The analytic structure of the reflection coefficient, a sum rule, and a complete description of the Weyl $m-$function of half-line Schrödinger operators with L_2-type potentials

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Abstract. We prove that the reflection coefficient of one-dimensional Schrödinger operators with potentials supported on a half-line can be represented in the upper half plane as the quotient of a contractive analytic function and a properly regularized Blaschke product. We apply this fact to obtain a new sum rule and sum inequality for the reflection coefficient that yields an exhaustive description of the Weyl $m-$function of Dirichlet half-line Schrödinger operators with potentials q subject to:

$$\int\int e^{-|x-y|}q(x)\,q(y)\,dxdy<\infty.$$

Among others, we also refine the $3/2-$Lieb-Thirring inequality.

1. Introduction

This paper continues the recent author's note [**16**]. Let (H,H_0), where

$$H_0=-d^2/dx^2, H=H_0+q(x) \text{ on } L_2(\mathbb{R}), \mathbb{R}:=(-\infty,\infty), \tag{1.1}$$

be a pair of the free H_0 and perturbed Schrödinger operators on the line. In the sequel we will always assume that q is real. In the short-range setting the scattering matrix $S(k)$ for (H,H_0) exists and given by

$$S(k)=\begin{pmatrix} T(k) & R_+(k) \\ R_-(k) & T(k)\end{pmatrix}, \tag{1.2}$$

where $T(k)$ and $R_\pm(k)$ denote the transmission and reflection coefficients form left (right) incidence. The function $T(k)$ is analytic in the upper half plane $\mathbb{C}_+$ except for purely imaginary simple poles $\{i\kappa_n\}_{n\geq 1}$ accumulating at most to 0. Note that $\{-\kappa_n^2\}_{n\geq 1}$ are bound states of H. The reflection coefficients $R_\pm(k)$, in general, are not analytic on $\mathbb{C}_+$. However in the case when the potential q vanishes to the left from some point, say 0, $R_+(k)$ continues analytically into the upper half plane $\mathbb{C}_+$ except for simple poles $\{i\kappa_n\}_{n\geq 1}$. Moreover, for such potentials q one can develop, following Pavlov [**20**], a scattering theory even without any decay assumption at

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∞. Although, the usual scattering matrix (1.2) need not exist but the reflection coefficient $R_+(k) =: R(k)$ can be properly defined and

$$R(k) = \frac{ik - m\left(k^2 + i0\right)}{ik + m\left(k^2 + i0\right)}, k \in \mathbb{R}, \tag{1.3}$$

where $m(z)$ is the Weyl $m-$function associated with $(z \in \mathbb{C}_+)$

$$-u'' + q(x)u = zu, u(0) = 0, \text{ on } L_2(\mathbb{R}_+), \mathbb{R}_+ := [0, \infty).$$

It immediately follows from (1.3) and the basic properties of the Weyl $m-$function that $|R(k)| \leq 1, k \in \mathbb{R}$, and analytic in k: $k^2 \in \mathbb{C}_+$. If $q(x) \geq 0$ then [**20**] $R(k)$ is analytic and contractive ($|R(k)| \leq 1$) on the whole $\mathbb{C}_+$. As Pavlov indicated in [**21**] if $q_- := \frac{1}{2}(|q| - q) \in L_1(\mathbb{R}_+)$ (summable over $\mathbb{R}_+$) then

$$R(k) = B^{-1}(k) S(k), k \in \mathbb{C}_+, \tag{1.4}$$

where

$$B(k) = \prod_{n \geq 1} \frac{k - i\kappa_n}{k + i\kappa_n} \tag{1.5}$$

is a Blaschke product (κ_n are as above) and $S(k)$ is some analytic contractive function on $\mathbb{C}_+$. We emphasize that the Blaschke condition

$$\sum_{n \geq 1} \kappa_n < \infty \tag{1.6}$$

is satisfied for $q_- \in L_1(\mathbb{R}_+)$ due to the 1996 result by Weidl [**22**].

In Section 2 of the present paper (Theorem 1) we establish that factorization (1.4) holds without the condition $q_- \in L_1(\mathbb{R}_+)$ if one replaces B in (1.4) with a

regularized Blaschke product recently discovered by Simon [**19**].

Let us introduce a L_2-type class $\mathcal{L}_2(\mathbb{R})$ of potentials q satisfying to

$$\|q\|^2_{\mathcal{L}_2} := \int \int \frac{e^{-2a|x-y|}}{4a^2} q(x) q(y) dxdy, a > 0, \tag{1.7}$$

(adopting the convention $\int := \int_{-\infty}^{\infty}$). One immediately observes that $L_2 \subset \mathcal{L}_2$ but $\mathcal{L}_2$ admits L_1 local singularities (as opposed to L_2) and it is wider than $L_1 + L_2$.

In Section 3 for potentials from $\mathcal{L}_2$ we derive a new stronger version of the Buslaev-Faddeev-Zhakharov second trace formula [**5**] for full-line Schrodinger operators with q's vanishing on the left half-line and a two-sided trace inequality for half-line Schrodinger operators with a Dirichlet boundary condition at 0. Following Killip-Simon [**10**], we will refer the former as to a *sum rule*. The latter will accordingly be called a *sum inequality*. Our proofs crucially rely on Theorem 1.

Section 4 is devoted to some corollaries of the main results of Section 3. Among others, we will give a complete description of the Weyl $m-$function (and hence the spectrum) of half-line Dirichlet Schrodinger operators with $\mathcal{L}_2$ potentials (Theorem 3). Note that Corollary 1 to Theorem 3 improves on a result of Damanik-Killip-Simon just announced in [**2**].

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2. The analytic structure of the reflection coefficient

The main result of this section is relied upon two lemmas. The first one is trivial and is a simple exercise to prove.

LEMMA 1. *Let $f(z)$ be a Heglotz function (i.e. $f:\mathbb{C}_+\to\mathbb{C}_+$) and*

$$f(z)=a+bz+\int\left(\frac{1}{s-z}-\frac{s}{1+s^2}\right)d\mu(s)$$

be its Herglotz representation. If the measure $\mu(s)$ is singular for $s<0$ with pure points λ_n subject to $\{|\lambda_n|\}\searrow 0$ then the function $f(x)=f(x+i0)$ is real for $x<0$, monotonic and continuous inside every interval $(\lambda_n,\lambda_{n+1})$ and $\lim_{x\to\lambda_n\pm 0}f(x)=\mp\infty$.

The following important lemma belongs to Simon [**19**].

LEMMA 2. *Let $\{\kappa_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ be two positive decreasing sequences such that $\kappa_n<\nu_n<\kappa_{n+1}, n\in\mathbb{N}$, and $\lim_{n\to\infty}\kappa_n=\lim_{n\to\infty}\nu_n=0$. Then the product*

$$B_N(z):=\left\{\prod_{n=1}^{N}\frac{z-i\kappa_n}{z+i\kappa_n}\right\}\left\{\prod_{n=1}^{N}\frac{z-i\nu_n}{z+i\nu_n}\right\}^{-1} \tag{2.1}$$

converges on $\overline{\mathbb{C}}_+\backslash clos\{\nu_n\}_{n\geq 1}$ to some analytic function $B(z)$ as $N\to\infty$. Moreover

$$|B(z)|=1, z\in\mathbb{R}\setminus\{0\}. \tag{2.2}$$

As it was pointed out in [**19**], condition (1.6) is not necessary for B to exists. B_N converges as long as κ_n and ν_n interlace and $\kappa_n,\nu_n\to 0$.

In the sequel, given self-adjoint operator A, $\sigma(A)$ denotes its spectrum and $\sigma_{\mathrm{ac}}(A),\sigma_{\mathrm{pp}}(A)$ denotes the absolutely continuous, pure point spectrum respectively. The theorem below is a generalization of (1.4)

THEOREM 1. *Let q be a real potential such that $q(x)=0, x<0$. Assume that q is in the limit point case at $+\infty$ and such*

$$\sigma\left(-d^2/dx^2+q(x)\right)\cap(-\infty,0)=\left\{-\kappa_n^2\right\}_{n\geq 1}\to 0, n\to\infty.$$

Then there is a sequence $\{\nu_n\}_{n\geq 1}$ subject to $\kappa_n<\kappa_n<\kappa_{n+1}, n\in\mathbb{N}$, such that the representation

$$R(z)=B^{-1}(z)S(z), z\in\mathbb{C}_+, \tag{2.3}$$

holds, where

$$B(z)=\lim_{N\to\infty}\left\{\prod_{n=1}^{N}\frac{z-i\kappa_n}{z+i\kappa_n}\right\}\left\{\prod_{n=1}^{N}\frac{z-i\nu_n}{z+i\nu_n}\right\}^{-1} \tag{2.4}$$

and $S(z)$ is analytic and contractive on $\mathbb{C}_+$, i.e.

$$|S(z)|\leq 1, z\in\mathbb{C}_+.$$

PROOF. Introduce the truncated potential $\widetilde{q}$ defined by

$$\widetilde{q}(x,b)=q(x)\chi_b(x),$$

where $\chi_b(x)$ is the characteristic function of the interval $[0,b], b>0$. It is clear that $\widetilde{q}(x,b)\to q(x), b\to\infty$, which we agree to write just as $\widetilde{q}\to q$. In the sequel, we

will drop b in $\widetilde{q}(x,b)$ and put $\widetilde{}$ on top of every object related to $\widetilde{q}$. Formula (1.3) is clearly applicable and

$$\widetilde{R}(z) = -\frac{\widetilde{m}(z^2) - iz}{\widetilde{m}(z^2) + iz} =: -\frac{\widetilde{R}_1(z)}{\widetilde{R}_2(z)}, z \in \mathbb{C}_+, \tag{2.5}$$

where $\widetilde{m}$ is the Weyl $m-$function associated with the half-line Schrodinger operator $\widetilde{H} = -d^2/dx^2 + \widetilde{q}(x)$ with a Dirichlet boundary condition $u(0) = 0$. The function $\widetilde{m}$ is subject to Lemma 1 with a finite number of poles at the points $\left\{\widetilde{\lambda}_n\right\}$discrete spectrum of $\widetilde{H}$. Hence by Lemma, $\widetilde{R}_1(\sqrt{z}) = \widetilde{m}(z) - i\sqrt{z}$ has exactly one real zero $-\nu_n^2 \in \left(\widetilde{\lambda}_n, \widetilde{\lambda}_{n+1}\right)$ and one easily sees that

$$\widetilde{R}_1(z) = \widetilde{B}_1(z)\widetilde{S}_1(z), \tag{2.6}$$

where

$$\widetilde{B}_1(z) := \prod_{n=1}^{\widetilde{N}} \frac{z - i\widetilde{\nu}_n}{z + i\widetilde{\nu}_n}$$

is a finite Blaschke product and $\widetilde{S}_1(z)$ is some analytic on $\mathbb{C}_+$ with simple poles at $i\left|\widetilde{\lambda}_n\right|^{1/2}$. Similarly one can easily show that the function $\widetilde{R}_1(\sqrt{z}) = \widetilde{m}(z) + i\sqrt{z}$ has only real zeros $-\kappa_n^2 \in \left(\widetilde{\lambda}_n, \widetilde{\lambda}_{n+1}\right)$ and

$$\widetilde{\kappa}_n < \widetilde{\nu}_n < \widetilde{\kappa}_{n+1}.$$

We then get

$$\widetilde{R}_2(z) = \widetilde{B}_2(z)\widetilde{S}_2(z), \tag{2.7}$$

where

$$\widetilde{B}_2(z) := \prod_{n=1}^{\widetilde{N}} \frac{z - i\widetilde{\kappa}_n}{z + i\widetilde{\kappa}_n}$$

and $\widetilde{S}_2(z)$ is some analytic on $\mathbb{C}_+$ with the same simple poles $i\left|\widetilde{\lambda}_n\right|^{1/2}$ as $\widetilde{S}_1$. Inserting (2.6) and (2.7) into (2.5), we get

$$\widetilde{R}(z) = \widetilde{B}_1(z)\widetilde{B}_2^{-1}(z)\widetilde{S}(z), \tag{2.8}$$

where $\widetilde{S} = -\widetilde{S}_1\widetilde{S}_2^{-1}$. Since $i\left|\widetilde{\lambda}_n\right|^{1/2}$ are clearly removable singularities, one observe that $\widetilde{S}(z)$ is analytic on the whole $\mathbb{C}_+$. Due to $\left|\widetilde{R}(z)\right| \leq 1$, $z \in \mathbb{R}$, and (2.8)

$$\left|\widetilde{S}(z)\right| = \left|\widetilde{R}(z)\right| \leq 1, z \in \mathbb{R},$$

and therefore

$$\left|\widetilde{S}(z)\right| \leq 1, z \in \mathbb{C}_+.$$

Setting

$$\widetilde{B}(z) := \widetilde{B}_1^{-1}(z)\widetilde{B}_2(z),$$

we arrive at (2.3) for $\widetilde{q}$:

$$\widetilde{R}(z) = \widetilde{B}^{-1}(z)\widetilde{S}(z), z \in \mathbb{C}_+. \tag{2.9}$$

It is left to pass in (2.9) to the limit as $\widetilde{q} \to q$. It is well-known that $\widetilde{m} \to m$ on every compact set in $\mathbb{C}_+$ as $\widetilde{q} \to q$ and hence

$$\widetilde{R}(z) \to R(z) \tag{2.10}$$

in $\mathbb{C}_+$ away from points $i\widetilde{\kappa}_n$ and $i\kappa_n$. Next, since $\widetilde{H} \to H$ in the strong resolvent sense, we conclude [**14**], [**15**] that $\widetilde{\kappa}_n \to \kappa_n$ and $\widetilde{\nu}_n \to \nu_n$ and

$$\kappa_n < \nu_n < \kappa_{n+1}. \tag{2.11}$$

Lemma 2 then applies and

$$\lim_{\widetilde{q}\to q} \widetilde{B}(z) = \lim_{\widetilde{q}\to q} \widetilde{B}_1^{-1}(z)\, \widetilde{B}_2(z) = \lim_{N\to\infty} B_N(z) = B(z). \tag{2.12}$$

Combining (2.9), (2.10), and (2.12) we conclude that $\widetilde{S}(z)$ converges in $\mathbb{C}_+$ to some analytic contractive function $S(z)$. The theorem is proven. □

Remark 1. *The contractive function $S(z)$ in* (2.3) *admits the standard canonical factorization with some Blaschke product which zeros can be interpreted as resonances (see* [**20**]*, and* [**18**]*). Formulas* (2.3) *shows that there are also purely imaginary resonances which number is the same as the number of bound states.*

Remark 2. *Similar to* (2.5) *expressions for the reflection coefficient for potentials supported on the whole line were found and effectively used in* [**9**] *and* [**8**]*. The reflection coefficient involves the Weyl $m-$functions corresponding to both half-lines $\mathbb{R}_\pm$.*

3. A sum rule for the reflection coefficient

We will need two technical lemmas. Denote by $\mathfrak{S}_2$ the Hilbert-Schmidt class of compact operators.

Lemma 3. *For any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$*

$$q \in \mathcal{L}_2 \Longleftrightarrow \left(-d^2/dx^2 - \lambda I\right)^{-1/2} q \left(-d^2/dx^2 - \lambda I\right)^{-1/2} \in \mathfrak{S}_2,$$

where the square root is chosen so that $\operatorname{Im}\left(-d^2/dx^2 + \lambda I\right)^{-1/2} \geq 0$.

Proof. It is enough to prove it for $\lambda = -a^2$.The Hilbert-Schmidt norm of the operator $\left(-d^2/dx^2 + a^2\right)^{-1/2} q \left(-d^2/dx^2 + a^2\right)^{-1/2}$ admits an explicit straightforward evaluation:

$$\begin{aligned} &\left\| \left(-d^2/dx^2 + a^2\right)^{-1/2} q \left(-d^2/dx^2 + a^2\right)^{-1/2} \right\|_{\mathfrak{S}_2}^2 \\ = \;& \frac{1}{(2a)^2} \int\int e^{-2a|x-y|} q(x)\, q(y)\, dxdy \end{aligned} \tag{3.1}$$

and the conclusion of the lemma follows. □

Let $H^p, p > 0$, be the Hardy class of analytic on $\mathbb{C}_+$ functions. The next lemma is a simple exercise.

Lemma 4. *If $f_n \in H^p, p > 0$, $sup_n \|f_n\|_{H^p} < \infty$ and $f_n(z) \to f(z)$ for any $z \in \mathbb{C}_+$. Then $f \in H^p$ and*

$$\|f_n\|_{H^p} \to \|f\|_{H^p}, n \to \infty.$$

The following theorem is the main theorem of this section.

THEOREM 2. *Let $q(x)=0, x<0$, and $q\in\mathcal{L}_2$. If $\left\{-\kappa_m^2\right\}$ are negative eigenvalues of $H=-d^2/dx^2+q(x)$ $\left(H_0=-d^2/dx^2\right)$ and $R(k)$ is the reflection coefficient. Choose $a>0$ so that*

$$\|A\|<1, A=A_a:=\left(H_0+a^2\right)^{-1/2}q\left(H_0+a^2\right)^{-1/2}. \tag{3.2}$$

Then the the following sum rule holds:

$$\begin{aligned} &2\sum_{n\geq 1}\frac{1}{2n+1}\sum_m\left(\frac{\kappa_m}{a}\right)^{2n+1}+\frac{1}{\pi a}\int\frac{k^2}{k^2+a^2}\log\left(1-|R(k)|^2\right)^{-1}dk \\ &= \log\det\left\{(1+A_a)^{-1}e^{A_a}\right\}. \end{aligned} \tag{3.3}$$

The right hand side of (3.3) *admits the following bounds*

$$\frac{1}{2}\|q\|_{\mathcal{L}_2}^2\leq\log\det\left\{(1+A_a)^{-1}e^{A_a}\right\}\leq\left(\frac{1}{2}+\log\frac{1}{1-\|A_a\|}\right)\|q\|_{\mathcal{L}_2}^2, \tag{3.4}$$

where

$$\|q\|_{\mathcal{L}_2}^2=\frac{1}{2a^2}\int\limits_{x\geq y\geq 0}e^{-2a(x-y)}q(x)\,q(y)\,dxdy.$$

PROOF. We follow the same as in Theorem 1 convention on the use of $\widetilde{\cdot}$. Let us evaluate the transmission coefficient $\tilde{T}(z)$ for $z\in\mathbb{C}_+$ in two different ways. Following Faddeev-Zhakharov [**5**]

$$\tilde{T}(z)=\prod_m\frac{z+i\widetilde{\kappa}_m}{z-i\widetilde{\kappa}_m}\exp\left\{i\int\frac{\tilde{f}(k)}{k-z}dk\right\}, \tag{3.5}$$

where $-\widetilde{\kappa}_j^2$ are negative eigenvalues of $\tilde{H}$ and

$$\tilde{f}(k):=\frac{1}{\pi}\log\left|\tilde{T}(k)\right|^{-1}=\frac{1}{2\pi}\log\left(1-\left|\widetilde{R}(k)\right|^2\right)^{-1}. \tag{3.6}$$

Due to the general properties of the transmission coefficient, the function $\tilde{f}$ is integrable, non-negative, and even and hence

$$\begin{aligned} \int\frac{\tilde{f}(k)}{k-z}dk &= -\frac{1}{z}\int\tilde{f}(k)\,dk+\frac{1}{z^2}\int\frac{k^2\tilde{f}(k)}{k-z}dk \\ &= -\frac{1}{z}\int\tilde{f}(k)\,dk+\frac{2}{z}\int_0^\infty\frac{k^2\tilde{f}(k)}{k^2-z^2}dk. \end{aligned} \tag{3.7}$$

We have used here the obvious identity

$$\frac{1}{k-z}=-\frac{k}{z^2}-\frac{1}{z}+\frac{1}{z^2}\frac{k^2}{k-z}.$$

By the first Buslaev-Faddeev-Zhakharov trace formula [**5**]

$$-2\sum_m\widetilde{\kappa}_m+\int\tilde{f}(k)\,dk=\frac{1}{2}\int\tilde{q}(x)\,dx, \tag{3.8}$$

and (3.7) can be continued

$$\begin{aligned}\int \frac{\tilde{f}(k)}{k-z}dk &= -\frac{1}{z}\int \tilde{f}(k)\,dk + \frac{2}{z}\int_0^\infty \frac{k^2\tilde{f}(k)}{k^2-z^2}dk \\ &= -\frac{2}{z}\sum_m \widetilde{\kappa}_j - \frac{1}{2z}\int \tilde{q}(x)\,dx + \frac{1}{z}\int \frac{k^2\tilde{f}(k)}{k^2-z^2}dk. \qquad (3.9)\end{aligned}$$

Plugging (3.9) into (3.5) one has

$$\tilde{T}(z) = \prod_m \frac{z+i\widetilde{\kappa}_m}{z-i\widetilde{\kappa}_m} e^{\frac{2i}{z}\widetilde{\kappa}_m} \cdot \exp\left\{\frac{i}{z}\int \frac{k^2\tilde{f}(k)}{k^2-z^2}dk\right\} \exp\left\{\frac{1}{2iz}\int \tilde{q}(x)\,dx\right\} \qquad (3.10)$$

and taking $z = ia$ in (3.10) one has

$$\tilde{T}(ia) = \prod_m \frac{a+\widetilde{\kappa}_j}{a-\widetilde{\kappa}_j} e^{\frac{2}{a}\widetilde{\kappa}_j} \cdot \exp\left\{\frac{1}{a}\int \frac{k^2\tilde{f}(k)}{k^2+a^2}dk\right\} \cdot \exp\left\{-\frac{1}{2a}\int \tilde{q}(x)\,dx\right\}. \qquad (3.11)$$

On the other hand, it is well-known that

$$\tilde{T}(z) = \det\left(1 + \left(H_0 - z^2 I\right)^{-1/2} q \left(H_0 - z^2 I\right)^{-1/2}\right)^{-1}$$

and setting $z = ia$, we have

$$\begin{aligned}\tilde{T}(ia) &= \det\left(1+\widetilde{A}\right)^{-1} = \det\left\{\left(1+\widetilde{A}\right)^{-1} e^{\widetilde{A}}\right\} \det e^{-\widetilde{A}} \\ &= \det\left\{\left(1+\widetilde{A}\right)^{-1} e^{\widetilde{A}}\right\} \exp\left\{-\frac{1}{2a}\int \tilde{q}(x)\,dx\right\} \qquad (3.12)\end{aligned}$$

and comparing (3.11) and (3.12) we arrive at

$$\det\left\{\left(1+\widetilde{A}\right)^{-1} e^{\widetilde{A}}\right\} = \prod_m \frac{a+\widetilde{\kappa}_m}{a-\widetilde{\kappa}_m} e^{\frac{2}{a}\widetilde{\kappa}_m} \cdot \exp\left\{\frac{1}{a}\int \frac{k^2\tilde{f}(k)}{k^2+a^2}dk\right\}. \qquad (3.13)$$

Under condition (3.2) $\sup_m |\widetilde{\kappa}_m| < a$ and one can pass in (3.13) to the logarithms

$$\begin{aligned}&\log\det\left\{\left(1+\widetilde{A}\right)^{-1} e^{\widetilde{A}}\right\} \qquad (3.14) \\ &= \sum_m \left(\log\frac{1+\widetilde{\kappa}_m/a}{1-\widetilde{\kappa}_m/a} - 2\widetilde{\kappa}_m/a\right) + \frac{1}{a}\int \frac{k^2}{k^2+a^2}\tilde{f}(k)\,dk.\end{aligned}$$

Noticing that

$$\log\frac{1+\alpha}{1-\alpha} - 2\alpha = 2\sum_{n\geq 1}\frac{\alpha^{2n+1}}{2n+1}$$

(3.14) reads

$$\log\det\left\{\left(1+\widetilde{A}\right)^{-1} e^{\widetilde{A}}\right\} = 2\sum_{n\geq 1}\frac{1}{2n+1}\sum_m \left(\frac{\widetilde{\kappa}_m}{a}\right)^{2n+1} + \frac{1}{\pi a}\int \frac{k^2}{k^2+a^2}\tilde{f}(k)\,dk. \qquad (3.15)$$

Recalling (3.6), we arrive at (3.3) with the finitely supported potential $\widetilde{q}$. Obtain now bounds (3.4)

$$\begin{aligned}\log\det\left\{\left(1+\widetilde{A}\right)^{-1}e^{\widetilde{A}}\right\} &= \operatorname{tr}\log\left(1+\widetilde{A}\right)^{-1}e^{\widetilde{A}}=\sum_{n\geq 2}\frac{(-1)^n}{n}\operatorname{tr}\widetilde{A}^n\\ &\geq \sum_{n\geq 1}\frac{1}{2n}\operatorname{tr}\widetilde{A}^{2n}\geq\frac{1}{2}\operatorname{tr}\widetilde{A}^2=\frac{1}{2}\left\|\widetilde{q}\right\|_{\mathcal{L}_2}^2,\end{aligned}$$

$$\begin{aligned}\log\det\left\{\left(1+\widetilde{A}\right)^{-1}e^{\widetilde{A}}\right\} &= \sum_{n\geq 2}\frac{(-1)^n}{n}\operatorname{tr}\widetilde{A}^n\leq\sum_{n\geq 2}\frac{1}{n}\operatorname{tr}\widetilde{A}^n\leq\left\|\widetilde{q}\right\|_{\mathcal{L}_2}^2\sum_{n\geq 0}\frac{1}{n+2}\left\|\widetilde{A}\right\|^n\\ &\leq \left(\frac{1}{2}+\log\left(1-\left\|\widetilde{A}\right\|\right)^{-1}\right)\left\|\widetilde{q}\right\|_{\mathcal{L}_2}^2.\end{aligned}$$

We pass now to the limit in (3.15) as $\widetilde{q}\to q$. It is clear that $\|q-\widetilde{q}\|_{\mathcal{L}_2}\to 0$ and hence, due to (3.1), $\left\|A-\widetilde{A}\right\|_{\mathfrak{S}_2}\to 0$. One then easily concludes

$$\det\left\{\left(1+\widetilde{A}\right)^{-1}e^{\widetilde{A}}\right\}\to\det\left\{(1+A)^{-1}e^{A}\right\},\widetilde{q}\to q.$$

Using the standard $\varepsilon/3-$procedure, one proves that $\sum_m\widetilde{\kappa}_m^{2n+1}\to\sum_m\kappa_m^{2n+1}, n\geq 1$ (see also [**1**],[**7**]). The least trivial part is taking the limit under the integral sign in (3.15). Observe that

$$\int\frac{k^2}{k^2+a^2}\log\left(1-\left|\widetilde{R}(k)\right|^2\right)^{-1}dk=\sum_{n\geq 1}\frac{1}{n}\int\frac{k^2}{k^2+a^2}\left|\widetilde{R}(k)\right|^{2n}dk \tag{3.16}$$

and consider ($n\geq 1$)

$$\int\frac{k^2}{k^2+a^2}\left|\widetilde{R}(k)\right|^{2n}dk=\int\left|\frac{k}{k+ia}\right|^2\left|\widetilde{R}(k)\right|^{2n}dk. \tag{3.17}$$

The function $\frac{k}{k+ia}$ can be analytically continued into $\mathbb{C}_+$ and has no zeros there. Hence $\left(\frac{z}{z+ia}\right)^{1/n}$ is uniquely defined on $\mathbb{C}_+$ and contractive. By Theorem 1, on the real line $\left|\widetilde{R}(k)\right|=\left|\widetilde{S}(k)\right|$, where $\widetilde{S}(k)$ is the boundary value of $\widetilde{S}(z)$ which is analytic and contractive on $\mathbb{C}_+$. It follows from (3.15) and (3.17) that

$$\left(\frac{z}{z+ia}\right)^{1/n}\widetilde{S}(z)\in H^{2n}$$

and by a standard fact [**6**] on Hardy spaces

$$\int\left|\frac{k}{k+ia}\right|^2\left|\widetilde{R}(k)\right|^{2n}dk=\left\|\left(\frac{z}{z+ia}\right)^{1/n}\widetilde{S}(z)\right\|_{H^{2n}}^{2n}.$$

By Theorem 1, $\widetilde{S}(z)\to S(z), z\in\mathbb{C}_+$, and by Lemma 4

$$\begin{aligned}\lim_{\widetilde{q}\to q}\left\|\left(\frac{z}{z+ia}\right)^{1/n}\widetilde{S}(z)\right\|_{H^{2n}}^{2n} &= \left\|\left(\frac{z}{z+ia}\right)^{1/n}S(z)\right\|_{H^{2n}}^{2n}\\ &=\int\left|\frac{k}{k+ia}\right|^2|S(k)|^{2n}dk=\int\left|\frac{k}{k+ia}\right|^2|R(k)|^{2n}dk,\end{aligned} \tag{3.18}$$

where we have used again Theorem 1. From (3.16) and (3.18) we finally arrive at (3.3) . The theorem is proven. □

REMARK 3. *One can easily see that the class*

$$l_2(L_1) := \left\{ q : \sum_n \left(\int_n^{n+1} |q(x)|\, dx \right)^2 < \infty \right\}$$

is included in $\mathcal{L}_2$ *and Theorem 2 quickly recovers the main result of* [**16**] *stating that*

$$q \in l_2(L_1) \Longrightarrow \sigma_{ac}(H) = \mathbb{R}_+.$$

Note that (3.3) *yields this result without references to* [**3**] *or* [**?**] *since* $\left| R\left(\sqrt{E}\right) \right| < 1$ *almost everywhere on some* $\Delta \subset \mathbb{R}$ *if and only if* $\Delta \subset \sigma_{ac}(H)$ *. We believe that actually* $\mathcal{L}_2 = l_2(L_1)$ *but do not have a formal proof.*

REMARK 4. *If we multiply relation* (3.3) *by* a^3 *and then take* $a \to \infty$*, it is not hard to show that* (3.3) *transforms into*

$$\frac{2}{3} \sum_m \kappa_m^3 + \frac{1}{\pi} \int k^2 \log\left(1 - |R(k)|^2\right)^{-1} dk = \frac{1}{8} \int q^2(x)\, dx, \tag{3.19}$$

which is the second (sic) Buslaev-Faddeev-Zhakharov trace formula [**5**]*. In* [**5**] *however,* (3.19) *was derived for Schwartz potentials only. We can therefore call* (3.3) *a trace formula but, following Killip-Simon, we will refer to* (3.3) *as a sum rule. We believe that* (3.3) *can also be viewed as a Nevanlinna-type characteristic of a function from a generalized Nevanlinna class of analytic function. Note in this connection that a similar to ours Poisson-type representation for* $T(z)$ *was recently used by Denisov* [**4**] *in the case of Krein systems.*

REMARK 5. *A similar to* (3.19) *formula also holds for Jacobi matrices. In Killip-Simon* [**10**] *such formula used for a complete description of all self-adjoint Hilbert-Schmidt perturbation of the discrete free Schrödinger operators (see* [**19**] *for simpler proofs).*

REMARK 6. *Identity* (3.3) *can actually be viewed as the first formula in a certain chain of sum rules. In the context of Jacobi matrices, similar relations were recently obtained by Laptev-Naboko-Safronov* [**11**]*. We will discuss these issues in* [**17**] *where we also plan to improve on the results of* [**?**] *devoted to the absolutely continuous spectrum of Schrödinger operators with* L_p-*potentials with* $p > 2$*.*

4. A sum inequality for the Weyl $m-$function and the $3/2-$Lieb-Thirring bound

This section is actually devoted to a number of important corollaries to Theorem 2. Their proofs are simple but a bit technical and we do not present them here.

THEOREM 3. *Let* $H = -d^2/dx^2 + q(x)$ *on* $\mathbb{R}_+$ *with a Dirichlet boundary condition at* 0*. Let* $m(z)$ *be its Weyl* $m-$*function and* $\{\lambda_n\}$ *be its negative discrete spectrum. Choose* $a > 0$ *so that*

$$\|A\| < 1, A = A_a := \left(H_0 + a^2\right)^{-1/2} q \left(H_0 + a^2\right)^{-1/2}.$$

Then the following sum inequality holds:

$$\begin{aligned} & \|A_a\|^{-3}\left(\log\frac{1+\|A_a\|}{1-\|A_a\|}-2\|A_a\|\right)Q_a \\ \leq\ & \frac{2}{3}\sum_m|\lambda_m|^{3/2}+\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\lambda}}{1+\lambda/a^2}\log\frac{\left|m(\lambda+i0)+i\sqrt{\lambda}\right|^2}{4\sqrt{\lambda}\operatorname{Im}m(\lambda+i0)}d\lambda \qquad (4.1)\\ \leq\ & \left(1+2\log\frac{1}{1-\|A_a\|}\right)Q_a, \end{aligned}$$

where

$$Q_a:=\frac{a}{4}\int\limits_{x\geq y\geq 0}e^{-2a(x-y)}q(x)\,q(y)\,dxdy.$$

COROLLARY 1. *Let $m(z)$ be the Weyl $m-$function of $H=-d^2/dx^2+q(x)$ on $\mathbb{R}_+$ with a Dirichlet boundary condition at 0. Then $q\in\mathcal{L}_2(\mathbb{R}_+)$ if and only if the following three conditions are satisfied:*

(1): $\sigma(H)=\{\lambda_n\}\cup\mathbb{R}_+,\lambda_n<0,|\lambda_n|\searrow 0$
(2): $\sum_m|\lambda_m|^{3/2}<\infty$
(3): $\int_0^\infty\frac{\sqrt{\lambda}}{1+\lambda}\log\frac{\left|m(\lambda+i0)+i\sqrt{\lambda}\right|^2}{4\sqrt{\lambda}\operatorname{Im}m(\lambda+i0)}d\lambda<\infty.$

COROLLARY 2. *If in (4.1) we let $a\to\infty$ then (4.1) transforms into*

$$C_1\int_0^\infty q^2(x)\,dx\leq\frac{2}{3}\sum_m|\lambda_m|^{3/2}+\frac{1}{\pi}\int_0^\infty\sqrt{\lambda}\log\frac{\left|m(\lambda+i0)+i\sqrt{\lambda}\right|^2}{4\sqrt{\lambda}\operatorname{Im}m(\lambda+i0)}d\lambda\leq C_2\int_0^\infty q^2(x)\,dx$$

with some universal constants $0<C_1$, $C_2<\infty$.

REMARK 7. *Corollary 2 extends the class $L_2(\mathbb{R}_+)$ in Damanik-Killip-Simon* [**2**] *to $\mathcal{L}_2(\mathbb{R}_+)$.*

THEOREM 4. *Let $H=-d^2/dx^2+q(x)$ on $L_2(\mathbb{R})$ and $q\in\mathcal{L}_2$. Then the negative eigenvalues $\{\lambda_n\}$ of H satisfy*

$$\sum_m|\lambda_m|^{3/2}\leq\frac{3}{8}\left(1+2\log(1-\|A_a\|)^{-1}\right)\cdot a\int\limits_{x\geq y}e^{-2a(x-y)}q_-(x)\,q_-(y)\,dxdy, \qquad (4.2)$$

where a is chosen as in Theorem 1 and $q_-(x)$ is the negative part of q.

REMARK 8. *Inequality (4.2) is a generalization of the Lieb-Thirring bound (see, e.g.* [**12**], [**22**]*)for the case of $\mathcal{L}_2$ potentials. One can easily see that the standard Lieb-Thirring inequality*

$$\sum_m|\lambda_m|^{3/2}\leq C\int q^2(x)\,dx$$

can be obtained from (4.2) by passing to the limit in (4.2) as $a\to\infty$. This procedure however does not provide the best possible constant C.

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