Equation (1.27) possesses an universal attractor of finite radius in $\mathrm{L}^2([-L/2, L/2])$ with periodic boundary conditions (see e.g. [CEE93]), hence we can expect $\hat{\mu}$ to be of size ε^3 times a typical solution in that attractor.

From (1.23), we get (the μ–dependence of s_1 is implicit here for concision)

$$s_1(x,t) = -\frac{\hat{\varepsilon}^4}{32}\, \hat{G}\, \left(4\hat{\mu}'(\hat{x},\hat{t}) + \hat{\varepsilon}^2\, \hat{\mu}(\hat{x},\hat{t})^2\right) \equiv \hat{\varepsilon}^4\, \hat{s}_1(\hat{x},\hat{t}) \ , \tag{1.28}$$

where $\hat{G}$ is the convolution operator with the fundamental solution of $\hat{\mathcal{G}}(x) - \frac{\varepsilon^2}{2}\hat{\mathcal{G}}''(x) = \delta(x)$. As above, $\hat{G}$ acts multiplicatively in Fourier space, with symbol $\hat{G}(k) = (1 + \frac{\varepsilon^2\, k^2}{2})^{-1}$. Equation (1.28) motivates the scalings

$$s(x,t) = \hat{\varepsilon}^4\, \hat{s}(\hat{x},\hat{t}) \ , \tag{1.29}$$

for s (see also (1.4)).

We now apply (1.26) and (1.29) to (1.21) and (1.22). From now on we drop the hats. Then s and μ satisfy the following equations

$$\partial_t s = -\frac{\chi}{\varepsilon^4}\, \mathcal{L}_s\, s + \frac{\chi}{\varepsilon^4}\, r_1(\mu) - \tfrac{\alpha^2\, \chi}{8}(2s'\mu + s\mu') + F_3(s,\mu) \ , \tag{1.30}$$

$$\partial_t \mu = -\mathcal{L}_\mu\, \mu - \mu\mu' + \varepsilon^2\, F_0(s,\mu)' + \varepsilon^2\, \chi\, \mathcal{L}_{\mu,r}\, r_2' \ , \tag{1.31}$$

where $\mathcal{L}_s$, $\mathcal{L}_\mu$ and $\mathcal{L}_{\mu,r}$ are multiplicative operators in Fourier space, with symbols given by

$$\mathcal{L}_s(k) = 1 + \frac{\varepsilon^2\, k^2}{2} \ ,$$

$$\mathcal{L}_\mu(k) = \frac{k^4 - k^2}{1 + \frac{\varepsilon^2\, k^2}{2}}\,,$$
$$\mathcal{L}_{\mu,r}(k) = 2\,\frac{2 + \varepsilon^2\,(1+\alpha^2) - \alpha^2\,\varepsilon^2\,k^2}{1 + \frac{\varepsilon^2\, k^2}{2}}\,, \tag{1.32}$$

while r_1, r_2, F_3 and F_0 are defined by

$$\begin{aligned}
r_1(\mu) &= -\frac{1}{32}(4\mu' + \varepsilon^2\,\mu^2)\,,\\
r_2 &= \frac{1}{\varepsilon^4}\left(s - \frac{\varepsilon^2}{2}\,s''\right) - \frac{r_1(\mu)}{\varepsilon^4}\\
F_3(s,\mu) &= -\alpha^2\,\chi\,\left(\tfrac{3}{2}\,s^2 + \varepsilon^2\,\tfrac{1}{32}\,s\,\mu^2 + \tfrac{\alpha^2}{2}\,\varepsilon^4\,s^3\right)\,,\\
F_0(s,\mu) &= \alpha^2\,\chi\left((2 + \varepsilon^2\,(1+\alpha^2))\,s^2 + \frac{s'\,\mu}{1+\varepsilon^4\,\alpha^2\,s} - \frac{2\,\varepsilon^2\,\alpha^2\,s\,s''}{1+\varepsilon^4\,\alpha^2\,s}\right)\\
&\quad - \frac{1}{4}\,G\,\mu^2 - \frac{1}{4}\,G\,(\mu^2)''\,,
\end{aligned}$$

where G is the operator with symbol

$$G(k) = \frac{1}{1 + \frac{\varepsilon^2\,k^2}{2}}\,.$$

We will prove that (1.30) defines a map $\mu \mapsto s(\mu)$ for all μ in an open ball of $\mathcal{W}_\sigma$, and that this map has indeed 'the same properties' as $G\,r_1(\mu)$, e.g. in terms of regularity. This is so essentially because for $\varepsilon \ll 1$, we have $\frac{\chi}{\varepsilon^4}\mathcal{L}_s \gg 1$, so that by Duhamel's formula, $s \sim \mathcal{L}_s^{-1}\,r_1(\mu) + \mathcal{O}(\varepsilon^4) = G\,r_1(\mu) + \mathcal{O}(\varepsilon^4)$ (see Section 3). At the same time, as a dynamical variable, r_2 satisfies

$$\partial_t r_2 = -\frac{\chi}{\varepsilon^4}\,G\,\mathcal{L}_r\,r_2 + \frac{\chi}{16}\mu\,\mathcal{L}_{\mu,r}\,r_2' + \frac{1}{\varepsilon^4}\,F_6(s,\mu)\,, \tag{1.33}$$

where $\mathcal{L}_r$ is the multiplicative operator in Fourier space with symbol

$$\mathcal{L}_r(k) = 1 + \left(\tfrac{3}{2} + \varepsilon^2(\tfrac{1+\alpha^2}{4})\right)\,\varepsilon^2\,k^2 + \left(\tfrac{1-\alpha^2}{4}\right)\,\varepsilon^4\,k^4\,,$$

and

$$\begin{aligned}
F_6(s,\mu) &= \mathcal{L}_s\,\Big(F_3(s,\mu) + F_4(s,\mu)\Big) + F_7(s,\mu) + F_8(\mu)\,,\\
F_4(s,\mu) &= -\tfrac{\alpha^2\,\chi}{8}(2s'\mu + s\mu')\,,\\
F_7(s,\mu) &= \frac{\varepsilon^2}{8}\,(\partial_x + \frac{\varepsilon^2\,\mu}{2})\Big(F_0(s,\mu)'\Big)\,,\\
F_8(s,\mu) &= -\frac{1}{8}\,(\partial_x + \frac{\varepsilon^2\,\mu}{2})\Big(\mathcal{L}_\mu\,\mu + \mu\,\mu'\Big)\,.
\end{aligned}$$

Once s is considered as a given map $\mu \mapsto s(\mu)$, (1.33) defines the map $\mu \mapsto r_2(\mu)$ through a *linear* equation for r_2. By the same mechanism as for s, we have $r_2 \sim (G\,\mathcal{L}_r)^{-1}\,F_6(s,\mu) \sim$

$G\ F_6(s,\mu)$ if $\alpha^2 < 1$ (see Section 4). The restriction $\alpha^2 < 1$ is necessary here to make $\mathcal{L}_r$ positive definite. For technical reasons, we have in fact to restrict $\alpha^2 < 1/2$ to prove Theorems 1.2 and 1.3. We believe that the results of these Theorems could be extended to part of the $\alpha^2 > 1/2$ region by exploiting the following argument. If $\alpha^2 > 1$, equation (1.33) for r_2 is linearly unstable at high frequencies. However, the linear coupling of r_2 to μ through (1.31) stabilizes r_2. To see this, we introduce the vector $\mathbf{v} = (\mu, r_2)$, and consider (1.31) and (1.33) simultaneously, as a vector dynamical system of the form

$$\partial_t \mathbf{v} = \mathcal{L}_M \mathbf{v} + f(\mathbf{v})\ , \tag{1.34}$$

for a (nonlinear) vector map f, where $\mathcal{L}_M$ is the operator with (matrix) symbol

$$\mathcal{L}_M(k) = \begin{pmatrix} -\mathcal{L}_\mu(k) & \varepsilon^2\ \chi\ \mathcal{L}_{\mu,r}(k)\ ik \\ -\frac{1}{8\ \varepsilon^4}\ \mathcal{L}_\mu(k)\ ik & -\frac{\chi}{\varepsilon^4} G\mathcal{L}_r(k) \end{pmatrix} .$$

The stability of (1.34) at high frequency is then determined by the eigenvalues $\lambda_\pm(k)$ of $\mathcal{L}_M(k)$ for large k. Since[4]

$$\lambda_\pm(k) \to -(1 \pm i|\alpha|)\ \frac{k^2}{\hat{\varepsilon}^2}$$

as $k \to \infty$, (1.34) is stable at high frequency, the real part of the eigenvalues $\lambda_\pm(k)$ of $\mathcal{L}(k)$ being negative for large k. However to exploit this would force us to solve (1.31) and (1.33) simultaneously, which is technically (and notationally) more difficult, see [GvB02] for a similar problem. Instead, in our approach the system (1.30), (1.31) and (1.33) is considered as a 'main' equation, (1.31), of the form

$$\partial_t \mu = -\mathcal{L}_\mu\ \mu - \mu\mu' + \varepsilon^2\ F(\mu)'\ , \tag{1.35}$$

supplied with two 'auxiliary' equations, (1.30) and (1.33), which can be solved independently.

We will first study (1.35) for a general class of map $F(\mu)$ in Section 2 below, because it explains the choice of the functional space, and which properties of the solutions of the amplitude equations (1.30) and (1.33) are needed. Then, in Sections 3 and 4, we will show that the solutions of the amplitude equations (1.30) and (1.33) exist and satisfy the 'right' properties.

2 The phase equation

2.1 Strategy

Having argued that $r_2 = r_2(\mu)$, we rewrite (1.31) as

$$\partial_t \mu = -\mathcal{L}_\mu\ \mu - \mu\mu' + \varepsilon^2\ F(\mu)'\ , \qquad \mu(x,0) = \mu_0(x)\ , \tag{2.1}$$

where μ_0 is a given (odd) space periodic function of period L for some given L. Since (2.1) preserves the mean of μ over $[-L/2, L/2]$, and since μ_0 is the space derivative of a space periodic function, we restrict ourselves to μ_0 which have zero mean over $[-L/2, L/2]$.

[4] This is the analogon of $(1 + i\alpha)\ u''$ in (CGL).

We will show that the term $\varepsilon^2\ F(\mu)'$ is in some sense negligible. If $\varepsilon = 0$, then $\mathcal{L}_\mu = \partial_x^4 + \partial_x^2 \equiv \mathcal{L}_{\mu,c}$, and (2.1) is the Kuramoto–Sivashinsky equation. If $F = 0$ and $\varepsilon > 0$, (in this case, $\mathcal{L}_\mu$ is of smaller order than $\partial_x^4 + \partial_x^2$), this situation can still be easily handled by the techniques of [CEE93] or [NST85], which show that equation (2.1) possesses a universal attractor of finite radius in $\mathrm{L}^2([-L/2, L/2])$ if $F = 0$. A key ingredient of that proof is the observation that the trilinear form $\int \mathrm{d}x\ \mu^2\mu'$ vanishes for periodic functions. However, in general, $\varepsilon^2 \int \mathrm{d}x\ \mu\ F(\mu)'$ will not vanish, and might even not exist at all for $\mu \in \mathrm{L}^2$.

We will explain precisely below how we circumvent this, but the mechanism is indeed quite simple. If the n–th Fourier coefficients of μ were vanishing for all $n \geq \frac{\delta}{q}$ with $1 \ll \delta \ll 1/\varepsilon$, we would have e.g. $\|\mu''\|_{\mathrm{L}^2} \leq \delta^2\ \|\mu\|_{\mathrm{L}^2}$, which would (presumably) give $\varepsilon^2 \int \mathrm{d}x\ \mu\ F(\mu)' \sim \varepsilon^2\delta^2\ \|\mu\|_{\mathrm{L}^2}^2$. For ε sufficiently small, this would only give a small blur to the attractor of the true Kuramoto–Sivashinsky equation.

Evidently, we cannot expect the high–n Fourier modes to vanish, so we will have to treat them separately. On that matter, we want to point out that contrary to the 'true' Kuramoto–Sivashinsky equation (1.27), where the linear operator $\mathcal{L}_{\mu,c}$ acting on μ on the r.h.s. is of fourth order, $\mathcal{L}_\mu$ is only of second order due to the regularizing properties of G. From the point of view of derivatives of μ, it is easy to see that $\varepsilon^2\ \hat{s}_1'$ and $\varepsilon^2\ \hat{s}_1''$ contain at most first derivatives of μ, hence we expect $\varepsilon^2\ F(\mu)'$ to contain at most second order derivatives of μ, and we see that at high frequencies, (1.35) is more similar to the well studied equation $\dot{u} = u'' + f(u, u', u'')$ (see e.g. [BKL94]) than to the Kuramoto–Sivashinsky equation.

Note that the term $F(\mu)'$ is 'irrelevant' due to its prefactor ε^2, while $\mu\ \mu'$ is certainly not. Indeed, it would be catastrophic to solve (2.1) by successive approximations, beginning with the solution of the equation with $-\mu\mu' + \varepsilon^2\ F(\mu) = 0$, inserting that solution into the nonlinear terms and solving again the linear inhomogeneous problem. This would lead to (apparently) exponentially growing modes, because the linear operator $\mathcal{L}_\mu$ is not positive definite at small frequencies. Solving (2.1) iteratively as

$$\partial_t\mu_{n+1} = -\mathcal{L}_\mu\ \mu_{n+1} - \mu_{n+1}\mu_{n+1}' + \varepsilon^2\ F(\mu_n)'\ ,$$

for $n \geq 0$ is a much better choice. We therefore consider the following class of equations

$$\partial_t\mu = -\mathcal{L}_\mu\ \mu - \mu\mu' + \varepsilon^2\ g'\ , \qquad \mu(x,0) = \mu_0(x)\ , \tag{2.2}$$

for some given time dependent and spatially periodic perturbation g and periodic initial data μ_0.

From this (informal) discussion, we see that we should treat the small n Fourier coefficients with an L^2–like norm as in [CEE93] or [NST85], and the high n modes as in e.g. [BKL94]. In the next three subsections, we implement this idea. We first show L^2 estimates for (2.2) in Subsection 2.2. Then in Subsection 2.3 we define functional spaces similar to those of [BKL94], and prove inequalities in these spaces, which will allow us to prove the 'high frequency estimates' in Subsection 2.4. In subsection 2.5, we will prove that the full phase equation has a solution if $\mu \mapsto F(\mu)$ is a well behaved Lipschitz map, and finally, in subsection 2.6, we will show how the phase equation relates to the Kuramoto–Sivashinsky equation.

2.2 Coercive functional method, L^2 estimates

The initial value problem (2.2) is globally well posed in $\mathrm{L}^2([-L/2, L/2])$ if the perturbation g is periodic and satisfies $\|g(\cdot,t)\|_{\mathrm{L}^2} < \infty$ for all $t \geq 0$. The local uniqueness/existence theory follows from standard techniques (see e.g. [Tem97]), whereas the global existence follows from the a priori estimate

$$\|\mu(\cdot,t)\|_{\mathrm{L}^2}^2 \le e^t \,\|\mu(\cdot,0)\|_{\mathrm{L}^2}^2 + 2\,\varepsilon^4\,\left(e^t-1\right)\sup_{0\le s\le t}\|g(\cdot,s)\|_{\mathrm{L}^2}^2\,. \tag{2.3}$$

This estimate can be obtained by multiplying (2.2) with μ and averaging over $[-L/2, L/2]$. Denoting by $\int$ the integral over a (space) period, we have, using integration by parts

$$\frac{1}{2}\partial_t \int \mu^2 = -\int \mu\,\mathcal{L}_\mu\,\mu - \int \mu^2\mu' - \varepsilon^2 \int \mu'\,g\,.$$

Since μ is periodic, we have $\int \mu^2\mu' = 0$. Using Young's inequality, we have

$$\partial_t \int \mu^2 \le -2\int \mu\,\mathcal{L}_\mu\,\mu + \frac{1}{2}\int (\mu')^2 + 2\,\varepsilon^4 \int g^2 \le \int \mu^2 + 2\,\varepsilon^4 \int g^2\,,$$

from which (2.3) follows immediately. As a much stronger result, we can in fact prove that the L^2–norm of the solution stays bounded for all $t \ge 0$. Introducing the operator $\mathcal{L}_v$, whose symbol is

$$\mathcal{L}_v(k) = \sqrt{\frac{1}{3}\frac{1+k^4}{1+\frac{\varepsilon^2\,k^2}{2}}}\,, \tag{2.4}$$

we have the following theorem.

Theorem 2.1 *There exist a constant K such that the solution μ of*

$$\partial_t \mu = -\mathcal{L}_\mu\,\mu - \mu\mu' + \varepsilon^2\,g'\,, \quad \mu(x,0) = \mu_0(x)$$

is periodic, antisymmetric, and satisfies

$$\sup_{t\ge 0}\|\mu(\cdot,t)\|_{\mathrm{L}^2} \le \rho + \|\mu_0\|_{\mathrm{L}^2} + 4\,\varepsilon^2 \sup_{t\ge 0}\|\mathcal{L}_v^{-1}\,g(\cdot,t)'\|_{\mathrm{L}^2}\,,$$

where $\rho = K\,L^{8/5}$, if μ_0 and g' are antisymmetric (spatially) periodic functions of period L,

Proof. Note first that $\mathcal{L}_v^{-1}\,\partial_x$ is a bounded operator on L^2 with norm ≤ 2 (see Lemma F.1 in Appendix F), then local existence in L^2 follows from the above argument.

Following [NST85] with the modifications of [CEE93], we write $\mu(x,t) = v(x,t) + \phi(x)$ for some constant periodic function ϕ to be chosen later on. Denoting by $\int$ the integral over $[-L/2, L/2]$, we get from (2.2)

$$\frac{1}{2}\partial_t \int v^2 = -\int v\,\mathcal{L}_\mu\,v - \int v\,\mathcal{L}_\mu\,\phi - \int v^2 v' - \frac{1}{2}\int v^2\phi' - \int v\phi\phi' + \varepsilon^2 \int v\,g'\,. \tag{2.5}$$

The term $\int v^2 v'$ vanishes because v is periodic, giving a much more compact form for (2.5):

$$\frac{1}{2}\partial_t(v,v) = -(v,v)_{\phi/2} - (v,\phi)_\phi + \varepsilon^2\,(v,g')\,, \tag{2.6}$$

where we have introduced the inner products

$$(v,w)=\int vw \quad \text{and} \quad (v,w)_{\gamma\phi}=\int v\,(\mathcal{L}_\mu+\gamma\phi')\,w\ .$$

This decomposition is helpful because we have the following nondegeneracy result which is proved in Appendix A

Proposition 2.2 *For all $L\geq 2\pi$, there exist a constant K and an antisymmetric periodic function ϕ such that for all $\gamma\in[\frac{1}{4},1]$ and $\varepsilon\leq L^{-2/5}$, and for every antisymmetric periodic function v, one has*

$$\begin{gathered}\frac{3}{4}\,(\mathcal{L}_v\,v,\mathcal{L}_v\,v)\leq(v,v)_{\gamma\phi}\leq\|\phi'\|_\infty\,(v,v)+(v'',v'')\ ,\\(\phi,\phi)_{\gamma\phi}\leq K\,L^{16/5}\ ,\\(\phi,\phi)\leq\tfrac{4}{3}\,L^3\ ,\end{gathered}$$

where $\mathcal{L}_v$ is defined in (2.4).

We first note that

$$(\mathcal{L}_v\,v,\mathcal{L}_v\,v)\geq\frac{4}{3}\Big(\frac{\sqrt{\varepsilon^4+4}-2}{\varepsilon^4}\Big)\,(v,v)\equiv c_v^2\,(v,v)\ .$$

Using Young's inequality and Proposition 2.2, we get from (2.6),

$$\begin{aligned}\partial_t(v,v)&\leq-2\,(v,v)_{\phi/2}+\tfrac{2}{3}(v,v)_\phi+\tfrac{3}{2}(\phi,\phi)_\phi+2\,\varepsilon^2\,(v,g')\\&\leq-\tfrac{4}{3}\,(v,v)_{\phi/4}+\tfrac{3}{2}(\phi,\phi)_\phi+2\,\varepsilon^2\,(v,g')\\&\leq-(\mathcal{L}_v\,v,\mathcal{L}_v\,v)+\tfrac{3}{2}(\phi,\phi)_\phi+2\,\varepsilon^2\,(\mathcal{L}_v\,v,\mathcal{L}_v^{-1}\,g')\\&\leq-\frac{1}{2}\,(\mathcal{L}_v\,v,\mathcal{L}_v\,v)+\tfrac{3}{2}(\phi,\phi)_\phi+2\,\varepsilon^4\,\|\mathcal{L}_v^{-1}\,g'\|_{\mathrm{L}^2}^2\\&\leq-\frac{c_v^2}{2}\,(v,v)+\tfrac{3}{2}(\phi,\phi)_\phi+2\,\varepsilon^4\,\|\mathcal{L}_v^{-1}\,g'\|_{\mathrm{L}^2}^2\ .\end{aligned}\tag{2.7}$$

Since $v(x,t)=\mu(x,t)-\phi(x)$ we conclude that

$$\|\mu(\cdot,t)-\phi(\cdot)\|_{\mathrm{L}^2}^2\leq\|\mu_0-\phi\|_{\mathrm{L}^2}^2+\frac{3}{c_v^2}(\phi,\phi)_\phi+\frac{4\,\varepsilon^4}{c_v^2}\sup_{t\geq0}\,\|\mathcal{L}_v^{-1}\,g(\cdot,t)'\|_{\mathrm{L}^2}^2\ .$$

Finally, since $\frac{2}{c_v}\leq 4$, we have

$$\sup_{t\geq0}\|\mu(\cdot,t)\|_{\mathrm{L}^2}\leq\|\mu_0\|_{\mathrm{L}^2}+\rho+4\,\varepsilon^2\sup_{t\geq0}\,\|\mathcal{L}_v^{-1}\,g(\cdot,t)'\|_{\mathrm{L}^2}\ ,$$

where

$$\rho=2\,\|\phi\|_{\mathrm{L}^2}+4\,\sqrt{(\phi,\phi)_\phi}\ .$$

Furthermore, by Proposition 2.2, we have $\rho<\infty$, since $\|\phi\|_{\mathrm{L}^2}=\sqrt{(\phi,\phi)}<\infty$ and $(\phi,\phi)_\phi<\infty$. This completes the proof of the theorem. ■

Corollary 2.3 *The antisymmetric solution of the Kuramoto–Sivashinsky equation with periodic boundary conditions on* $[-L/2, L/2]$

$$\partial_t \mu = -\mu'''' - \mu'' - \mu\, \mu' \ , \quad \mu(x,0) = \mu_0(x) \ , \tag{2.8}$$

stays in a ball of radius $\mathcal{O}(L^{8/5})$ *in* L^2 *as* $L \to \infty$*.*

Proof. This result was already established in [NST85] and [CEE93]. To prove it, we only have to note that (2.8) corresponds to (2.2) with $\varepsilon = 0$, and that Theorem 2.1 is uniformly valid in $\varepsilon \leq 1$. ■

Remark 2.4 *The proof of Theorem 2.1 is the only point in this paper where we need* s*, respectively* μ*, to be spatially even, resp. odd, functions. The theorem holds also in the general (non symmetric) case. The proof can be obtained as a straightforward extension of the result of [CEE93] for the Kuramoto–Sivashinsky equation in the non symmetric case.*

If $\varepsilon = 0$, Theorem 2.1 shows that the solution of (2.2) stays in a ball in L^2, centered on 0 and of radius $\|\mu_0\|_{\mathrm{L}^2} + \rho$ for all $t \geq 0$, with $\rho = \mathcal{O}(L^{8/5})$ as $L \to \infty$. When $\varepsilon \neq 0$, the radius of the ball widens to lowest order like $\varepsilon^2 \sup_{t\geq 0} \|g(\cdot,t)\|_{\mathrm{L}^2}$.

2.3 Functional spaces, definitions and properties

In this section, we explain how to treat the high frequency part of the solution of (2.2). This development is inspired by [BKL94] (see also [GvB02] for similar definitions). We will need some technical estimates which are proven in Appendix B.

2.3.1 Basic Definitions

Let $L \geq 2\pi$ and $q \equiv \frac{2\pi}{L} \leq 1$. We define the Fourier coefficients f_n of a function $f : [-L/2, L/2] \to \mathbf{R}$ by

$$f_n = \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{d}x\, e^{-iqnx} f(x) \ , \qquad \text{so that} \qquad f(x) = \sum_{n\in\mathbf{Z}} e^{iqnx}\, f_n \ ,$$

and $P_<$, $P_>$, the projectors on the small/high frequency part by

$$P_< f(x) = \sum_{|n|\leq \frac{\delta}{q}} e^{iqnx} f_n \quad , \quad P_> f(x) = \sum_{|n|> \frac{\delta}{q}} e^{iqnx} f_n \ ,$$

where the parameter $\delta \geq 2$ will be chosen later. We also define the L^p and l^p norms as

$$\|f\|_{\mathrm{L}^p} = \left(\int_{-L/2}^{L/2} \mathrm{d}x\, |f(x)|^p \right)^{1/p} \ , \quad \|f\|_{\mathrm{L}^\infty} = \operatorname*{ess\,sup}_{x\in[-L/2,L/2]} |f(x)| \ ,$$

$$\|f\|_{l^p} = \left(\sum_{n\in\mathbf{Z}} |f_n|^p \right)^{1/p} \ , \qquad \|f\|_{l^\infty} = \sup_{n\in\mathbf{Z}} |f_n| \ .$$

We will use repeatedly Plancherel's equality without notice

$$\|f\|_{\mathrm{L}^2} = \sqrt{L}\,\|f\|_{l^2} \,.$$

Finally, for $\sigma \geq 0$, $\delta \geq 2$, we define the norm $\|\cdot\|_{\mathcal{N},\sigma}$ by

$$\|f\|_{\mathcal{N},\sigma} = \tfrac{\sqrt{\delta}}{q} \sup_{n\in\mathbf{Z}} (1 + (\tfrac{qn}{\delta})^2)^{\frac{\sigma}{2}} \, |f_n| \,.$$

With a different normalization, the norm $\|\cdot\|_{\mathcal{N},\sigma}$ was introduced in [BKL94] to study the long time asymptotics of solutions of $\dot{u} = u'' + f(u, u', u'')$, where f is some (polynomial) nonlinearity. From the point of view of the nonlinearity, our situation is similar to the case treated there, but our linear operator $\mathcal{L}_\mu$ is not positive definite as $-\Delta$ was in their case. The potentially exponentially growing modes correspond to $|n| \leq \frac{1}{q}$, and we saw in Section 2.2 that their l^2 norm was bounded. Since there are only a finite (but large) number of linearly unstable modes, changing the definition of the $\|\cdot\|_{\mathcal{N},\sigma}$–norm on these modes to an l^2–like norm will give an equivalent norm which is better suited to our case. Thus we define the norms $\|\cdot\|_{\mathcal{W},\sigma}$ and $\|\cdot\|_\sigma$ by

$$\|f\|_{\mathcal{W},\sigma} = \tfrac{\sqrt{\delta}}{q} \sup_{|n|>\frac{\delta}{q}} (1 + (\tfrac{qn}{\delta})^2)^{\frac{\sigma}{2}} \, |f_n| \,, \tag{2.9}$$

$$\|f\|_\sigma = \|f\|_{\mathrm{L}^2} + \|f\|_{\mathcal{W},\sigma} \,. \tag{2.10}$$

While $\|\cdot\|_{\mathcal{W},\sigma}$ is clearly *not* a norm, $\|\cdot\|_\sigma$ is a norm which is equivalent to $\|\cdot\|_{\mathcal{N},\sigma}$ for $\sigma \geq 1$. Indeed, easy calculations lead to

$$\|f\|_{\mathcal{N},\sigma} \leq \sqrt{2^\sigma\,L\,\delta}\,\|f\|_{\mathrm{L}^2} + \|f\|_{\mathcal{W},\sigma} \leq (1 + \sqrt{2^\sigma\,L\,\delta})\,\|f\|_\sigma \,, \tag{2.11}$$

$$\|f\|_\sigma \leq \left(1 + \pi\,\sqrt{2}\right)\,\|f\|_{\mathcal{N},\sigma} \,. \tag{2.12}$$

We point out that if $\sigma > \frac{1}{2}$, the $\|\cdot\|_{\mathcal{W},\sigma}$–semi–norm is a *decreasing* function of δ. Indeed, we have (here the norms carry an additional index to specify the value of δ)

$$\|f\|_{\mathcal{W},\sigma,\delta_1} \leq \sqrt{\frac{\delta_1}{\delta_0}} \left(\frac{2}{1 + (\frac{\delta_1}{\delta_0})^2}\right)^{\frac{\sigma}{2}} \|f\|_{\mathcal{W},\sigma,\delta_0} \leq 2^{\frac{\sigma}{2}} \left(\frac{\delta_0}{\delta_1}\right)^{\sigma-\frac{1}{2}} \|f\|_{\mathcal{W},\sigma,\delta_0} \,, \tag{2.13}$$

for all $\delta_1 \geq \delta_0 \geq 2$. As δ will be fixed later on, the additional index is suppressed to simplify the notation. On the other hand, $\|\cdot\|_\sigma$ is an *non–decreasing* function of σ:

$$\|f\|_{\sigma_0} \leq \|f\|_{\sigma_1} \,, \tag{2.14}$$

for all $\sigma_1 \geq \sigma_0$.

Definition 2.5 *Denoting by $\mathcal{C}^\infty_{0,\mathrm{per}}([-L/2, L/2], \mathbf{R})$ the set of infinitely differentiable periodic real valued functions on $[-L/2, L/2]$, we define the (Banach) space $\mathcal{W}_{0,\sigma}$ as the completion of $\mathcal{C}^\infty_{0,\mathrm{per}}([-L/2, L/2], \mathbf{R})$ under the norm $\|\cdot\|_\sigma$, and $\mathcal{B}_{0,\sigma}(r) \subset \mathcal{W}_{0,\sigma}$ the open ball of radius r centered on $0 \in \mathcal{W}_{0,\sigma}$.*

Up to now, we considered functions depending on the space variable only. We extend the definition (2.10) to functions $f : [-L/2, L/2] \times [0, \infty) \to \mathbf{R}$ by

$$|\!|\!| f |\!|\!|_\sigma = \sup_{t \geq 0} \| f(\cdot, t) \|_\sigma \ .$$

We will use the same convention for the L^p and l^p norms, e.g.

$$|\!|\!| f |\!|\!|_{\mathrm{L}^2} = \sup_{t \geq 0} \| f(\cdot, t) \|_{\mathrm{L}^2} \ .$$

Finally, we make the following definition.

Definition 2.6 *Let $\Omega = [-L/2, L/2] \times \mathbf{R}^+$ and $\mathcal{C}^\infty_{\text{per}}(\Omega, \mathbf{R})$ denote the set of infinitely differentiable functions on Ω compactly supported on $\mathbf{R}^+$ and satisfying $f(-L/2, t) = f(L/2, t)$ for all $t \in \mathbf{R}^+$. We define the (Banach) space $\mathcal{W}_\sigma$ as the completion of $\mathcal{C}^\infty_{\text{per}}(\Omega, \mathbf{R})$ under the norm $|\!|\!| \cdot |\!|\!|_\sigma$, and $\mathcal{B}_\sigma(r) \subset \mathcal{W}_\sigma$ the open ball of radius r centered on $0 \in \mathcal{W}_\sigma$.*

We then have the

Proposition 2.7 *For all $\sigma > \frac{5}{2}$, $\mathcal{W}_\sigma$ is a Banach space included in the Banach space $\mathrm{L}^\infty(\mathbf{R}^+, W_{2,2}([-L/2, L/2]$ of functions (and their space derivatives up to order 2) on Ω uniformly (in time) bounded in $\mathrm{L}^2([-L/2, L/2])$.*

Proof. The proof follows directly from the Lemma 2.8 below. ■

Lemma 2.8 *Let $\sigma \geq \frac{3}{2}$. There exists a constant C such that for all $n \leq \sigma - \frac{3}{2}$ and for all $m \leq \sigma - 1$, we have*

$$\begin{aligned} \| f^{(m)} \|_{\sigma - m} + \| G \, f^{(m)} \|_{\sigma - m} &\leq C \, \delta^m \quad \| f \|_\sigma \ , &(2.15)\\ \| f^{(m)} \|_{\mathrm{L}^\infty} + \| G \, f^{(n)} \|_{\mathrm{L}^\infty} &\leq C \, \delta^{n + \frac{1}{2}} \, \| f \|_\sigma \ , &(2.16) \end{aligned}$$

where $f^{(m)}$ is the m–th order spatial derivative of f.

Proof. See Appendix B. ■
Although the indices m and n make the reading of Lemma 2.8 a bit cumbersome, it merely says that G is 'transparent' for the norms, and that each derivative 'cost' a factor δ.

2.3.2 Properties

The $\| \cdot \|_{\mathcal{N},\sigma}$ and $\| \cdot \|_{\mathcal{W},\sigma}$ norms can be used to control the nonlinear term $F_0(\mu)'$. For concision, all the proofs of this section are relegated in Appendix B. The map $F_0(\mu)$ admits the following decomposition (see Appendix C)

$$F_0(\mu) = F_1(\mu) + G \, F_2(\mu) \ , \tag{2.17}$$

where

$$F_1(\mu) = (1 + \varepsilon^2) \, \hat{s}^2 - \frac{\alpha^2 \, \hat{s} \, (\varepsilon^2 \hat{s}'')}{1 + \varepsilon^4 \, \hat{s}} - \frac{\alpha^2}{2} \, \frac{\mu \, \hat{s} \, (\varepsilon^4 \, \hat{s}')}{1 + \varepsilon^4 \, \hat{s}} \ ,$$

$$F_2(\mu) = -\frac{1}{4}\mu^2 - \frac{1}{4}(\mu^2)'' + \frac{\alpha^2}{2}\,\mu\,\hat{r}' + \alpha^2\,\mu'\,\hat{r} - \alpha^2\,\mu'\,\hat{s} - \frac{\alpha^2}{4}\mu''\,(\varepsilon^2\,\hat{s}')\,,$$
$$\hat{r}(\mu) = \left(1 - \frac{\varepsilon^2}{2}\partial_x^2\right)\,\hat{s}(\mu)\,.$$

To bound the contribution of F_0 to μ, we need a sequence of easy Propositions and Lemmas. The first result concerns the various terms appearing in Duhamel's formula.

Proposition 2.9 *Let $\delta \geq 2$, then*

$$\begin{aligned}\left\|e^{-\mathcal{L}_\mu t}\,f(\cdot)\right\|_{\mathcal{W},\sigma} &\leq e^{-4t}\,\|f(\cdot)\|_{\mathcal{W},\sigma}\,,\\ \left\|\int_0^t \mathrm{d}s\,e^{-\mathcal{L}_\mu(t-s)}\,g'(\cdot,s)\right\|_{\mathcal{W},\sigma} &\leq \sup_{0\leq s\leq t}\left\|\frac{g'(\cdot,s)}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma}\,,\end{aligned} \tag{2.18}$$

where $e^{-\mathcal{L}_\mu t}$ is the propagation Kernel associated with $\partial_t f = -\mathcal{L}_\mu f$, and $\mathcal{L}_\mu$ is defined in (2.1).

Then, on the r.h.s. of (2.18), we have the

Lemma 2.10 *Let $\delta \geq 2$, then*

$$\begin{aligned}\left\|\frac{g'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} &\leq \frac{\sqrt{2}}{\delta}\,\|g\|_{\mathcal{W},\sigma-1}\,,\\ \left\|\frac{G\,g'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} &\leq \frac{2^{7/2}}{3\,\delta^3}\,\|g\|_{\mathcal{W},\sigma-3}\,.\end{aligned}$$

This shows that we need only control $\|F_1\|_{\mathcal{W},\sigma-1}$ and $\|F_2\|_{\mathcal{W},\sigma-3}$. These will be bounded using Propositions 2.11 and 2.12 below, which show that multiplication and division of functions are well defined in $\mathcal{W}_\sigma$.

Proposition 2.11 *Let $\|u\|_{\sigma_1} < \infty$, $\|v\|_{\sigma_2} < \infty$ and $\sigma = \min(\sigma_1,\sigma_2) \geq \frac{3}{2}$. Then there exists a constant C_m depending only on σ such that*

$$\|uv\|_\sigma \leq C_m\,\sqrt{\delta}\,\|u\|_{\sigma_1}\,\|v\|_{\sigma_2}\,,$$

and if $\sigma \leq 1$, we have the two particular cases

$$\begin{aligned}\|uv\|_{\mathcal{W},\frac{1}{2}} &\leq C_m\,\sqrt{\delta}\,\|u\|_1\,\|v\|_1\,,\\ \|uv\|_{\mathcal{W},0} &\leq C_m\,\sqrt{\delta}\,\|u\|_{\mathrm{L}^2}\,\|v\|_{\mathrm{L}^2}\,.\end{aligned}$$

The following proposition shows that the $\|\cdot\|_\sigma$–norm of $\frac{u}{1+v}$ is essentially given by $\|u\|_\sigma$ if $\|v\|_\sigma \ll 1$.

Proposition 2.12 *Let $\|u\|_{\sigma_1} < \infty$, $\|v\|_{\sigma_2} < \infty$ and $\sigma = \min(\sigma_1,\sigma_2) \geq \frac{3}{2}$. Then*

$$\left\|\frac{u}{1+v}\right\|_\sigma \leq \frac{\|u\|_{\sigma_1}}{1 - C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2}}\,.$$

for all v satisfying $C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2} < 1$, where C_m is the constant of Proposition 2.11.

2.4 High frequency estimates

Here, we study the high frequency part of the solution of

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, g' \; , \quad \mu(x,0) = \mu_0(x) \; .$$

The solution of this equation exists by Theorem 2.1, and is bounded in L^2 for all $t \geq 0$ if $\|\mu_0\|_{\mathrm{L}^2} + \|\mathcal{L}_v^{-1} \, g'\|_{\mathrm{L}^2} < \infty$. We will now show that upon further restrictions on μ_0 and g, the solution has bounded $\|\cdot\|_\sigma$–norm for all $t \geq 0$.

We first need some definitions. We set

$$\begin{aligned} c_0 &= 1 + \frac{\|\mu_0\|_\sigma}{\rho} + \frac{4\,\varepsilon^2}{\rho} \, \|\mathcal{L}_v^{-1} \, g'\|_{\mathrm{L}^2} + \frac{\varepsilon^2}{\rho} \, \left\| \frac{g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \; , \\ \xi &= \frac{C_m \, c_0 \, \rho}{\sqrt{2}\,\delta} \; . \end{aligned} \tag{2.19}$$

Then we have

Theorem 2.13 *Assume that the initial condition μ_0 and g satisfy (2.19) with $\xi < \frac{1}{4}$. Then the solution μ of*

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, g' \; , \quad \mu(x,0) = \mu_0(x) \tag{2.20}$$

satisfies

$$\|\mu\|_\sigma \leq \left(\frac{1 - \sqrt{1 - 4\xi}}{2\xi} \right) \left(\rho + \|\mu_0\|_\sigma + 4\,\varepsilon^2 \, \|\mathcal{L}_v^{-1} \, g'\|_{\mathrm{L}^2} + \varepsilon^2 \, \left\| \frac{g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \right) \; . \tag{2.21}$$

Remark 2.14 *Note that c_0 is implicitly dependent of δ (because the norm $\|\cdot\|_\sigma$ is). If μ_0 and g are given, c_0 is a* non increasing *function of δ (see (2.13)). Hence we can surely satisfy $\xi < \frac{1}{4}$ by taking δ sufficiently large.*

Proof of Theorem 2.13. We first note that by Theorem 2.1, we have

$$\|\mu\|_{\mathrm{L}^2} \leq \rho + \|\mu_0\|_{L^2} + 4\,\varepsilon^2 \, \|\mathcal{L}_v^{-1} \, g'\|_{\mathrm{L}^2} \equiv c_1 \, \rho \; . \tag{2.22}$$

To bound $\|\mu\|_{\mathcal{W},\sigma}$, we use Duhamel's formula for the solution of (2.20)

$$\mu(x,t) = e^{-\mathcal{L}_\mu t} \, \mu_0(x) - \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, (\mu\mu')(x,s) + \varepsilon^2 \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, g'(x,s) \; . \tag{2.23}$$

Next, we define

$$T(\mu)(x,t) = - \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, (\mu\mu')(x,s) \; .$$

Since $\mu \, \mu' = \frac{1}{2}(\mu^2)'$, using Proposition 2.9 and Lemma 2.10, we get for all $\sigma' \leq \sigma$, the bound

$$\|T(\mu)\|_{\mathcal{W},\sigma'} \leq \frac{1}{\sqrt{2}\,\delta} \|\mu^2\|_{\mathcal{W},\sigma'-1} \; ,$$

and using again Proposition 2.9, we get

$$\|\!|\mu|\!\|_{\mathcal{W},\sigma'} \leq \|\mu_0\|_{\mathcal{W},\sigma'} + \frac{1}{\sqrt{2}\,\delta}\,\|\!|\mu^2|\!\|_{\mathcal{W},\sigma'-1} + \varepsilon^2 \left\|\!\left|\frac{g'}{\mathcal{L}_\mu}\right|\!\right\|_{\mathcal{W},\sigma'} . \tag{2.24}$$

Using that $\|f\|_{\sigma_1} \leq \|f\|_{\sigma_2}$ if $\sigma_1 \leq \sigma_2$, inequality (2.22), and dividing (2.24) by $c_0\,\rho$ (note that $c_0\rho \geq \rho > 0$), we get

$$\frac{1}{c_0\,\rho}\|\!|\mu|\!\|_{\sigma'} \leq 1 + \frac{1}{\sqrt{2}\,\delta\,c_0\rho}\,\|\!|\mu^2|\!\|_{\mathcal{W},\sigma'-1} . \tag{2.25}$$

We use this equation inductively in σ' to show (2.21). As a first step, notice that by Proposition 2.11, we have $\|\!|\mu^2|\!\|_{\mathcal{W},0} \leq C_m\sqrt{\delta}\,\|\!|\mu|\!\|^2_{\mathrm{L}^2}$, so that from (2.25) with $\sigma' = 1$ and the definition of c_1 (see (2.22)), we get

$$\begin{aligned}\frac{1}{c_0\,\rho}\|\!|\mu|\!\|_1 &\leq 1 + \frac{C_m}{\sqrt{2}\,\delta\,c_0\,\rho}\,(c_1\,\rho)^2 \leq 1 + \frac{C_m\,c_0\,\rho}{\sqrt{2}\,\delta}\left(\frac{c_1}{c_0}\right)^2 \\ &\leq 1 + \xi\left(\frac{c_1}{c_0}\right)^2 \equiv \frac{c_2}{c_0} .\end{aligned} \tag{2.26}$$

Using this inequality, (2.25) with $\sigma' = \frac{3}{2}$ and Proposition 2.11, we get

$$\begin{aligned}\frac{1}{c_0\,\rho}\|\!|\mu|\!\|_{\frac{3}{2}} &\leq 1 + \frac{C_m}{\sqrt{2}\,\delta\,c_0\,\rho}\,\|\!|\mu|\!\|_1^2 \leq 1 + \frac{C_m\,c_0\,\rho}{\sqrt{2}\,\delta}\left(\frac{c_2}{c_0}\right)^2 \\ &\leq 1 + \xi\left(\frac{c_2}{c_0}\right)^2 \equiv \frac{c_3}{c_0} .\end{aligned} \tag{2.27}$$

Let now $\sigma_3 = \frac{3}{2}$, $\sigma_n = \sigma_{n-1} + 1$ for all $4 \leq n \leq n_0$, where the integer n_0 is defined by $\sigma - 1 \leq \sigma_{n_0} < \sigma$. Using (2.25) and Proposition 2.11, we have

$$\begin{aligned}\frac{1}{c_0\,\rho}\|\!|\mu|\!\|_{\sigma_n} &\leq 1 + \frac{C_m}{\sqrt{2}\,\delta\,c_0\,\rho}\,(\|\!|\mu|\!\|_{\sigma_{n-1}})^2 \leq 1 + \frac{C_m\,c_0\,\rho}{\sqrt{2}\,\delta}\left(\frac{c_{n-1}}{c_0}\right)^2 \\ &\leq 1 + \xi\left(\frac{c_{n-1}}{c_0}\right)^2 \equiv \frac{c_n}{c_0} ,\end{aligned} \tag{2.28}$$

for all $4 \leq n \leq n_0$. Let now $\tilde{c}_n = \frac{c_n}{c_0}$ for $n \geq 2$. We can write (2.26)–(2.28) as $\tilde{c}_{n+1} = 1 + \xi\,\tilde{c}_n^2$ for $n \geq 2$. Furthermore, since $\frac{c_1}{c_0} \leq 1$, if we set $\tilde{c}_1 = 1$, we will also get an upper bound for $\|\mu\|_\sigma$. And now, since $\xi < \frac{1}{4}$, the (infinite) sequence $\tilde{c}_{n+1} = 1 + \xi\,\tilde{c}_n^2$, $\tilde{c}_1 = 1$, is increasing and satisfies $\tilde{c}_n \leq \lim\limits_{n\to\infty} \tilde{c}_n = \tilde{c}_\infty \equiv \frac{1-\sqrt{1-4\xi}}{2\xi}$, hence

$$\begin{aligned}\|\!|\mu|\!\|_\sigma &\leq \tilde{c}_\infty\,c_0\,\rho \\ &= \left(\frac{1-\sqrt{1-4\xi}}{2\xi}\right)\left(\rho + \|\mu_0\|_\sigma + 4\,\varepsilon^2\,\|\!|\mathcal{L}_v^{-1}\,g'|\!\|_{\mathrm{L}^2} + \varepsilon^2\left\|\!\left|\frac{g'}{\mathcal{L}_\mu}\right|\!\right\|_{\mathcal{W},\sigma}\right) .\end{aligned} \tag{2.29}$$

This completes the proof. ■

2.5 Existence and unicity of the solution of the phase equation

Let $\tilde{\mu} \in \mathcal{W}_\sigma$ and $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\mu_0}\,\rho) \subset \mathcal{W}_{0,\sigma}$. We consider the equation

$$\partial_t f = -\mathcal{L}_\varepsilon f - f\, f' + \varepsilon^2\, F(\tilde{\mu})' \;, \quad f(x,0) = \mu_0(x) \;. \tag{2.30}$$

By Theorem 2.13, f exists if $\|\mu_0\|_\sigma + \|\mathcal{L}_v^{-1}\, F(\tilde{\mu})'\|_{\mathrm{L}^2} + \|\mathcal{L}_\mu^{-1}\, F(\tilde{\mu})'\|_{\mathcal{W},\sigma} < \infty$, in which case, we define the map $(\tilde{\mu}, \mu_0) \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$, by $\mathcal{F}(\tilde{\mu}, \mu_0) \equiv f$. We will show that for fixed μ_0, $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$ is a contraction in the ball $\mathcal{B}_\sigma(c_\mu\,\rho)$ if the following condition holds.

Condition 2.15 *There exists a constant $\lambda_1 < 1$ such that for all $c_\mu > \frac{c_{\mu_0}+1}{1-\lambda_1}$, there exists a constant ε_0 such that for all $\varepsilon \le \varepsilon_0$ and for all $\mu_1, \mu_2 \in \mathcal{B}_\sigma(c_\mu\,\rho)$ the following bounds hold*

$$4\,\varepsilon^2\,\|\mathcal{L}_v^{-1}\, F(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2\,\left\|\frac{F(\mu_i)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le \lambda_1\, c_\mu\, \rho \;, \tag{2.31}$$

$$\varepsilon^2\,\|\mathcal{L}_v^{-1}\,(F(\mu_1) - F(\mu_2))'\|_{\mathrm{L}^2} + \varepsilon^2\,\left\|\frac{F(\mu_1)' - F(\mu_2)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le \lambda_1\,\|\mu_1 - \mu_2\|_\sigma \;, \tag{2.32}$$

$$\varepsilon^4\,\|r_2(\mu_i)\|_{\mathrm{L}^2} + \varepsilon^2\,\|\mathcal{L}_v^{-1}\, F(\mu_i)'\|_{\mathrm{L}^2} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_\mu\,\rho \;, \tag{2.33}$$

where $\mathcal{L}_v$ is defined in (1.32).

We prove that this condition holds in Section 4. The proof requires bounds on s and r_2. At this point, we note that if $s = s_1(\mu)$, or equivalently $r_2 = 0$, we have $F(\mu) = F_0(s_1(\mu), \mu)$, and that (see Appendix C or the beginning of Section 4) we can satisfy Condition 2.15 for any $\lambda_1 < 1$ and $\varepsilon_0 = c_\varepsilon\,\delta^{-5/4}\,\rho^{-1/2}$ if c_ε is sufficiently small (depending on λ_1). To apply Theorem 2.13, we need $\xi = \frac{C_m\, c_0\,\rho}{\sqrt{2\,\delta}} < \frac{1}{4}$, and from (2.31), we have $c_0 < c_\mu$, hence we can satisfy $\xi < \frac{1}{4}$ by choosing $\delta = c_\delta\,\rho^2$ for some constant c_δ. This implies also that we should take (at least) $\varepsilon_0 = c_\varepsilon\,\rho^{-m_\varepsilon}$ with $m_\varepsilon \ge 3$.

We then have the following Proposition

Proposition 2.16 *Let $c_\mu > \frac{1+c_{\mu_0}}{1-\lambda_1}$, and assume that Condition 2.15 holds with ε_0 sufficiently small. Then there exists a constant c_δ such that if $\delta = c_\delta\,\rho^2$ and $\varepsilon \le \varepsilon_0$, then*

$$\|\mathcal{F}(\tilde{\mu}_i, \mu_0)\|_\sigma < c_\mu\,\rho \tag{2.34}$$

for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\mu_0}\,\rho)$.

Proof. Let

$$c_0(\tilde{\mu}) = 1 + c_{\mu_0} + 4\,\varepsilon^2\,\frac{\|\mathcal{L}_v^{-1}\, F(\tilde{\mu})'\|_{\mathrm{L}^2}}{\rho} + \frac{\varepsilon^2}{\rho}\,\left\|\frac{F(\tilde{\mu})'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \;,$$

$$\xi(\tilde{\mu}) = \frac{C_m\, c_0(\tilde{\mu})\,\rho}{\sqrt{2\,\delta}} \;.$$

for all $\mu \in \mathcal{B}_\sigma(c_\mu\,\rho)$ and $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\mu_0}\,\rho)$, we have $c_0(\tilde{\mu}) < \lambda\, c_\mu$ with $\lambda = \lambda_1 + \frac{1+c_{\mu_0}}{c_\mu} < 1$. Choosing

$$c_\delta > \frac{C_m^2\, c_\mu^2}{2}\,\max\Bigl(4\,,\; \frac{1}{1-\lambda^2}\Bigr)^2 \;,$$

we have $\xi(\tilde\mu) < \frac{1}{4}$ and $\left(\frac{1-\sqrt{1-4\xi(\tilde\mu)}}{2\xi(\tilde\mu)}\right)\lambda < 1$, so that by Theorem 2.13, we have

$$\|\mathcal{F}(\tilde\mu,\mu_0)\|_\sigma \le \left(\frac{1-\sqrt{1-4\xi(\tilde\mu)}}{2\xi(\tilde\mu)}\right) c_0(\tilde\mu)\,\rho < c_\mu\,\rho\,.$$

This completes the proof. ■

Proposition 2.17 *Let c_μ, c_δ and ε_0 be given by Proposition 2.16, and assume that for all $\tilde\mu_1, \tilde\mu_2 \in \mathcal{B}_\sigma(c_\mu\,\rho)$ the following bound holds*

$$\|\mathcal{F}(\tilde\mu_i,\mu_0)\|_\sigma < c_\mu\,\rho \tag{2.35}$$

for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\mu\,\rho)$. Then there exists a time t_0 such that

$$\sup_{0\le t\le t_0} \|\mathcal{F}(\tilde\mu_1,\mu_0)(\cdot,t)-\mathcal{F}(\tilde\mu_2,\mu_0)(\cdot,t)\|_\sigma < \sup_{0\le t\le t_0} \|\tilde\mu_1(\cdot,t)-\tilde\mu_2(\cdot,t)\|_\sigma\,. \tag{2.36}$$

Proof. The proof of (2.36), being very similar to the estimates leading to (2.34), can be found in Appendix D. Note that here we only asked for $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\mu\,\rho)$ and not for $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\mu_0}\,\rho)$. ■

We now deduce from Propositions 2.16 and 2.17 existence, unicity, and estimates for the solution of the phase equation.

Theorem 2.18 *Let c_μ, c_δ and ε_0 be given by Proposition 2.16. Then for all $T \ge 0$, the solution $\mu_\star$ of*

$$\partial_t\mu_\star = -\mathcal{L}_\mu\,\mu_\star - \mu_\star\,\mu_\star' + \varepsilon^2\,F(\mu_\star)'\,, \quad \mu(x,0)=\mu_0(x) \tag{2.37}$$

exists for all $0 \le t \le T$ and satisfies

$$\sup_{0\le t\le T} \|\mu_\star(\cdot,t)\|_\sigma \le c_\mu\,\rho\,, \tag{2.38}$$

for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\mu_0}\,\rho)$.

Proof. Let $\mathcal{F}(\tilde\mu,\mu_0)$ be the solution of

$$\partial_t\mu = -\mathcal{L}_\mu\,\mu - \mu\,\mu' + \varepsilon^2\,F(\tilde\mu)'\,, \quad \mu(x,0)=\mu_0(x)\,.$$

By Proposition 2.16, we know that $\|\mathcal{F}(\tilde\mu)\|_\sigma < c_\mu\,\rho$ if $\|\tilde\mu\|_\sigma \le c_\mu\,\rho$. Hence, we can apply Proposition 2.17 and get that $\tilde\mu \mapsto \mathcal{F}(\tilde\mu,\mu_0)$ is a contraction for $0 \le t \le t_0$ in the ball of radius $c_\mu\,\rho$. Thus $\tilde\mu \mapsto \mathcal{F}(\tilde\mu,\mu_0)$ has a unique fixed point $\mu_\star$ in that ball. By easy arguments (see e.g. [GvB02]), this fixed point is the unique *strong* solution of (2.37) for $0 \le t \le t_0$. Furthermore, since the image of $\tilde\mu \mapsto \mathcal{F}(\tilde\mu,\mu_0)$ is in a ball of radius $c_\mu\,\rho$, $\mu_\star$ satisfies (2.38) with $T = t_0$.

We can now show inductively that $\mu_\star$ exists for all $t \ge 0$ and satisfies (2.38) for all $T \ge 0$. Define $t_n = (n+1)t_0$ for $n \ge 1$, and suppose that $\mu_\star$ exists on $0 \le t \le t_{n-1}$ and satisfies (2.38) with $T = t_{n-1}$. By Proposition 2.16, we know that for $t_{n-1} \le t \le t_n$, the solution $\mathcal{F}(\tilde\mu,\mu_\star(\cdot,t_{n-1}))$ of

$$\partial_t\mu = -\mathcal{L}_\mu\,\mu - \mu\,\mu' + \varepsilon^2\,F(\tilde\mu)'\,, \quad \mu(x,t_0)=\mu_\star(x,t_{n-1}) \tag{2.39}$$

is in a ball of size $c_\mu\ \rho$ if $\tilde{\mu}$ is in a ball of size $c_\mu\ \rho$ for $t_{n-1} \leq t \leq t_n$, because it is the *continuation* of a solution of (2.30), beginning with μ_0 in $t = 0$, with $\tilde{\mu}(x,t) = \mu_\star(x,t)$ for $0 \leq t \leq t_{n-1}$. Shifting the origin of time to t_{n-1} and replacing μ_0 by $\mu_\star(\cdot, t_{n-1})$, we see that the conditions of Proposition 2.17 are satisfied, hence $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_\star(\cdot, t_{n-1}))$ is a contraction for $t_{n-1} \leq t \leq t_n$. As above, this implies that there exists an unique fixed point $\mu_\star$ which is the unique *strong* solution of 2.37 on $0 \leq t \leq t_n$ and satisfies (2.38) with $T = t_n$. ■

2.6 Consequences

Up to now, we did not use (2.33) of Condition 2.15. This inequality has two important consequences. The first one is that s (if it exists) and η are related by $s = -\frac{1}{8}\ \eta'' = -\frac{1}{8}\ \mu'$ up to corrections of order ε^2 (see Theorem 2.19 below), and the second one concerns the relation with the Kuramoto–Sivashinsky equation (see Theorem 2.20 below). Once these theorems are proved, we will only have to prove the bound on $\hat{s}$ to complete the proof of Theorems 1.2 and 1.3.

Theorem 2.19 *There exists a constant $c_\varepsilon > 0$ such that if Condition 2.15 holds with $\varepsilon_0 \leq c_\varepsilon\ \rho^{-3}$, and δ is given by Proposition 2.16, then*

$$\frac{\|s + \frac{1}{8}\ G\ \mu' + \frac{\varepsilon^2}{32}\ G\ (\mu)^2\|_{\mathrm{L}^2}}{c_\mu\ \rho} \leq \left(\frac{\varepsilon}{\varepsilon_0}\right)^2$$

if $\varepsilon \leq \varepsilon_0$.

Proof. The proof is very simple. We use that $s + \frac{1}{32}G\ (4\mu' + \varepsilon^2\ \mu^2) = \varepsilon^4\ G\ r_2$, and that by assumption (see (2.33)), we have $\varepsilon^4\ \|r_2\|_{\mathrm{L}^2} \leq \left(\frac{\varepsilon}{\varepsilon_0}\right)^2\ c_\mu\ \rho$. ■

We next show that the solution μ_c of the Kuramoto–Sivashinsky equation (in derivative form) captures the dynamics of the (derivative of the) phase for short times (then $-\frac{1}{8}\ \mu_c'$ captures the dynamics of the amplitude by Theorem 2.19). To state the result, we introduce the operator $\mathcal{L}_{\mu,c} = \partial_x^4 + \partial_x^2$. We have the following Theorem.

Theorem 2.20 *Let μ and μ_c be the solutions of*

$$\begin{aligned} \partial_t \mu &= -\mathcal{L}_\mu\ \mu - \mu\mu' + \varepsilon^2\ F(\mu)' \ , \qquad & \mu(x,0) &= \mu_0(x) \ , \\ \partial_t \mu_c &= -\mathcal{L}_{\mu,c}\ \mu_c - \mu_c\mu_c' \ , & \mu_c(x,0) &= \mu_0(x) \ , \end{aligned}$$

There exist constants c_ε and c_t such that if Condition 2.15 holds with $\varepsilon_0 \leq c_\varepsilon\ \rho^{-4}$, then

$$\sup_{0 \leq t \leq t_0} \frac{\|\mu(\cdot,t) - \mu_c(\cdot,t)\|_{\mathrm{L}^2}}{c_\mu\ \rho} \leq \frac{\varepsilon}{\varepsilon_0} \ , \tag{2.40}$$

for all $t_0 \leq c_t\ \rho^{-4}$ and for all $\varepsilon \leq \varepsilon_0$.

This theorem implies directly Theorem 1.3.

Proof of Theorem 2.20. Let $\mu_\pm = \mu \pm \mu_c$ and $\mathcal{L}_\pm = \mathcal{L}_{\mu,c} \pm \mathcal{L}_\mu$. Note that μ_c exists and satisfies $\|\mu_c\|_\sigma \leq c_\mu\ \rho$. Furthermore, we have

$$\partial_t \mu_- = -\frac{\mathcal{L}_+}{2}\mu_- + \mathcal{L}_-\mu_+ - \frac{1}{2}(\mu_+\mu_-)' + \varepsilon^2\ F(\mu)' \ .$$

Multiplying this equation by μ_- and integrating over $[-L/2, L/2]$, we get

$$\frac{1}{2}\,\partial_t\,(\mu_-,\mu_-) = -\frac{1}{2}(\mu_-,\mathcal{L}_+\mu_-) + (\mu_-,\mathcal{L}_-\mu_+) - \frac{1}{4}(\mu_-,\mu'_+\mu_-) + \varepsilon^2(\mu_-,F(\mu)')\;.$$

Next, we use the Cauchy–Schwartz inequality, the identity $(\mathcal{L}_+ - \mathcal{L}_-) = 2\mathcal{L}_\mu$ and $\mathcal{L}_- \geq 0$, (this follows from $\mathcal{L}_{\mu,c}(k) = k^4 - k^2$), to get

$$\begin{aligned}\partial_t(\mu_-,\mu_-) &\leq -(\mu_-,(2\mathcal{L}_\mu - \mathcal{L}_v^2)\mu_-) + (\mu_+,\mathcal{L}_-\mu_+)\\ &\quad + \frac{\|\mu'_+\|_{\mathrm{L}^\infty}}{2}\,(\mu_-,\mu_-) + \varepsilon^4\,\|\mathcal{L}_v^{-1}\,F(\mu)'\|^2_{\mathrm{L}^2}\\ &\leq (1+\frac{\|\mu'_+\|_{\mathrm{L}^\infty}}{2})(\mu_-,\mu_-) + (\mu_+,\mathcal{L}_-\mu_+) + \varepsilon^4\,\|\mathcal{L}_v^{-1}\,F(\mu)'\|^2_{\mathrm{L}^2}\;.\end{aligned}$$

Since the Fourier coefficients of $\mathcal{L}_-$ satisfy $(\mathcal{L}_-)_n \leq \varepsilon^2\,(qn)^6$ we get, using Lemma 2.8

$$(\mu_+,\mathcal{L}_-\mu_+) \leq \varepsilon^2\,\|\mu'''_+\|^2_{\mathrm{L}^2} \leq C\,\varepsilon^2\,\delta^6\,\|\mu_+\|^2_\sigma \leq C\,\varepsilon^2\,\delta^6\,(c_\mu\,\rho)^2\;.$$

Let $\zeta = 1 + C\,c_\mu\,\rho\,\delta^{3/2} = c_\zeta\,\rho^4$. We have

$$\begin{aligned}\partial_t(\mu_-,\mu_-) &\leq \zeta\,(\mu_-,\mu_-) + C\,\varepsilon^2\,\delta^6\,(c_\mu\,\rho)^2 + \varepsilon^4\,\|\mathcal{L}_v^{-1}\,F(\mu)'\|^2_{\mathrm{L}^2}\\ &\leq \zeta\,(\mu_-,\mu_-) + C\,\varepsilon^2\,\delta^6\,(c_\mu\,\rho)^2 + \left(\frac{\varepsilon}{\varepsilon_0}\right)^4(c_\mu\,\rho)^2\;,\end{aligned}$$

where we used (2.33). Let $\varepsilon_0 \leq c_\varepsilon\,\rho^{-4}$, and $t_0 \leq c_t\,\rho^{-4}$. We have

$$\sup_{0\leq t\leq t_0}\frac{\|\mu(\cdot,t)-\mu_c(\cdot,t)\|_{\mathrm{L}^2}}{c_\mu\,\rho} \leq \left(\frac{\varepsilon}{\varepsilon_0}\right)\left(\frac{2\,C_m\,c_\delta^3\,c_\varepsilon}{\sqrt{c_\zeta}} + \frac{1}{\sqrt{c_\zeta}\,\rho^2}\right)\sqrt{e^{c_\zeta\,c_t}-1} \leq \frac{\varepsilon}{\varepsilon_0}\;, \tag{2.41}$$

if c_ε and c_t are sufficiently small. ■

3 The amplitude equation

In (1.30), we showed that in terms of the amplitude s and the (derivative of the) phase μ of the perturbation of $e^{i\,\phi_0 - i\beta t}$, the 'amplitude' part of the Complex Ginzburg Landau equation becomes

$$\partial_t s = -\frac{\chi}{\varepsilon^4}(s - \frac{\varepsilon^2}{2}s'') + \frac{\chi}{\varepsilon^4}\,r_1(\mu) - \tfrac{\alpha^2\,\chi}{8}(2s'\mu + s\mu') + F_3(s,\mu)\;, \tag{3.1}$$

with $s(x,0) = s_0(x)$ and

$$\begin{aligned}r_1(\mu) &= -\frac{1}{32}(4\mu' + \varepsilon^2\,\mu^2)\;,\\ F_3(s,\mu) &= -\alpha^2\,\chi\,\left(\tfrac{3}{2}\,s^2 + \varepsilon^2\,\tfrac{1}{32}\,s\,\mu^2 + \tfrac{\alpha^2}{2}\,\varepsilon^4\,s^3\right)\;.\end{aligned}$$

We now show that for given μ with $|\!|\!|\mu|\!|\!|_\sigma$ not too large, the solution of (3.1) is determined by a well defined Lipschitz map of μ. We will use the definitions and properties of the norms $|\!|\!|\cdot|\!|\!|_\sigma$

of Subsections 2.3.1 and 2.3.2. We proceed as we did for the phase equation, that is, we first show L^2 estimates, and then $\|\cdot\|_\sigma$ estimates. Equation (3.1) suggests that we study

$$\partial_t s = -\frac{\chi}{\varepsilon^4}(s - \frac{\varepsilon^2}{2}s'') - \tfrac{\alpha^2\,\chi}{8}(2s'\nu + s\nu') + f\ , \qquad s(x,0) = s_0(x)\ , \tag{3.2}$$

for given s_0, ν and f. If $\|\nu\|_{\mathrm{L}^\infty} + \|\nu'\|_{\mathrm{L}^\infty}$ is sufficiently small, this equation is a linear (in s) inhomogeneous damped heat equation, hence the local existence and unicity of the solution in L^2 is known by classical arguments (see e.g. [Tem97]). Furthermore, the solution satisfies the

Lemma 3.1 *If s is the solution of (3.2) then*

$$\|s\|_{\mathrm{L}^2} \le \|s_0\|_{\mathrm{L}^2} + \frac{\varepsilon^4}{\chi}\,\|f\|_{\mathrm{L}^2}\ . \tag{3.3}$$

Proof. Multiplying (3.2) with s, integrating over one period, using Young's inequality, and using that $\int s(2s'\nu + s\nu') = \int (s^2\nu)' = 0$ because s and ν are periodic, we get

$$\partial_t \int s^2 \le -\frac{2\chi}{\varepsilon^4}\int s^2 + \int s\,f \le -\frac{\chi}{\varepsilon^4}\int s^2 + \frac{\varepsilon^4}{\chi}\int f^2\ ,$$

from which (3.3) follows immediately. ■

Proposition 3.2 *If s is the solution of (3.2), then*

$$\|s\|_{\sigma-1} \le 2\|s_0\|_{\sigma-1} + \frac{2\,\varepsilon^4}{\chi}\|f\|_{\sigma-1} \tag{3.4}$$

for all ν satisfying $\varepsilon^3\,\sqrt{\delta}\,\alpha^2\,C_m\,(4+\varepsilon\delta)\,\|\nu\|_\sigma \le 4$.

Proof. The idea is again to use Duhamel's representation formula for s. Let $\mathcal{L}$ be the operator with symbol

$$\mathcal{L}(k) = \frac{\chi}{\varepsilon^4}\Big(1 + \frac{\varepsilon^2\,k^2}{2}\Big)\ ,$$

then s satisfies

$$s(x,t) = e^{-\mathcal{L}t}s_0(x) + \mathcal{T}(s,f)(s,t)\ ,$$

with

$$\begin{aligned}\mathcal{T}(s,f)(s,t) &= -\int_0^t \mathrm{d}\tau\ e^{-\mathcal{L}(t-\tau)}\ \Big(\tfrac{\alpha^2\,\chi}{8}(2\,\nu(x,\tau)\,\partial_x s(x,\tau) + s(x,\tau)\,\partial_x\nu(x,\tau)) - f(x,\tau)\Big)\\ &= \int_0^t \mathrm{d}\tau\ e^{-\mathcal{L}(t-\tau)}\ \Big(\tfrac{\alpha^2\,\chi}{8}(2\,\partial_x(s(x,\tau)\,\nu(x,\tau)) - s(x,\tau)\,\partial_x\nu(x,\tau)) - f(x,t_0)\Big)\ .\end{aligned}$$

Using the inequality

$$\Big\|\int_0^t \mathrm{d}\tau\ e^{-\mathcal{L}(t-\tau)}\ f(x,\tau)\Big\|_{\mathcal{W},\sigma} \le \frac{\varepsilon^4}{\chi}\|G\,f\|_{\mathcal{W},\sigma}\ ,$$

we get for any $\sigma' \leq \sigma - 1$,

$$\|s\|_{\mathcal{W},\sigma'} \leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{4}\|G\,(s\,\nu)'\|_{\mathcal{W},\sigma'} + \frac{\varepsilon^4\,\alpha^2}{8}\|G\,(s\,\nu')\|_{\mathcal{W},\sigma'} \tag{3.5}$$

$$\leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^2\,\alpha^2}{2}\|s\,\nu\|_{\mathcal{W},\sigma'-1} + \frac{\varepsilon^3\,\alpha^2}{4}\|s\,\nu'\|_{\mathcal{W},\sigma'-1}\,. \tag{3.6}$$

We now use (3.6) inductively in σ' to show that $\|s\|_{\mathcal{W},\sigma-1}$ is bounded. Then we will use (3.5) to show that $\|s\|_{\mathcal{W},\sigma-1}$ satisfies the bound (3.4).

Using (3.6) with $\sigma' = 1$, we get $\|s\|_{\mathcal{W},1} < \infty$, and $\|s\|_1 < \infty$, because

$$\|s\,\nu\|_{\mathcal{W},0} \leq C_m\,\sqrt{\delta}\|s\|_{\mathrm{L}^2}\,\|\nu\|_{\mathrm{L}^2} \leq C_m\,\sqrt{\delta}\|s\|_{\mathrm{L}^2}\,\|\nu\|_\sigma\,,$$
$$\|s\,\nu'\|_{\mathcal{W},0} \leq C_m\,\sqrt{\delta}\|s\|_{\mathrm{L}^2}\,\|\nu'\|_{\mathrm{L}^2} \leq C\,\delta^{3/2}\|s\|_{\mathrm{L}^2}\,\|\nu\|_\sigma\,.$$

Then, using (3.6) with $\sigma' = \frac{3}{2}$, we get $\|s\|_{\mathcal{W},\frac{3}{2}} < \infty$ because

$$\|s\|_{\mathcal{W},\frac{3}{2}} \leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^2\,\alpha^2}{2}\|s\,\nu\|_{\mathcal{W},1} + \frac{\varepsilon^3\,\alpha^2}{4}\|s\,\nu'\|_{\mathcal{W},1}$$
$$\leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^2\,\alpha^2\,C_m\,\delta^{3/2}}{2}\Big(\frac{1}{\delta}+\frac{1}{2}\Big)\|s\|_1\,\|\nu\|_\sigma\,.$$

Then, for any $\sigma' \geq \frac{5}{2}$, we get

$$\|s\|_{\mathcal{W},\sigma'} \leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^2\,\alpha^2}{2}\|s\,\nu\|_{\mathcal{W},\sigma'-1} + \frac{\varepsilon^3\,\alpha^2}{4}\|s\,\nu'\|_{\mathcal{W},\sigma'-1}$$
$$\leq \|s_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^3\,\alpha^2\,C_m\,\delta^{3/2}}{2}\Big(\frac{1}{\delta}+\frac{1}{2}\Big)\|s\|_{\sigma'-1}\,\|\nu\|_\sigma\,.$$

Using this last inequality with $\sigma' = \frac{5}{2}, \frac{7}{2}, \ldots$ until we reach $\sigma - 1$ shows that $\|s\|_{\mathcal{W},\sigma-1} < \infty$. Then, from (3.5) and Lemma 3.1, we also get

$$\|s\|_{\sigma-1} \leq \|s_0\|_{\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{4}\|G\,(s\,\nu)'\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{8}\|G\,(s\,\nu')\|_{\mathcal{W},\sigma-1}$$
$$\leq \|s_0\|_{\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\sigma-1} + \frac{\varepsilon^3\,\alpha^2}{2}\|s\,\nu\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{8}\|s\,\nu'\|_{\mathcal{W},\sigma-1}$$
$$\leq \|s_0\|_{\sigma-1} + \frac{\varepsilon^4}{\chi}\|f\|_{\sigma-1} + \|s\|_{\sigma-1}\frac{\varepsilon^3\,\alpha^2\,\sqrt{\delta}\,C_m\,(4+\varepsilon\delta)\,\|\nu\|_\sigma}{8}\,.$$

Since by hypothesis $(\varepsilon^3\,\alpha^2\,\sqrt{\delta}\,C_m\,(4+\varepsilon\delta)\,\|\nu\|_\sigma)/8 \leq 1/2$, the proof is completed. ■

Corollary 3.3 *Let s_1, resp. s_2 be the solution of (3.2) with $f = f_1$ resp. f_2, and assume that $s_1(x,0) = s_2(x,0)$. Then*

$$\|s_1 - s_2\|_{\sigma-1} \leq \frac{2\,\varepsilon^4}{\chi}\,\|f_1 - f_2\|_{\sigma-1}\,. \tag{3.7}$$

Proof. Since (3.2) is linear in s, $s_1 - s_2$ satisfies (3.2) with $s_0 = 0$ and $f \equiv f_1 - f_2$. ■

Corollary 3.4 *Let $s(\nu)$ be the solution of (3.2) with $f = f(\nu)$ satisfying $\|f(\nu)\|_{\sigma-1} < \infty$. Then*

$$\|s(\nu_1) - s(\nu_2)\|_{\sigma-1} \leq \frac{2\,\varepsilon^4}{\chi} \|f(\nu_1) - f(\nu_2)\|_{\sigma-1} \, . \tag{3.8}$$

Proof. Since (3.2) is linear in s, $s(\nu_1) - s(\nu_2)$ satisfies (3.2) with $s_0 = 0$ and $f \equiv f(\nu_1) - f(\nu_2)$. ■

We are now in position to prove that the solution of (3.1) exists if ε_0 is sufficiently small.

Theorem 3.5 *Let c_{r_1} and c_μ be given by Proposition C.1 and Theorem 2.18, and $c_{s_0} > 0$. Let $c_s > 2(c_{r_1} + c_{s_0})$. There exists a constant c_ε such that for all $\varepsilon \leq c_\varepsilon\, \delta^{-5/4}\, \rho^{-1/2}$, for all $\mu \in \mathcal{B}_\sigma(c_\mu\, \rho)$ and for all $s_0 \in \mathcal{B}_{0,\sigma-1}(c_{s_0}\, \delta\, \rho)$, the solution s of*

$$\partial_t s = -\frac{\chi}{\varepsilon^4}(s - \frac{\varepsilon^2}{2} s'') + \frac{\chi}{\varepsilon^4}\, r_1(\mu) - \tfrac{\alpha^2\,\chi}{8}(2s'\mu + s\mu') + F_3(s, \mu) \, , \qquad s(x, 0) = s_0(x) \, , \tag{3.9}$$

exists and is unique in $\mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho)$. As such, it defines the map $\mu \mapsto s(\mu)$, which satisfies

$$\|s(\mu_i)\|_{\sigma-1} \leq c_s\, \delta\, \rho \, , \tag{3.10}$$

$$\|s(\mu_1) - s(\mu_2)\|_{\sigma-1} \leq c_s\, \delta\, \|\mu_1 - \mu_2\|_\sigma \, , \tag{3.11}$$

for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\, \rho)$.

Proof. Fixing $\tilde{s} \in \mathcal{W}_{\sigma-1}$, we consider the equation

$$\partial_t f = -\frac{\chi}{\varepsilon^4}(f - \frac{\varepsilon^2}{2} f'') + \frac{\chi}{\varepsilon^4}\, r_1(\mu) - \tfrac{\alpha^2\,\chi}{8}(2f'\mu + f\mu') + F_3(\tilde{s}, \mu) \, , \qquad f(x, 0) = s_0(x) \, .$$

By Proposition 3.2, f exists if $\|s_0\|_{\sigma-1} + \|r_1(\mu)\|_{\sigma-1} + \|F_3(\tilde{s}, \mu)\|_{\sigma-1} < \infty$, in which case we define the map $\tilde{s} \mapsto T(\tilde{s}, \mu)$ by $T(\tilde{s}, \mu) \equiv f$. To show that $s(\mu)$ exists, is unique and satisfies (3.10), we only have to show that if ε is sufficiently small, $\tilde{s} \mapsto T(\tilde{s}, \mu)$ is a contraction in $\mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho) \subset \mathcal{W}_{\sigma-1}$. Using Propositions 3.2 and C.1 and the assumption on s_0, we have

$$\begin{aligned} \|T(s, \mu)\|_{\sigma-1} &\leq 2\|r_1(\mu)\|_{\sigma-1} + 2\|s_0\|_{\sigma-1} + \frac{2\,\varepsilon^4}{\chi} \|F_3(s, \mu)\|_{\sigma-1} \\ &\leq 2\,(c_{r_1} + c_{s_0})\, \delta\, \rho + \frac{2\,\varepsilon^4}{\chi} \|F_3(s, \mu)\|_{\sigma-1} \, . \end{aligned} \tag{3.12}$$

Similarly, using Corollary 3.3, we have

$$\|T(s_1, \mu) - T(s_2, \mu)\|_{\sigma-1} \leq \frac{2\,\varepsilon^4}{\chi} \|F_3(s_1, \mu) - F_3(s_2, \mu)\|_{\sigma-1} \, . \tag{3.13}$$

By Proposition C.5, there exists a constant c_ε such that for all $\varepsilon \leq c_\varepsilon\, \delta^{-5/4}\, \rho^{-1/2}$, for all $\mu \in \mathcal{B}_\sigma(c_\mu\, \rho)$, and for all $s_i \in \mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho)$, we have

$$\frac{2\,\varepsilon^4}{\chi} \|F_3(s_i, \mu)\|_{\sigma-1} < \Big(1 - \frac{2(c_{r_1} + c_{s_0})}{c_s}\Big)\, c_s\, \delta\, \rho \, , \tag{3.14}$$

$$\frac{2\,\varepsilon^4}{\chi}\,\|F_3(s_1,\mu)-F_3(s_2,\mu)\|_{\sigma-1}\le\Big(1-\frac{2(c_{r_1}+c_{s_0})}{c_s}\Big)\|s_1-s_2\|_{\sigma-1}\,. \tag{3.15}$$

From (3.12) and (3.14), and since $c_s>2(c_{r_1}+c_{s_0})$, we conclude that $s\mapsto T(s,\mu)$ maps the ball $\mathcal{B}_{\sigma-1}(c_s\,\delta\,\rho)$ (strictly) inside itself, whereas from (3.13) and (3.15) we see that it is a contraction in that ball. Hence, the map $s\mapsto T(s,\mu)$ has a unique fixed point $s^\star(\mu)$. This fixed point satisfies (3.10) and is a strong solution of (3.9) (see also [GvB02]).

For (3.11), using Corollary 3.4, we have

$$\begin{aligned}\|s^\star(\mu_1)-s^\star(\mu_2)\|_{\sigma-1}&\le 2\|r_1(\mu_1)-r_1(\mu_2)\|_{\sigma-1}\\ &\le\quad+\frac{2\varepsilon^4}{\chi}\,\|F_3(s^\star(\mu_1),\mu_1)-F_3(s^\star(\mu_2),\mu_2)\|_{\sigma-1}\,.\end{aligned}$$

Using that

$$F_3(s_1,\mu_1)-F_3(s_2,\mu_2)=F_3(s_1,\mu_1)-F_3(s_1,\mu_2)+F_3(s_1,\mu_2)-F_3(s_2,\mu_2)$$

and that by Proposition C.5 we have

$$\begin{aligned}&\frac{2\varepsilon^4}{\chi}\,\|F_3(s_1,\mu_1)-F_3(s_1,\mu_2)\|_{\sigma-1}\le 2\,c_{s_0}\,\delta\,\|\mu_1-\mu_2\|_\sigma\,,\\ &\frac{2\varepsilon^4}{\chi}\,\|F_3(s_1,\mu_2)-F_3(s_2,\mu_2)\|_{\sigma-1}\le\Big(1-\frac{2(c_{r_1}+c_{s_0})}{c_s}\Big)\,\|s_1-s_2\|_{\sigma-1}\,,\end{aligned}$$

we conclude, using Proposition C.1, that

$$\begin{aligned}\|s^\star(\mu_1)-s^\star(\mu_2)\|_{\sigma-1}&\le 2\,(c_{r_1}+c_{s_0})\,\delta\,\|\mu_1-\mu_2\|_\sigma\\ &\quad+\frac{2\varepsilon^4}{\chi}\,\|F_3(s^\star(\mu_1),\mu_1)-F_3(s^\star(\mu_2),\mu_2)\|_{\sigma-1}\\ &\le 2\,(c_{r_1}+c_{s_0})\,\delta\,\|\mu_1-\mu_2\|_\sigma\\ &\quad+\Big(1-\frac{2(c_{r_1}+c_{s_0})}{c_s}\Big)\,\|s^\star(\mu_1)-s^\star(\mu_2)\|_{\sigma-1}\,.\end{aligned}$$

Since $c_s>2(c_{r_1}+c_{s_0})$, we have

$$\frac{2(c_{r_1}+c_{s_0})}{c_s}\|s^\star(\mu_1)-s^\star(\mu_2)\|_{\sigma-1}\le 2(c_{r_1}+c_{s_0})\,\delta\,\|\mu_1-\mu_2\|_\sigma\,.$$

Therefore, $\tilde s\mapsto T(\tilde s,\mu)$ is a contraction. ■

We define $r(\mu)=s(\mu)-\frac{\varepsilon^2}{2}s(\mu)''$ and $r_0=s_0-\frac{\varepsilon^2}{2}s_0''$, and prove that $\mu\mapsto r(\mu)$ satisfies essentially the same bounds as $\mu\mapsto s(\mu)$.

Corollary 3.6 *Assume that* $r_0\in\mathcal{B}_{0,\sigma-1}(c_{r_1}\,\delta\,\rho)$. *Then* $\mu\mapsto r(\mu)$ *satisfies*

$$\|r(\mu)\|_{\sigma-1}\le 8\,c_{r_1}\,\delta\,\rho\,, \tag{3.16}$$

$$\|r(\mu_1)-r(\mu_2)\|_{\sigma-1}\le 8\,c_{r_1}\,\delta\,\|\mu_1-\mu_2\|_\sigma\,, \tag{3.17}$$

if the conditions of Theorem 3.5 are satisfied.

Proof. The proof, being very similar to the ones of Proposition 3.2 and Theorem 3.5 is outlined in Appendix E. ■

4 The Condition 2.15, properties of $\mu \mapsto F(\mu)$ and $\mu \mapsto r_2(\mu)$

We recall that

$$F(\mu) = F_0(s(\mu), \mu) + \chi\, \mathcal{L}_{\mu,r}\, r_2(\mu)\ ,$$

where

$$r_2(\mu) = \frac{r(\mu)}{\varepsilon^4} - \frac{r_1(\mu)}{\varepsilon^4}\ . \tag{4.1}$$

For Condition 2.15 to hold, we need to show that there exists a constant $\lambda_1 < 1$ such that for all $\mu_1, \mu_2 \in \mathcal{B}_\sigma(c_\mu\, \rho)$,

$$\varepsilon^4\, \|r_2(\mu_i)\|_{\mathrm{L}^2} + \varepsilon^2\, \|\mathcal{L}_v^{-1}\, F(\mu_i)'\|_{\mathrm{L}^2} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_\mu\, \rho\ , \tag{4.2}$$

$$4\,\varepsilon^2\, \|\mathcal{L}_v^{-1}\, F(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2 \left\|\frac{F(\mu_i)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le \lambda_1\, c_\mu\, \rho\ , \tag{4.3}$$

$$\varepsilon^2\, \|\mathcal{L}_v^{-1}\, (F(\mu_1) - F(\mu_2))'\|_{\mathrm{L}^2} + \varepsilon^2 \left\|\frac{F(\mu_1)' - F(\mu_2)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le \lambda_1\, \|\mu_1 - \mu_2\|_\sigma\ , \tag{4.4}$$

for all $\varepsilon \le \varepsilon_0$ if ε_0 is sufficiently small. If $r_2 = 0$, these conditions can be satisfied if $\varepsilon_0 \le c_\varepsilon\, \rho^{-3}$ with c_ε sufficiently small. Namely, from Theorem C.2, Appendix C, using also $\|\mathcal{L}_v^{-1}\, f'\|_{\mathrm{L}^2} \le 2\, \|f\|_{\mathrm{L}^2}$, we have

$$\varepsilon^2\, \|\mathcal{L}_v^{-1}\, F_0(\mu_i)'\|_{\mathrm{L}^2} \le 2\,\varepsilon^2\, \|F_0(\mu_i)\|_{\mathrm{L}^2} \le 2\,\varepsilon^2\, c_{F_0}\, \delta^{5/2}\, \rho^2\ , \tag{4.5}$$

$$4\,\varepsilon^2\, \|\mathcal{L}_v^{-1}\, F_0(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2 \left\|\frac{F_0(\mu_i)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le 9\,\varepsilon^2\, c_{F_0}\, \delta^{5/2}\, \rho^2\ , \tag{4.6}$$

$$\varepsilon^2\, \|\mathcal{L}_v^{-1}\, \Delta F_0(\mu_1,\mu_2)'\|_{\mathrm{L}^2} + \varepsilon^2 \left\|\frac{\Delta F_0(\mu_1,\mu_2)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le 5\,\varepsilon^2\, c_{F_0}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma\ , \tag{4.7}$$

where $\Delta F_0(\mu_1, \mu_2) = F_0(\mu_1) - F_0(\mu_2)$. Since $\delta = c_\delta\, \rho^2$, we see that for $\varepsilon_0 = c_\varepsilon\, \rho^{-3}$, the contribution of F_0 to the bounds (4.2)–(4.4) can be made arbitrarily small, choosing c_ε sufficiently small, independently of ρ, or of the size of the system L. So what we need is more detailed information on r_2. Note that r_2 inherits the bounds of r and r_1, but with a factor ε^{-4}, so that we have to work a little more to show that the bounds on r_2 are finite as $\varepsilon \to 0$, and that (4.2)–(4.4) are also satisfied when the contribution of r_2 is taken into account. The essential input will be (1.33), where we showed that, as a dynamical variable, r_2 satisfies

$$\partial_t r_2 = -\frac{\chi}{\varepsilon^4}\, G\, \mathcal{L}_r\, r_2 + \frac{\chi}{16}\mu\, \mathcal{L}_{\mu,r}\, r_2' + \frac{1}{\varepsilon^4}\, F_6(s,\mu)\ , \quad r_2(x,0) = r_{2,0}(x)\ , \tag{4.8}$$

with $r_{2,0} = \frac{r_0}{\varepsilon^4} - \frac{r_1(\mu_0)}{\varepsilon^4}$. Since we know that $s(\mu)$ exists, we can view this equation as a linear inhomogeneous equation for r_2 and derive bounds from it. We first have the following lemma:

Lemma 4.1 *If r_2 solves (4.8), one has*

$$\|r_2(\mu)\|_{\mathrm{L}^2} \le \|r_{2,0}\|_{\mathrm{L}^2} + \|F_6(s(\mu),\mu)\|_{\mathrm{L}^2}\ , \tag{4.9}$$

$$\|\chi \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2(\mu)'\|_{\mathrm{L}^2} \le 64\, \|r_{2,0}\|_{\mathrm{L}^2} + \sqrt{2}\, \|\mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, F_6(s(\mu),\mu)'\|_{\mathrm{L}^2}\ , \tag{4.10}$$

for all μ with $\|\mu\|_\sigma \le c_\mu\, \rho$.

Remark 4.2 *Without the term $\mu\ \mathcal{L}_{\mu,r}\ r_2'$ in (4.8), the proof of (4.9) and (4.10) would follow immediately by positivity of $G\ \mathcal{L}_r$, and we would have the same estimates for $r_2(\mu_1) - r_2(\mu_2)$, but with $r_{2,0} = 0$ and $F_6(\mu)$ replaced by $F_6(\mu_1) - F_6(\mu_2)$. The term $\mu\ \mathcal{L}_{\mu,r}\ r_2'$ cannot destroy the positivity of $G\ \mathcal{L}_r$ if ε is sufficiently small (see (4.11) below or Proposition F.2), and it will add a (small) correction to the norm of the difference.*

Proof. From (4.8), using Young's inequality and Proposition F.2 (see subsection F.1 of Appendix F), we get

$$\begin{aligned} \partial_t \int r_2^2 &= -\frac{2\,\chi}{\varepsilon^4} \left(\int r_2\ G\ \mathcal{L}_r\ r_2 - \frac{\varepsilon^4}{16} \int r_2\ \mu\ \mathcal{L}_{\mu,r}\ r_2' \right) + \frac{2}{\varepsilon^4} \int r_2\ F_6(s,\mu)\ , \\ &\le -\frac{\chi}{\varepsilon^4} \int r_2^2 + \frac{2}{\chi\ \varepsilon^4} \int F_6(s,\mu)^2\ . \end{aligned} \tag{4.11}$$

The proof of (4.9) is completed integrating this differential inequality and noting that $\frac{2}{\chi^2} \le 1$. The proof of (4.10) can be found in subsection F.2. ■

Corollary 4.3 *Assume that $256\ \varepsilon^2\ \|r_{2,0}\|_{\mathrm{L}^2} \le \lambda_{2,2}\ c_\mu\ \rho$. Then there exists a constant C such that*

$$\begin{aligned} \varepsilon^4\ \|r_2(\mu)\|_{\mathrm{L}^2} &\le \varepsilon^2\ \lambda_{2,2}\ c_\mu\ \rho + \varepsilon^2\ C\ (\delta^{5/2}\ \rho^2 + \delta^4\ \rho)\ , \\ 4\ \varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ r_2(\mu)'\|_{\mathrm{L}^2} &\le \lambda_{2,2}\ c_\mu\ \rho + \varepsilon^2\ C\ (\delta^{5/2}\ \rho^2 + \delta^4\ \rho)\ , \end{aligned} \tag{4.12}$$

for all $\mu \in \mathcal{B}_\sigma(c_\mu\ \rho)$.

Proof. This is really a statement on F_6. For the proof, see Section C.5. ■
Choosing $\varepsilon_0 = c_\varepsilon\ \rho^{-4}$ with c_ε sufficiently small and a $\lambda_{2,2}$ sufficiently small, the r.h.s. of (4.12) is bounded by $c_\mu\ \rho$. Thus the hypotheses of the following lemma can be fulfilled.

Lemma 4.4 *Let r_2 be the solution of (4.8) with $\mu \in \mathcal{B}_\sigma(c_\mu\ \rho)$, and let $b(\alpha) = \max(2, \frac{\alpha^2}{1-\alpha^2})$. Assume that r_2 satisfies*

$$\varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ r_2(\mu)'\|_{\mathrm{L}^2} \le c_\mu\ \rho\ . \tag{4.13}$$

Then there exists a constant C such that

$$\left\| \frac{\chi\ \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ r_2(\mu)' \right\|_{\mathcal{W},\sigma} \le \frac{\frac{4\,\chi}{\delta}\ \|r_{2,0}\|_{\mathcal{W},\sigma-1} + \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ \frac{F_6(s(\mu),\mu)'}{G\ \mathcal{L}_r} \right\|_{\mathcal{W},\sigma} + C\ b(\alpha)\ \varepsilon^2\ c_\mu\ \rho}{1 - \varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2}}\ ,$$

for all ε satisfying $\varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2} < 1$.

Proof. See Section F.2. ■

Corollary 4.5 *Assume that* $256\ \varepsilon^2\ \|r_{2,0}\|_{\mathcal{W},\sigma-1} \leq \lambda_{2,\mathcal{W}}\ c_\mu\ \rho$. *Then there exists a constant* C *such that*

$$\varepsilon^2 \left\| \frac{\chi \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ r_2(\mu)' \right\|_{\mathcal{W},\sigma} \leq \frac{\max\left(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\right) \left((1 + C\ \varepsilon^2)\ c_\mu\ \rho + \varepsilon^2\ C\ \delta^{5/2}\ \rho^2\right)}{1 - \varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2}} + \frac{\lambda_{2,\mathcal{W}}\ c_\mu\ \rho}{1 - \varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2}}\,, \tag{4.14}$$

for all $\mu \in \mathcal{B}_\sigma(c_\mu\ \rho)$, *and for all* ε *satisfying* $\varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2} < 1$.

Proof. This is also a statement on F_6. For the proof, see Section C.5. ■
Choosing $\lambda_{2,\mathcal{W}}$ and c_ε sufficiently small, we bound the r.h.s. of (4.14) by $c_\mu\ \rho$. Thus, the hypotheses of the next Lemma can also be fulfilled.

Lemma 4.6 *Let* $r_2(\mu_i)$ *be the solution of (4.8) with* $\mu = \mu_i$, *and define* $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ *and* $\Delta F_6 = F_6(s(\mu_1), \mu_1) - F_6(s(\mu_2), \mu_2)$. *Assume that* $\mu_i \in \mathcal{B}_\sigma(c_\mu\ \rho)$, *and that*

$$\varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ r_2(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2 \left\| \frac{\chi\ \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ r_2(\mu_i)' \right\|_{\mathcal{W},\sigma} \leq c_\mu\ \rho\,. \tag{4.15}$$

Then there exists a constant C *such that*

$$\|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ \Delta r_2'\|_{\mathrm{L}^2} \leq \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ \Delta F_6'\|_{\mathrm{L}^2} + C\ \delta^{5/2}\ \rho\ \|\mu_1 - \mu_2\|_\sigma\,. \tag{4.16}$$

Corollary 4.7 *The map* $\mu \mapsto r_2(\mu)$ *satisfies*

$$\varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ (r_2(\mu_1) - r_2(\mu_2))'\|_{\mathrm{L}^2} \leq \varepsilon^2\ C\ (\delta^{5/2}\ \rho + \delta^4)\ \|\mu_1 - \mu_2\|_\sigma \tag{4.17}$$

for some constant C.

Proof. This is again a statement on F_6. For the proof, see Section C.5. ■
Again, choosing c_ε sufficiently small, the r.h.s. of (4.17) will be bounded by $\|\mu_1 - \mu_2\|_\sigma$. Thus the hypotheses of the next lemma can be fulfilled.

Lemma 4.8 *Let* $r_2(\mu_i)$ *be the solution of (4.8) with* $\mu = \mu_i$, *and define* $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ *and* $\Delta F_6 = F_6(s(\mu_1), \mu_1) - F_6(s(\mu_2), \mu_2)$. *Assume that* $\mu_i \in \mathcal{B}_\sigma(c_\mu\ \rho)$, *and that*

$$\varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ r_2(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2 \left\| \frac{\chi\ \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ r_2(\mu_i)' \right\|_{\mathcal{W},\sigma} \leq c_\mu\ \rho\,, \tag{4.18}$$
$$\varepsilon^2\ \|\chi\ \mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ \Delta r_2(\mu_i)'\|_{\mathrm{L}^2} \leq \|\mu_1 - \mu_2\|_\sigma\,.$$

Then there exists a constant C *such that*

$$\left\| \frac{\chi\ \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\ \Delta r_2' \right\|_{\mathcal{W},\sigma} \leq \frac{\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\ G\ \mathcal{L}_r}\ \Delta F_6' \right\|_{\mathcal{W},\sigma} + C\ b(\alpha)\ \|\mu_1 - \mu_2\|_\sigma}{1 - \varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2}}\,, \tag{4.19}$$

for all ε *satisfying* $\varepsilon^2\ 2\ b(\alpha)\ C_m\ c_\mu\ c_\delta^{-1/2} < 1$.

Proof. See Section F.2. ■

Corollary 4.9 *There exists a constant C such that*

$$\varepsilon^2 \left|\!\left|\!\left| \frac{\chi \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} \, \Delta r_2' \right|\!\right|\!\right|_{\mathcal{W},\sigma} \le \frac{\max(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}) \left(1 + \varepsilon^2 \, C \, (1 + \delta^{5/2} \, \rho)\right)}{1 - \varepsilon^2 \, 2 \, b(\alpha) \, C_m \, c_\mu \, c_\delta^{-1/2}} \, \|\!|\mu_1 - \mu_2|\!\|_\sigma \,, \tag{4.20}$$

for all $\mu \in \mathcal{B}_\sigma(c_\mu \, \rho)$, and for all ε satisfying $\varepsilon^2 \, 2 \, b(\alpha) \, C_m \, c_\mu \, c_\delta^{-1/2} < 1$.

Proof. This is also a statement on F_6. For the proof, see Section C.5. ■
We are now in position to prove that the hypotheses (4.2)–(4.4) are satisfied if ε is sufficiently small.

Theorem 4.10 *Condition 2.15 is satisfied if $\varepsilon_0 \le c_\varepsilon \sqrt{1 - 2\,\alpha^2} \, \rho^{-4}$ with c_ε sufficiently small and if the initial data μ_0 and s_0 are in the class $\mathcal{C}(K, L, \alpha, \hat{\varepsilon}, \hat{\varepsilon}_0, c_{s_0}, c_{\eta_0}, \sigma)$.*

Proof. By Definition 1.1 (see also (4.1)), if the initial data μ_0 and s_0 are in the class $\mathcal{C}(K, L, \alpha, \hat{\varepsilon}, \hat{\varepsilon}_0, c_{s_0}, c_{\eta_0}, \sigma)$, then

$$256 \, \varepsilon^2 \, \|r_{2,0}\|_{\sigma-1} \le \lambda_2 \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_\mu \, \rho \,,$$

with $\lambda_2 < \min(\frac{2}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2})$. This means that the hypotheses of Corollaries 4.3, 4.5, 4.7 and 4.9 on $r_{2,0}$ are satisfied with $\lambda_{2,2} + \lambda_{2,\mathcal{W}} \le \lambda_2 \, (\frac{\varepsilon}{\varepsilon_0})^2$. Collecting the results of these corollaries, and using (4.5)–(4.7), we have

$$\begin{aligned} \varepsilon^4 \, \|\!|r_2(\mu_i)|\!\|_{\mathrm{L}^2} + \varepsilon^2 \, \|\!|\mathcal{L}_v^{-1} \, F(\mu_i)'|\!\|_{\mathrm{L}^2} &\le (\frac{1}{4} + \varepsilon^2) \, \lambda_{2,2} \, c_\mu \, \rho + \varepsilon^2 \, C(\delta^{5/2} \, \rho^2 + \delta^4 \, \rho) \\ &\le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_\mu \, \rho \, \tilde{\lambda}_2(\lambda_2, \varepsilon_0, \rho, \delta) \,, \end{aligned} \tag{4.21}$$

where

$$\tilde{\lambda}_2(\lambda_2, \varepsilon_0, \rho, \delta) \equiv \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_\mu \, \rho \left(\lambda_2 \; + \varepsilon_0^2 \, C(\delta^{5/2} \, \rho + \delta^4)\right) .$$

In the same way, using $\varepsilon \le \varepsilon_0$, we have

$$4 \, \varepsilon^2 \, \|\!|\mathcal{L}_v^{-1} \, F(\mu_i)'|\!\|_{\mathrm{L}^2} + \varepsilon^2 \left|\!\left|\!\left| \frac{F(\mu_i)'}{\mathcal{L}_\mu} \right|\!\right|\!\right|_{\mathcal{W},\sigma} \le \tilde{\lambda}_1(\alpha, \varepsilon_0, \rho, \delta) \, c_\mu \, \rho \,, \tag{4.22}$$

$$4 \, \varepsilon^2 \, \|\!|\mathcal{L}_v^{-1} \, \Delta F'|\!\|_{\mathrm{L}^2} + \varepsilon^2 \left|\!\left|\!\left| \frac{\Delta F'}{\mathcal{L}_\mu} \right|\!\right|\!\right|_{\mathcal{W},\sigma} \le \tilde{\lambda}_1(\alpha, \varepsilon_0, \rho, \delta) \, \|\!|\mu_1 - \mu_2|\!\|_\sigma \,, \tag{4.23}$$

where

$$\tilde{\lambda}_1(\alpha, \varepsilon_0, \rho, \delta) \equiv \frac{\max\left(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\right) \left(1 + \varepsilon_0^2 \, C \, (\delta^{5/2} \, \rho + \delta^4 + 1)\right) + \lambda_2}{1 - \varepsilon^2 \, 2 \, b(\alpha) \, C_m \, c_\mu \, c_\delta^{-1/2}} \,.$$

Now, let $\delta = c_\delta\,\rho^2$, $\varepsilon_0 = c_\varepsilon\,\sqrt{1-2\,\alpha^2}\,\rho^{-4}$. Since $\lambda_2 < \min(\frac{2}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2})$, we have

$$\tilde\lambda_1(\alpha,\varepsilon_0,\rho,\delta) \le \frac{\max\left(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2}\right)(1+c_\varepsilon^2\,(1-2\,\alpha^2)\,C_1)+\lambda_2}{1-c_\varepsilon^2\,(1-2\,\alpha^2)\,C_2}\,,$$
$$\tilde\lambda_2(\lambda_2,\varepsilon_0,\rho,\delta) \le \frac{2}{3} + C_3\,c_\varepsilon^2\,(1-2\,\alpha^2) \le \frac{2}{3} + C_3\,c_\varepsilon^2\,,$$

for some (positive) constants C_1, C_2 and C_3. We now choose

$$c_\varepsilon < \min_{\alpha^2\in[0,1/2]}\left(\frac{1}{C_3}\,,\,\frac{1}{1-2\,\alpha^2}\left(\frac{\min(\frac{2}{3},\frac{1-2\,\alpha^2}{1-\alpha^2})-\lambda_2}{C_1\,\max(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2})+C_2}\right)\right)^{1/2}\,, \tag{4.24}$$

and get $\tilde\lambda_1(\alpha,\varepsilon_0,\rho,\delta) < 1$ and $\tilde\lambda_2(\lambda_2,\varepsilon_0,\rho,\delta) < 1$ as requested for Condition 2.15 to hold. Note that there always exists a $c_\varepsilon > 0$ satisfying inequality (4.24), since $\lambda_2 < \min(\frac{2}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2})$. ■

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Appendix A: Coercive functional for the phase

In this section, we prove the Proposition 2.2. This is a nondegeneracy result on the operator $\mathcal{L}_\mu$ based on a similar result of [CEE93] for $\mathcal{L}_{\mu,c} = \partial_x^4 + \partial_x^2$. We will need some technical alterations of their proof to take into account that $\mathcal{L}_\mu$ is of lower order than $\mathcal{L}_{\mu,c}$. However, since the two operators are equal in the limit $\varepsilon \to 0$ ($\mathcal{L}_\mu = G\,\mathcal{L}_{\mu,c}$ and $\lim_{\varepsilon\to 0} G = \mathrm{Id}$ by (1.8) and (1.32)), we will recover their result as a particular case.

Proposition A.1 *For all $L \ge 2\pi$, there exist a constant K and an antisymmetric periodic function ϕ such that for all $\gamma \in [\frac{1}{4}, 1]$, all $\varepsilon \le (\pi L^{2/5})^{-1}$ and any antisymmetric periodic function v, one has*

$$\frac{3}{4}(\mathcal{L}_v\,v, \mathcal{L}_v\,v) \le (v,v)_{\gamma\phi} \le \|\phi'\|_\infty\,(v,v) + (v'',v'')\,,$$
$$(\phi,\phi)_{\gamma\phi} \le K\,L^{16/5}\,,$$
$$(\phi,\phi) \le \tfrac{4}{3}\,L^3\,,$$

where the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\gamma\phi}$ are defined by

$$(v,v) = \int_{-L/2}^{L/2} dx\;v(x)^2\,,\quad (v,v)_{\gamma\phi} = \int_{-L/2}^{L/2} dx\;v(x)\,(\mathcal{L}_\mu + \gamma\phi'(x))\,v(x)\,,$$

and $\mathcal{L}_v$ is multiplicative in Fourier space with symbol $\mathcal{L}_v(k) = \sqrt{\frac{1}{3}\frac{1+k^4}{1+\frac{\varepsilon^2\,k^2}{2}}}$.

Remark A.2 *The restriction $\varepsilon \leq (\pi L^{2/5})^{-1}$ is a convenient one because then we can use the same function ϕ as that defined in [CEE93]. We will see later that we need a much stronger restriction ($\varepsilon \leq \mathcal{O}(L^{-32/5})$) anyway.*

proof of Proposition A.1. The proof really amounts to construct the function ϕ. Let $q \equiv \frac{2\pi}{L} \leq 1$ and M the smallest integer (strictly) larger than $\frac{1}{2} L^{7/5}$. We define ϕ by

$$\phi(x) = \sum_{n \in \mathbf{Z}} e^{iqnx} \phi_n ,$$

where the Fourier coefficients ϕ_n are given by

$$\phi_n = \begin{cases} 0 , & n = 0 \\ \frac{4\ i}{qn} , & 1 \leq |n| \leq 2M \\ \frac{4\ i\ f(|n|/2M-1)}{qn} , & \text{otherwise} \end{cases} ,$$

where f is a non-increasing $\mathcal{C}^1$ function satisfying $f(0) = 1$, $f'(0) = 0$ and

$$f \geq 0, \quad \sup |f'| < 1, \quad \int_0^\infty \mathrm{d}k\ (1+k)^2\ |f(k)|^2 < \infty .$$

The proof then follows from the three technical lemmas below. ■

Lemma A.3 *There exists a constant K such that the function ϕ defined above satisfies*

$$\begin{aligned} (\phi, \phi) &\leq \tfrac{4}{3}\ L^3 , \\ (\phi, \phi)_{\gamma\phi} &\leq K\ L^{16/5} , \\ (v, v)_{\gamma\phi} &\leq K\ L^{7/5}\ \|v\|_{\mathrm{L}^2}^2 + \|v''\|_{\mathrm{L}^2}^2 . \end{aligned}$$

for all periodic antisymmetric functions v.

Proof. For the first inequality, we have

$$(\phi, \phi) = \frac{4\pi}{q} \sum_{n=1}^\infty |\phi_n|^2 \leq \frac{4^3\ \pi}{q^3} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{6} \left(\frac{4\pi}{q} \right)^3 = \frac{4}{3}\ L^3 .$$

For the second inequality, we use that ϕ is periodic, so that $\int \phi^2 \phi' = 0$, giving

$$(\phi, \phi)_{\gamma\phi} = (\phi, \mathcal{L}_\mu \phi) = \frac{4\pi}{q} \sum_{n=1}^\infty \mathcal{L}_\mu(qn)\ |\phi_n|^2 ,$$

where

$$\mathcal{L}_\mu(qn) = \frac{(qn)^4 - (qn)^2}{1 + \frac{\varepsilon^2 (qn)^2}{2}} .$$

Since $\mathcal{L}_\mu(qn) \le (qn)^4$ and $M < L^{7/5}$, we get

$$\begin{aligned}(\phi,\phi)_{\gamma\phi} = (\phi, \mathcal{L}_\mu \phi) &= \frac{4\pi}{q} \sum_{n=1}^{\infty} \mathcal{L}_\mu(qn)\, |\phi_n|^2 \\ &\le 4^3 \pi\, q \left(\sum_{n=1}^{2M} n^2 + (2M)^2 \sum_{n=1}^{\infty} \Big(1 + \frac{n}{2M}\Big)^2 f\Big(\frac{n}{2M}\Big)^2 \right) \\ &\le C\, L^{16/5} \left(1 + \int_0^\infty \mathrm{d}k\, (1+k)^2\, f(k)^2\right) .\end{aligned}$$

Finally, using again $\mathcal{L}_\mu(qn) \le (qn)^4$, we have

$$(v,v)_{\gamma\phi} \le \|\phi'\|_{\mathrm{L}^\infty}\, \|v\|^2_{\mathrm{L}^2} + \|v''\|^2_{\mathrm{L}^2} .$$

Using the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\|\phi'\|_{\mathrm{L}^\infty} &\le 2 \sum_{n=1}^{\infty} |qn|\, |\phi_n| \le 16M + 2 \sum_{n=1}^{\infty} \Big| f\Big(\frac{n}{2M}\Big)\Big| \\ &\le 16M + 4M \int_0^\infty \mathrm{d}k \left(\frac{1+k}{1+k}\right) |f(k)| \\ &\le C\, L^{7/5} \left(1 + \sqrt{\int_0^\infty \mathrm{d}k\, (1+k)^2\, |f(k)|^2}\right) .\end{aligned}$$

This completes the proof of the Lemma. ■

Lemma A.4 *For all $L \ge 2\pi$, for all $\gamma \in [\frac{1}{4}, 1]$ and for all $\varepsilon \le \frac{1}{Mq}$, one has*

$$(v,v)_{\gamma\phi} \ge \frac{3}{4} (\mathcal{L}_v\, v, \mathcal{L}_v\, v) . \tag{A.1}$$

Proof. Following [CEE93] one shows first that

$$(v,v)_{\gamma\phi} = 2\, L \left[\sum_{n>0} (\mathcal{L}_\mu(qn) + \gamma \psi_{2n}) v_n^2 + 2\gamma \sum_{k>m>0} v_k\, v_m (\psi_{|k+m|} - \psi_{|k-m|}) \right] ,$$

where $\psi_n = -iqn\, \phi_n$. Then one notices that for $0 \le \varepsilon \le 1$, one has

$$\mathcal{L}_\mu(qn) + \gamma \psi_{2n} \ge \tau(qn)^2 \equiv \frac{1}{2} \frac{1 + (qn)^4}{1 + \frac{\varepsilon^2 (qn)^2}{2}} \ge \tau_1(qn)^2 \equiv \frac{1}{2} \frac{(qn)^4}{1 + \frac{\varepsilon^2 (qn)^2}{2}} .$$

The definition of τ here is different from that of [CEE93], except in the $\varepsilon = 0$ limit. Set now $w_n = v_n \tau_n$ (in particular $w = \sqrt{\frac{3}{2}}\, \mathcal{L}_v\, v$), so that

$$(v,v)_{\gamma\phi} \ge 2\, L \left[\sum_{n>0} w_n^2 + 2\gamma \sum_{0<m<k} w_k \frac{\psi_{|k+m|} - \psi_{|k-m|}}{\tau_k \tau_m} w_m \right] \equiv (w\ ,\ (\mathrm{Id} + 2\gamma\Gamma)\, w) .$$

In Lemma A.5 below, we prove that the Hilbert–Schmidt norm of $2\gamma\Gamma$ is less than $\frac{1}{2}$, hence

$$(v,v)_{\gamma\phi} \geq \frac{1}{2}\,(w,w) = \frac{3}{4}\,(\mathcal{L}_v\,v, \mathcal{L}_v\,v)\;,$$

which proves inequality (A.1). ■

Lemma A.5 *For all $L \geq 2\pi$, for all $\gamma \in [\frac{1}{4}, 1]$ and for all $\varepsilon \leq \frac{1}{Mq}$, the Hilbert–Schmidt norm of $2\gamma\Gamma$ is smaller than $\frac{1}{2}$, where the operator Γ is defined by*

$$(w, \Gamma w) = L \sum_{0<m<k} w_k \frac{\psi_{|k+m|} - \psi_{|k-m|}}{\tau_k \tau_m} w_m\;.$$

Proof. Note again that $\varepsilon \leq \frac{1}{Mq} = (\pi L^{2/5})^{-1}$ is only a convenient restriction (see remark A.2). To prove this lemma, it is sufficient (see [CEE93]) to show that

$$\|\Gamma\|_{\text{HS}}^2 \equiv \sum_{0<m<k} \left| \frac{\psi_{|k+m|} - \psi_{|k-m|}}{\tau_k\;\tau_m} \right|^2 < \frac{1}{16}\;. \tag{A.2}$$

By definition of ϕ, for all $k > m > 0$, we have

$$|\psi_{k-m} - \psi_{k+m}| = 0\;,\quad \text{if } k+m \leq 2M\;,$$

and

$$|\psi_{k-m} - \psi_{k+m}| \leq 4 \min\left\{1, \frac{m}{M}\right\}\;,\quad \text{for all } k > m\;.$$

We distinguish two sets of summation indices $S = S_{\text{I}} \cup S_{\text{II}}$ in the sum (A.2),

$$S_{\text{I}} = \left\{(m,k) \in \mathbf{N}^2 \text{ s.t. } M+1 \leq m \text{ and } m+1 \leq k\right\}\;,$$
$$S_{\text{II}} = \left\{(m,k) \in \mathbf{N}^2 \text{ s.t. } 1 \leq m \leq M \text{ and } 2M-m+1 \leq k\right\}\;,$$

and write $\|\Gamma\|_{\text{HS}}^2 = T_{\text{I}} + T_{\text{II}}$ accordingly.

In the region I, we have $m > M$, and using $\varepsilon \leq \frac{1}{Mq}$ and $\frac{1}{\tau(k)} \leq \frac{1}{\tau_1(k)}$, we get

$$T_{\text{I}} \leq 16 \sum_{m=M+1}^{\infty} \frac{1}{\tau(qm)^2} \sum_{k=m+1}^{\infty} \frac{1}{\tau(qk)^2} \leq 16 \int_M^{\infty} \mathrm{d}m\, \frac{1}{\tau_1(qm)^2} \int_m^{\infty} \mathrm{d}k\, \frac{1}{\tau_1(qk)^2} \leq \frac{200}{9} \frac{1}{q^8\,M^6}\;,$$

whereas in the region II, we have $m \leq M$ and $k \geq M+1$, and using again $\varepsilon \leq \frac{1}{Mq}$ and $\frac{1}{\tau(k)} \leq \frac{1}{\tau_1(k)}$, we get

$$\begin{aligned}
T_{\text{II}} &\leq \frac{16}{M^2} \sum_{m=1}^{M} \frac{m^2}{\tau(qm)^2} \sum_{k=2M-m+1}^{\infty} \frac{1}{\tau(qk)^2} \leq \frac{16}{M^2} \sum_{m=1}^{M} \frac{m^2}{\tau(qm)^2} \int_M^{\infty} \mathrm{d}k\, \frac{1}{\tau_1(qk)^2} \\
&\leq \frac{160}{3} \frac{1}{M^5\,q^4} \sum_{m=1}^{M} \left(\frac{1}{q^2} \frac{q^2\,m^2}{1+m^4\,q^4} + \frac{1}{2\,M^2\,q^4} \frac{q^4\,m^4}{1+m^4\,q^4} \right)
\end{aligned}$$

$$\le \frac{160}{3}\frac{1}{M^5\,q^4}\left(\frac{1}{q^2}\int_0^\infty \frac{\mathrm{d}m}{1+q^2\,m^2}+\frac{1}{2\,M\,q^4}\right)\,.$$

Collecting these results, we get

$$\|\Gamma\|_{\mathrm{HS}}^2 \le \frac{80\,\pi}{3}\,\frac{1}{q^7\,M^5}+\frac{440}{9}\,\frac{1}{q^8\,M^6}\,.$$

Note that this bound is worse than that of [CEE93] by numerical factors only (in their bound $\frac{80\,\pi}{3}$ is replaced by $\frac{128}{3}$ and $\frac{440}{9}$ by $\frac{16}{3}$), but is uniform in $\varepsilon \le \frac{1}{Mq}$. This motivates the restriction $\varepsilon \le \frac{1}{Mq}$. The proof is then completed using $M > \frac{1}{2}\,L^{7/5}$. ■

Appendix B: Proofs for Section 2.3

Lemma B.1 *Let $\sigma \ge \frac{3}{2}$. There exists a constant C such that for all $n \le \sigma - \frac{3}{2}$ and for all $m \le \sigma - 1$, we have*

$$\|f^{(m)}\|_{\sigma-m} + \|G\,f^{(m)}\|_{\sigma-m} \;\le\; c_\infty\,\delta^m \quad \|f\|_\sigma\,, \tag{B.1}$$

$$\|f^{(n)}\|_{\mathrm{L}^\infty} + \|G\,f^{(n)}\|_{\mathrm{L}^\infty} \quad\le\; c_\infty\,\delta^{n+\frac{1}{2}}\,\|f\|_\sigma\,, \tag{B.2}$$

where $f^{(m)}$ is the m–th order spatial derivative of f.

Proof. Throughout the proof, we use that G acts multiplicatively in Fourier space, $(G\,f)_n = G(qn)\,f_n$ with

$$G(k) = \frac{1}{1+\frac{\varepsilon^2\,k^2}{2}} \le 1\,,$$

so that G is a bounded operator in the l^p and $\|\cdot\|_\sigma$ norms. Using that $\|f\|_{\mathrm{L}^\infty} \le \|f\|_{l^1}$, and that the space derivative commutes with G, we see that we need only prove (B.1) and (B.2) for the terms without G, and with L^∞ replaced by l^1 in (B.2). In the sequel, we denote by K the operator with symbol $K(k) = |k|$.

For (B.1), we use that $|x| \le \sqrt{1+x^2}$ and that $\|\cdot\|_{\mathrm{L}^2} = \sqrt{L}\|\cdot\|_{l^2}$ to show that

$$\begin{aligned}
\|f^{(m)}\|_{\sigma-m} &= \|f^{(m)}\|_{\mathcal{W},\sigma-m} + \|f^{(m)}\|_{\mathrm{L}^2} \le \delta^m\,\|f\|_{\mathcal{W},\sigma} + \|f^{(m)}\|_{\mathrm{L}^2}\\
&\le \delta^m\,\|f\|_{\mathcal{W},\sigma} + \sqrt{L}\,\|K^m\,P_< f\|_{l^2} + \sqrt{L}\,\|K^m\,P_> f\|_{l^2}\\
&\le \delta^m\,\|f\|_{\mathcal{W},\sigma} + \delta^m\,\sqrt{L}\,\|P_< f\|_{l^2} + \sqrt{L}\,\delta^m\|(1+(K/\delta)^2)^{m/2}\,P_> f\|_{l^2}\\
&\le \delta^m\,\|f\|_{\mathcal{W},\sigma} + \delta^m\,\|f\|_{\mathrm{L}^2} + \delta^m\sqrt{\int_{-\infty}^\infty \frac{2\pi\,\mathrm{d}x}{(1+x^2)^{\sigma-m}}}\,\|f\|_{\mathcal{W},\sigma}\\
&\le \delta^m\,\|f\|_\sigma\left(1+\sqrt{\int_{-\infty}^\infty \frac{2\pi\,\mathrm{d}x}{(1+x^2)^{\sigma-m}}}\right)\,.
\end{aligned}$$

For (B.2), using the Cauchy–Schwartz inequality, we have

$$\|P_< f\|_{l^1} \le \sqrt{\frac{2\,\delta}{q}}\,\|P_< f\|_{l^2} \le \sqrt{\frac{\delta}{\pi}}\,\sqrt{L}\,\|P_< f\|_{l^2} \le \sqrt{\frac{\delta}{\pi}}\,\|P_< f\|_{\mathrm{L}^2}\,, \tag{B.3}$$

so that

$$\begin{aligned}\|f^{(n)}\|_{l^1} &\leq \|K^n\, P_< f\|_{l^1} + \|K^n\, P_> f\|_{l^1}\\ &\leq \frac{\delta^{n+\frac{1}{2}}}{\sqrt{\pi}}\, \|f\|_{\mathrm{L}^2} + \|K^n\, P_> f\|_{l^1}\\ &\leq \delta^{n+\frac{1}{2}}\, \|f\|_{\mathrm{L}^2} + \delta^{n+\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{\sigma-n}{2}}}\, \|f\|_{\mathcal{W},\sigma}\\ &\leq \delta^{n+\frac{1}{2}}\, \|f\|_\sigma \left(1 + \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{\sigma-n}{2}}}\right) .\end{aligned}$$

Since $\sigma - n \geq \frac{3}{2}$ and $\sigma - m \geq 1$, setting

$$c_\infty = 1 + \max\left\{ \sqrt{\int_{-\infty}^{\infty} \frac{2\pi\, \mathrm{d}x}{1+x^2}}\ ,\ \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{4}}} \right\} ,$$

the proof is completed. ■

Before proving Propositions 2.11 and 2.12, we prove a simpler lemma (see also [BKL94]).

Lemma B.2 *Let $\sigma_1, \sigma_2 \geq \frac{3}{2}$ and $\sigma = \min(\sigma_1, \sigma_2) \geq \frac{3}{2}$, then there exists a constant c_b depending only on σ such that*

$$\|uv\|_{\mathcal{N},\sigma} \leq c_b\, \sqrt{\delta}\, \|u\|_{\mathcal{N},\sigma_1}\, \|v\|_{\mathcal{N},\sigma_2}\ , \tag{B.4}$$

and if $\sigma < 1$, we have the two particular cases

$$\|uv\|_{\mathcal{N},\frac{1}{2}} \leq c_b\, \sqrt{\delta}\, \|u\|_{\mathcal{N},1}\, \|v\|_{\mathcal{N},1}\ , \tag{B.5}$$

$$\|uv\|_{\mathcal{N},0} \leq c_b\, \sqrt{\delta}\, \|u\|_{\mathrm{L}^2}\, \|v\|_{\mathrm{L}^2}\ . \tag{B.6}$$

Proof. We begin with (B.6). We have

$$\|uv\|_{\mathcal{N},0} = \frac{\sqrt{\delta}}{q} \sup_{n\in\mathbf{Z}} \sum_{m\in\mathbf{Z}} |u_n|\, |v_{m-n}| \leq \frac{\sqrt{\delta}\, L}{2\pi}\, \|u\|_{l^2}\, \|v\|_{l^2} \leq \frac{\sqrt{\delta}}{2\pi}\, \|u\|_{\mathrm{L}^2}\, \|v\|_{L^2}\ .$$

For the other inequalities, we proceed as follows. Let $p = \frac{q}{\delta}$, then we have

$$\begin{aligned}\|uv\|_{\mathcal{N},\sigma} &\leq \frac{1}{\sqrt{\delta}} \sup_{n\in\mathbf{Z}} \frac{(1+(pn)^2)^{\frac{\sigma}{2}}}{p} \sum_{m\in\mathbf{Z}} |u_m||v_{n-m}|\\ &\leq \left(\sup_{n\in\mathbf{Z}} \sum_{m\in\mathbf{Z}} \frac{p}{(1+(pm)^2)^{\frac{\sigma_1}{2}}} \frac{(1+(pn)^2)^{\frac{\sigma}{2}}}{(1+(p(m-n))^2)^{\frac{\sigma_2}{2}}}\right) \sqrt{\delta}\, \|u\|_{\mathcal{N},\sigma_1}\, \|v\|_{\mathcal{N},\sigma_2}\\ &\leq \left(\sup_{x\in\mathbf{R}} \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{(1+y^2)^{\frac{\sigma}{2}}} \frac{(1+x^2)^{\frac{\sigma}{2}}}{(1+(x-y)^2)^{\frac{\sigma}{2}}}\right) \sqrt{\delta}\, \|u\|_{\mathcal{N},\sigma_1}\, \|v\|_{\mathcal{N},\sigma_2}\ ,\\ \|uv\|_{\mathcal{N},\frac{1}{2}} &\leq \left(\sup_{x\in\mathbf{R}} \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{(1+y^2)^{\frac{1}{2}}} \frac{(1+x^2)^{\frac{1}{4}}}{(1+(x-y)^2)^{\frac{1}{2}}}\right) \sqrt{\delta}\, \|u\|_{\mathcal{N},1}\, \|v\|_{\mathcal{N},1}\ .\end{aligned}$$

Thus we define

$$c_b = \max\left\{\frac{1}{2\pi}\ ,\ S\Big(1,\frac{1}{2}\Big)\ ,\ S(\sigma,\sigma)\right\}\ ,$$

where

$$S(\sigma,\sigma') = \sup_{x\in\mathbf{R}} \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{(1+y^2)^{\frac{\sigma}{2}}} \frac{(1+x^2)^{\frac{\sigma'}{2}}}{(1+(x-y)^2)^{\frac{\sigma}{2}}}\ .$$

To see that $c_b < \infty$, we can assume without loss of generality that $x \geq 0$, and split the y integration into two pieces, $y \in (-\infty, x/2]$ and $y \in [x/2, \infty)$. We then have

$$\begin{aligned} y \in (-\infty, x/2] \ &\Rightarrow\ \frac{1+x^2}{1+(x-y)^2} \leq 4\ ,\ \frac{1}{1+(x-y)^2} \leq \frac{1}{1+y^2}\ , \\ y \in [x/2, \infty) \ &\Rightarrow\ \frac{1+x^2}{1+y^2} \leq 4\ ,\ \frac{1}{1+y^2} \leq \frac{1}{1+(x-y)^2}\ , \end{aligned}$$

from which we get

$$\begin{aligned} S(\sigma,\sigma) &\leq \int_{-\infty}^{x/2} \mathrm{d}y\, \frac{2^\sigma}{(1+y^2)^{\frac{\sigma}{2}}} + \int_{x/2}^{\infty} \mathrm{d}y\, \frac{2^\sigma}{(1+(x-y)^2)^{\frac{\sigma}{2}}} \leq \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{2\, 2^\sigma}{(1+y^2)^{\frac{\sigma}{2}}} \leq 2^\sigma\, 15\ , \\ S\Big(1,\frac{1}{2}\Big) &\leq \int_{-\infty}^{x/2} \mathrm{d}y\, \frac{2^{1/2}}{(1+y^2)^{\frac{3}{4}}} + \int_{x/2}^{\infty} \mathrm{d}y\, \frac{2^{1/2}}{(1+(x-y)^2)^{\frac{3}{4}}} \leq \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{2\, 2^{1/2}}{(1+y^2)^{\frac{3}{4}}} \leq 2^\sigma\, 15\ . \end{aligned}$$

The proof of the Lemma is completed. ■

Proposition B.3 *Let $\|u\|_{\sigma_1} < \infty$, $\|v\|_{\sigma_2} < \infty$ and $\sigma = \min(\sigma_1,\sigma_2) \geq \frac{3}{2}$, then there exists a constant C_m depending only on σ such that*

$$\|uv\|_\sigma \leq C_m \sqrt{\delta}\, \|u\|_{\sigma_1}\, \|v\|_{\sigma_2}\ , \tag{B.7}$$

and if $\sigma < 1$, we have the two particular cases

$$\|uv\|_{\mathcal{W},\frac{1}{2}} \leq C_m \sqrt{\delta}\, \|u\|_1\, \|v\|_1\ , \tag{B.8}$$

$$\|uv\|_{\mathcal{W},0} \leq C_m \sqrt{\delta}\, \|u\|_{\mathrm{L}^2}\, \|v\|_{\mathrm{L}^2}\ . \tag{B.9}$$

Proof. We first note that if $\sigma = \min(\sigma_1,\sigma_2) \geq \frac{3}{2}$, by Lemma B.1, we have

$$\|uv\|_{\mathrm{L}^2} \leq \|u\|_{\mathrm{L}^\infty}\, \|v\|_{\mathrm{L}^2} \leq C\sqrt{\delta}\, \|u\|_{\sigma_1}\, \|v\|_{\sigma_2}\ .$$

So the L^2 part of (B.7) is proved. For the $\|\cdot\|_{\mathcal{W},\sigma}$ part of (B.7), for (B.8) and for (B.9), we write $u = u_< + u_>$, where $u_< = P_< u$ and $u_> = P_> u$ and the same for v. Then we have

$$\|uv\|_{\mathcal{W},\sigma} \leq \|uv\|_{\mathcal{N},\sigma} \leq \|u_<\, v_<\|_{\mathcal{N},\sigma} + \|u_<\, v_>\|_{\mathcal{N},\sigma} + \|u_>\, v_<\|_{\mathcal{N},\sigma} + \|u_>\, v_>\|_{\mathcal{N},\sigma}\ .$$

Clearly, $\|P_> f\|_{\mathcal{N},\sigma} \leq \|f\|_{\mathcal{W},\sigma} \leq \|f\|_\sigma$, and we can apply directly Lemma B.2 to the last term. The first three terms are bounded using Lemmas B.4 and B.5 below. ■

Lemma B.4 *Let $\sigma \geq 0$, then there exists a constant C depending only on σ such that*

$$\|(P_{>}u)\,(P_{<}v)\|_{\mathcal{N},\sigma} \leq C\,\sqrt{\delta}\,\|u\|_{\sigma}\,\|v\|_{\sigma}\,. \tag{B.10}$$

Proof. Let $p = \frac{q}{\delta}$. By the Cauchy–Schwartz inequality, we have $\|P_{<}v\|_{l^1} \leq \sqrt{\delta}\,\|v\|_{\mathrm{L}^2}$ (see (B.3) above), so that

$$\begin{aligned}
\|(P_{>}u)\,(P_{<}v)\|_{\mathcal{N},\sigma} &\leq \frac{1}{\sqrt{\delta}} \sup_{n\in\mathbf{Z}} \frac{(1+(pn)^2)^{\frac{\sigma}{2}}}{p} \sum_{|m|\leq\frac{1}{p}} |v_m|\,|u_{n-m}| \\
&\leq \|u\|_{\mathcal{W},\sigma}\,\|P_{<}v\|_{l^1} \sup_{n\in\mathbf{Z}} \sup_{|m|\leq 1} \left(\frac{1+n^2}{1+(m-n)^2}\right)^{\frac{\sigma}{2}} \\
&\leq C\,\sqrt{\delta}\,\|u\|_{\mathcal{W},\sigma}\,\|v\|_{\mathrm{L}^2} \leq C\,\sqrt{\delta}\,\|u\|_{\sigma}\,\|v\|_{\sigma}\,.
\end{aligned}$$

This completes the proof of the Lemma. ∎

Lemma B.5 *Let $\sigma \geq 0$, then there exists a constant C depending only on σ such that*

$$\|(P_{<}u)\,(P_{<}v)\|_{\mathcal{N},\sigma} \leq C\,\sqrt{\delta}\,\|u\|_{\sigma}\,\|v\|_{\sigma}\,. \tag{B.11}$$

Proof. Let $p = \frac{q}{\delta}$, we have

$$\begin{aligned}
\|(P_{<}u)(P_{<}v)\|_{\mathcal{N},\sigma} &\leq \frac{1}{\sqrt{\delta}} \sup_{n\in\mathbf{Z}} \frac{(1+(pn)^2)^{\frac{\sigma}{2}}}{p} \sum_{\substack{|m|\leq\frac{1}{p} \\ |m-n|\leq\frac{1}{p}}} |u_m|\,|v_{n-m}| \\
&\leq \frac{5^{\frac{\sigma}{2}}}{2\pi}\,\sqrt{\delta}\,L \sum_{\substack{|m|\leq\frac{1}{p} \\ |m-n|\leq\frac{1}{p}}} |u_m|\,|v_{n-m}| \leq 5^{\frac{\sigma}{2}}\,\sqrt{\delta}\,L\,\|u\|_{l^2}\,\|v\|_{l^2} \\
&\leq 5^{\frac{\sigma}{2}}\,\sqrt{\delta}\;\;\|u\|_{\mathrm{L}^2}\,\|v\|_{\mathrm{L}^2} \leq C\,\sqrt{\delta}\,\|u\|_{\sigma}\,\|v\|_{\sigma}\,.
\end{aligned}$$

This completes the proof. ∎

Proposition B.6 *Let $\|u\|_{\sigma_1} < \infty$, $\|v\|_{\sigma_2} < \infty$ and $\sigma = \min(\sigma_1, \sigma_2) \geq \frac{3}{2}$, then*

$$\left\|\frac{u}{1+v}\right\|_{\sigma} \leq \frac{\|u\|_{\sigma_1}}{1 - C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2}}\,.$$

for all v satisfying $C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2} < 1$, where C_m is the constant of Proposition B.3.

Proof. The idea is to write a geometric series for $\frac{1}{1+v}$, and to use that since $C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2} < 1$, the series is convergent. Indeed, using Proposition B.3 inductively, we have

$$\left\|\frac{u}{1+w}\right\|_{\sigma} \leq \sum_{m\geq 0} \|uv^m\|_{\sigma} \leq \|u\|_{\sigma_1} \sum_{m\geq 0} \Big(C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2}\Big)^m \leq \frac{\|u\|_{\sigma_1}}{1 - C_m\,\sqrt{\delta}\,\|v\|_{\sigma_2}}\,. \tag{B.12}$$

This completes the proof. ∎

Proposition B.7 *Let $\delta \geq 2$, then*

$$\left\| e^{-\mathcal{L}_\mu t}\, f(\cdot) \right\|_{\mathcal{W},\sigma} \leq e^{-4t}\, \|f(\cdot)\|_{\mathcal{W},\sigma} \,,$$

$$\left\| \int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu (t-s)}\; g'(\cdot,s) \right\|_{\mathcal{W},\sigma} \leq \sup_{0\leq s\leq t} \left\| \frac{g'(\cdot,s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \,,$$

where $e^{-\mathcal{L}_\mu t}$ is the propagation Kernel associated with $\partial_t f = -\mathcal{L}_\mu f$.

Proof. In Fourier space, the propagation Kernel $e^{-\mathcal{L}_\mu t}$ acts as

$$\left(e^{-\mathcal{L}_\mu t}\, f \right)_n = e^{-\mathcal{L}_\mu (qn) t}\, f_n \,, \quad \text{with} \quad \mathcal{L}_\mu(k) = \frac{k^4 - k^2}{1 + \frac{\varepsilon^2 k^2}{2}} \,.$$

For $\varepsilon \leq 1$, $\delta \geq 2$ and $|k| \geq \delta$, one has $\mathcal{L}_\mu(k) \geq \mathcal{L}_\mu(\delta) \geq 4$, which gives

$$\sup_{t\geq 0} \left\| e^{-\mathcal{L}_\mu t}\; f(\cdot) \right\|_{\mathcal{W},\sigma} \leq e^{-\mathcal{L}_\mu(\delta)\; t}\; \|f(\cdot)\|_{\mathcal{W},\sigma} \leq e^{-4t}\; \|f(\cdot)\|_{\mathcal{W},\sigma} \,.$$

Next, we use that for $q|n| \geq \delta$, we have

$$\begin{aligned}
\left| \int_0^t \mathrm{d}s \left(e^{-\mathcal{L}_\mu (t-s)}\; g'(\cdot,s) \right)_n \right| &\leq \int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu(qn)\;(t-s)}\;\; |qn|\; |g_n(s)| \\
&\leq \frac{q}{\sqrt{\delta}} \left(1 + (\frac{qn}{\delta})^2 \right)^{-\sigma/2} \mathcal{L}_\mu(qn) \int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu(qn)\;(t-s)}\; \left\| \frac{g'(\cdot,s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \\
&\leq \frac{q}{\sqrt{\delta}} \left(1 + (\frac{qn}{\delta})^2 \right)^{-\sigma/2} \left(1 - e^{-\mathcal{L}_\mu(qn)\; t} \right) \sup_{0\leq s\leq t} \left\| \frac{g'(\cdot,s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \,.
\end{aligned}$$

Since $1 - e^{-\mathcal{L}_\mu(qn)\; t} \leq 1$ for $qn \geq \delta \geq 2$, the proof is completed. ■

Lemma B.8 *Let $\delta \geq 2$, then*

$$\left\| \frac{g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \leq \frac{\sqrt{2}}{\delta}\, \|g\|_{\mathcal{W},\sigma-1} \,,$$

$$\left\| \frac{G\, g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \leq \frac{2^{7/2}}{3\,\delta^3}\, \|g\|_{\mathcal{W},\sigma-3} \,.$$

Proof. We have

$$\begin{aligned}
\left\| \frac{g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} &\leq \frac{\sqrt{\delta}}{q} \sup_{|n|\geq \frac{\delta}{q}} \left(1 + (\frac{qn}{\delta})^2 \right)^{\sigma/2} \left(1 + \frac{\varepsilon^2 (qn)^2}{2} \right) \frac{|qn|\; |g_n|}{(qn)^4 - (qn)^2} \\
&\leq \frac{1}{\delta} \frac{\sqrt{\delta}}{q} \sup_{|n|\geq \frac{\delta}{q}} \frac{\left(1 + (\frac{qn}{\delta})^2 \right)^{1/2}}{\frac{|qn|}{\delta}} \frac{\left(1 + \frac{(qn)^2}{2} \right)}{(qn)^2 - 1} \left(1 + (\frac{qn}{\delta})^2 \right)^{\frac{\sigma-1}{2}} |g_n|
\end{aligned}$$

$$\leq \frac{1}{\delta}\, \|g\|_{\mathcal{W},\sigma-1} \left(\sup_{x\geq 1} \frac{\sqrt{1+x^2}}{x}\right) \left(\sup_{x\geq 2} \frac{1+\frac{x^2}{2}}{x^2-1}\right) \leq \frac{\sqrt{2}}{\delta}\, \|g\|_{\mathcal{W},\sigma-1} \,,$$

and similarly

$$\begin{aligned} \left\| \frac{G\, g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} &\leq \frac{\sqrt{\delta}}{q} \sup_{|n|\geq \frac{\delta}{q}} \Big(1+(\frac{qn}{\delta})^2\Big)^{\sigma/2} \frac{|qn|\,|g_n|}{(qn)^4-(qn)^2} \\ &\leq \frac{1}{\delta^3}\, \|g\|_{\mathcal{W},\sigma-3} \left(\sup_{x\geq 1} \frac{\sqrt{1+x^2}}{x}\right)^3 \left(\sup_{x\geq 2} \frac{x^4}{x^4-x^2}\right) \leq \frac{2^{7/2}}{3\,\delta^3}\, \|g\|_{\mathcal{W},\sigma-3} \,. \end{aligned}$$

This completes the proof. ∎

Appendix C: Bounds on nonlinear terms

We begin by recalling that

$$\begin{aligned} F_0(s,\mu) = \alpha^2\, \chi\Big(&(2+\varepsilon^2(1+\alpha^2))\; s^2 + \frac{s'\,\mu}{1+\varepsilon^4\,\alpha^2\, s} - \frac{2\,\varepsilon^2\,\alpha^2\, s\, s''}{1+\varepsilon^4\,\alpha^2\, s}\Big) \\ &- \frac{1}{4}\, G\,\mu^2 - \frac{1}{4}\, G\,(\mu^2)'' \,. \end{aligned}$$

Let $r = s - \frac{\varepsilon^2}{2}s''$. We will now prove that we can write $F_0(s,\mu)$ as

$$F_0(s,\mu) = F_1(s,\mu) + G\; F_2(s,r,\mu) \,, \tag{C.1}$$

with

$$\begin{aligned} F_1(s,\mu) &= \chi\,\alpha^2\, \left((2+\varepsilon^2(1+\alpha^2))\; s^2 - \frac{\alpha^2\,(\varepsilon^4\, s')\, s\,\mu}{1+\varepsilon^4\,\alpha^2\, s} - \frac{2\,\alpha^2\, s\,(\varepsilon^2\, s'')}{1+\varepsilon^4\,\alpha^2\, s}\right) \,, \\ F_2(s,r,\mu) &= -\frac{1}{4}\,\mu^2 - \frac{1}{4}\,(\mu^2)'' + \frac{\chi\,\alpha^2}{2}\Big(2\,\mu\, r' + 4\,\mu'\, r - 4\,\mu'\, s - \mu''\,(\varepsilon^2\, s')\Big) \,. \end{aligned}$$

To prove (C.1) it is sufficient to show that

$$s'\mu = G\, \left(\mu\, r' + 2\mu'\, r - 2\mu'\, s - \frac{1}{2}\mu''\,(\varepsilon^2\, s')\right) \,.$$

But this is true because acting on both sides of this equation with $\left(1-\frac{\varepsilon^2}{2}\partial_x^2\right)$ gives

$$\Big(1-\frac{\varepsilon^2}{2}\partial_x^2\Big)(\mu\, s') = \mu\, s' - \frac{\varepsilon^2}{2}\Big(\mu\, s''' + 2\mu'\, s'' + \mu''\, s'\Big) \,,$$

and using $\varepsilon^2\, s'' = 2s - 2r$ and $\varepsilon^2\, s''' = 2s' - 2r'$, we get the desired result.

C.1 Bounds on r_1

In this subsection we prove bounds on $r_1(\mu)$ and $r_1(\mu_1) - r_1(\mu_2)$ in terms of $\|\mu\|_\sigma$. We recall that $r_1(\mu)$ is defined by

$$r_1(\mu) = -\frac{1}{32}\,(4\mu' + \varepsilon^2\,\mu^2)\;, \tag{C.2}$$

and assume that the following holds

$$\|\mu\|_\sigma \le c_\mu\,\rho\;, \quad \|\mu_i\|_\sigma \le c_\mu\,\rho\;, \quad \delta = c_\delta\,\rho^2\;, \quad \varepsilon \le 1\;, \tag{C.3}$$

where $c_\delta > 1$.

Proposition C.1 *Assume that (C.3) hold, and that r_1 is defined by (C.2). Then there exists a constant c_{r_1} such that*

$$\|r_1(\mu)\|_{\sigma-1} \le c_{r_1}\,\delta\,\rho\;, \tag{C.4}$$

$$\|r_1(\mu_1) - r_1(\mu_2)\|_{\sigma-1} \le c_{r_1}\,\delta\,\|\mu_1 - \mu_2\|_{\sigma-1}\;. \tag{C.5}$$

Proof. Using Lemma 2.8, Proposition 2.11 and the assumptions (C.3), we have

$$\|r_1(\mu)\|_{\sigma-1} \le \frac{\delta}{8}\,\|\mu\|_\sigma + \frac{\varepsilon^2}{32}\,\|\mu^2\|_\sigma \le \frac{\delta}{8}\,\|\mu\|_\sigma(1 + \frac{C_m}{4\sqrt{\delta}}\,\|\mu\|_\sigma) \le \frac{1}{8}\Big(1 + \frac{C_m\,c_\mu}{4\sqrt{c_\delta}}\Big)\,\delta\,\rho\;,$$

and since $\mu_1^2 - \mu_2^2 = (\mu_1 - \mu_2)(\mu_1 + \mu_2)$, we have

$$\begin{aligned}
\|r_1(\mu_1) - r_1(\mu_2)\|_\sigma &\le \frac{\delta}{8}\,\|\mu_1 - \mu_2\|_\sigma + \frac{\varepsilon^2}{32}\,\|\mu_1^2 - \mu_2^2\|_\sigma \\
&\le \frac{\delta}{8}\,\|\mu_1 - \mu_2\|_\sigma(1 + \frac{C_m}{4\sqrt{\delta}}\,\|\mu_1 + \mu_2\|_\sigma) \\
&\le \frac{1}{8}\Big(1 + \frac{C_m\,c_\mu}{2\sqrt{c_\delta}}\Big)\,\delta\,\|\mu_1 - \mu_2\|_\sigma\;.
\end{aligned}$$

Setting $c_{r_1} = \frac{1}{8}\Big(1 + \frac{C_m\,c_\mu}{2\sqrt{c_\delta}}\Big)$ completes the proof. ∎

C.2 Bounds on F_0, F_1 and F_2

In this subsection, we define

$$\begin{aligned}
F_0(\mu) &= F_0(G\,r(\mu), \mu)\;, \\
F_1(\mu) &= F_1(G\,r(\mu), \mu)\;, \\
F_2(\mu) &= F_2(G\,r(\mu), r(\mu), \mu)\;,
\end{aligned}$$

and we suppose that for all μ, μ_1 and μ_2 in $\mathcal{B}_\sigma(c_\mu\,\rho)$, we have

$$\begin{aligned}
\|r(\mu)\|_{\sigma-1} &\le c_r\,\delta\,\rho\;, \\
\|r(\mu_1) - r(\mu_2)\|_{\sigma-1} &\le c_r\,\delta\,\|\mu_1 - \mu_2\|_\sigma\;,
\end{aligned}$$

for some constant c_r of order c_{r_1}, where c_{r_1} is given by Proposition C.1. See also Section 3.

Theorem C.2 *Let $\delta = c_\delta\, \rho^2$. There exist constants c_ε and c_{F_0} such that*

$$\|F_0(\mu)\|_{\mathrm{L}^2} \leq c_{F_0}\, \delta^{5/2}\, \rho^2\ , \tag{C.6}$$

$$\|F_0(\mu)\|_{\sigma-2} \leq c_{F_0}\, \delta^{5/2}\, \rho^2\ , \tag{C.7}$$

$$\left\| \frac{F_0(\mu)'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \leq c_{F_0}\, \delta^{5/2}\, \rho^2\ , \tag{C.8}$$

for all $\varepsilon \leq c_\varepsilon\, \delta^{-3/8}\, \rho^{-1/4}$ and for all $\mu \in \mathcal{B}_\sigma(c_\mu\, \rho)$.

Proof. We recall that, defining $s(\mu) = G\, r(\mu)$, we have

$$\begin{aligned}
F_0(\mu) &= \chi\, \alpha^2 \left((2+\varepsilon^2(1+\alpha^2))\, s^2 + \frac{s'\, \mu}{1+\varepsilon^4\, \alpha^2\, s} - \frac{2\, \varepsilon^2\, \alpha^2\, s\, s''}{1+\varepsilon^4\, \alpha^2\, s} \right) - \frac{1}{4}\, G\, \mu^2 - \frac{1}{4}\, G\, (\mu^2)'' \\
&= F_1(\mu) + G\, F_2(\mu)\ , \\
F_1(\mu) &= \chi\, \alpha^2 \left((2+\varepsilon^2(1+\alpha^2))\, s^2 - \frac{\alpha^2\, (\varepsilon^4\, s')\, s\, \mu}{1+\varepsilon^4\, \alpha^2\, s} - \frac{2\, \alpha^2\, s\, (\varepsilon^2\, s'')}{1+\varepsilon^4\, \alpha^2\, s} \right)\ , \\
F_2(\mu) &= -\frac{1}{4}\, \mu^2 - \frac{1}{4}\, (\mu^2)'' + \frac{\chi\, \alpha^2}{2} \Big(2\, \mu\, r' + 4\, \mu'\, r - 4\, \mu'\, s - \mu''\, (\varepsilon^2\, s') \Big)\ ,
\end{aligned}$$

where we omitted the μ dependence of s and r for concision. Note that from the definition of G, we have

$$\|s\|_{\sigma-1} \leq \|r\|_{\sigma-1}\ , \quad \|\varepsilon s'\|_{\sigma-1} \leq \sqrt{2}\, \|r\|_{\sigma-1} \quad \text{and} \quad \left\|\varepsilon^2 s''\right\|_{\sigma-1} \leq 2\, \|r\|_{\sigma-1}\ ,$$

since $s = G\, r$. Using these inequalities and Propositions 2.11 and 2.12, we have

$$\left\|s^2\right\|_{\sigma-1} \leq C_m\, \sqrt{\delta}\, \|r\|_{\sigma-1}^2\ , \tag{C.9}$$

$$\left\| \frac{s\, (\varepsilon^2 s)''}{1+\varepsilon^4\, \alpha^2\, s} \right\|_{\sigma-1} \leq \frac{C\, \sqrt{\delta}\, \|r\|_{\sigma-1}^2}{1 - C_m\, \varepsilon^4\, \alpha^2\, \sqrt{\delta}\, \|s\|_{\sigma-1}}\ , \tag{C.10}$$

$$\left\| \frac{\mu\, s'}{1+\varepsilon^4\, \alpha^2\, s} \right\|_{\sigma-2} \leq \frac{C\, \delta^{3/2}\, \|\mu\|_\sigma\, \|r\|_{\sigma-1}}{1 - C_m\, \varepsilon^4\, \alpha^2\, \sqrt{\delta}\, \|r\|_{\sigma-1}}\ , \tag{C.11}$$

$$\left\| \frac{\mu\, s\, (\varepsilon^4\, s)'}{1+\varepsilon^4\, \alpha^2\, s} \right\|_{\sigma-1} \leq \frac{C\, \varepsilon^3\, \delta\, \|\mu\|_{\sigma-1}\, \|r\|_{\sigma-1}^2}{1 - C\, \varepsilon^4\, \alpha^2\, \sqrt{\delta}\, \|r\|_{\sigma-1}}\ , \tag{C.12}$$

$$\left\|\mu''\, (\varepsilon^2\, s')\right\|_{\sigma-3} \leq \varepsilon\, C\, \delta^{5/2}\, \|\mu\|_\sigma\, \|r\|_{\sigma-1}\ , \tag{C.13}$$

$$\|\mu^2\|_{\sigma-3} \leq C\, \sqrt{\delta}\, \|\mu\|_\sigma^2\ , \tag{C.14}$$

$$\|(\mu^2)''\|_{\sigma-3} \leq C\, \delta^{5/2}\, \|\mu\|_\sigma^2\ , \tag{C.15}$$

$$\|\mu\, r'\|_{\sigma-3} \leq C\, \delta^{3/2}\, \|\mu\|_\sigma\, \|r\|_{\sigma-1}\ , \tag{C.16}$$

$$\|\mu'\, r\|_{\sigma-3} + \|\mu'\, s\|_{\sigma-3} \leq C\, \delta^{3/2}\, \|\mu\|_\sigma \|r\|_{\sigma-1}\ . \tag{C.17}$$

By hypothesis, we have $\|r\|_{\sigma-1} \leq c_r\, \delta\, \rho$. We now choose $c_\varepsilon = \left(\frac{1}{2\, C_m\, c_r\, \alpha^2}\right)^{1/4}$, and get that for all $\mu \in \mathcal{B}_\sigma(c_\mu\, \rho)$, and for all $\varepsilon \leq c_\varepsilon\, \delta^{-3/8}\, \rho^{-1/4} \leq c_\varepsilon\, c_\delta^{-3/8}\, \rho^{-1}$,

$$\frac{1}{1 - C_m\, \varepsilon^4\, \alpha^2\, \sqrt{\delta}\, \|r\|_{\sigma-1}} \leq 2\ ,$$

$$\varepsilon^3 \sqrt{\delta}\, \|\mu\|_\sigma \leq c_\mu\, \varepsilon^3 \sqrt{\delta}\, \rho = c_\mu \sqrt{c_\delta}\, \varepsilon^3\, \rho^2 \leq c_\mu \sqrt{c_\delta}\, c_\varepsilon^{3/2}\, c_\delta^{-9/8} .$$

Hence the r.h.s. of the inequalities (C.9)–(C.17) are all bounded by some constant times $\delta^{5/2}\,\rho^2$, except (C.13) which is bounded by a constant times $\delta^{7/2}\,\rho^2$. From (C.9)–(C.12) and Lemma 2.8 for the two last terms of $F_0(\mu)$, we see that (C.7) holds, then (C.6) also holds because $\|F_0(\mu)\|_{\mathrm{L}^2} \leq \|F_0(\mu)\|_{\sigma-2}$. For (C.8), we use first Lemma 2.10 which gives

$$\left\|\frac{F_0(\mu)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \leq \frac{\sqrt{2}}{\delta}\, \|F_1(\mu)\|_{\mathcal{W},\sigma-1} + \frac{2^{7/2}}{3\,\delta^3}\, \|F_2(\mu)\|_{\mathcal{W},\sigma-3} .$$

Using (C.13)–(C.17) for the F_2–term and (C.9), (C.10) and (C.12) for the F_1–term completes the proof. ■

Theorem C.3 *Let $\delta = c_\delta\,\rho^2$, and let c_ε be given by Theorem C.2. There exists a constant c_{F_0} such that*

$$\|F_0(\mu_1) - F_0(\mu_2)\|_{\mathrm{L}^2} \leq c_{F_0}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma , \tag{C.18}$$

$$\|F_0(\mu_1) - F_0(\mu_2)\|_{\sigma-2} \leq c_{F_0}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma , \tag{C.19}$$

$$\left\|\frac{F_0(\mu_1)' - F_0(\mu_2)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \leq c_{F_0}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma , \tag{C.20}$$

for all $\varepsilon \leq c_\varepsilon^{1/4}\,\rho^{-1/4}\,\delta^{-3/8}$ and for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\,\rho)$.

Proof. We use the three following equalities

$$a_1 b_1 - a_2 b_2 = (a_1 - a_2) b_1 + (b_1 - b_2) a_2 , \tag{C.21}$$

$$a_1 b_1 c_1 - a_2 b_2 c_2 = (a_1 - a_2) b_1 c_1 + (b_1 - b_2) a_2 c_1 + (c_1 - c_2) a_2 b_2 , \tag{C.22}$$

$$\frac{f(\mu_1)}{1 + \varepsilon^4\, s(\mu_1)} - \frac{f(\mu_2)}{1 + \varepsilon^4\, s(\mu_2)} = \frac{f(\mu_1) - f(\mu_2)}{1 + \varepsilon^4\, s(\mu_1)} + \frac{f(\mu_2)}{1 + \varepsilon^4\, s(\mu_2)}\, \frac{\varepsilon^4\, \Delta s}{1 + \varepsilon^4\, s(\mu_1)} , \tag{C.23}$$

where $\Delta s = s(\mu_2) - s(\mu_1)$. Then we proceed exactly as we did in the proof of Theorem C.2. For example, in (C.14), we used that

$$\|\mu^2\|_{\sigma-3} \leq C_m \sqrt{\delta}\, \|\mu\|_\sigma^2 \leq C_m\, c_\mu^2 \sqrt{\delta}\, \rho^2 , \tag{C.24}$$

then here, this bound is replaced by

$$\|\mu_1^2 - \mu_2^2\|_{\sigma-3} \leq C_m \sqrt{\delta}\, \|\mu_1 + \mu_2\|_\sigma\, \|\mu_1 - \mu_2\|_\sigma \leq 2\, C_m\, c_\mu \sqrt{\delta}\, \rho\, \|\mu_1 - \mu_2\|_\sigma . \tag{C.25}$$

All other estimates are similar. ■

Corollary C.4 *Let $\delta = c_\delta\,\rho$, and define $F_7(\mu) = \frac{\varepsilon^2}{8}\,(\partial_x + \frac{\varepsilon^2\,\mu}{2}) F_0(\mu)'$. Then there exist constants c_ε and c_{F_7} such that for all $\varepsilon \leq c_\varepsilon\,\rho^{-2}$ and $\mu_i \in \mathcal{B}_\sigma(c_\mu\,\rho)$ the following bounds hold*

$$\|F_7(\mu_i)\|_{\mathrm{L}^2} + \left\|\mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, F_7(\mu_i)'\right\|_{\mathrm{L}^2} \leq c_{F_7}\, \delta^{5/2}\, \rho^2 , \tag{C.26}$$

$$\left\|\mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, \Delta F_7'\right\|_{\mathrm{L}^2} \leq c_{F_7}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma , \tag{C.27}$$

$$\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\, G\, \mathcal{L}_r}\, F_7(\mu_i)' \right\|_{\mathcal{W},\sigma} \le c_{F_7}\ \max\Big(2, \frac{\alpha^2}{1-\alpha^2}\Big)\ \delta^{5/2}\ \rho^2\ , \tag{C.28}$$

$$\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\, G\, \mathcal{L}_r}\, \Delta F_7' \right\|_{\mathcal{W},\sigma} \le c_{F_7}\ \max\Big(2, \frac{\alpha^2}{1-\alpha^2}\Big)\ \delta^{5/2}\ \rho\ \|\mu_1-\mu_2\|_\sigma\ , \tag{C.29}$$

where $\Delta F_7 = (F_7(\mu_1) - F_7(\mu_2))$.

Proof. We first use that $\|\mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ f'\|_{\mathrm{L}^2} \le 16\|f\|_{\mathrm{L}^2}$ (see Lemma F.1 in Appendix F), so that for the L^2 bounds (C.26)–(C.27), we need only bound $\|F_7(\mu_i)\|_{\mathrm{L}^2}$ and $\|F_7(\mu_1) - F_7(\mu_2)\|_{\mathrm{L}^2}$. Then we have also $\|f\|_{\mathrm{L}^2} \le \|f\|_{\sigma'}$ for any $\sigma' > 0$, from which we get

$$\begin{aligned}
\|F_7(\mu_i)\|_{\mathrm{L}^2} &\le \varepsilon^2 \|F_0(\mu_i)''\|_{\mathrm{L}^2} + \frac{\varepsilon^4}{2}\|\mu_i\ F_0(\mu_i)'\|_{\mathrm{L}^2} \\
&\le \varepsilon^2\ \|F_0(\mu_i)''\|_{\sigma-4} + \frac{\varepsilon^4\ C_m\ \sqrt{\delta}}{2}\|\mu_i\|_{\sigma-3}\ \|F_0(\mu_i)'\|_{\sigma-3} \\
&\le \varepsilon^2\ \delta^2\ \Big(C^2 + \frac{\varepsilon^2\ C_m\ C\ c_\mu}{\sqrt{c_\delta}}\Big)\ \|F_0(\mu_i)\|_{\sigma-2} \le C\ \delta^{5/2}\ \rho^2\ ,
\end{aligned}$$

since $\|F_0(\mu_i)\|_{\sigma-2} \le c_{F_0}\ \delta^{5/2}\ \rho^2$ and $\varepsilon \le c_\varepsilon\ \rho^{-2}$. Similarly, since

$$\mu_1\ F_0(\mu_1)' - \mu_2\ F_0(\mu_2)' = \frac{1}{2}\Delta\mu\ (F_0(\mu_1) + F_0(\mu_2))' + \frac{1}{2}(\mu_1+\mu_2)\ \Delta F_0'\ , \tag{C.30}$$

where $\Delta\mu = \mu_1 - \mu_2$ and $\Delta F_0 = F_0(\mu_1) - F_0(\mu_2)$, we also have

$$\|\Delta F_7\|_{\mathrm{L}^2} \le C_1\ \|\Delta F_0\|_{\sigma-2} + C_2\ \varepsilon^2\ \|F_0(\mu_1) + F_0(\mu_2)\|_{\sigma-2}\ \|\mu_1-\mu_2\|_\sigma\ .$$

The proof of (C.27) is completed noting that $\|\Delta F_0\|_{\sigma-2} \le c_{F_0}\ \delta^{5/2}\ \rho\ \|\mu_1-\mu_2\|_\sigma$, and using again $\varepsilon \le c_\varepsilon\ \rho^{-2}$.

For the proof of (C.28) and (C.29), we define $b(\alpha) = \max(2, \frac{\alpha^2}{1-\alpha^2})$, then (see Lemma F.1 of Appendix F)

$$\begin{aligned}
\varepsilon^2\ \left\| \frac{\mathcal{L}_{\mu,r}}{G\,\mathcal{L}_r}\ f'' \right\|_{\mathcal{W},\sigma} &\le 8\ \max\Big(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\Big)\ \|f\|_{\mathcal{W},\sigma} \le C\ b(\alpha)\ \|f\|_{\mathcal{W},\sigma}\ , \\
\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\ G\,\mathcal{L}_r}\ f' \right\|_{\mathcal{W},\sigma} &\le C\ b(\alpha)\ \delta^{-3}\ \|f\|_{\mathcal{W},\sigma-3}
\end{aligned}$$

for some constant C. Using this, we conclude that

$$\begin{aligned}
\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\ G\,\mathcal{L}_r}\ F_7(\mu_i)' \right\|_{\mathrm{L}^2} &\le C\ b(\alpha)\ \left(\left\| \frac{F_0(\mu_i)'}{\mathcal{L}_\mu} \right\| + \frac{\varepsilon^4\ \delta^{-3}}{2}\|\mu_i\ F_0(\mu_i)'\|_{\sigma-3} \right) \\
&\le C\ b(\alpha)\ \left(\left\| \frac{F_0(\mu_i)'}{\mathcal{L}_\mu} \right\| + \frac{\varepsilon^4\ \delta^{-3/2}}{2}\|\mu_i\|_\sigma\ \|F_0(\mu_i)\|_{\sigma-2} \right) \\
&\le C'\ b(\alpha)\ \left(\left\| \frac{F_0(\mu_i)'}{\mathcal{L}_\mu} \right\| + \|F_0(\mu_i)\|_{\sigma-2} \right) \le C''\ b(\alpha)\ \delta^{5/2}\ \rho^2\ .
\end{aligned}$$

The proof of (C.29) is very similar (use e.g. (C.30) and proceed as for (C.28) above). ∎

C.3 Bounds on F_3

We begin by recalling that

$$F_3(s,\mu) = -\alpha^2\, \chi \left(\tfrac{3}{2}\, s^2 + \varepsilon^2\, \tfrac{1}{32}\, s\, \mu^2 + \tfrac{\alpha^2}{2}\, \varepsilon^4\, s^3\right) .$$

The map F_3 satisfies the two following propositions. The first one is used for the properties of s, and the second one for those of r_2.

Proposition C.5 *Let $c_\mu, c_{r_1}, c_{s_0}, \rho > 0$, $\delta > 2$, and $c_s > 2(c_{r_1} + c_{s_0})$. There exists a constant c_ε such that for all $\varepsilon \le c_\varepsilon\, \delta^{-5/4}\, \rho^{-1/2}$, for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\, \rho)$ and for all $s_i \in \mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho)$ the following bounds hold*

$$\frac{2\,\varepsilon^4}{\chi}\, \|F_3(s_i,\mu_i)\|_{\sigma-1} < \left(1 - \frac{2(c_{r_1}+c_{s_0})}{c_s}\right) c_s\, \delta\, \rho\, , \tag{C.31}$$

$$\frac{2\,\varepsilon^4}{\chi}\, \|F_3(s_1,\mu_i) - F_3(s_2,\mu_i)\|_{\sigma-1} \le \left(1 - \frac{2(c_{r_1}+c_{s_0})}{c_s}\right) \|s_1 - s_2\|_{\sigma-1}\, , \tag{C.32}$$

$$\frac{2\,\varepsilon^4}{\chi}\, \|F_3(s_i,\mu_1) - F_3(s_i,\mu_2)\|_{\sigma-1} \le 2\, c_{s_0}\, \delta\, \|\mu_1 - \mu_2\|_{\sigma-1}\, . \tag{C.33}$$

Proof. We have

$$\begin{aligned}
\varepsilon^4 \|s_i^2\|_{\sigma-1} &\le \varepsilon^4\, C_m\, \sqrt{\delta}\, (c_s\, \delta\, \rho)^2 = (\varepsilon^4\, \delta^{3/2}\, \rho\, C_m\, c_s)\, c_s\, \delta\, \rho\, ,\\
\varepsilon^6 \|s_i\, \mu_i^2\|_{\sigma-1} &\le (\varepsilon^6\, \delta\, \rho^2\, C_m^2\, c_\mu^2)\, c_s\, \delta\, \rho\, ,\\
\varepsilon^8 \|s_i^3\|_{\sigma-1} &\le \varepsilon^8\, C_m^2\, \delta\, (c_s\, \delta\, \rho)^3 = (\varepsilon^8\, \delta^3\, \rho^2\, C_m^2\, c_s^2)\, c_s\, \delta\, \rho\, ,\\
\varepsilon^4 \|s_1^2 - s_2^2\|_{\sigma-1} &\le \varepsilon^4\, C_m\, \sqrt{\delta}\, (\|s_1\|_{\sigma-1} + \|s_2\|_{\sigma-1}) \|s_1 - s_2\|_{\sigma-1}\\
&\le (\varepsilon^4\, \delta^{3/2}\, \rho\, 2\, C_m\, c_s)\, \|s_1 - s_2\|_{\sigma-1}\, ,\\
\varepsilon^8 \|s_1^3 - s_2^3\|_{\sigma-1} &\le \varepsilon^8\, C_m^2\, \delta\, (\|s_1\|_{\sigma-1} + \|s_2\|_{\sigma-1})^2 \|s_1 - s_2\|_{\sigma-1}\\
&\le (\varepsilon^8\, \delta^3\, \rho^2\, 4\, C_m^2\, c_s^2)\, \|s_1 - s_2\|_{\sigma-1}\, ,\\
\varepsilon^6 \|s_i\, \mu_1^2 - s_i\, \mu_2^2\|_{\sigma-1} &\le \varepsilon^6\, C_m^2\, \delta\, \|s_i\|_{\sigma-1}\, (\|\mu_1\|_\sigma + \|\mu_2\|_\sigma) \|\mu_1 - \mu_2\|_\sigma\\
&\le (\varepsilon^4\, \delta\, \rho^2\, C_m^2\, c_\mu\, c_s)\, \delta\, \|\mu_1 - \mu_2\|_\sigma\, .
\end{aligned}$$

Choosing c_ε independent of δ and ρ and sufficiently small, we can satisfy (C.31)–(C.33). ■

Proposition C.6 *Let $\delta = c_\delta\, \rho^2 > 2$, $c_\mu, c_s, \rho > 0$ and $\mathcal{L}_s = 1 - \frac{\varepsilon^2}{2}\partial_x^2$. There exist constants c_ε and m_ε such that for all $\varepsilon \le c_\varepsilon\, \rho^{-m_\varepsilon}$, for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\, \rho)$ and for all maps s satisfying $\|s(\mu_i)\|_{\sigma-1} \le c_s\, \delta\, \rho$ and $\|s(\mu_1) - s(\mu_2)\|_{\sigma-1} \le c_s\, \delta\, \|\mu_1 - \mu_2\|$ the following bounds hold*

$$\|\mathcal{L}_s\, F_3(s(\mu_i),\mu_i)\|_{\sigma-3} \le c_{F_3}\, \delta^{5/2}\, \rho^2\, , \tag{C.34}$$

$$\|\mathcal{L}_s\, F_3(s(\mu_1),\mu_1) - \mathcal{L}_s\, F_3(s(\mu_2),\mu_2)\|_{\sigma-3} \le c_{F_3}\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma\, , \tag{C.35}$$

for some constant c_{F_3}.

Proof. We first note that since $\mathcal{L}_s = 1 - \frac{\varepsilon^2}{2}\partial_x^2$, we have

$$\|\mathcal{L}_s\, f\|_{\sigma-3} \le \left(1 + \frac{C\,\varepsilon^2\,\delta^2}{2}\right)\|f\|_{\sigma-1} \le \left(1 + \frac{C\,c_\varepsilon^2\,c_\delta^2}{2}\right)\|f\|_{\sigma-1} \le C\,\|f\|_{\sigma-1}\ ,$$

if $m_\varepsilon \ge 2$. Let $s(\mu_i) = s_i$, then, as in the proof of Proposition C.5, we have

$$\begin{aligned}\|s_i^2\|_{\sigma-1} + \varepsilon^2\|s_i\,\mu_i^2\|_{\sigma-1} + \varepsilon^4\|s_i^3\|_{\sigma-1} &\le \delta^{5/2}\,\rho^2\left(C_m\,c_s^2 + \frac{\varepsilon^2\,C_m^2\,c_\mu^2\,c_s\,\rho}{\sqrt{\delta}} + \varepsilon^4\,\delta^{3/2}\,\rho\,C_m^2\,c_s^3\right)\\ &\le \delta^{5/2}\,\rho^2\left(C_m\,c_s^2 + \frac{c_\varepsilon^2\,C_m^2\,c_\mu^2\,c_s}{\sqrt{c_\delta}} + c_\varepsilon^4\,c_\delta^{3/2}\,C_m^2\,c_s^3\right)\ ,\end{aligned}$$

if $m_\varepsilon \ge 1$. The proof of (C.35) is similar, we omit the details. ■

C.4 Bounds on F_4

We begin by recalling that

$$F_4(s,\mu) = -\tfrac{\alpha^2\,\chi}{8}(2s'\mu + s\mu')\ ,$$

then we have the

Proposition C.7 *Let $\delta = c_\delta\,\rho^2 > 2$, $c_\mu, c_s, \rho > 0$ and $\mathcal{L}_s = 1 - \frac{\varepsilon^2}{2}\partial_x^2$. There exist constants c_ε and m_ε such that for all $\varepsilon \le c_\varepsilon\,\rho^{-m_\varepsilon}$, for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\,\rho)$ and for all maps s satisfying $\|\mathcal{L}_s\,s(\mu_i)\|_{\sigma-1} \le c_s\,\delta\,\rho$ and $\|\mathcal{L}_s\,s(\mu_1) - \mathcal{L}_s\,s(\mu_2)\|_{\sigma-1} \le c_s\,\delta\,\|\mu_1 - \mu_2\|$ the following bounds hold*

$$\|\mathcal{L}_s\,F_4(s(\mu_i),\mu_i)\|_{\sigma-3} \le c_{F_4}\,\delta^{5/2}\,\rho^2\ , \tag{C.36}$$

$$\|\mathcal{L}_s\,F_4(s(\mu_1),\mu_1) - \mathcal{L}_s\,F_4(s(\mu_2),\mu_2)\|_{\sigma-3} \le c_{F_4}\,\delta^{5/2}\,\rho\,\|\mu_1 - \mu_2\|_\sigma\ , \tag{C.37}$$

for some constant c_{F_4}.

Proof. We first note that $\|f\|_\sigma \le \|\mathcal{L}_s\,f\|_\sigma$, and that (see Proposition C.6) $\|\mathcal{L}_s\,f\|_{\sigma-3} \le C\,\|f\|_{\sigma-1}$ if $m_\varepsilon \ge 2$. We thus have

$$\|\mathcal{L}_s\,(s(\mu_i)\,\mu_i')\|_{\sigma-3} \le C\,\sqrt{\delta}\,\|s(\mu_i)\|_{\sigma-1}\,\|\mu_i'\|_{\sigma-1} \le C\,\delta^{5/2}\,\rho^2\ ,$$

and for the other term, we use

$$\mathcal{L}_s(s'\,\mu) = \mu\,(\mathcal{L}_s s') + 2\,\mu'\,(\mathcal{L}_s s) + s'\,(\mathcal{L}_s\,\mu) - 2\,s\,\mu' - s'\,\mu\ ,$$

which gives

$$\|\mathcal{L}_s\,(s(\mu_i)'\,\mu_i)\|_{\sigma-3} \le C\,\delta^{5/2}\,\rho^2\ .$$

The proof of (C.37) is similar, we omit the details. ■

C.5 Bounds on F_6

We recall that

$$F_6(s,\mu) = \mathcal{L}_s\,\Big(F_3(s,\mu)+F_4(s,\mu)\Big)+F_7(\mu)+F_8(\mu)\;,$$
$$F_7(\mu) = \frac{\varepsilon^2}{8}\,(\partial_x+\frac{\varepsilon^2\,\mu}{2})\;F_0(\mu)'\;,$$
$$F_8(\mu) = -\frac{1}{8}\,(\partial_x+\frac{\varepsilon^2\,\mu}{2})\Big(\mathcal{L}_\mu\,\mu+\mu\,\mu'\Big)\;.$$

We will now prove the estimates of Corollaries 4.3, 4.5, 4.7 and 4.9. These estimates follow immediately from the bounds on F_6 of the next proposition.

Proposition C.8 *Let $\delta = c_\delta\,\rho$ then there exist constants c_ε and c_{F_6} such that for all $\varepsilon \le c_\varepsilon\,\rho^{-2}$ and for all $\mu_i \in \mathcal{B}_\sigma(c_\mu\,\rho)$ the following bounds hold*

$$\varepsilon^2\,\left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\,G\,\mathcal{L}_r}\;F_6(\mu_i)'\right\|_{\mathcal{W},\sigma} \le \max\Big(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2}\Big)\Big(c_\mu\,\rho+\varepsilon^2\,c_{F_6}\,\delta^{5/2}\,\rho^2\Big)\;, \tag{C.38}$$
$$\varepsilon^2\,\left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\,G\,\mathcal{L}_r}\;\Delta F_6'\right\|_{\mathcal{W},\sigma} \le \max\Big(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2}\Big)(1+\varepsilon^2\,c_{F_6}\,\delta^{5/2}\,\rho)\;\|\mu_1-\mu_2\|_\sigma\;, \tag{C.39}$$

and

$$\varepsilon^2\;\|F_6(\mu_i)\|_{\mathrm{L}^2}+\left\|\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\;F_6(\mu_i)'\right\|_{\mathrm{L}^2} \le c_{F_6}\,\Big(\delta^4\,\rho+\delta^{5/2}\,\rho^2\Big)\;, \tag{C.40}$$
$$\left\|\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\;\Delta F_6'\right\|_{\mathrm{L}^2} \le c_{F_6}\,\Big(\delta^4+\delta^{5/2}\,\rho\Big)\;\|\mu_1-\mu_2\|_\sigma\;, \tag{C.41}$$

where $\Delta F_6 = F_6(\mu_1)-F_6(\mu_2)$.

Proof. For the proof of (C.38) and (C.39), we define $b(\alpha) = \max(2,\frac{\alpha^2}{1-\alpha^2})$, then we have (see Lemma F.1 of Appendix F)

$$\frac{\varepsilon^2}{8}\,\left\|\frac{\mathcal{L}_{\mu,r}}{G\,\mathcal{L}_r}\;f''\right\|_{\mathcal{W},\sigma} \le \max\Big(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2}\Big)\;\|f\|_{\mathcal{W},\sigma}\;,$$
$$\left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\,G\,\mathcal{L}_r}\;f'\right\|_{\mathcal{W},\sigma} \le C\,b(\alpha)\,\delta^{-3}\;\|f\|_{\mathcal{W},\sigma-3}\;,$$

for some constant C. On the other hand, we have

$$\left\|\frac{\varepsilon^2}{2}\mu_i\mathcal{L}_\mu\,\mu_i\right\|_{\mathcal{W},\sigma-3} \le C_m\,\sqrt{\delta}\;\|\mu_i\|_\sigma\;\|\varepsilon^2\mathcal{L}_\mu\,\mu_i\|_{\sigma-2} \le C\,\delta^{5/2}\,\rho^2\;,$$
$$\left\|(\partial_x+\frac{\varepsilon^2\,\mu_i}{2})\mu_i\mu_i'\right\|_{\mathcal{W},\sigma-3} \le C_1\,\delta^{5/2}\,\rho^2+\varepsilon^2\,C_2\,\delta^2\,\rho^3 \le C\,\delta^{5/2}\,\rho^2\;,$$

from which we get, using Corollary C.4, Propositions C.6 and C.7 above for the contributions of F_0, F_3 and F_4,

$$\varepsilon^2\,\left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\varepsilon\,G\,\mathcal{L}_r}\;F_6(\mu_i)'\right\|_{\mathcal{W},\sigma} \le \max\Big(\frac{1}{3},\frac{\alpha^2}{1-\alpha^2}\Big)\;\|\mu_i\|_{\mathcal{W},\sigma}+\varepsilon^2\,C\,b(\alpha)\,\delta^{5/2}\,\rho^2\;.$$

The proof of (C.38) follows since $b(\alpha) \le 6\max(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2})$. The proof of (C.39) being very similar, we omit the details.

For the proof of (C.40) and (C.41), we first show that

$$\begin{aligned}
\frac{1}{8}\|\mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, \mathcal{L}_\mu\, \mu''\|_{\mathrm{L}^2} &\le 2\|(1+\partial_x^4)\, \mu\|_{\mathrm{L}^2} \le C\, \delta^4\, \|\mu\|_\sigma \le C\, \delta^4\, \rho\ ,\\
\frac{\varepsilon^2}{8}\, \|\mathcal{L}_\mu\, \mu'\|_{\mathrm{L}^2} &\le \frac{\varepsilon^2}{16}\|(1-\partial_x^2)^{5/2}\, \mu\|_{\mathrm{L}^2} \le \frac{\varepsilon^2\, \delta^5}{16}\|\mu\|_\sigma\\
&\le \delta^{5/2}\, \rho^2 \left(\frac{c_\mu\, \varepsilon^2\, \delta^{5/2}}{16\, \rho}\right) \le C\, \delta^{5/2}\, \rho^2\ ,
\end{aligned}$$

since by hypothesis $\delta = c_\delta\, \rho^2$ and $\varepsilon \le c_\varepsilon\, \rho^{-2}$. To complete the proof of (C.40), we first note that $\|\mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, f'\|_{\mathrm{L}^2} \le 16\|f\|_{\mathrm{L}^2}$, then using also Corollary C.4, Propositions C.6 and C.7 above, we see that it is in fact sufficient to show that $\|F_6(\mu_i) - \frac{1}{8}\, \mathcal{L}_\mu\, \mu_i'\|_{\mathrm{L}^2} \le C\, \delta^{5/2}\, \rho^2$, but we have

$$\begin{aligned}
\|F_6(\mu_i) - \frac{1}{8}\, \mathcal{L}_\mu\, \mu_i'\|_{\mathrm{L}^2} &\le \frac{\varepsilon^2}{16}\|\mu_i \mathcal{L}_\mu\, \mu_i\|_{\mathrm{L}^2} + \frac{1}{8}\|(\mu_i\, \mu_i')'\|_{\mathrm{L}^2} + \frac{\varepsilon^2}{8}\|\mu_i^2\, \mu_i'\|_{\mathrm{L}^2}\\
&\le C_1\, \sqrt{\delta}\, \|\mu_i\|_\sigma\, \|\varepsilon^2\, \mathcal{L}_\mu\, \mu_i\|_{\sigma-2} + C_2\, \delta^{5/2}\, \|\mu_i\|_\sigma^2 + \varepsilon^2\, C_3\, \delta^2\, \|\mu_i\|_\sigma^3\\
&\le C\, \delta^{5/2}\, \|\mu_i\|_\sigma^2 \left(1 + \frac{\|\mu_i\|_\sigma}{\sqrt{\delta}}\right) \le C\, \delta^{5/2}\, \rho^2\ .
\end{aligned}$$

The proof of (C.41) follows easily using Corollary C.4, Propositions C.6 and C.7, and equalities like

$$\mu_1\, f(\mu_1) - \mu_2\, f(\mu_2) = \frac{1}{2}(\Delta\mu)\, (f(\mu_1) + f(\mu_2)) + \frac{1}{2}(\mu_1 + \mu_2)\, \Delta f\ , \tag{C.42}$$

for $\Delta\mu = \mu_1 - \mu_2$ and $\Delta f = f(\mu_1) - f(\mu_2)$. ■

Appendix D: Proof of Proposition 2.16

Before proving Proposition 2.16, we prove a simpler Lemma.

Lemma D.1 *Let δ and ε_0 be given by Proposition 2.16, and let $\mathcal{F}(\tilde{\mu}, \mu_0)$ be the solution of*

$$\partial_t \mu = -\mathcal{L}_\mu\, \mu - \mu\, \mu' + \varepsilon^2\, F(\tilde{\mu})'\ , \quad \mu(x,0) = \mu_0(x)\ .$$

Assume that

$$\|\!|\mathcal{F}(\tilde{\mu}, \mu_0)|\!\|_\sigma \le c_\mu\, \rho\ ,$$

and that (2.32) holds with $\lambda_1 < 1$ for all $\varepsilon \le \varepsilon_0$, for all $\tilde{\mu} \in \mathcal{B}_\sigma(c_\mu\, \rho)$, and for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\mu\, \rho)$. Then for all $0 < c_\lambda < 1$, there exists a $t_1 > 0$ such that

$$\sup_{0\le t\le t_1}\ \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_{\mathrm{L}^2} \le c_\lambda\, \lambda_1\, \sup_{0\le t\le t_1}\ \|\tilde{\mu}_1(\cdot, t) - \tilde{\mu}_2(\cdot, t)\|_\sigma\ ,$$

for all $\tilde{\mu}_i \in \mathcal{B}_\sigma(c_\mu\, \rho)$.

Proof. Let $\mu_i = \mathcal{F}(\tilde{\mu}_i, \mu_0)$, $i = 1, 2$ and

$$\mu_\pm = \mathcal{F}(\tilde{\mu}_1, \mu_0) \pm \mathcal{F}(\tilde{\mu}_2, \mu_0) \, .$$

We have

$$\begin{aligned} \partial_t \mu_1 &= -\mathcal{L}_\mu \, \mu_1 - \frac{1}{2}(\mu_1^2)' + \varepsilon^2 \, F(\tilde{\mu}_1)' \, , \quad \mu_1(x, 0) = \mu_0(x) \, , \\ \partial_t \mu_2 &= -\mathcal{L}_\mu \, \mu_2 - \frac{1}{2}(\mu_2^2)' + \varepsilon^2 \, F(\tilde{\mu}_2)' \, , \quad \mu_2(x, 0) = \mu_0(x) \, . \end{aligned}$$

Subtracting these two equations, we get

$$\partial_t \mu_- = -\mathcal{L}_\mu \, \mu_- - \frac{1}{2}(\mu_+ \mu_-)' + \varepsilon^2 \left(F(\tilde{\mu}_1) - F(\tilde{\mu}_2) \right)' \, , \quad \mu_-(x, 0) = 0 \, . \tag{D.1}$$

Let $\Delta F = F(\tilde{\mu}_1) - F(\tilde{\mu}_2)$, then multiplying (D.1) by μ_-, integrating over $[-L/2, L/2]$ and using Young's inequality, we get

$$\begin{aligned} \partial_t(\mu_-, \mu_-) &= -2(\mu_-, \mathcal{L}_\mu \, \mu_-) - \frac{1}{2}(\mu_-, \mu_+' \, \mu_-) + 2 \, \varepsilon^2 \, (\mu_-, \Delta F') \\ &\leq (\mu_-, (\mathcal{L}_v^2 - 2\mathcal{L}_\mu)\mu_-) + \frac{1}{2} \|\mu_+'\|_{\mathrm{L}^\infty} (\mu_-, \mu_-) + \varepsilon^4 \, \|\mathcal{L}_v^{-1} \, \Delta F'\|_{\mathrm{L}^2}^2 \\ &\leq (1 + \frac{1}{2}\|\mu_+'\|_{\mathrm{L}^\infty}) \, (\mu_-, \mu_-) + \varepsilon^4 \, \|\mathcal{L}_v^{-1} \, \Delta F'\|_{\mathrm{L}^2}^2 \, . \end{aligned}$$

Now we use that

$$1 + \frac{1}{2}\|\mu_+'\|_{\mathrm{L}^\infty} \leq 1 + \frac{\|\mu_1'\|_{\mathrm{L}^\infty} + \|\mu_2'\|_{\mathrm{L}^\infty}}{2} \leq 1 + C \, c_\mu \, \delta^{3/2} \, \rho \equiv \zeta \, ,$$

and that by (2.32), for all $\varepsilon \leq \varepsilon_0$

$$\varepsilon^2 \, \|\mathcal{L}_v^{-1} \, (F(\tilde{\mu}_1) - F(\tilde{\mu}_2))'\|_{\mathrm{L}^2} \leq \lambda_1 \, \|\tilde{\mu}_1 - \tilde{\mu}_2\|_\sigma \, ,$$

with $\lambda_1 < 1$ to conclude that

$$\|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_{\mathrm{L}^2} \leq \lambda_1 \sqrt{\frac{e^{\zeta t} - 1}{\zeta}} \sup_{0 \leq s \leq t} \|\tilde{\mu}_1(\cdot, s) - \tilde{\mu}_2(\cdot, s)\|_\sigma \, .$$

Setting

$$t_1 = \frac{1}{\zeta} \ln\Big(1 + c_\lambda^2 \, \zeta\Big)$$

completes the proof. ■

Proposition 2.16 is then an easy consequence of the following proposition.

Proposition D.2 *There exist constants c_δ sufficiently large and c_λ sufficiently small such that if t_1 is given by Lemma D.1, and $\mathcal{F}(\tilde{\mu}, \mu_0)$, the solution of*

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu \, \mu' + \varepsilon^2 \, F(\tilde{\mu})' \,, \quad \mu(x,0) = \mu_0(x) \,,$$

satisfies

$$\|\!| \mathcal{F}(\tilde{\mu}, \mu_0) \|\!|_\sigma \leq c_\mu \, \rho \,,$$

and (2.32) holds with $\lambda_1 < 1$ for all $\varepsilon \leq \varepsilon_0$, for all $\tilde{\mu} \in \mathcal{B}_\sigma(c_\mu \, \rho)$, and for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\mu \, \rho)$, then there exists a constant $0 < \lambda < 1$ such that

$$\sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma \leq \lambda \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_1(\cdot, t) - \tilde{\mu}_2(\cdot, t)\|_\sigma \,, \tag{D.2}$$

for all $\tilde{\mu}_i \in \mathcal{B}_\sigma(c_\mu \, \rho)$.

Proof. We will use the same definitions as in Lemma D.1 above, and $\Delta F = F(\tilde{\mu}_1) - F(\tilde{\mu}_2)$. We first note that we have

$$\begin{aligned}
\sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma &= \sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma \leq 2 \, c_\mu \, \rho < \infty \,, \\
\sup_{0 \leq t \leq t_1} \|\mu_+(\cdot, t)\|_\sigma &= \sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) + \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma \leq 2 \, c_\mu \, \rho < \infty \,.
\end{aligned}$$

To prove (D.2), the idea is to use Duhamel's representation formula

$$\mu_-(x,t) = -\frac{1}{2} \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, (\mu_- \mu_+)'(x,s) + \varepsilon^2 \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, \Delta F'(x,s) \,, \tag{D.3}$$

for the solution of (D.1). We have

$$\varepsilon^2 \left\| \int_0^t \mathrm{d}s \; e^{-\mathcal{L}_\mu (t-s)} \, \Delta F'(\cdot, s) \right\|_{\mathcal{W},\sigma} \leq \varepsilon^2 \sup_{0 \leq s \leq t_1} \left\| \frac{\Delta F'(\cdot, s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \leq \lambda_1 \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_-(\cdot, t)\|_{\mathcal{W},\sigma} \,,$$

with $\lambda_1 < 1$. Then, from (D.3), Proposition 2.9 and Lemma D.1, we have

$$\begin{aligned}
\sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma &\leq \lambda_1 \, (1 + c_\lambda) \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_-(\cdot, t)\|_{\mathcal{W},\sigma} + \frac{\sqrt{2}}{\delta} \sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t) \, \mu_+(\cdot, t)\|_{\mathcal{W},\sigma - 1} \\
&\leq \lambda_1 \, (1 + c_\lambda) \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_-(\cdot, t)\|_{\mathcal{W},\sigma} + \frac{\sqrt{2} \, C_m}{\sqrt{\delta}} \sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma \, \|\mu_+(\cdot, t)\|_\sigma \\
&\leq \lambda_1 \, (1 + c_\lambda) \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_-(\cdot, t)\|_{\mathcal{W},\sigma} + \frac{2^{3/2} \, C_m \, c_\mu \, \rho}{\sqrt{\delta}} \sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma \,.
\end{aligned}$$

Since $\sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma < \infty$, and $\delta = c_\delta \, \rho^2$, with $c_\delta > 8 \, C_m \, c_\mu$, we have

$$\sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_\sigma \leq \lambda_1 \frac{(1 + c_\lambda)}{1 - \frac{2^{3/2} \, C_m \, c_\mu}{\sqrt{c_\delta}}} \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_-(\cdot, t)\|_{\mathcal{W},\sigma} \,.$$

Hence, we finally get

$$\sup_{0\le t\le t_1} \|\mathcal{F}(\tilde{\mu}_1,\mu_0)(\cdot,t)-\mathcal{F}(\tilde{\mu}_2,\mu_0)(\cdot,t)\|_\sigma \le \lambda \sup_{0\le t\le t_1} \|\tilde{\mu}_1(\cdot,t)-\tilde{\mu}_2(\cdot,t)\|_\sigma \ ,$$

with

$$\lambda = \lambda_1 \, \frac{1+c_\lambda}{1-\frac{2^{3/2}\,C_m\,c_\mu}{\sqrt{c_\delta}}} \ .$$

Choosing c_δ sufficiently large and c_λ sufficiently small we can certainly make λ arbitrarily close to $\lambda_1 < 1$, in particular, we can make $\lambda < 1$, which completes the proof. ■

Appendix E: Further properties of the amplitude equation

Corollary E.1 *Assume that $\|r_0\|_{\sigma-1} \le c_{r_1}\,\delta\,\rho$. Then $\mu \mapsto r(\mu)$ satisfies*

$$\|\!|r(\mu)|\!\|_{\sigma-1} \le 8\,c_{r_1}\,\delta\,\rho \ , \tag{E.1}$$

$$\|\!|r(\mu_1)-r(\mu_2)|\!\|_{\sigma-1} \le 8\,c_{r_1}\,\delta\,\|\!|\mu_1-\mu_2|\!\|_\sigma \ , \tag{E.2}$$

if the conditions of Theorem 3.5 are satisfied.

Proof. As a first step, we note that $\|\!|r(\mu)|\!\|_{\sigma-3}$ is finite, because

$$\|\!|r(\mu)|\!\|_{\sigma-3} \le \|\!|s(\mu)|\!\|_{\sigma-3} + \frac{\varepsilon^2}{2}\|\!|s(\mu)''|\!\|_{\sigma-3} \le (1+\frac{\varepsilon^2\,\delta^2}{2})\|\!|s(\mu)|\!\|_{\sigma-1} \ .$$

On the other hand, we have

$$\|\!|r(\mu)|\!\|_{\sigma-1} \le \|\!|r(\mu)|\!\|_{\sigma-3} + \|\!|r(\mu)|\!\|_{\mathcal{W},\sigma-1} \ , \tag{E.3}$$

and with the same arguments as the proof of Proposition 3.2, we have that for all $\sigma' \le \sigma-1$,

$$\begin{aligned}
\|\!|r(\mu)|\!\|_{\mathcal{W},\sigma'} &\le \|r_0\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4}{\chi}\|\!|F_3(s,\mu)|\!\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{8}\|\!|(s\,\mu')|\!\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{4}\|\!|(s\,\mu)'|\!\|_{\mathcal{W},\sigma'} \\
&\le \frac{3}{2}\,c_{r_1}\,\delta\,\rho + \frac{\varepsilon^4\,\alpha^2\,C_m\,\delta^{3/2}}{8}\|\!|s|\!\|_{\sigma-1}\,\|\!|\mu|\!\|_\sigma + \frac{\varepsilon^4\,\alpha^2}{4}\|\!|(s\,\mu)'|\!\|_{\mathcal{W},\sigma'} \\
&\le 2\,c_{r_1}\,\delta\,\rho + \frac{\varepsilon^4\,\alpha^2}{4}\|\!|(s\,\mu)'|\!\|_{\mathcal{W},\sigma'} \ ,
\end{aligned}$$

since $\varepsilon \le c_\varepsilon\,\delta^{-5/4}\,\rho^{-1/2}$. And now, we use that

$$s\mu = G\,\left(r\mu - \varepsilon^2 s'\mu' - \frac{\varepsilon^2}{2}s\mu''\right) \ ,$$

from which we get

$$\|\!|r(\mu)|\!\|_{\mathcal{W},\sigma'} \le 2\,c_{r_1}\,\delta\,\rho + \frac{\varepsilon^6\,\alpha^2}{8}\|\!|G\,(2s'\,\mu' + s\,\mu'')'|\!\|_{\mathcal{W},\sigma-1} + \frac{\varepsilon^4\,\alpha^2}{4}\|\!|G\,(r(\mu)\,\mu)'|\!\|_{\mathcal{W},\sigma'}$$

$$\leq 2\, c_{r_1}\, \delta\, \rho + \frac{\varepsilon^4\, \alpha^2}{4}\, (2\|s'\, \mu'\|_{\mathcal{W},\sigma-2} + \|s\, \mu''\|_{\mathcal{W},\sigma-2}) + \frac{\varepsilon^4\, \alpha^2}{4} \|G\, (r(\mu)\, \mu)'\|_{\mathcal{W},\sigma'}$$
$$\leq 2\, c_{r_1}\, \delta\, \rho + \frac{\varepsilon^4\, \alpha^2\, C_m\, \delta^{5/2}\, 3}{4} \|s\|_{\sigma-1}\, \|\mu\|_{\sigma} + \frac{\varepsilon^4\, \alpha^2}{4} \|G\, (r(\mu)\, \mu)'\|_{\mathcal{W},\sigma'}\, .$$

Using this inequality and (E.3), we finally have

$$\|r(\mu)\|_{\sigma'} \leq 7\, c_{r_1}\, \delta\, \rho + \frac{\varepsilon^4\, \alpha^2}{4} \|G\, (r(\mu)\, \mu)'\|_{\mathcal{W},\sigma'}\, . \tag{E.4}$$

Since $\|G\, (r(\mu)\, \mu)'\|_{\mathcal{W},\sigma'} \leq 2\|r(\mu)\, \mu\|_{\mathcal{W},\sigma'-1}$, we use (E.4) with $\sigma' = \sigma - 2$, and then with $\sigma' = \sigma - 1$ to conclude that $\|r(\mu)\|_{\mathcal{W},\sigma-1}$ is finite, and then we have

$$\|r(\mu)\|_{\sigma-1} \leq 7\, c_{r_1}\, \delta\, \rho + \frac{\varepsilon^3\, \alpha^2\, \sqrt{2}}{4} \|r(\mu)\, \mu\|_{\mathcal{W},\sigma-1}$$
$$\leq 7\, c_{r_1}\, \delta\, \rho + (\varepsilon^3\, \sqrt{\delta}\, \alpha^2\, C_m\, \|\mu\|_{\sigma}) \|r(\mu)\|_{\sigma-1}\, .$$

Since $\varepsilon \leq c_{\varepsilon}\, \delta^{-5/4}\, \rho^{-1/2}$, this last parenthesis is smaller than $\frac{1}{8}$, and the proof of (E.1) is completed. The proof of (E.2) is similar, we omit the details. ■

Appendix F: The $\mu \mapsto r_2(\mu)$ map.

We begin with a preliminary lemma.

Lemma F.1 *We have*

$$\frac{\varepsilon^2}{8} \left\| \frac{\mathcal{L}_{\mu,r}}{G\, \mathcal{L}_r}\, f'' \right\|_{\mathcal{W},\sigma} \leq \max\Big(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\Big)\, \|f\|_{\mathcal{W},\sigma}\, , \tag{F.1}$$

$$\|\mathcal{L}_{\mu,r}\, f\|_{\sigma} \leq 8\, \|f\|_{\sigma}\, , \tag{F.2}$$

$$\|\mathcal{L}_v^{-1}\, f'\|_{\mathrm{L}^2} \leq 2\, \|f\|_{\mathrm{L}^2}\, , \tag{F.3}$$

$$\|(1-\partial_x^2)^{-1}\, \mathcal{L}_v\, f\|_{\mathrm{L}^2} \leq \|f\|_{\mathrm{L}^2}\, , \tag{F.4}$$

$$\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_{\mu}\, G\, \mathcal{L}_r}\, f' \right\|_{\mathcal{W},\sigma} \leq 11\, \max\Big(2, \frac{\alpha^2}{1-\alpha^2}\Big)\, \delta^{-3}\, \|f\|_{\mathcal{W},\sigma-3}\, , \tag{F.5}$$

for all $\varepsilon^2 \leq 1$ *and* $\alpha^2 < 1/2$.

Proof. In terms of the Fourier coefficients, we have

$$\frac{\varepsilon^2}{8} \left(\frac{\mathcal{L}_{\mu,r}}{G\, \mathcal{L}_r}\, f'' \right)_n = -\left(\frac{\varepsilon^2\, (qn)^2}{8}\, \frac{\mathcal{L}_{\mu,r}(qn)}{G(qn)\, \mathcal{L}_r(qn)} \right)\, f_n\, ,$$

and

$$\frac{\varepsilon^2\, k^2}{8}\, \frac{\mathcal{L}_{\mu,r}(k)}{G(k)\, \mathcal{L}_r(k)} = \Big(\frac{\varepsilon^2\, k^2}{2}\Big)\, \frac{1 + \varepsilon^2\, (\frac{1+\alpha^2}{2}) - \alpha^2\, (\frac{\varepsilon^2\, k^2}{2})}{1 + (3 + \varepsilon^2\, \frac{1+\alpha^2}{2})\, (\frac{\varepsilon^2\, k^2}{2}) + (1-\alpha^2)\, (\frac{\varepsilon^4\, k^4}{4})}$$
$$= \frac{\xi^2\, (\lambda^2 - \alpha^2\, \xi^2)}{1 + (2+\lambda^2)\, \xi^2 + (1-\alpha^2)\, \xi^4}\, ,$$

with $\xi = \frac{\varepsilon^2 k^2}{2}$ and $\lambda^2 = 1 + \varepsilon^2 (\frac{1+\alpha^2}{2})$. Then as a function of ξ, we have

$$-\frac{\alpha^2}{1-\alpha^2} \leq \frac{\xi^2 (\lambda^2 - \alpha^2 \xi^2)}{1 + (2+\lambda^2)\xi^2 + (1-\alpha^2)\xi^4} \leq \frac{\lambda^4}{\lambda^4 + 4\lambda^2 + 4\alpha^2} \leq \frac{1}{3} ,$$

where the last inequality comes from from the fact that $\varepsilon^2 \leq 1$ and $\alpha^2 < 1$ imply that $1 \leq \lambda^2 \leq 2$. This proves (F.1). For (F.2), we have $(\mathcal{L}_{\mu,r} f)_n = \mathcal{L}_{\mu,r}(qn) f_n$, and with the above notations,

$$|\mathcal{L}_{\mu,r}(k)| = 4 \frac{|\lambda^2 - \alpha^2 \xi^2|}{1+\xi^2} \leq 4 \max(\alpha^2, \lambda^2) \leq 8 ,$$

while for (F.3) and (F.4), we use that

$$|ik\, \mathcal{L}_v(k)^{-1}| \leq \sqrt{\frac{3k^2(1+\frac{k^2}{2})}{1+k^4}} \leq 2 ,$$
$$\left|\frac{\mathcal{L}_v(k)}{1+k^2}\right| \leq 1 .$$

For (F.5), we have

$$\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\, G\, \mathcal{L}_r} f'\right)_n = i\, qn \left(\frac{\mathcal{L}_{\mu,r}(qn)}{\mathcal{L}_\mu(qn)\, G(qn)\, \mathcal{L}_r(qn)}\right) f_n ,$$

then for $|qn| = |k| \geq \delta \geq 2$, we have

$$\left|\frac{\mathcal{L}_{\mu,r}(k)}{\mathcal{L}_\mu(k)\, G(k)\, \mathcal{L}_r(k)}\right| \leq \frac{8}{k^4} \sup_{|k|\geq\delta}\left|\frac{k^4}{k^4-k^2}\right| \sup_{|\xi|\geq 0}\left|\frac{(1+\xi^2)(\lambda^2-\alpha^2\xi^2)}{1+(2+\lambda^2)\xi^2+(1-\alpha^2)\xi^4}\right|$$
$$\leq \frac{32}{3k^4} \max\Big(\lambda^2 , \frac{\alpha^2}{1-\alpha^2}\Big) \leq \frac{11}{k^4} \max\Big(2 , \frac{\alpha^2}{1-\alpha^2}\Big) .$$

The proof of (F.5) is completed noting that $\|K^{-4} f'\|_{\mathcal{W},\sigma} = \delta^{-3} \|f\|_{\mathcal{W},\sigma-3}$, if $(K^{-4} f)_n \equiv (qn)^{-4} f_n$. ∎

F.1 Coercive functionals for the amplitude

Proposition F.2 *Let $c_\mu > 0$ and $\alpha^2 < 1$. There exists a constant c_ε such that*

$$\int r_2\, G\, \mathcal{L}_r\, r_2 - \frac{\varepsilon^4}{16} \int r_2\, \mu\, \mathcal{L}_{\mu,r}\, r_2' \geq \frac{3}{4} \int r_2^2 , \tag{F.6}$$

$$\int r_4\, G\, \mathcal{L}_r\, r_4 - \frac{\varepsilon^4}{16} \int r_4\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\Big(\mu\, \mathcal{L}_v\, r_4\Big)' \geq \frac{3}{4} \int r_4^2 . \tag{F.7}$$

for all $\varepsilon \leq c_\varepsilon\, \delta^{-5/4}\, \rho^{-1/2}$ and for all $\mu \in \mathcal{B}_\sigma(c_\mu\, \rho)$.

Proof. We notice first that $\mathcal{L}_{\mu,r} r_2' = a_1\, G\, r_2' - a_2 \frac{\varepsilon^2}{2}\, G\, r_2'''$ with $a_1 = 4 + 2\,\varepsilon^2 (1+\alpha^2)$ and $a_2 = 4\alpha^2$. Then we have

$$\left|\int \mu\, r_2\, G\, r_2'\right| \leq \|\mu\|_{\mathrm{L}^\infty} \|r_2\|_{\mathrm{L}^2} \|r_2'\|_{\mathrm{L}^2} \leq \frac{C\, c_\mu\, \rho \sqrt{\delta}}{2}\Big(\|r_2\|_{\mathrm{L}^2}^2 + \|r_2'\|_{\mathrm{L}^2}^2\Big) ,$$

$$\left| \int \mu \, r_2 \, \frac{\varepsilon^2}{2} \, G \ r_2''' \right| \leq \|\mu\|_{\mathrm{L}^\infty} \, \|r_2\|_{\mathrm{L}^2} \, \|r_2'\|_{\mathrm{L}^2} \leq \frac{C \, c_\mu \, \rho \, \sqrt{\delta}}{2} \Big(\|r_2\|_{\mathrm{L}^2}^2 + \|r_2'\|_{\mathrm{L}^2}^2 \Big) \ ,$$

since (by Fourier Transform) we have $\|G\ f\|_{\mathrm{L}^2} \leq \|f\|_{\mathrm{L}^2}$ and $\|\frac{\varepsilon^2}{2}\ G\ f''\|_{\mathrm{L}^2} \leq \|f\|_{\mathrm{L}^2}$. We thus get

$$\begin{aligned} \left| \frac{\varepsilon^4}{16} \int r_2 \, \mu \, \mathcal{L}_{\mu,r} \, r_2' \right| &\leq \varepsilon^4 \ \tfrac{(a_1+a_2)\ C\ c_\mu\ \rho\ \sqrt{\delta}}{32} \left(\int r_2^2 + \int (r_2')^2 \right) \\ &\leq \varepsilon^2 \ \tfrac{(a_1+a_2)\ C\ c_\mu\ \rho\ \sqrt{\delta}}{16} \left(\int r_2^2 + \frac{\varepsilon^2}{2} \int (r_2')^2 \right) \ . \end{aligned}$$

Let now $a_3 = 3 + \varepsilon^2 \Big(\frac{1+\alpha^2}{2})$ and $a_4 = 1 - \alpha^2$. We have

$$\int r_2 \, G \, \mathcal{L}_r \, r_2 - \frac{\varepsilon^4}{16} \int r_2 \, \mu \, \mathcal{L}_{\mu,r} \, r_2' \geq \gamma \int r_2^2 \ ,$$

where

$$\gamma = \min_{\xi \in \mathbf{R}} \left(\frac{1 + a_3 \, \xi^2 + a_4 \, \xi^4}{1+\xi^2} - \varepsilon^2 \, \rho \, \sqrt{\delta} \, (\tfrac{(a_1+a_2)\ C\ c_\mu}{16})(1+\xi^2) \right) \ .$$

Since $a_3 \geq 3$ and $a_4 > 0$, choosing c_ε sufficiently small completes the proof of (F.6). The proof of (F.7) is similar. We first use

$$\begin{aligned} \int r_4 \, \mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \Big(\mu \, \mathcal{L}_v^{1/2} \, r_4 \Big)' &= - \int \Big(\mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \, r_4 \Big)' \, \mu \, \mathcal{L}_v \, r_4 = \int f \, \mu \, (1 - \partial_x^2) \, g \\ &= \int f \, \mu \, g + f' \, \mu \, g' + f \, \mu' \, g \ , \end{aligned}$$

where $f = \mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \, r_4'$ and $g = (1 - \partial_x^2)^{-1} \, \mathcal{L}_v \, r_4$. Let $f^{(m)}$ be the m–th order spatial derivative of f. Then we have $\|f^{(m)}\|_{\mathrm{L}^2} \leq 16 \, \|r_4^{(m)}\|_{\mathrm{L}^2}$ and $\|g^{(m)}\|_{\mathrm{L}^2} \leq \|r_4^{(m)}\|_{\mathrm{L}^2}$. Furthermore, we have $\|\mu'\|_{\mathrm{L}^\infty} \leq C \, c_\mu \, \delta^{3/2} \, \rho$ and $\|\mu\|_{\mathrm{L}^\infty} \leq C \, c_\mu \, \delta^{1/2} \, \rho \leq C \, c_\mu \, \delta^{3/2} \, \rho$. Using these inequalities, we have

$$\begin{aligned} \frac{\varepsilon^4}{16} \left| \int r_4 \, \mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \Big(\mu \, \mathcal{L}_v^{1/2} \, r_4 \Big)' \right| &\leq \varepsilon^4 \, C \, c_\mu \, \delta^{3/2} \, \rho \, (\|r_4\|_{\mathrm{L}^2}^2 + \|r_4\|_{\mathrm{L}^2} \, \|r_4'\|_{\mathrm{L}^2} + \|r_4'\|_{\mathrm{L}^2}^2) \\ &\leq 3 \, \varepsilon^2 \, C \, c_\mu \, \delta^{3/2} \, \rho \left(\int r_4^2 + \frac{\varepsilon^2}{2} \int (r_4')^2 \right) \ . \end{aligned}$$

As above, choosing c_ε sufficiently small completes the proof of (F.7). ■

F.2 Various bounds on r_2

Lemma F.3 *Let r_2 be the solution of (4.8). Then r_2 satisfies*

$$\|\chi \mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \, r_2'\|_{\mathrm{L}^2} \leq 64 \, \|r_{2,0}\|_{\mathrm{L}^2} + \sqrt{2} \, \|\mathcal{L}_{\mu,r} \, \mathcal{L}_v^{-1} \, F_6(s(\mu), \mu)'\|_{\mathrm{L}^2} \ . \tag{F.8}$$

Proof. Let $r_4 = \chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2'$, then r_4 satisfies

$$\partial_t r_4 = -\frac{\chi}{\varepsilon^4}\left(G\,\mathcal{L}_r\, r_4 + \frac{\varepsilon^4}{16}\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,(\mu\,\mathcal{L}_v\, r_4)'\right) + \frac{\chi}{\varepsilon^4}\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,F_6(s(\mu),\mu)'\;, \qquad \text{(F.9)}$$

with initial condition $r_4(x,0) = \chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1/2}\, r_{2,0}(x)' \equiv r_{4,0}(x)$. Using Proposition F.2, we then have

$$\begin{aligned}
\partial_t(r_4,r_4) &\leq -\frac{\chi}{\varepsilon^4}\,\frac{3}{2}\,(r_4,r_4) + \frac{2\,\chi}{\varepsilon^4}(r_4,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,F_6(s(\mu),\mu)')\\
&\leq -\frac{\chi}{\varepsilon^4}\,(r_4,r_4) + \frac{2\,\chi}{\varepsilon^4}\|\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,F_6(s(\mu),\mu)'\|_{\mathrm{L}^2}^2\;.
\end{aligned}$$

Integrating this differential inequality and using that $\|r_{4,0}\|_{\mathrm{L}^2} \leq 64\,\|r_{2,0}\|_{\mathrm{L}^2}$ completes the proof. ■

Lemma F.4 *Let r_2 be the solution of (4.8) with $\mu \in \mathcal{B}_\sigma(c_\mu\,\rho)$, and assume that r_2 satisfies*

$$\varepsilon^2\,\|\!|\chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,r_2'\|\!|_{\mathrm{L}^2} \leq c_\mu\,\rho\;.$$

Then there exists a constant C such that

$$\left\|\!\left|\frac{\chi\,\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,r_2'\right\|\!\right|_{\mathcal{W},\sigma} \leq \frac{\frac{4\,\chi}{\delta}\,\|r_{2,0}\|_{\mathcal{W},\sigma-1} + \left\|\!\left|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,\frac{F_6(s(\mu),\mu)'}{G\,\mathcal{L}_r}\right\|\!\right|_{\mathcal{W},\sigma} + C\,\varepsilon^2\,\max(2\,,\,\frac{\alpha^2}{1-\alpha^2})\,c_\mu\,\rho}{1-\varepsilon^2\,2\,C_m\,c_\mu\,c_\delta^{-1/2}\,\max(2,\frac{\alpha^2}{1-\alpha^2})}\;,$$

for all ε satisfying $\varepsilon^2\,2\,C_m\,c_\mu\,c_\delta^{-1/2}\,\max(2,\frac{\alpha^2}{1-\alpha^2}) < 1$.

Proof. We define

$$r_3 = P_>\left(\frac{\chi\,\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,r_2'\right)\;, \qquad \text{(F.10)}$$

and we note that $\|\!|r_3\|\!|_{\mathcal{W},\sigma} < \infty$, because

$$\|\!|r_3\|\!|_{\mathcal{W},\sigma} = \left\|\!\left|\frac{\chi\,\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,r_2'\right\|\!\right|_\sigma \leq \frac{4\,\chi}{\delta}\,\|\!|r_2\|\!|_{\sigma-1} \leq \frac{4\,\chi}{\delta\,\varepsilon^4}\left(\|\!|r(\mu)\|\!|_{\sigma-1} + \|\!|r_1(\mu)\|\!|_{\sigma-1}\right)\;.$$

On the other hand, r_3 satisfies

$$\begin{aligned}
\partial_t r_3 &= -\frac{\chi}{\varepsilon^4}\,G\,\mathcal{L}_r\,r_3 + \frac{\chi}{16}\,P_>\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,(\mu\,\chi\,\mathcal{L}_{\mu,r}\,r_2')'\right) + \frac{\chi}{\varepsilon^4}\,P_>\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,F_6(s(\mu),\mu)'\right)\\
&= -\frac{\chi}{\varepsilon^4}\,G\,\mathcal{L}_r\,r_3 + \frac{\chi}{16}\,P_>\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,(\mu\,\mathcal{L}_\mu\,r_3)'\right) + \frac{\chi}{\varepsilon^4}\,P_>\,F_9(\mu,P_<\,r_2)\;,
\end{aligned}$$

with $r_3(x,0) = r_{3,0}(x)$ and

$$F_9(\mu,P_<\,r_2) = \frac{\chi\,\varepsilon^4}{16}\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,(\mu\,\mathcal{L}_{\mu,r}\,P_<\,r_2')'\right) + \frac{1}{\varepsilon^4}\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,F_6(s(\mu),\mu)'\right)\;.$$

Then we use Duhamel's formula for the solution to conclude that

$$\begin{aligned}\|r_3\|_{\mathcal{W},\sigma} &\leq \|r_{3,0}\|_{\mathcal{W},\sigma} + \frac{\varepsilon^4}{16} \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\, G\, \mathcal{L}_r} \, (\mu\, \mathcal{L}_\mu\, r_3)' \right\|_{\mathcal{W},\sigma} + \left\| \frac{F_9(\mu, P_<\, r_2)}{G\, \mathcal{L}_r} \right\|_{\mathcal{W},\sigma} \\ &\leq \frac{4\,\chi}{\delta} \|r_{2,0}\|_{\mathcal{W},\sigma-1} + \varepsilon^2\, \frac{11}{16}\, \max\Big(2\,,\, \frac{\alpha^2}{1-\alpha^2}\Big)\, \varepsilon^2\, \delta^{-3}\, \|\mu\, \mathcal{L}_\mu\, r_3\|_{\mathcal{W},\sigma-3} \\ &\quad + \left\| \frac{F_9(\mu, P_<\, r_2)}{G\, \mathcal{L}_r} \right\|_{\mathcal{W},\sigma} ,\end{aligned}$$

where we used Lemma F.1. Then we have

$$\varepsilon^2\, \delta^{-3}\, \|\mu\, \mathcal{L}_\mu\, r_3\|_{\mathcal{W},\sigma-3} \leq C_m\, \delta^{-5/2}\, \|\mu\|_\sigma\, \|\varepsilon^2\, \mathcal{L}_\mu\, r_3\|_{\sigma-3} \leq 2\, C_m\, c_\mu\, c_\delta^{-1/2}\, \|r_3\|_\sigma ,$$

from which we get

$$\|r_3\|_{\mathcal{W},\sigma} \leq \frac{\frac{4\,\chi}{\delta}\|r_{2,0}\|_{\mathcal{W},\sigma-1} + \|\frac{F_9(\mu,P_<\, r_2)}{G\,\mathcal{L}_r}\|_{\mathcal{W},\sigma}}{1 - \varepsilon^2\, 2\, C_m\, c_\mu\, c_\delta^{-1/2}\, \max(2\,,\, \frac{\alpha^2}{1-\alpha^2})} ,$$

since by hypothesis $\varepsilon^2\, 2\, C_m\, c_\mu\, c_\delta^{-1/2}\, \max(2\,,\, \frac{\alpha^2}{1-\alpha^2}) < 1$. Using twice Lemma F.1, we also have

$$\begin{aligned}\frac{\chi\,\varepsilon^4}{16} \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\, G\, \mathcal{L}_r} \, (\mu\, \mathcal{L}_{\mu,r}\, P_<\, r_2')' \right\|_{\mathcal{W},\sigma} &\leq \varepsilon^4\, \frac{11}{16}\, m(\alpha)\, \delta^{-3}\, \|\mu\, (\chi\, P_<\, \mathcal{L}_{\mu,r}\, r_2')\|_{\mathcal{W},\sigma-3} \\ &\leq \varepsilon^4\, \frac{11}{2}\, m(\alpha)\, C_m\, c_\mu\, \delta^{-5/2}\, \rho\, \|\chi\, P_<\, \mathcal{L}_{\mu,r}\, r_2'\|_\sigma ,\end{aligned}$$

where $m(\alpha) = \max\Big(2\,,\, \frac{\alpha^2}{1-\alpha^2}\Big)$. But we have $\|P_<\, \mathcal{L}_v\, f\|_\sigma = \|P_<\, \mathcal{L}_v\, f\|_{\mathrm{L}^2} \leq 4\, \delta^2\, \|f\|_{\mathrm{L}^2}$, so that

$$\|\chi\, P_<\, \mathcal{L}_{\mu,r}\, r_2'\|_\sigma \leq \|\chi\, P_<\, \mathcal{L}_v\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2'\|_{\mathrm{L}^2} \leq 4\, \delta^2\, \|\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2'\|_{\mathrm{L}^2} .$$

Hence we have

$$\begin{aligned}\frac{\chi\,\varepsilon^4}{16} \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu\, G\, \mathcal{L}_r} \, (\mu\, \mathcal{L}_{\mu,r}\, P_<\, r_2')' \right\|_{\mathcal{W},\sigma} &\leq \varepsilon^2\, \left(22\, m(\alpha)\, \frac{C_m\, c_\mu}{\sqrt{c_\delta}}\right)\, \varepsilon^2\, \|\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2'\|_{\mathrm{L}^2} \\ &\leq \varepsilon^2\, \left(22\, m(\alpha)\, \frac{C_m\, c_\mu}{\sqrt{c_\delta}}\right)\, c_\mu\, \rho .\end{aligned}$$

This completes the proof. ■

Lemma F.5 *Let $r_2(\mu_i)$ be the solution of (4.8) with $\mu = \mu_i \in \mathcal{B}_\sigma(c_\mu\, \rho)$, and assume that*

$$\varepsilon^2\, \|\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2\, \left\| \frac{\chi\, \mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\, r_2(\mu_i)' \right\|_{\mathcal{W},\sigma} \leq c_\mu\, \rho . \tag{F.11}$$

Let $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ and $\Delta F_6 = F_6(s(\mu_1), \mu_1) - F_6(s(\mu_2), \mu_2)$. Then there exists a constant C such that

$$\|\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, \Delta r_2'\|_{\mathrm{L}^2} \leq \|\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, \Delta F_6'\|_{\mathrm{L}^2} + C\, \delta^{5/2}\, \rho\, \|\mu_1 - \mu_2\|_\sigma . \tag{F.12}$$

Proof. The proof relies on the fact that Δr_2 satisfies the same equation as r_2, but with initial data $\Delta r_2(x,0)=0$, $F_6(s(\mu),\mu)$ replaced by $\Delta F_6 = F_6(s(\mu_1),\mu_1) - F_6(s(\mu_2),\mu_2)$ and $\mu\,\mathcal{L}_{\mu,r}\,r_2'$ replaced by

$$\mu_1\,\mathcal{L}_{\mu,r}\,r_2(\mu_1)' - \mu_2\,\mathcal{L}_{\mu,r}\,r_2(\mu_2)' = \Delta\mu\,\mathcal{L}_{\mu,r}\,r_2^{+\prime} + \Big(\frac{\mu_1+\mu_2}{2}\Big)\,\mathcal{L}_{\mu,r}\,\Delta r_2'\,, \tag{F.13}$$

where $\Delta\mu = \mu_1 - \mu_2$ and $r_2^+ = \frac{r_2(\mu_1)+r_2(\mu_2)}{2}$. Since $\frac{\mu_1+\mu_2}{2}$ satisfies the same bound as μ in Lemma F.3, we see that the conclusion of this Lemma holds with the replacements $r_{2,0}=0$, $r_2 \leftrightarrow \Delta r_2$ and $f_6 \leftrightarrow \Delta F_6$ and an additional term given by $\frac{\varepsilon^4}{8}\,\|\chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1/2}\,(\Delta\mu\,\mathcal{L}_{\mu,r}\,r_2^{+\prime})'\|_{\mathrm{L}^2}$, on which we have

$$\frac{\varepsilon^4}{8}\,\|\chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1/2}\,(\Delta\mu\,\mathcal{L}_{\mu,r}\,r_2^{+\prime})'\|_{\mathrm{L}^2} \le 8\,\varepsilon^4\,\|\Delta\mu\,\mathcal{L}_{\mu,r}\,r_2^{+\prime}\|_{\mathrm{L}^2}\,.$$

Using (F.11), defining r_3 as in (F.10) and writing r_2 instead of r_2^+ to simplify the notation, we have

$$\begin{aligned}\varepsilon^4\,\|\Delta\mu\,\mathcal{L}_{\mu,r}\,r_2'\|_{\mathrm{L}^2} &\le \varepsilon^4\,\|\Delta\mu\,\mathcal{L}_{\mu,r}\,P_<\,\mathcal{L}_v\,\mathcal{L}_v^{-1}\,r_2'\|_{\mathrm{L}^2} + \varepsilon^4\,\|\Delta\mu\,\mathcal{L}_\mu\,r_3\|_{\mathrm{L}^2}\\ &\le (3\,\varepsilon^2\,C\,\delta^{5/2})\,\|\Delta\mu\|_\sigma\,(\varepsilon^2\,\|\mathcal{L}_v^{-1}\,r_2'\|_{\mathrm{L}^2}) + \varepsilon^2\,\|\Delta\mu\,(\varepsilon^2\mathcal{L}_\mu)\,r_3\|_{\sigma-2}\\ &\le (3\,\varepsilon^2\,C\,\delta^{5/2})\,\|\Delta\mu\|_\sigma\,c_\mu\,\rho + 2\,C_m\,\delta^{5/2}\,\|\Delta\mu\|_\sigma\,(\varepsilon^2\,\|r_3\|_\sigma)\,,\end{aligned}$$

since $\|P_<\,\mathcal{L}_v^{-1}\,f\|_{\mathrm{L}^2} \le 3\,\|f\|_{\mathrm{L}^2}$ and $\varepsilon^2\|\mathcal{L}_\mu\,f\|_{\sigma-2} \le 2\,\delta^2\,\|f\|_\sigma$. By hypothesis, we have $\varepsilon^2\,\|r_3\|_\sigma \le c_\mu\,\rho$ and the proof is completed. ■

Lemma F.6 *Let $r_2(\mu_i)$ be the solution of (4.8) with $\mu=\mu_i$ and define $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ and $\Delta F_6 = F_6(s(\mu_1),\mu_1) - F_6(s(\mu_2),\mu_2)$. Assume that $\mu_i \in \mathcal{B}_\sigma(c_\mu\,\rho)$, and that*

$$\begin{aligned}\varepsilon^2\,\|\chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,r_2(\mu_i)'\|_{\mathrm{L}^2} + \varepsilon^2\,\Big\|\frac{\chi\,\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,r_2(\mu_i)'\Big\|_{\mathcal{W},\sigma} &\le c_\mu\,\rho\,,\\ \varepsilon^2\,\|\chi\,\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,\Delta r_2(\mu_i)'\|_{\mathrm{L}^2} &\le \|\mu_1-\mu_2\|_\sigma\,.\end{aligned} \tag{F.14}$$

Then there exists a constant C such that

$$\Big\|\frac{\chi\,\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,\Delta r_2'\Big\|_{\mathcal{W},\sigma} \le \frac{\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\,\frac{\Delta F_6'}{G\,\mathcal{L}_r}\|_{\mathcal{W},\sigma} + C\,\max(2,\frac{\alpha^2}{1-\alpha^2})\,\|\mu_1-\mu_2\|_\sigma}{1-\varepsilon^2\,2\,C_m\,c_\mu\,c_\delta^{-1/2}\,\max(2,\frac{\alpha^2}{1-\alpha^2})}\,, \tag{F.15}$$

for all ε satisfying $\varepsilon^2\,2\,C_m\,c_\mu\,c_\delta^{-1/2}\,\max(2,\frac{\alpha^2}{1-\alpha^2}) < 1$.

Proof. The proof relies again on the fact that Δr_2 satisfies the same equation as r_2, with initial data $\Delta r_2(x,0)=0$, $F_6(s(\mu),\mu)$ replaced by $\Delta F_6 = F_6(s(\mu_1),\mu_1) - F_6(s(\mu_2),\mu_2)$ and $\mu\,\mathcal{L}_{\mu,r}\,r_2'$ replaced by

$$\mu_1\,\mathcal{L}_{\mu,r}\,r_2(\mu_1)' - \mu_2\,\mathcal{L}_{\mu,r}\,r_2(\mu_2)' = \Delta\mu\,\mathcal{L}_{\mu,r}\,r_2^{+\prime} + \Big(\frac{\mu_1+\mu_2}{2}\Big)\,\mathcal{L}_{\mu,r}\,\Delta r_2'\,. \tag{F.16}$$

The proof can be done as that of Lemma F.4, hence we omit the details. ■

Appendix G: Discussion

The proofs of this section follow from definitions and proofs of Section 2.3 which should be read first. By (1.18), we have

$$\eta(x,t) = \frac{\hat{\varepsilon}^2}{4} \int_0^{\hat{\varepsilon}\, x} \mathrm{d}z\; \hat{\mu}(z,\hat{t}) ,$$

and we get

$$\begin{aligned} \|\eta\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} &\le \hat{\varepsilon}^2\, \frac{\hat{\varepsilon}\, L_0}{2}\, \|\hat{\mu}\|_{\mathrm{L}^\infty} \le c_\mu\, \hat{\varepsilon}^2\, L\, \rho \le C\, \hat{\varepsilon}^2\, \rho^{13/8} , \\ \|s\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} &\le \hat{\varepsilon}^4\, \|\hat{s}\|_{\mathrm{L}^\infty} \le C\, \varepsilon^4\, \delta^{3/2}\, \rho . \end{aligned}$$

If $\hat{\varepsilon} \le c_\varepsilon\, \rho^{-m_\varepsilon}$ with $m_\varepsilon \ge 4$, we get

$$\begin{aligned} \|\eta\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} &\le C\, \varepsilon^{2-13/(8\, m_\varepsilon)} , \\ \|s\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} &\le C\, \varepsilon^{4-4/m_\varepsilon} , \end{aligned} \tag{G.1}$$

since $\rho \le c_\varepsilon\, \hat{\varepsilon}^{-1/m_\varepsilon}$. We also have

$$\|\eta'\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \le \hat{\varepsilon}^{5/2}\, \|\hat{\mu}\|_{\mathrm{L}^2} \le C\, \varepsilon^{5/2}\, \rho \le C\, \varepsilon^{5/2-1/m_\varepsilon} , \tag{G.2}$$

$$\|\eta'\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} \le \hat{\varepsilon}^{3}\, \|\hat{\mu}\|_{\mathrm{L}^\infty} \le C\, \varepsilon^{3}\, \sqrt{\delta}\, \rho \le C\, \varepsilon^{3-2/m_\varepsilon} , \tag{G.3}$$

$$\|s\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \le \hat{\varepsilon}^{7/2}\, \|\hat{s}\|_{\mathrm{L}^2} \le C\, \varepsilon^{7/2}\, \delta\, \rho \le C\, \varepsilon^{7/2-3/m_\varepsilon} . \tag{G.4}$$

Various other estimates, e.g. on higher order derivatives can be obtained in a similar way.

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