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Local Limit Theorem for the Lorentz Process and Its Recurrence in the Plane

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Abstract. For Young systems, i. e. for hyperbolic systems without/with singularities satisfying Young's axioms [**You 98**] (which imply exponential decay of correlation and the CLT) a local CLT is proven. In fact, a unified version of the local CLT is found, covering among others the absolutely contionuous and the arithmetic cases. For the planar Lorentz process with a finite horizon this result implies a.) the local CLT and b.) the recurrence. For the latter case ($d = 2$, finite horizon), combining

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the global CLT with abstract ergodic theoretic ideas, K. Schmidt, [**Sch 98**] and J.-P. Conze, [**Con 99**], could already establish recurrence.

1. *Introduction*

1.1. *Motivation* The *Lorentz process* is a physically utmost interesting mechanical model of Brownian motion (cf. [**Sz 00**]). It is the deterministic motion of a point particle starting from a random phase point and undergoing specular reflections on the boundaries of strictly convex scatterers. Throughout this paper we will only consider a $\mathbb{Z}^d$-periodic configuration of scatterers. Once it had been established that the diffusion limit of the planar Lorentz process is, indeed, the Wiener process ([**BS 81**], see also [**BChS 91**]), the question of its *recurrence* was immediately raised by Ya. G. Sinai. Here recurrence means that the process almost surely returns to any fixed bounded domains of the configuration space. In fact, for Lorentz processes the exact analogue of Pólya's theorem known for random walks is strongly expected. The first positive result was obtained in [**KSz 85**], where a slightly weaker form of recurrence was demonstrated: the process almost surely returns infinitely often to a moderately (actually logarithmically) increasing sequence of domains. The authors used a probabilistic method combined with the dynamical tools of Markov approximations. The weaker form of the recurrence was the consequence of the weaker form of their local limit theorem: they could only control the probabilities that the Lorentz process S_n in the moment of n^{th} collision falls into a sequence of moderately increasing domain rather than into a domain of fixed size. These results, moreover, were restricted to the finite horizon case, i. e. to the case when there is no orbit without any collision.

A novel - and surprising - approach appeared in 1998-1999, when independently Schmidt [**Sch 98**] and Conze [**Con 99**] were, indeed, able to deduce the recurrence from the global central limit theorem (CLT) of [**BS 81**] by adding (abstract) ergodic theoretic ideas. Their approach seems to be essentially restricted to the finite horizon case and to $d = 2$. Our main aim is to return to the probabilistic-dynamical approach and - still for the finite horizon case - we can first prove a true local central limit theorem (LCLT) for the planar Lorentz process S_n.

1.2. *Statement of theorems* As a matter of fact, beyond treating just the Lorentz process we are also able to obtain a LCLT in a much wider setup. Namely our LCLT is valid whenever Young obtains a CLT. Her systems, called in our paper as *Young systems*, are introduced in subsection 2.1. Roughly speaking, these are systems (X, T, ν)

1. whose every power is ergodic;
2. which satisfy several technical assumptions well-known from hyperbolic theory;
3. whose phase space X contains a subset Λ with a hyperbolic product structure;
4. where the return time into Λ has an exponentially decaying tail.

For stating our main theorem we have to fix some notations first. For a fixed $f : X \to \mathbb{R}^d$ denote the average $\nu(f) = a$, and

$$S_n(x) = \sum_{k=0}^{n-1} f(T^k x)$$

the Birkhoff sum. Consider the smallest translated closed subgroup $V + r \subseteq \mathbb{R}^d$ which supports the values of f (V is the group and r is the translation). By ergodicity of all powers of T, the support of S_n is $V + nr$.

THEOREM 1.1. *Suppose that*

1. (X, T, ν) *is a Young system (cf. subsection 2.1);*
2. f *is minimal: i. e. it is not cohomologous to a function for which the support in the above sense is strictly smaller.*
3. f *is nondegenerate: i. e. span* $\langle V \rangle = \mathbb{R}^d$, *and*
4. f *is bounded and Hölder-continuous.*

Let $k_n \in V + nr$ *be such that* $\frac{k_n - na}{\sqrt{n}} \to k$. *Denote the distribution of* $S_n - k_n$ *by* υ_n. *Then*

$$n^{\frac{d}{2}} \upsilon_n \to \varphi(k) l$$

where φ *is a non-degenerate normal density function with zero expectation, and* l *is the uniform measure on* V*: product of counting measures and Lebesgue measures. The convergence is meant in the weak topology.*

Remark For non-minimal functions we can obtain an analogous result. The limit measure on the right hand side in this case is not necessarily uniform.

Remark Traditionally one formulates the LCLT for the absolutely continuous and for the arithmetic case separately. An advantage of our statement is that it is unified and beyond these two cases it also contains the mixed ones. Though for the absolutely continuous case it is slightly weaker than the LCLT for densities, nevertheless our variant, for instance, is still amply sufficient to treat recurrence properties.

Turning to the Lorentz process, let us denote by $(M, S^{\mathbb{R}}, \mu)$ a two-dimensional *dispersing billiard* dynamical system with a finite horizon, the usual factor of the Lorentz process, where μ is the natural invariant probability measure (the Liouville-one), and consider its Poincaré section $(\partial M, T, \mu_1)$ (for formal definitions of billiards cf. section 5).

In case one takes f as $\kappa : \partial M \to \mathbb{R}^2$, the discrete free flight function of the planar Lorentz process, then this result combined with considerations of [**KSz 85**], and an asymptotic independence statement proved right after the main theorem immediately provide the recurrence of S_n as well. It will be shown in section 5 that κ satisfies the conditions of the main theorem.

COROLLARY 1.1. *The planar Lorentz process with a finite horizon is almost surely recurrent.*

1.3. *Some history* LCLT's for functions of a Markov chain were first obtained by Kolmogorov in 1955 using probabilistic ideas. Then, in 1957, Nagaev, [**Nag 57**] – by using operator valued Fourier transforms and perturbation theory – could find a general form of LCLT's for functions of a Markov chain. Independently, variants of this method got later rediscovered and/or applied A) by Krámli and Szász [**KSz 83**] to prove a LCLT for random walks with internal states, B) by Guivarch and Hardy [**GH 88**] in the setting of Anosov diffeomorphisms C) by Roussean-Egele, [**R-E 83**], Morita [**Mor 94**] and Broise [**Bro 96**] for expanding maps of the interval and finally D) by Aaronson and Denker, [**AD 01**] in the setting of Gibbs-Markov maps.

Beyond establishing LCLT's for the planar Lorentz process for the first time, the technical interest and achievement of this paper is the following: [**BS 81**] and [**KSz 85**] used a Markov approximation scheme of the Lorentz process based upon the Markov partition of the Sinai billiard. Several later works demonstrated that a Markov partition for a hyperbolic system with singularities is a too rigid construction, and introduced Markov sieves [**BChS 91**] and finally Markov returns [**You 98**] instead. Our aim therefore is to work out *how Markov returns can be used to prove probabilistic statements* (e. g. to a large deviation result we return in a forthcoming paper).

1.4. *Probabilistic ideas: What is a local CLT and what is its relation to recurrence?* For illustrating the probabilistic ideas, take a simple symmetric random walk (SSRW) on $\mathbb{Z}^d$. So let $W_n = X_1 + \dots X_n$, where $X_1, \dots, X_n, \dots$ are independent, identically distributed random variables with the common distribution $P(X_i = \pm e_j) = \frac{1}{2d}$; $1 \le j \le d$ for all $i \in \mathbb{Z}_+$ (here the e_js are the standard unit vectors of $\mathbb{Z}^d$). To investigate whether the SSRW is recurrent or not one turns to the Borel-Cantelli lemma: if $\sum_n P\left(W_n = 0\right) = \infty$, then

$P(\exists n_k \to \infty \text{ such that } \forall k\ W_{n_k} = 0) = 1$. If the sum is convergent, then $P(\exists N \text{ such that } \forall n > N \quad W_n \neq 0) = 1$. To apply the lemma we need to calculate the asymptotics of the probability $P(W_n = 0)$.

The CLT says that $P(W_n \in \sqrt{n}A) \to \Phi(A)$ as $n \to \infty$, where $\Phi(A) = \int_A \phi(s)ds$ and $\phi(s) = \exp(-\frac{s^2}{2})$ is the d-dimensional Gaussian density. In other words it describes the asymptotics of a sequence of sets increasing like $\sqrt{n}$. In contrast, for the SSRW the local CLT says that, $n^{d/2}P(W_n = [s\sqrt{n}]) \to 2\phi(s)$ as $n \to \infty$, if the sum of the coordinates of the vector $[s\sqrt{n}] - n$ is even. In other words, the LCLT describes the asymptotics of a sequence of sets of fixed size (in this case: a point), consequently it is, indeed, local! As an application we get $P(W_n = 0) \sim 2n^{-d/2}$, consequently for the line and the plane SSRW is recurrent, for higher dimension it is transient.

This paper is organized as follows. Primarily, in section 2, we will formulate the abstract setting, define the notion of Young-systems and recall our basic spectral tool: the Doeblin-Fortet (in the theory of dynamical systems also known as Lasota-Yorke) inequality. Section 3 is devoted to important spectral properties of the Fourier transform: quasicompactness, arithmeticity and a useful Nagaev-type theorem on a one-dimsnional approximation of the Fourier transform in a neighbourhood of the origin. In section 4 we establish our local limit theorem for Young systems and, in addition, a certain asymptotic independence statement necessary to prove the recurrence. In the fifth section we turn our attention to billiards and to the Lorentz-process (with prerequisites in subsection 5.1) to get recurrence of planar Lorentz-process as an application of the abstract theorems in subsection 5.3. In subsection 5.2 we analyze the arithmeticity of the discrete free flight function.

2. *Prerequisites*

Since local central limit theorems are refined versions of (global) central limit theorems, it is not surprising that our approach relies heavily on Young's work [**You 98**], where – among others – an exponential decay of correlations and a central limit theorem were proved for 2-D dispersing billiards with a finite horizon. Here we present a concise summary of the main points of Young's paper, which are necessary for our consideration.

2.1. *Young systems*

Let T be a $C^{1+\epsilon}$ diffeomorphism with singularities of a compact Riemannian manifold X with boundary. More precisely, there exists a finite or countably infinite number of pairwise disjoint open regions $\{X_i\}$ whose boundaries are C^1 submanifolds of codimension 1, and finite volume such that $\cup X_i = X$, $T\big|_{\cup X_i}$ is $1-1$ and $T\big|_{X_i}$ can be extended to a $C^{1+\epsilon}$-diffeomorphism of $\bar{X}_i$ onto its image. Then $\breve{S} = X \setminus \cup X_i$ is the *singularity set*. Later, for billiards, we will also use the notation $S = \breve{S} \cup T^{-1}\breve{S}$. The Riemannian measure will be denoted by μ, and if $W \subset X$ is a submanifold, then μ_W will denote the induced measure. The invariant Borel probability measure will be denoted by ν.

Definition An embedded disk $\gamma \subset X$ is called an *unstable manifold* or an *unstable disk* if $\forall x, y \in \gamma,\ d(T^{-n}x, T^{-n}y) \to 0$ exponentially fast as $n \to \infty$; it is called a *stable manifold* or a *stable disk* if $\forall x, y \in \gamma,\ d(T^n x, T^n y) \to 0$ exponentially fast as $n \to \infty$. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable disks* if the following hold:

- K^s is an arbitrary compact set; D^u is the unit disk of some $\mathbb{R}^n$;
- $\Phi^u \colon K^s \times D^u \to X$ is a map with the property that
  - Φ^u maps $K^s \times D^u$ homeomorphically onto its image,
  - $x \to \Phi^u \mid (\{x\} \times D^u)$ is a continuous map from K^s into the space of C^1 embeddings of D^u into X,
  - γ^u, the image of each $\{x\} \times D^u$, is an unstable disk.

Continuous families of C^1 stable disks are defined similarly.

Definition We say that $\Lambda \subset X$ has a *hyperbolic product structure* if there exist a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ such that

(i) $\dim \gamma^u + \dim \gamma^s = \dim X$

(ii) the γ^u-disks are transversal to the γ^s-disks with the angles between them bounded away from 0;

(iii) each γ^u-disk meets each γ^s-disk in exactly one point;

(iv) $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$.

Definition Suppose Λ has a hyperbolic product structure. Let Γ^u and Γ^s be the defining families for Λ. A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if Λ_0 also has a hyperbolic product structure and its defining families can be chosen to be Γ^u and Γ_0^s with $\Gamma_0^s \subset \Gamma^s$; *u-subsets* are defined analogously. For $x \in \Lambda$, let $\gamma^u(x)$ denote the element of Γ^u containing x.

In general a measurable bijection $M : (X_1, m_1) \to (X_2, m_2)$ between two finite measure spaces is called *nonsingular* if it maps sets of m_1-measure 0 to sets of m_2-measure 0. If M is nonsingular, we define the Jacobian of M wrt m_1 and m_2, written $J_{m_1,m_2}(M)$ or simply $J(M)$, to be the Radon-Nikodym derivative $\frac{d(M_*^{-1}m_2)}{dm_1}$. To denote $J(T)$ wrt μ_{γ^u} we will use $\det DT^u$.

Definition We call (X, T, ν) a *Young system*, if the following Properties **(P1)-(P8)** are true:

(P1) There exists a $\Lambda \subset X$ with a hyperbolic product structure and with $\mu_\gamma\{\gamma \cap \Lambda\} > 0$ for every $\gamma \in \Gamma^u$.

(P2) There is a countable number of disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

- on each γ^u-disk $\mu_{\gamma^u}\{(\Lambda \setminus \cup \Lambda_i) \cap \gamma^u\} = 0$;
- for each i, $\exists R_i \in \mathbb{Z}^+$ such that $T^{R_i}\Lambda_i$ is a u-subset of Λ;
- for each n there are at most finitely many i's with $R_i = n$;
- $\min R_i \geq$ some R_0 depending only on T

(P3) For every pair $x, y \in \Lambda$, we have a notion of *separation time* denoted by $s_0(x, y)$. If $s_0(x, y) = n$, then the orbits of x and y are thought of as being "indistinguishable" or "together" through their n^{th} iterates, while $T^{n+1}x$ and $T^{n+1}y$ are thought of as having been "separated." (This could mean that the points have moved a certain distance apart, or have landed on opposite sides of a discontinuity manifold, or that their derivatives have ceased to be comparable.) We assume:

(i) $s_0 \geq 0$ and depends only on the γ^s-disks containing the two points;

(ii) the number of "distinguishable" n-orbits starting from Λ is finite for each n;

(iii) for $x, y \in \Lambda_i$, $s_0(x,y) \geq R_i + s_0(T^{R_i}x, T^{R_i}y)$;

(P4) Contraction along γ^s disks. There exist $C > 0$ and $\alpha < 1$ such that for $y \in \gamma^s(x)$, $d(T^n x, T^n y) \leq C\alpha^n \ \forall n \geq 0$.

(P5) Backward contraction and distorsion along γ^u. For $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x,y)$, we have

(a) $d(T^n x, T^n y) \leq C\alpha^{s_0(x,y)-n}$;

(b)

$$\log \prod_{i=k}^{n} \frac{\det DT^u(T^i x)}{\det DT^u(T^i y)} \leq C\alpha^{s_0(x,y)-n}.$$

(P6) Convergence of $D(T^i|\gamma^u)$ and absolute continuity of Γ^s.

(a) for $y \in \gamma^s(x)$,

$$\log \prod_{i=n}^{\infty} \frac{\det T^u(T^i x)}{\det T^u(T^i y)} \leq C\alpha^n \quad \forall n \geq 0.$$

(b) for $\gamma, \gamma' \in \Gamma^u$, if $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det DT^u(T^i x)}{\det DT^u(T^i \Theta x)}.$$

(P7) $\exists C_0 > 0$ and $\theta_0 < 1$ such that for some $\gamma \in \Gamma^u$,

$$\mu_\gamma\{x \in \gamma \cap \Lambda : R(x) > n\} \leq C_0 \theta_0^n \quad \forall n \geq 0;$$

(P8) (T^n, ν) is ergodic $\forall n \geq 1$.

Now we will define the *Markov extension.* Let $R : \Lambda \to \mathbb{Z}_+$ be the function which is R_i on Λ_i, and let

$$\Delta \stackrel{\text{def}}{=} \{(x, l) : x \in \Lambda; \ \ l = 0, 1, \ldots, R(x) - 1\}$$

and define

$$F(x, l) = \begin{cases} (x, l+1) & \text{if} \quad l + 1 < R(x) \\ (T^R x, 0) & \text{if} \quad l + 1 = R(x) \end{cases}$$

We will refer to Δ_l as the l^{th} level of the tower Δ. Young also has a construction for $\tilde{\nu}$, the SRB-measure of the extension, for which the pushforward is ν, and $J(F) \equiv 1$ except on $F^{-1}(\Delta_0)$.

On the tower a Markov partition $\mathcal{D}$ can be defined, with the following properties:

(a) $\mathcal{D}$ is a refinement of the partition Δ_l. ($\mathcal{D}_l$ denotes $\mathcal{D}|\Delta_l$.)

(b) $\mathcal{D}_l$ has only a finite number of elements and each one is the union of a collection of Λ_i's;

(c) $\mathcal{D}_l$ is a refinement of $F\mathcal{D}_{l-1}$;

(d) if x and y belong to the same element of $\mathcal{D}_l$, then $s_0(F^{-l}x, F^{-l}y) \geq l$;

(e) if $R_i = R_j$ for some $i \neq j$, then Λ_i and Λ_j belong to different elements of $\mathcal{D}_{R_i - 1}$.

Let $\Delta^*_{l,j} = \Delta_{l,j} \cap F^{-1}(\Delta_0)$. We think of $\Delta_{l,j} \setminus \Delta^*_{l,j}$ as "moving upward" under F, while $\Delta^*_{l,j}$ returns to the base.

It is natural to *redefine the separation time* to be $s(x,y) \stackrel{\text{def}}{=}$ the largest n such that for all $i \le n$, $F^i x$ and $F^i y$ lie in the same element of $\{\Delta_{l,j}\}$. We claim that **(P5)** is valid for $x, y \in \gamma^u \cap \Delta_{l,j}$ with s in the place of s_0. To verify this, first consider $x, y \in \Lambda$. We claim that $s(x,y) \le s_0(x,y)$. If x, y do not belong to the same Λ_i, then this follows from rule (d) in the construction of $\mathcal{D}_l$; if $x, y \in \Lambda_i$, but $T^R x, T^R y$ are not contained in the same Λ_j, then $s(x,y) = R_i + s(T^R x, T^R y)$, which is $\le s_0(x,y)$ by property **(P3)**,(iii) of s_0, and so on. In general, for $x, y \in \Delta_{l,j}$, let $x_0 = F^{-l}x,\ y_0 = F^{-l}y$ be the unique inverse images of x and y in Δ_0. Then by definition $s(x,y) = s(x_0,y_0) - l$, and what is said earlier on about x_0 and y_0 is equally valid for x and y.

*From here on s_0 is replaced by s and **(P5)** is modified accordingly.*

Now we recall an important distorsion property of the so called sliding map. Fix an arbitrary $\hat\gamma \in \Gamma^u$. For $x \in \Lambda$, let $\hat x$ denote the point in $\gamma^s(x) \cap \hat\gamma$, and define

$$u_n(x) = \sum_{i=0}^{n-1} (\varphi(T^i x) - \varphi(T^i \hat x))$$

where $\varphi = \log|\det DT^u|$. From **(P6)**(a) it follows that u_n converges uniformly to some function u. On each $\gamma \in \Gamma^u$, we let m_γ be the measure, whose density wrt μ_γ is $e^u \cdot 1_{\gamma\cap\Lambda}$. Clearly, $T^{R_i}|(\Lambda_i \cap \gamma)$ is nonsingular wrt these reference measures. If $T^{R_i}(\Lambda_i \cap \gamma) \subset \gamma'$, then for $x \in \Lambda_i \cap \gamma$ we write $J(T^R)(x) = J_{m_\gamma, m_{\gamma'}}(T^{R_i}|(\Lambda\cap\gamma))(x)$.

Lemma 2.1. *(1) Let $\Theta_{\gamma,\gamma'}\colon \gamma\cap\Lambda \to \gamma'\cap\Lambda$ be the sliding map along Γ^s. Then*

$\Theta_* m_\gamma = m_{\gamma'}$.

(2) $J(T^R)(x) = J(T^R)(y)\ \ \forall y \in \gamma^s(x)$.

(3) $\exists C_1 > 0$ *such that* $\forall i$ *and* $\forall x, y \in \Lambda_i \cap \gamma$,

$$\left| \frac{J(T^R)(x)}{J(T^R)(y)} - 1 \right| \leq C_1 \alpha^{\frac{1}{2} s(T^R x, T^R y)}.$$

Next Young uses a factorised dynamics with a factorisation along stable manifolds of Δ. The advantage is that this dynamics will behave as an expanding map, a simpler object to study. Let $\bar{\Delta} := \Delta / \sim$ where $x \sim y$ iff $y \in \gamma^s(x)$. Since F takes γ^s-leaves to γ^s-leaves, the quotient dynamical system $\bar{F} \colon \bar{\Delta} \to \bar{\Delta}$ is clearly well defined.

Let us define $\bar{m}$ in the following way: let $\bar{m}|\bar{\Delta}_l$ be the measure induced from the natural identification of $\bar{\Delta}_l$ with a subset of $\bar{\Delta}_0$, so that $J(\bar{F}) \equiv 1$ except on $\bar{F}^{-1}(\bar{\Delta}_0)$, where $J(\bar{F}) = J(\overline{T^R} \circ \bar{F}^{-(R-1)})$.

We now define $\bar{m}$ on $\bar{\Lambda}$ following the ideas that have been used for Axiom A. Lemma 2.1 (1) allows us to define $\bar{m}$ on $\bar{\Lambda}$ to be the measure whose representative on each $\gamma \in \Gamma^u$ is m_γ. Statement (2) says that $J(T^R)$ is well defined wrt $\bar{m}$, and (3) says that $\log J(T^R)$ has a dynamically defined Hölder type property, in the sense that $\alpha^{s(T^R x, T^R y)}$ could be viewed as a notion of distance between $T^R x$ and $T^R y$ (see **(P5)**). By using this lemma Young obtains a distorsion property of the factorised map with a weaker constant β. Let β be such that $\alpha^{\frac{1}{2}} \leq \beta < 1$, and let C_1 be as in Lemma 2.1 (3).

(I) Height of tower.

  (i) $R \geq N$ for some N satisfying $C_1 e^{C_1} \beta^N \leq \frac{1}{100}$;

  (ii) $\bar{m}\{R \geq n\} \leq C_0' \theta_0^n \;\; \forall n \geq 0$ for some $C_0' > 0$ and $\theta_0 < 1$.

(II) Regularity of the Jacobian.

(i) $J\bar{F} \equiv 1$ on $\bar{\Delta} - \bar{F}^{-1}(\bar{\Delta}_0)$,

(ii)

$$\left|\frac{J\bar{F}(\bar{x})}{J\bar{F}(\bar{y})} - 1\right| \leq C_1 \beta^{s(\bar{F}\bar{x},\bar{F}\bar{y})} \quad \forall \bar{x}, \bar{y} \in \bar{\Delta}^*_{l,j}.$$

Young proves [**You 98**], that there exists an invariant probability measure $\bar{\nu}$, absolutely continuous wrt $\bar{m}$, such that $\rho = \frac{d\bar{\nu}}{d\bar{m}}$ is bounded away from zero and infinity, and is Lipschitz-continuous wrt the distance β^s.

2.2. *The Doeblin-Fortet inequality and spectral properties* *Definition* Let $(\mathcal{C}, \mathcal{L})$ be a pair of Banach spaces, such that $\mathcal{L} \leq \mathcal{C}$ is a linear subspace, $\|\,.\,\|_{\mathcal{L}} \geq \|\,.\,\|_{\mathcal{C}}$. We call this pair *adapted* if each $\mathcal{L}$-bounded set is precompact in $\mathcal{C}$.

Definition Let $(\mathcal{C}, \mathcal{L})$ be an adapted pair. We call an $A\colon \mathcal{C} \to \mathcal{C}$ bounded linear operator a *Doeblin-Fortet operator*, if $\exists \tau < 1, \exists K > 0, \exists n \in \mathbb{N} \quad \forall \varphi \in \mathcal{L}$,

$$\|A^n \varphi\|_{\mathcal{L}} \leq \tau \|\varphi\|_{\mathcal{L}} + K \|\varphi\|_{\mathcal{C}}.$$

This latter is called the *Doeblin-Fortet inequality.*

THEOREM 2.1. *[**I-TM 50**] If A is a Doeblin-Fortet operator on the adapted pair $(\mathcal{C}, \mathcal{L})$, then $\exists \vartheta < 1, N \geq 1$, projections $E_1, \ldots, E_N$ onto finite dimensional subspaces of $\mathcal{L}$, and $\lambda_1, \ldots, \lambda_N \in \{z \in \mathbb{C} : |z| = 1\}$ such that $\forall \varphi \in \mathcal{L}, n \in \mathbb{N}$*

$$\left\|A^n \varphi - \sum_{k=1}^{N} \lambda_k^n E_k \varphi\right\|_{\mathcal{L}} \leq K \vartheta^n \|\varphi\|_{\mathcal{L}}.$$

Now we will define the *function spaces.* Let $\epsilon > 0$ be such that

(ϵi) $e^{2\epsilon}\theta_0 < 1$,

(ϵii) $\bar{m}(\bar{\Delta}_0)^{-1} \sum_{l,j} \bar{m}(\bar{\Delta}^*_{l,j}) e^{l\epsilon} \leq 2.$

Now we are ready to define the function spaces. The elements will be functions $\bar{\varphi}\colon \bar{\Delta} \to \mathbb{C}$ and the $\mathcal{C}$ norm is

$$\|\bar{\varphi}\|_{\mathcal{C}} \stackrel{\text{def}}{=} \sup_{l,j} \left\| \bar{\varphi}|_{\bar{\Delta}_{l,j}} \right\|_{\infty} e^{-l\epsilon}$$

where $\| \, . \, \|_{\infty}$ is the essential supremum wrt $\bar{m}$. By (ϵi) it is clear that constant multiple of this norm dominates the L_1-norm wrt $\bar{m}$. Let us introduce

$$\|\bar{\varphi}\|_{h} \stackrel{\text{def}}{=} \sup_{l,j} \left(\sup_{\bar{x},\bar{y}\in\bar{\Delta}_{l,j}} \frac{|\bar{\varphi}(\bar{x}) - \bar{\varphi}(\bar{y})|}{\beta^{s(\bar{x},\bar{y})}} \right) e^{-l\epsilon};$$

where the inner sup is again essential supremum wrt $\bar{m} \times \bar{m}$ and $\mathcal{L}$-norm is

$$\|\bar{\varphi}\|_{\mathcal{L}} \stackrel{\text{def}}{=} \|\bar{\varphi}\|_{\mathcal{C}} + \|\bar{\varphi}\|_{h} \, .$$

$\mathcal{C}$ resp. $\mathcal{L}$ consist of functions for which the $\mathcal{C}$-norm resp. $\mathcal{L}$-norm is finite. The adaptedness is an easy consequence of the Arzela-Ascoli theorem. The Perron-Frobenius operator acting on these spaces is defined as follows:

$$P(\bar{\varphi})(\bar{x}) = \sum_{\bar{x}^{-1}:\bar{F}\bar{x}^{-1}=\bar{x}} \frac{\bar{\varphi}(\bar{x}^{-1})}{J\bar{F}(\bar{x}^{-1})}.$$

This is the adjungate operator of $\bar{\varphi} \mapsto \bar{\varphi} \circ \bar{F}$ on $L_2(\bar{m})$. By (ϵi) both $\mathcal{C}$ and $\mathcal{L}$ is contained in $L_2(\bar{m})$. The fact, that P is a bounded operator on $\mathcal{C}$ follows from (ϵii). The similar statement for $\mathcal{L}$ is proved in [**You 98**], where Young deduces that

(i) P is a contraction in $\mathcal{L}$.

(ii) it satisfies the D-F inequality,

(iii) by Theorem 2.1 it has a spectral gap,

(iv) and by **(P8)** its only eigenvalue on the unit circle is 1 and it is simple. (The eigenfunction is the invariant density ρ.)

Later we will need the adjungate of $\bar\varphi \mapsto \bar\varphi \circ \bar F$ on $L_2(\bar\nu)$, this is $P^\rho(\bar\varphi) \overset{\text{def}}{=} \frac{1}{\rho}P(\rho\bar\varphi)$. Note that the spectrum of P and P^ρ is the same, just the eigenfunctions are divided by ρ.

3. *Spectral properties of the Fourier-transform*

In this section we are working with Young systems throughout. Let $f\colon X \to \mathbb{R}^d$ be a bounded, piecewise η-Hölder function i. e. $f(x) - f(y) \le C_f d(x,y)^\eta$ whenever $x, y \in X_i$. We are going to associate a function $\bar f\colon \bar\Delta \to \mathbb{R}^d$ of the symbolic space. First we *pull back* f along the projection map $\pi\colon \Delta \to \cup T^n\Lambda$ to a function $\tilde f\colon \Delta \to \mathbb{R}^d$. This is clearly bounded and by **(P5)** $\tilde f(x) - \tilde f(y) \le C_f \left(C\alpha^{s(x,y)}\right)^\eta$ meaning $\tilde f$ is η-Hölder wrt the metric α^s. Next we use a standard method described for example in [**PP 90**]. We choose an unstable manifold in each Markov-rectangle $\Delta_{l,j}$, and consider the projection Ξ which sends each point along its stable manifold to our preferred unstable manifold. Consider the function $h \overset{\text{def}}{=} \sum_{n=0}^\infty \left(\tilde f \circ F^n - \tilde f \circ F^n \circ \Xi\right)$! The defining series converges since $\tilde f F^n x - \tilde f F^n \Xi x \le C_f d(T^n\pi x, T^n\pi\Xi x)^\eta$ and by **(P4)** $\le C_f(C\alpha^n)^\eta$.

$$
\begin{aligned}
h - h\circ F &= \sum_{n=0}^\infty \left(\tilde f \circ F^n - \tilde f \circ F^n \circ \Xi\right) - \sum_{n=0}^\infty \left(\tilde f \circ F^n + 1 - \tilde f \circ F^n \circ \Xi \circ F\right) \\
&= \tilde f - \left[\tilde f \circ \Xi + \sum_{n=0}^\infty \tilde f \circ F^{n+1} \circ \Xi - \tilde f \circ F^n \circ \Xi \circ F\right].
\end{aligned}
$$

This can be rewritten as $h - h\circ F = \tilde f - \bar f$, where $\bar f$ is defined by the expression in square brackets. Evidently $\bar f$ is constant when restrticted to any stable manifold, so it can be regarded as a function defined on $\bar\Delta$.

LEMMA 3.1. *If $f\colon X \to \mathbb{R}^d$ is piecewise η-Hölder, and β satisfies $1 > \beta \ge \alpha^{\eta/2}$,* *then the associated function $\bar{f}\colon \bar{\Delta} \to \mathbb{R}^d$ is bounded and Lipschitz-continuous wrt the metric β^s:*

$$\left|\bar{f}(\bar{x}) - \bar{f}(\bar{y})\right| \leq C\beta^{s(\bar{x},\bar{y})}.$$

Proof Let $\bar{x}, \bar{y} \in \bar{\Delta}$ such that $s(x,y) \geq 2n$ then **(P5)** ensures

$$\left|\tilde{f}F^k x - \tilde{f}F^k y\right|, \left|\tilde{f}F^k \Xi x - \tilde{f}F^k \Xi y\right| \leq C_f (C\alpha^{2n-k})^\eta, \quad 0 \leq k \leq n.$$

For all $k > 0$ **(P4)** gives

$$\left|\tilde{f}F^n x - \tilde{f}F^n \Xi x\right|, \left|\tilde{f}F^n y - \tilde{f}F^n \Xi y\right| \leq C_f (C\alpha^n)^\eta.$$

Hence $|h(x) - h(y)| \leq 2C_f \sum_{k=0}^{n} (C\alpha^{2n-k})^\eta + 2C_f \sum_{k=n+1}^{\infty} (C\alpha^n)^\eta \leq \mathrm{const} C_f \alpha^{n\eta}$ given $1 > \beta \geq \alpha^{\eta/2}$ the latter estimate $\leq \bar{C}_f \beta^s$. $\square$

3.1. *Quasicompactness* The purpose of this subsection is to prove the Doeblin-Fortet inequality for the Fourier transform of the Perron-Frobenius operator:

$$P_t(\bar{\varphi}) := P(e^{it\bar{f}}\bar{\varphi}) \quad (\bar{\varphi} \in \mathcal{C})$$

where $f\colon X \to \mathbb{R}^d$ measurable, and $t \in \mathbb{R}^d$. Simpifying the notations for a fixed t denote $\omega = e^{i\langle t, \bar{f}\rangle}$, so $P_t(\bar{\varphi}) = P(\omega\bar{\varphi})$. For to prove the inequality we need the assumption of Hölder continuity for the measurable f.

Lemma 3.2. *If f and β satisfies the conditions of the previous lemma, then the operator P_t satisfies the Doeblin-Fortet inequality $\forall t \in \mathbb{R}^d$.*

Proof

$$\|P_t^n \bar{\varphi}\|_{\mathcal{L}} = \|P_t^n \bar{\varphi}\|_{\mathcal{C}} + \|P_t^n \bar{\varphi}\|_h \leq \|P_t^n\|_{\mathcal{C}} \|\bar{\varphi}\|_{\mathcal{C}} + \|P_t^n \bar{\varphi}\|_h.$$

By (ϵii) $\|P_t^n\|_{\mathcal{C}} \leq 2$, so we only have to bound the continuity modulus.

$$P_t^n(\bar{\varphi}) = P^n(\omega_n \bar{\varphi}) \text{ where } \quad \omega_n(\bar{x}) := \prod_{k=0}^{n-1} \omega(\bar{F}^k \bar{x}).$$

It follows that

$$P_t^n(\bar{\varphi})(\bar{x}) = \sum_{\bar{x}^{-n}:T^n\bar{x}^{-n}=\bar{x}} \frac{\omega_n(\bar{x}^{-n})\bar{\varphi}(\bar{x}^{-n})}{J\bar{F}^n(\bar{x}^{-n})}$$

If $\bar{x}$ and $\bar{y}$ lie in the same element $\bar{\Delta}_{l,j}$, then the inverse images can be coupled: $\bar{x}_i^{-n}$ and $\bar{y}_j^{-n}$ form a pair if $\forall k, 0 \leq k \leq n \quad \bar{F}^k(\bar{x}_i^{-n})$ and $\bar{F}^k(\bar{y}_j^{-n})$ belong to the same element of the Markov partition $\{\bar{\Delta}_{l,j}\}$. That this is really a coupling is ensured by (e) in the definition of $\mathcal{D}$. For notational simplicity suppose that the inverse images are numbered according to the coupling. We have then the following expression for the continuity modulus:

$$|P_t^n(\bar{\varphi})(\bar{x}) - P_t^n(\bar{\varphi})(\bar{y})| = \left| \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \frac{\omega_n(\bar{x}_i^{-n})\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\omega_n(\bar{y}_i^{-n})\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right|.$$

The right hand side can be written as $|I + II|$ where

$$I = \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \omega_n(\bar{x}_i^{-n}) \left(\frac{\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right),$$

and

$$II = \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \left(\omega_n(\bar{x}_i^{-n}) - \omega_n(\bar{y}_i^{-n}) \right).$$

The first quantity can be estimated as follows:

$$|I| \leq \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \left| \frac{\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right|$$

Young [**You 98**] gets her D-F inequality by estimating the same quantity in the case where $n = N$. For the estimate of the second term we have to say something about the continuity modulus of ω:

$$|\omega(a) - \omega(b)| = \left| e^{i\left\langle t, \bar{f}(a)\right\rangle} - e^{i\left\langle t, \bar{f}(b)\right\rangle} \right| \leq |t| \left| \bar{f}(a) - \bar{f}(b) \right|.$$

By lemma 3.1 this latter is

$$\leq |t|\, C\beta^{s(a,b)}.$$

Then the continuity modulus of ω_N:

$$\begin{aligned} \left|\omega_N(\bar{x}_i^{-N}) - \omega_N(\bar{y}_i^{-N})\right| &= \sum_{k=0}^{N-1} \left|\omega(\bar{F}^k(\bar{x}_i^{-N})) - \omega(\bar{F}^k(\bar{y}_i^{-N}))\right| \\ &\leq \sum_{k=0}^{N-1} |t|\, C\beta^{s(\bar{F}^k(\bar{x}_i^{-N}), \bar{F}^k(\bar{y}_i^{-N}))} \\ &= \sum_{k=0}^{N-1} |t|\, C\beta^{s(\bar{x},\bar{y})+N-k} \\ &\leq \frac{\beta\, |t|\, C\beta^{s(\bar{x},\bar{y})}}{1-\beta} \end{aligned}$$

II can be estimated by taking absolute value term by term. Then the continuity modulus is multiplied by $P^N|\varphi|y \leq e^{\epsilon l} \left\| P^N|\varphi| \right\|_{\mathcal{C}} \leq e^{l\epsilon} 2^N \|\varphi\|_{\mathcal{C}}$. From these it is easy to see, that in the D-F inequality this estimate of II contributes to the coefficient of $\|\varphi\|_{\mathcal{C}}$ by $\frac{2^N \beta C|t|}{1-\beta}$, so it doesn't bother Young's estimate of I. □

3.2. *Minimality* Next we have to investigate the t values, for which P_t has an eigenvalue on the unit circle. Otherwise P_t is strictly contractive by quasicompactness. As we will see, this is the question of minimality. First we give the basic definitions for an arbitrary dynamical system (X, T, ν). Then we will investigate Young's symbolic system $(\bar{\Delta}, \bar{F}, \bar{\nu})$ to get a characterisation of the abovementioned t-values. Finally, we will prove that the definitions for a Young system (X, T, ν), and for the associated symbolic system $(\bar{\Delta}, \bar{F}, \bar{\nu})$ provide the same answer. Thus we can characterise the "bad" t-values, by concentrating on the minimality of our function on the original system.

Definition We say that f is *cohomologous* to g (notation: $f \sim g$) if $\exists h$ measurable such that $f - g = h - h \circ T$. Under the *minimal support* of a function f (notation: $S(f)$) we mean the minimal translated closed subgroup of $\mathbb{R}^d$, which supports its values. We call a translated closed subgroup the *minimal lattice* of f if it is the intersection of minimal supports in the cohomology class of f ($M(f) = \cap_{g:g\sim f} S(g)$). We call f *minimal* if $S(f) = M(f)$. We call f *degenerate* if $M(f)$ is contained in a smaller dimensional affine subspace of $\mathbb{R}^d$.

LEMMA 3.3. *Fix the function f. Then $P_t^{\rho}\bar{g} = \lambda\bar{g}$ with $|\lambda| = 1 \iff e^{it\bar{f}}\bar{g} = \lambda\bar{g} \circ \bar{F}$. Moreover $\bar{g}$ can be supposed to take values on the unit circle.*

Proof

$\Longrightarrow$ If $P_t^{\rho}\bar{g} = \lambda\bar{g}$ then by (ϵi) $\bar{g} \in \mathcal{L} \Longrightarrow \bar{g} \in L_2(\bar{m})$, and also $\bar{g} \in L_2(\bar{\nu})$ we can take:

$$\left\langle e^{it\bar{f}}\bar{g}, \bar{g} \circ \bar{F} \right\rangle_{\bar{\nu}} = \left\langle P\left(e^{it\bar{f}}\bar{g}\right), \bar{g} \right\rangle_{\bar{\nu}} = \langle \lambda\bar{g}, \bar{g} \rangle_{\bar{\nu}} = \lambda \left\|\bar{g}\right\|^2_{L_2(\bar{\nu})}.$$

From Cauchy-Schwartz inequality it follows that $e^{it\bar{f}}\bar{g} = \lambda\bar{g} \circ \bar{F}$. By ergodicity we can suppose $|\bar{g}| \equiv 1$.

$\Longleftarrow$ If $e^{it\bar{f}}\bar{g} = \lambda\bar{g} \circ \bar{F}$ then $P_t^\rho(\bar{g}) = \frac{1}{\rho}P(\rho e^{it\bar{f}}\bar{g}) = \frac{\lambda}{\rho}P(\bar{g} \circ \bar{F}\rho) = \lambda\bar{g}\frac{P(\rho)}{\rho} = \lambda\bar{g}$.

Since $|\bar{g}| = 1 \Longrightarrow \bar{g} \in \mathcal{C}$, then it follows that $\bar{g} \in \mathcal{L}$ [**I-TM 50**].

□

This lemma shows that the t values for which the abovementioned property holds form a closed subgroup of $\mathbb{R}^d$, moreover the eigenvalues and -functions preserve the group structure. If $P_{t_1}^\rho \bar{g}_1 = \lambda_1\bar{g}_1 \circ \bar{F}$ and $P_{t_2}^\rho \bar{g}_2 = \lambda_2\bar{g}_2 \circ \bar{F}$, then $P_{t_1+t_2}^\rho \bar{g}_1\bar{g}_2 = \lambda_1\lambda_2(\bar{g}_1\bar{g}_2) \circ \bar{F}$. Also, for $t \in G$, $t \mapsto \bar{g}_t$ and $t \mapsto \lambda_t$ are uniquely determined by ergodicity. (Here G denotes the subgroup of $\mathbb{R}^d$ formed by these t values.) This uniqueness can be easily derived from the multiplicative structure, and the already known spectral picture for $P = P_0$. Since λ_t is a multiplicative functional of t, so the logarithm is a linear one, and therefore $-i\log\lambda_t = tr$ for some r real vector. (Taking the adequate branch of the logarithm.)

Theorem 3.1. $M(\bar{f}) = \widehat{\mathbb{R}^d/G} + r$. *There exist minimal functions in each cohomology class. The minimal function is unique iff it is constant.*

Proof

$\subset$ We are going to prove that $\forall t \in G, \forall x \in M(\bar{f})$ one has $e^{itx} = e^{itr}$. Since $t \in G$ we have $e^{it\bar{f}}\bar{g} = \lambda\bar{g} \circ \bar{F}$. Taking the logarithm

$$t\bar{f} \equiv -i\log\lambda + i\log\bar{g} - i\log\bar{g} \circ \bar{F} \pmod{2\pi}. \tag{1}$$

Remember that the first term on the right hand side is tr. By denoting $h = i\log\bar{g}$ we get that $t\bar{f} - (Z + tr) = h - h \circ \bar{F}$ for some Z, which takes values in $2\pi\mathbb{Z}$. To lift it to vector valued equation let us denote

$\vec{h} = \frac{th}{|t|^2}$, $\vec{Z} = \frac{tZ}{|t|^2} + \bar{f}^{t^\perp} - r^{t^\perp}$, we get that $\bar{f} \sim \vec{Z} + r$, and the right hand side takes values in $H = t^\perp \oplus \frac{2\pi t}{|t|^2}\mathbb{Z} + r$. By definition $H \supset M(f)$, and since $\forall x \in H \;\; e^{itx} = e^{itr}$ this is true for $\forall x \in M(\bar{f})$.

$\supset$ We are going to prove that if for $t \in \mathbb{R}^d$ and $\forall x \in M(\bar{f})$ we have $e^{itx} = e^{itr}$, then $t \in G$. The condition means that $\exists Z,\;\; Z \sim \bar{f},\;\; S(Z) \subset t^\perp \oplus \frac{2\pi t}{|t|^2}\mathbb{Z} + r$. Combining the condition with the cohomological equation we get $e^{itZ} = e^{itr} = e^{it(\bar{f} - h + h\circ\bar{F})}$. After rearranging one obtains $e^{it\bar{f}}e^{-ith} = e^{itr}e^{-ith\circ\bar{F}}$, and by the previous lemma $t \in G$.

$\exists$ Let us revisit the congruence (1). Observe that $i\log\bar{g}$ is also a linear functional of t, so $i\log\bar{g} = ts$ for some $s : \bar{\Delta} \to \mathbb{R}^d$. The function Z derived from this congruence is also linear in t, so $Z = tz$. Denote by H the orthocomplement of the linear subspace generated by G. Recalling the definition of r, s and z we can see, that r^H, s^H and z^H can be arbitrary, so let the latter one agree with $\bar{f}^H$, and the others be 0. We get $\bar{f} - (z + r) = s - s \circ \bar{F}$. Consider now $S(z + r)$. In the definition of Z we said that it takes values in $2\pi\mathbb{Z}$, but $Z = tz$ gives $\forall t \in G \;\; e^{it(z+r)} = e^{itr}$, so from the already proven part of the theorem it follows that $S(z + r) = M(\bar{f})$. Uniqueness is obvious: if $M(\bar{f})$ is not a single point, then taking any $h : X \to M(\bar{f})$ nonconstant $\bar{f} - h + h \circ \bar{F}$ is also a minimal function, and by ergodicity is not equal to $\bar{f}$.

□

Let us remark, that $M(S_n) = M(\bar{f}) + (n-1)r$. One of the inclusions ($\subset$) is trivial, the other ($\supset$) follows from ergodicity of iterates. Now we turn our attention to the point, that neither the Markov extension, nor the factorisation changes minimality properties.

THEOREM 3.2. *If g is minimal in the class of f, then $\bar{g}$ is minimal in the class of $\bar{f}$. $M(f) = M(\bar{f})$.*

Proof First we prove, that $f \sim g \Longrightarrow \tilde{f} \sim \tilde{g}$. Indeed $f - g = h - h \circ T \Longrightarrow \tilde{f} - \tilde{g} = \tilde{h} - \tilde{h} \circ F$. Consider now the construction of $\bar{f}$. It is clear that this construction preserves addition, so it is enough to prove, that if f is null-cohomologous, then $\bar{f}$ also. This means $f = h - h \circ F$. Putting this in the definition of $\bar{f}$, we see, that $\bar{f} = h \circ \Xi - h \circ \Xi \circ F$. The function $h \circ \Xi$ is constant along stable lines, so $\bar{f}$ is null-cohomologous in the factorised system. So far we have reached $f \sim g \Longrightarrow \bar{f} \sim \bar{g}$. If g is minimal, then $S(g) = S(\bar{g})$. From the formula $\supset$ is trivial, and it cannot be strictly smaller, since $\tilde{g} \sim \bar{g}$ would contradict the minimality of g. The only thing remained is to show that $S(\bar{g}) = M(\bar{g})$. If not there would be an other function with smaller support in the same class. Consider this on Δ, as a function constant along stable manifolds! It would be in the class of $\tilde{g}$, which would again contradict the minimality. $\square$

3.3. *A Nagaev type theorem* Expand now P_t in a Taylor series around $t = 0$! $P_t(\bar{\varphi}) = P(e^{i\langle t, \bar{f}\rangle}\bar{\varphi}) = P(\bar{\varphi}) + itP\left(\bar{f}\bar{\varphi}\right) - \frac{t^2}{2}P\left(\bar{f}^2\bar{\varphi}\right) + o(t^2)\left\|\bar{f}^2\bar{\varphi}\right\|_{\mathcal{L}}$. From lemma 3.1 it follows that the norm exists, so the second order Taylor-expansion at zero makes sense. Let us denote the operator $\bar{\varphi} \mapsto P(\bar{f}\bar{\varphi})$ by M (mean) and $\bar{\varphi} \mapsto P(\bar{f}^2\bar{\varphi})$

by Σ (covariance).

Denote by λ_t the leading -also simple- eigenvalue of P_t, (we know that $\lambda_0 = 1$) and by τ_t the projection operator corresponding to λ_t. The invariant density ρ is known to be bounded away from zero and infinity, and is Hölder. We know that $\tau_0 = \rho\bar{m}$, since ρ is the invariant density. Consider the second order Taylor polynomial of these two objects:

$$\lambda_t = 1 + iat - b\frac{t^2}{2} + o(t^2)$$

$$\tau_t = \rho\bar{m} + \eta t + \chi t^2 + o(t^2)$$

By definition $\tau_t P_t = \lambda_t \tau_t$. Expressing the terms by the above equations and considering the coefficients of t and t^2 we get the following:

$$i\rho\bar{m}M + \eta P = \eta + ia\rho\bar{m}$$

$$-\frac{1}{2}\rho\bar{m}\Sigma + i\eta M + \chi P = \chi + ia\eta - \frac{b\rho\bar{m}}{2}$$

evaluating these on ρ we get from the first that $a = \bar{m}M(\rho)$. We are allowed to suppose that $M(\rho)$ is a constant. This is because if we change $\bar{f}$ to a cohomologous $\bar{f}'$ the maximal eigenvalue does not change. Just like in the case of P_t^ρ we will study a conjugated operator with the same spectrum. Let us solve the equation: $P(\bar{f}\rho) - \int f d\nu = Pu - u$. This is solvable since the left hand side $\in \ker \bar{m}$. Let us consider $\bar{f}' = \bar{f} - \frac{u}{\rho} + \frac{u \circ \bar{F}}{\rho \circ \bar{F}}$! This is clearly cohomologous to $\bar{f}$. Let us consider $M'(\rho) = P(\bar{f}'\rho) = P(\bar{f}\rho) - Pu + P(\frac{u\circ\bar{F}}{\rho\circ\bar{F}}\rho)$. This latter term is $\frac{u}{\rho}P\rho = u$. So by the definition of u $M'(\rho) = \int f d\nu$ constant. Evaluating the second equation on ρ we get $b = \bar{m}\Sigma'(\rho) = \int \bar{f}'^2 d\bar{\nu}$, remember, that a was the average of the function, now b is some second moment, and we can define covariance by $\sigma^2 = b - a^2$. It is also remarkable, that σ is the second central moment of a function cohomologous to $\bar{f}$. If f is nondegenerate, each such quadratic form (and consequently σ) is nondegenerate also. We have proved the following theorem:

Theorem 3.3. *There are constants $\epsilon > 0$, $K > 0$ and $\theta < 1$ and a function $\rho : (-\epsilon, \epsilon)^d \to \mathcal{L}$ such that*

$$\left\| P_t^n h - \lambda_t^n \rho_t \int_{\bar{\Delta}} h d\bar{m} \right\|_{\mathcal{L}} \leq K\theta^n \|h\|_{\mathcal{L}} \quad \forall |t| < \epsilon, \;\; n \geq 1, \;\; h \in \mathcal{L},$$

and $\rho_0 = \rho$, $\lambda_t = 1 + ait - (\sigma^2 + a^2)\frac{t^2}{2} + o(t^2)$.

4. *Proof of the Main Theorem*

In this section we are still going to consider Young systems, in general. Without loss of generality (by adding a scalar) we can suppose, that $r = 0$, which means, that $M(f)$ is a closed subgroup of $\mathbb{R}^d$. It also means, that $P_{t+u} = P_t$ if $u \in G$, so the t values are actually taken from $\widehat{M(f)} = \mathbb{R}^d/G$. Later we will concentrate on compact parts of this group.

Lemma 4.1 ([**AD 01**]) *Suppose that $\mathcal{K}$ is a compact set of $\mathcal{L}$ operators such that each element of $\mathcal{K}$ is a Doeblin-Fortet operator, and none of them has an $\mathcal{L}$-eigenvalue on the unit circle.Then $\exists K > 0$ and $\theta < 1$ such that*

$$\|Q^n\|_{\mathcal{L}} \leq K\theta^n \quad \forall n \geq 1, \quad Q \in \mathcal{K}.$$

For to apply this lemma we have to cut out a neighborhood of zero. In it, however, theorem 3.3 holds. Now we are able to prove our main theorem.

THEOREM 4.1. *Suppose that*

1. (X, T, ν) *is a Young system (cf. subsection 2.1);*
2. f *is minimal (cf. subsection 3.2);*
3. f *is nondegenerate (cf. subsection 3.2);*
4. f *is bounded and Hölder-continuous.*

Let $k_n \in M(f)$ *be such that* $\frac{k_n - na}{\sqrt{n}} \to k \in \mathbb{R}^d$*. Denote the distribution of* $S_n - k_n$ *by* υ_n*, then*

$$\lim_{n\to\infty} n^{\frac{d}{2}} \upsilon_n = \frac{e^{\frac{-k^2}{2\sigma^2}}}{\det\sigma\sqrt{(2\pi)}^d} l.$$

l *is the uniform distribution on* $M(f)$*, more exactly it is product of suitable counting measures and Lebesgue measures.*

Proof Suppose that we choose a random point of X according to the invariant distribution ν. Let the joint distribution of $(x, T^n x, S_n(x) - k_n)$ be denoted by Υ_n! We are going to prove, that

$$\lim_{n\to\infty} n^{\frac{d}{2}} \Upsilon_n \to \frac{e^{\frac{-k^2}{2\sigma^2}}}{\det\sigma\sqrt{(2\pi)}^d} \nu^2 \times l.$$

The definitions of $\tilde{\Upsilon}_n$, and $\bar{\Upsilon}_n$ are straightforward. In the following paragraph we are going to see that it is enough to prove that the limit of $n^{\frac{d}{2}}\bar{\Upsilon}_n$ is $\bar{\nu}^2 \times l$ multiplied by the gaussian density of covariance σ at k.

It is clear that a similar limit for $\tilde{\Upsilon}_n$ is sufficient. Remember the definition of $\bar{f}$! Nota bene if f is minimal then h takes values in $M(f)$. So in the language of Υ the factorisation means the application of the mapping $(x, y, \xi) \mapsto (x, y, \xi + h(x) - h(y))$. So the same mapping applies to the weak limit, which leaves it invariant so the uniform limits for the triples are equivalent. We have successfully changed the last variable in the triple. What remained to change are the two $\tilde{\nu}$ distributed variables to their $\bar{\nu}$ distributed versions. The σ-algebra $\bar{\mathcal{S}}$, generated by factorised functions, is the multiplication of the σ-algebra generated by the rectangles in Δ in the stable direction, and the Borel-algebra in the unstable direction (mod 0). The forthcoming limit theorem for $\bar{\Upsilon}_n$ proves the same for $F\bar{\mathcal{S}}$, because the application of F means the application of $(x, y, \xi) \mapsto (Fx, Fy, \xi - \bar{f}(x) + \bar{f}(y))$, and the limit is invariant under this action. Since $\bigvee_{n>0} F^n \bar{\mathcal{S}} = \mathcal{S}$ (mod 0) it is enough to prove the limit theorem for $\bar{\Upsilon}_n$.

For to do this we are going to integrate test functions: $w(\bar{x}, \bar{y}, \xi)$. We will restrict ourselves to functions which are in $\mathcal{L}$ as functions of x and y and are integrable (with respect to the prospective limit) as functions of ξ, moreover their Fourier transform is compactly supported. By Breiman [**Bre 68**] checking convergence for these functions proves weak convergence of measures. For simplicity we are going to use the inverse transform: $w(\bar{x}, \bar{y}, \xi) = \int \hat{w}(\bar{x}, \bar{y}, t) e^{it\xi} dt$.

$$
\begin{aligned}
n^{\frac{d}{2}} \int_{\bar{\Delta} \times \bar{\Delta} \times M(f)} w d\bar{\Upsilon}_n &= n^{\frac{d}{2}} \int w(\bar{x}, \bar{F}^n \bar{x}, S_n(\bar{x}) - k_n) d\bar{\nu} \\
&= n^{\frac{d}{2}} \int \int_{\widehat{M(f)}} \hat{w}(\bar{x}, \bar{F}^n \bar{x}, t) e^{it(S_n(\bar{x}) - k_n)} dt \ \ d\bar{\nu} \\
&= n^{\frac{d}{2}} \int \rho^{-1}(\bar{x}) P^n \rho(\bar{x}) \left(\int_{\widehat{M(f)}} \hat{w}(\bar{x}, \bar{F}^n \bar{x}, t) e^{it(S_n(\bar{x}) - k_n)} dt \right) d\bar{\nu} \\
&= n^{\frac{d}{2}} \int \int_{\mathrm{supp}\hat{w}} \rho^{-1}(\bar{x}) e^{-itk_n} P_t^n \left(\rho(\bar{x}) \hat{w}(\bar{x}, \bar{F}^n \bar{x}, t) \right) dt \ \ d\bar{\nu}
\end{aligned}
$$

Using lemma 4.1 and theorem 3.3 we can substitute $P_t^n \rho \hat{w}$ by $\lambda_t^n \rho_t \int_{\bar{\Delta}} \rho \hat{w} d\bar{m}$ in the domain $|t| < \delta$ and we get an error term $O(n^{\frac{d}{2}} \theta^n)$ inside the integration wrt $\bar{\nu}$.

This involves the error terms of lemma 4.1 and theorem 3.3. Since $\int \hat{w} d\bar{\nu}$ depends only on t we will use the shorter $\hat{w}(t)$ form.

$$
\begin{aligned}
n^{\frac{d}{2}} \int_{\bar{\Delta}\times\bar{\Delta}\times M(f)} w d\bar{\Upsilon}_n &= \int \rho^{-1}(\bar{x}) \int_{|t|<\delta\sqrt{n}} \hat{w}\left(\frac{t}{\sqrt{n}}\right) e^{-it\frac{k_n}{\sqrt{n}}} \lambda^n_{\frac{t}{\sqrt{n}}} \rho_{\frac{t}{\sqrt{n}}}(\bar{x}) dt + o(1) d\bar{\nu} \\
&\to \int_{\mathbb{R}^d} \int \hat{w}(\bar{x},\bar{y},0) d\bar{\nu} e^{-itk} e^{\frac{-\sigma^2 t^2}{2}} dt \\
&= \frac{1}{(2\pi)^d} \int_{M(f)} w(\bar{x},\bar{y},\xi) d\bar{\nu}^2 \times dl \quad \frac{1}{\det \sigma} \sqrt{2\pi}^d e^{-\frac{k^2}{2\sigma^2}}
\end{aligned}
$$

In the above limit the order of the error term is meant in $\mathcal{L}$-norm (cf. lemma 4.1 and theorem 3.3), this implies that limiting makes the error term vanish (cf. definition of $\mathcal{L}$-norm). The same applies for the $\bar{x}$ dependence of $\rho_{\frac{t}{\sqrt{n}}}$. The convergence in t is dominated, since $\exists C \quad \forall |t| \le \delta\sqrt{n} \quad \left|\lambda^n_{\frac{t}{\sqrt{n}}}\right| \le e^{-C|t|^2}$. □

Remark The case of nonminimal functions is obvious from the first argument of the proof. If $f - g = h - h\circ T$ then the limit measure for f differs from the limit measure for g by convolving the distribution of h and of $-h$.

THEOREM 4.2. *Let $k_n \in M(f)$ be such that $\frac{k_n - na}{\sqrt{n}} \to k \in \mathbb{R}^d$, and $\kappa_n \in M(f)$ be such that $\frac{\kappa_n - na}{\sqrt{n}} \to \kappa \in \mathbb{R}^d$. Denote the joint distribution of $S_n - k_n, S_m - \kappa_m$ by $\upsilon_{n,m}$! If f is minimal and nondegenerate, then*

$$
\lim_{n,m,n-m\to\infty} n^{\frac{d}{2}} m^{\frac{d}{2}} \upsilon_{n,m} \to \frac{e^{\frac{-k^2}{2\sigma^2}} e^{\frac{-\kappa^2}{2\sigma^2}}}{\det^2 \sigma (2\pi)^d} l \times l.
$$

Proof Again as in the previous proof if we consider the joint distribution $\Upsilon_{n,m}$ of the 5-tuple $(x, T^n x, T^m x, S_n(x) - k_n, S_m(x) - \kappa_m)$, then it is enough to prove, that

$$
\lim_{n,m,n-m\to\infty} n^{\frac{d}{2}} m^{\frac{d}{2}} \bar{\Upsilon}_{n,m} \to \frac{e^{\frac{-k^2}{2\sigma^2}} e^{\frac{-\kappa^2}{2\sigma^2}}}{\det^2 \sigma (2\pi)^d} \bar{\nu}^3 \times l^2.
$$

To prove convergence we are going to integrate test functions: $w(\bar{x}, \bar{y}, \bar{z}, \xi, \zeta)$. Again as in the previous proof we restrict ourselves to the same class of functions. We are going to use the inverse transform: $w(\bar{x}, \bar{y}, \bar{z}, \xi, \zeta) = \int \hat{w}(\bar{x}, \bar{y}, \bar{z}, t, u) e^{i(t\xi + u\zeta)} dt \ du$.

$$n^{\frac{d}{2}} m^{\frac{d}{2}} \int\limits_{\bar{\Delta}^3 \times M(f)^2} w d\bar{\Upsilon}_{n,m} = n^{\frac{d}{2}} m^{\frac{d}{2}} \int \int\limits_{\widehat{M(f)}^2} \rho^{-1} e^{-i(tk_n + u\kappa_n)} P_t^n \left(\rho e^{iuS_m} \hat{w}\right) dt \ du \ d\bar{\nu}$$

$$= n^{\frac{d}{2}} m^{\frac{d}{2}} \int \int\limits_{|t|<\delta} \rho^{-1}(\bar{x}) e^{-itk_n} \lambda_t^n \rho_t \int e^{-iu\kappa_n} \int\limits_{\bar{\Delta}} \rho e^{iuS_m} \hat{w} d\bar{m} \ du \ dt + O(n^{\frac{d}{2}} \theta^n) d\bar{\nu}$$

Again the inner integration is invariant under P, so

$$\begin{aligned} \int_{\bar{\Delta}} \rho e^{iuS_m} \hat{w} d\bar{m} &= \int_{\bar{\Delta}} P^m \rho e^{iuS_m} \hat{w} d\bar{m} \\ &= \int_{\bar{\Delta}} P_u^m \rho \hat{w} d\bar{m} \\ &= \int_{\bar{\Delta}} \lambda_u^m \rho_u \int_{\bar{\Delta}} \rho \hat{w} d\bar{m} + O(\theta^m) d\bar{m} \end{aligned}$$

From this point the variables can be handled separately and the argument of the previous proof should be repeated twice to get the statement of this theorem. □

5. *Recurrence of planar Lorentz-process*

5.1. *Semi-dispersing billiards* In this subsection we summarize some basic properties of semi-dispersing billiards. Our aim is to introduce the most important concepts and fix the notation. For a more detailed description see the literature, especially [**KSSz 90**].

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$. The boundary of the domain in assumed to be piecewise C^3-smooth. Inside Q the motion is uniform while the reflection at the boundary ∂Q is elastic. As the absolute value of the velocity is a first integral of motion, the phase space of the billiard flow is fixed as $M = Q \times S^{d-1}$ – in other words, every phase point x is of the form $x = (q, v)$ with $q \in Q$ and $v \in \mathbb{R}^d$, $|v| = 1$. The Liouville probability measure μ on M is essentially the product of the Lebesgue measures, i. e. $d\mu = \text{const.}\, dqdv$. The resulting dynamical system $(M, S^{\mathbb{R}}, \mu)$ is the (toric) *billiard flow*.

Let $n(q)$ denote the unit normal vector of a smooth component of the boundary ∂Q at the point q, directed inwards Q. Throughout the paper we restrict our attention to *semi-dispersing billiards*: we require for every $q \in \partial Q$ the second fundamental form $K(q)$ of the boundary component to be non-negative.

The boundary ∂Q defines a natural cross-section for the billiard flow. Namely consider

$$\partial M = \{(q, v) \mid q \in \partial Q, \langle v, n(q)\rangle \geq 0\}.$$

This set actually has a natural bundle structure (cf. [**BChSzT**]). The Poincaré section map T, also called the *billiard map* is defined as the first return map on ∂M. The invariant measure for the map is denoted by μ_1, and we have $d\mu_1 = \text{const.}\, |\langle v, n(q)\rangle|\, dqdv$. Throughout the paper we work with this discrete time dynamical system $(\partial M, T, \mu_1)$. Recall the usual notation: for $(q, v) \in M$ one denotes $\pi(q, v) = q$ the natural projection.

The *Lorentz process* is the natural $\mathbb{Z}^d$ cover of a toric billiard. More precisely: consider $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ the factorisation by $\mathbb{Z}^d$. Its fundamental domain D is a d-dimensional cube (semi-open, semi-closed) in $\mathbb{R}^d$, so $\mathbb{R}^d = \cup_{z \in \mathbb{Z}^d}(D + z)$, where $D + z$ is the translated fundamental domain.

By denoting $\tilde{Q} = \Pi^{-1}Q$, $\tilde{M} = \tilde{Q} \times S^{d-1}$, etc., the Lorentz dynamics is $(\tilde{M}, \{\tilde{S}^t \mid t \in \mathbb{R}\}, \tilde{\mu})$ and its Poincaré section map is $(\partial\tilde{M}, \tilde{T}, \tilde{\mu}_1)$. The *free flight function* $\tilde{\psi} : \partial\tilde{M} \to \mathbb{R}^d$ is defined as follows: $\tilde{\psi}(\tilde{x}) = \tilde{q}(T\tilde{x}) - \tilde{q}(\tilde{x})$. The *discrete free flight function* $\tilde{\kappa} : \partial\tilde{M} \to \mathbb{Z}^d$ is defined as follows: $\tilde{\kappa}(\tilde{x}) = \iota(\tilde{T}\tilde{x}) - \iota(\tilde{x})$, where $\iota(\tilde{x}) = z$ if $\tilde{x} \in Dz$. Observe finally, that $\tilde{\psi}$ and $\tilde{\kappa}$ are invariant under the $\mathbb{Z}^d$ action, so there are ψ and κ functions defined on ∂M, such that $\tilde{\psi} = \Pi^*\psi$ and $\tilde{\kappa} = \Pi^*\kappa$. Actually for our purposes it will be more convenient to choose the fundamental domain in such a way that $\partial\tilde{Q} \cap \partial D = \emptyset$. In this way κ will be continuous.

5.2. *Minimality of the free flight function* Start with a simple observation

LEMMA 5.1.

$$\kappa \sim \psi$$

Proof Fix an arbitrary point $w \in D$. For $x = (q, v) \in \partial M$ define $h(x) = w - q$. if $h(Tx) \in D + z$ for some $z \in \mathbb{Z}^d$, then $\kappa(x) = z$, and, of course,

$$\psi(x) = \kappa(x) + h(x) - h(Tx)$$

□

THEOREM 5.1. *κ is minimal in the class of ψ.*

Proof Suppose the contrary and denote the minimal function by κ'! Apply the factorisation by the minimal lattice: $\kappa_f : \partial M \to \mathbb{Z}^d/M(\kappa)$! Then $\kappa_f \sim \kappa'_f$, and κ'_f is the constant function. Denote by n the cardinality of this abelian group $\mathbb{Z}^d/M(\kappa)$! (We can suppose $n < \infty$.) In this case $\forall x$ periodic, such that $n|\mathrm{per}(x) = p$ the

Birkhoff sum $S_p(\kappa_f)x = 0$. The proof of the theorem is based on our forthcoming lemma 5.2. It is a variant of a statement which was originally applied in [**BChS 91**] to establish the non-singularity of the limiting covariance in the CLT. To contradict the non-minimality we are going to find a periodic point for each sublattice of finite index, not satisfying the above equation.

LEMMA 5.2. *For any finite index sublattice $Z \subset \mathbb{Z}^d$ there exists a periodic point x such that the period p is a multiple of $\left|\mathbb{Z}^d : Z\right|$ and $\sum_{i=0}^{p-1} \kappa(T^i x) \not\equiv 0 \pmod{Z}$*

Proof of lemma The idea is a suitably adapted, simplified and generalized version of an argument of [**BChS 91**]. The original idea is well explained in [**B 00**]. Fix the lattice Z, denote the index by i, and fix $\Lambda \subset \mathbb{T}_0^d$, the basic product set of the Young system of our billiard ($\mu_1(\Lambda) > 0$). Take a billiard in the elongated torus $\mathbb{T}(Z) = \mathbb{R}^d/Z$, which is an appropriate projection of our Lorentz process. Consider the images of Λ on the elongated torus. Take two of them Λ_0 and Λ_1. By using the ergodicity of powers of the billiard in $\mathbb{T}(Z)$ we see that there exists an $n \in \mathbb{Z}_+$ such that $\Lambda_0 \cap T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure where $T(Z)$ denotes the Poincaré section map of the billiard on $\mathbb{T}(Z)$. The fact that $\Lambda_0 \cup T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure requires a proof. This is the only part in our paper where we have to go beyond properties (P1-8) of Young systems formulated in subsection 2.1 and to use some more detailed arguments from her construction. To make the reading of the main body of this paper easier we will postpone until the Appendix the proof of the sublemma formulating this particular statement .

Sublemma For the billiard on $\mathbb{T}(Z)$ there exists an $n \in \mathbb{Z}_+$ such that $\Lambda_0 \cap T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure.

By identifying Λ with Λ_0, $\cap_{l=-\infty}^{\infty} T^{lni}\Lambda^*$ consists of exactly one point x^*. Clearly $T^n ix^* = x^*$ and, moreover, the claim of the lemma is also evident. □

To conclude the proof it is sufficient to observe that the relation $\kappa \sim \kappa'$ and the periodicity of x also imply that $\sum_{i=0}^{p-1} \kappa(T^i x) \not\equiv 0 \pmod{Z}$. Hence the theorem. □

5.3. *Proof of recurrence* In this subsection we want to apply the local limit theorem in order to get the recurrence for the planar Lorentz-process, a result already proved in [**Sch 98**] and in [**Con 99**]. Let the system be a billiard on the 2-dimensional torus, with strictly convex scatterers, and finite horizon. Such a system is always a Young system. This was proved in [**You 98**]. For the role of f in the main theorem let we choose $\kappa : X \to \mathbb{R}^2$ the discrete free flight function. Time reversion symmetry ensures zero average. We have just proved that κ is minimal. Its boundedness is equivalent with the finite horizon assumption, and the other conditions are trivial. Then theorem 4.1 ensures that $\nu(S_n \in D) > \frac{C}{n}$ for some $C > 0$. It immediately extends to any fixed domain.

THEOREM 5.2. *The planar Lorentz process with finite horizon is almost surely recurrent.*

Proof The proof follows the ideas used in [**KSz 85**]. The sequence of events

$$A_n = \{S_n \in D\}$$

fulfills the condition of Lamperti's Borel-Cantelli [**Spi 64**]:

$$\sum_{k=1}^{\infty} \nu\{A_k\} = \infty$$

is clear by the main theorem

$$\liminf_{n\to\infty} \frac{\sum\limits_{j,k=1}^{n} \nu(A_j A_k)}{\left(\sum\limits_{k=1}^{n} \nu(A_k)\right)^2} < c$$

the denominator is of order $\log^2 n$, the numerator will be decomposed as follows:

$$\sum_{j,k=1}^{n} \nu(A_j A_k) \le \sum_{\min(j,k)<\log n} \nu(A_j A_k) + \sum_{|j-k|<\log n} \nu(A_j A_k) + \sum_{j,k,|j-k|\ge \log n} \nu(A_j A_k).$$

The first sum can be estimated by $2\log n \sum_{k=1}^{n} m(A_k)$ which is of order $\log^2 n$. The same is true for the second term as well. Concerning the third one, by theorem 4.2 we know that the asymptotics of the summand is proportional to $\frac{1}{jk}$, so the sum is of order $\log^2 n$. Consequently, by Lamperti's lemma

$$\nu\{A_k \text{ i. o.}\} > \frac{1}{c}.$$

Since this event is invariant under the ergodic dynamics, it happens almost surely.

□

Finally it is interesting to note that, as observed by Simányi [**Sim 89**] the recurrence of the planar Lorentz process is equivalent to saying that the corresponding billiard in the whole plane (with an infinite invariant measure) is ergodic (see also [**Pen 00**]).

Appendix: Proof of sublemma

The only aim of this appendix to provide the proof of Sublemma.

Sublemma For the billiard on $\mathbb{T}(Z)$ there exists an $n \in \mathbb{Z}_+$ such that $\Lambda_0 \cap T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure.

Proof In order that our ideas be clear with a minimal knowledge of sections 7 and 8 of [**You 98**] we summarize some facts from this reference. First, let us note that often it is convenient to use the semi-metric p determined by the density $\cos\phi dr$. We will write $p(.)$ for the p-length of a curve, while $l(.)$ denotes its Euclidean length. Finally, as before, $d(.,.)$ denotes Euclidean distance. In particular, $\gamma^u_\delta(x)$ will denote that piece of a γ^u_{loc}-curve whose endpoints have p-distance δ from its 'center' x.

Facts:

(i) $\delta_1 > 0$ is a suitably small number, $\delta = \delta_1^4$ and $\alpha_1 = \alpha^{\frac{1}{4}}$.

(ii) The product set Λ has a sort of center $x_0 \in A_{\delta_0} = \{x \in M |\, \gamma^u_{3\delta_0}(x) exists\} \neq \emptyset$. Denote $\Omega = \gamma^u_{3\delta_0}(x_0)$. Moreover, let us fix a small, rectangular shaped neighbourhood U of x_0 such that $\Lambda \cap U$ itself is a product set with $\mu_1(\Lambda \cap U) > 0$.

(iii) For the product set Λ one has a simply connected, rectangular-shaped region $Q(x_0)$ such that $\partial Q(x_0)$ is made up of two u-curves and two s-curves. The two u-curves are roughly $2\delta_0$ in length and they are either from $\Gamma^u(x_0)$ or do not meet any element of $\Gamma^u(x_0)$. The two s-curves are approximately 2δ long and have the same properties wrt $\Gamma^s(x_0)$. $\hat{Q}(x_0)$ is a proper u-subrectangle of $Q(x_0)$, i. e. it shares the s-boundaries of $Q(x_0)$ and its u-boundaries, which must have the same properties as those of $Q(x_0)$, are strictly inside $Q(x_0)$.

(iv) Denote $\Omega_\infty = \{y \in \Omega |$ for $\forall\ n \geq 0\ \ d(T^n y, S) > \delta_1 \alpha^n\}$. There are unions of a finite number of closed connected curves ω such that $\Omega_n \supset \Omega_{n+1}$ and $\Omega = \cap_n \Omega_n$. In addition, if ω is a component of Ω_n, then $T^n\omega$ is a connected smooth curve with $d(T^n\omega, S) \geq \frac{1}{2}\delta_1\alpha^n$, and, in particular, $T^{n+1}\omega$ is also a connected smooth curve.

(v) If for a point x one has $R(x) = n$, then x belongs to an s-subrectangle Q_ω of $Q(x_0)$ (where ω is some component figuring in (iv)) such that $T^jQ_\omega \cap S = \emptyset$ for every $0 \leq j \leq n$. Also, $10\delta_0 \leq p(T^n\omega) \leq 20\delta_0$ and $T^n\omega$ u-crosses $\hat{Q}(x_0)$ with segments $2\delta_0$ in length sticking out on both sides.

(vi) Finally, for some $R_1 \geq R_0$ large enough it is true that if, for some $n \geq R_1$, a component ω of Ω_n u-crosses the middle half of Q under T^n , then the entire s-subrectangle of Q associated with ω u-crosses Q under T^n.

When now turning to the billiard on $\mathbb{T}(Z)$ we will extend our previous usage of notations: for instance, $x_0^{(0)}, \ldots, x_0^{(Z)}$ will denote the different copies of x_0, and similarly $U^{(0)}, \ldots, U^{(Z)}$ the different copies of U. $\mu_1(Z, .)$ will denote the invariant probability measure for our 'elongated' billiard system. We note that Young's construction uses powers of T which are multiple of some given natural number. Here, for simplicity, we take this number to be equal to one and use the ergodicity of T. However, for our billiard it is known that any power of T is also ergodic so our simplification is by no means a restriction.

In fact, claim (vi) is the main fact necessary for our purposes. Introduce the function

$$w(x) = \chi_{\{p(\gamma^u(x)) \geq 10\delta_0\}}(x)\chi_{\{x \in \Lambda^{(Z)} \cap U^{(Z)}\}}(x).$$

By ergodicity,

$$\frac{1}{n}\sum_{k=0}^{n-1}\int \chi_{\{x\in\Lambda^{(0)}\cap U^{(0)}\}}(x)w(T^k x)d\mu_1(Z,x) \to \mu(\Lambda^{(0)}\cap U^{(0)})\bar{w}$$

where $\bar{w} = \int w(x)d\mu_1(Z,x) > 0$. Therefore, for some $x \in \Lambda^{(0)} \cap U^{(0)}$ there exist arbitrarily large indices k such that $T^k x \in \Lambda^{(Z)} \cap U^{(Z)}$ and $p(\gamma^u(T^k x)) \geq 10\delta_0$. Since $x \in \Omega_\infty^{(0)} \subset \Omega_k^{(0)}$, by property (vi) we are done.

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