On the spectrum of semi-classical Witten-Laplacians and Schrödinger operators in large dimension

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Abstract

We investigate the low-lying spectrum of Witten-Laplacians on forms of arbitrary degree in the semi-classical limit and uniformly in the space dimension. We show that under suitable assumptions implying that the phase function has a unique minimum one obtains a number of bands of discrete eigenvalues at the bottom of the spectrum. Moreover we are able to count the number of eigenvalues in each band. We apply our results to certain sequences of Schrödinger operators having strictly convex potentials and show that some well-known results of semi-classical analysis hold also uniformly in the dimension.

Keywords: Semi-classical analysis, thermodynamic limit, maximum principles, Witten-Laplacian, Schrödinger operator.

1 Introduction and main results

In the last years the semi-classical analysis of Witten-Laplacians uniformly in the space dimension has attracted some attention in connection with models of classical statistical mechanics [3, 4, 5, 10, 11, 15, 22, 35, 36]. This is mainly due to an identity by Helffer and Sjöstrand [22] that represents truncated two-point

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correlation functions of continuous spin lattice models as matrix elements of the inverse of a Witten-Laplacian restricted to one-forms. Furthermore suitable Witten-Laplacians on zero-forms are unitarily equivalent to the generators of gradient Brownian motions whose stationary distributions are finite dimensional Gibbs measures. In these models the semi-classical parameter is interpreted as the temperature. The interest in uniform properties with respect to large dimensions is motivated by the fact that the physical behaviour of these models is observed in the thermodynamic limit. Relevant questions connected with Witten-Laplacians on zero-forms are for instance the existence of spectral gaps, Poincaré and logarithmic Sobolev inequalities. In fact they are related to the decay of correlations [5]. The Helffer-Sjöstrand representation allows one to derive sharp asymptotics of truncated two-point correlation functions. The crucial feature of the Witten-Laplacian on one-forms entering into the analysis of the correlation functions is the shape of its low-lying spectrum. For the first time this was made explicit in [35] where the leading asymptotics of correlations functions were computed. In this work the Hamilton function of the spin system, which replaces the Morse function in Witten's original paper [38], was assumed to be strictly convex. In a subsequent paper [3] this condition was relaxed allowing in general non-convex Hamilton functions having a unique non-degenerate minimum. The spectral information exploited in these papers is what we call the first band of eigenvalues and the gap above.

Here we observe that, combining the techniques of [3] with the Hodge theory of the Witten complex, we obtain a number n of bands at the bottom of the spectrum of Witten-Laplacians on forms of any degree supposing the semi-classical parameter $h > 0$ is small enough. The width and the separation of the bands is uniform with respect to the dimension, as is the upper bound for h which depends only on n. A similar picture of the spectrum of the Witten-Laplacians on zero and one forms was obtained at least implicitely in [36]. In this article Sjöstrand calculated the complete low-temperature asymptotics of two-point correlations in the thermodynamical limit. To this end he constructed approximate eigenforms of Witten-Laplacians on zero and one forms and used inductively higher order Grushin problems.

We remark that because of the one-well shape of the Hamilton function there are no phase transitions in the models considered in [3, 35, 36]. Rather the correlations decrease exponentially with the distance of the lattice sites. Using a more complicated deformation of the exterior derivative than Witten's deformation one can, however, also analyze correlation functions coming from Hamilton functions that may have several local minima but only one global minimum, for temperatures in the single-phase region below some possible critical value where phase transitions might occur [4]. See [13] for examples of phase transitions due to local minima of the Hamilton function.

Finally we note that in fixed dimensions there are of course numerous well-known results on low-lying eigenvalues of Witten-Laplacians. Originally Witten introduced his deformed complex on compact Riemannian manifolds, the deformation being induced by a Morse function [38]. Among other things he indicated that the semi-classical analysis of the Witten-Laplacian yields a proof of the Morse inequalities. Rigorously this was performed in the sequel in [29] (see also [9]) and also in [19] using WKB methods from [18]. An alternative presentation of Witten's proof can be found in [39]. Especially the techniques of [19] yield very precise asymptotics of the low-lying eigenvalues as h tends to zero. All these results are, however, obtained without any control with respect to the dimension.

One crucial point in Witten's proof is that the number of exponentially small w.r.t. $\frac{1}{h}$ eigenvalues of the Witten-Laplacian on k-forms, counted with multiplicity and including zero eigenvalues, is equal to the number of critical points of the Morse function with Morse index k. On the other hand the dimension of the kernel of the Witten-Laplacian on k-forms equals k^{th} Betti number of the manifold. In our considerations we observe that up to some h-dependent form degree l the Witten-Laplacian on k-forms, $0 \leqslant k \leqslant l$, is bounded from below by a constant times k. This is due to the fact that our functions $\Phi_{h,\Lambda}$ which replace the Hamilton resp. Morse functions and which we call phase functions have only one non-degenerate critical point, namely their minimum.

Our original aim when starting this work was to obtain estimates on the separation of low-lying eigenvalues measured relative to the ground state energy of Schrödinger operators in the semi-clssical limit for large dimensions. As is well-known a Schrödinger operator is apart from its simple ground state energy level identical to a suitable Witten-Laplacian on zero-forms. In order to construct this Witten-Laplacian one has to replace the Morse function in Witten's original deformation by minus the logarithm of the strictly positive ground state eigenfunction of the Schrödinger operator. Employing maximum principles as developped in [22, 33, 34], we are able to extract enough information on the latter in order to apply the techniques from [3]. The use of the maximum principles is the reason why we restrict our attention to a class of Schrödinger operators having strictly convex potentials which are in some sense small perturbations of harmonic oscillator potentials. As a special case of our results on more general Witten-Laplacians we hence obtain a band structure of the low-lying spectrum of Schrödinger operators in the semi-clssical limit uniformly in the space dimension. The number of eigenvalues in the n^{th} band equals the degeneracy of the n^{th} energy level of a simple isotropic harmonic oscillator. We do not get any information on the first eigenvalue by our methods.

For some classes of Schrödinger operators, for instance for the Kac model, Helffer and Sjöstrand prove that the thermodynamic limit, i.e. the limit as the dimension tends to infinity of the first eigenvalue divided by the dimension, exists and converges exponentially fast with respect to the dimension. Moreover they obtain complete semi-classical asymptotic expansions of the first eigenvalue and its thermodynamic limit. Furthermore Helffer and Sjöstrand give uniform estimates from below and above of the gap between the first and second eigenvalues for some Schrödinger operators with convex potentials. In the case of double-well potentials they show that the splitting between the first two eigenvalues is exponentially small with respect to the dimension for small values of the semi-classical parameter h supposing that the dimension is $\mathcal{O}(h^{-N})$, for some $N > 0$. These results are derived in [14, 20, 21, 30, 31, 32, 33]. A survey and further results and references are given in [14]. Finally we note that, employing classical infrared estimates from statistical physics, one can prove that the splitting between the first two eigenvalues in the double-well case vanishes for fixed $0 < h \ll 1$ as the dimension tends to infinity which indicates the existence of phase transitions in some models of quantum statistical mechanics [16].

1.1 The main result for the Witten-Laplacian

Let $\mathfrak{L}$ be some arbitrary family of finite sets. Furthermore let $\Phi_{h,\Lambda} \in C^\infty(\mathbb{R}^\Lambda)$, such that $\exp(-\frac{1}{h}\Phi_{h,\Lambda}) \in L^2(\mathbb{R}^\Lambda)$, for each $h \in (0,1]$ and $\Lambda \in \mathfrak{L}$. In the following we define and collect some basic properties of an associated family of Witten-Laplacians, $\Delta_{h,\Lambda}$, $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, acting in $\mathscr{H} \equiv \mathscr{H}(\Lambda) := L^2(\mathbb{R}^\Lambda) \otimes \bigwedge \mathbb{C}^\Lambda$. We are interested in the behaviour of the low-lying spectrum of $\Delta_{h,\Lambda}$, as the cardinality of $\Lambda \in \mathfrak{L}$ increases and for small values of the semi-classical parameter $h \in (0,1]$. Our first main result is formulated in Theorem 1.4 and is valid under Hypothesis 1 on $\Phi_{h,\Lambda}$ given below.

Let $\Omega_0^k \equiv \Omega_0^k(\Lambda) := C_0^\infty(\mathbb{R}^\Lambda) \otimes \bigwedge^{(k)}\mathbb{C}^\Lambda$, $0 \leqslant k \leqslant |\Lambda|$, $\Lambda \in \mathfrak{L}$, denote the space of compactly supported, smooth, complex-valued k-forms. The Witten-Laplacian $\Delta_{h,\Lambda}$ is just the Hodge-Laplacian associated to the complex

$$0 \longrightarrow \Omega_0^0 \xrightarrow{d_h^{(0)}} \Omega_0^1 \xrightarrow{d_h^{(1)}} \cdots \xrightarrow{d_h^{(|\Lambda|-1)}} \Omega_0^{|\Lambda|} \xrightarrow{d_h^{(|\Lambda|)}} 0\,.$$

Here d_h denotes Witten's deformation of the ordinary exterior derivative d,

$$d_h \;=\; h^{1/2}\exp(-\tfrac{1}{h}\,\Phi_{h,\Lambda}) \circ d \circ \exp(\tfrac{1}{h}\,\Phi_{h,\Lambda})\,,$$

and $d_h^{(k)}$ its restriction to Ω_0^k. We introduce the short-hand notation $\Phi_i' := \frac{\partial}{\partial x_i}\Phi_{h,\Lambda}$, $\Phi_{ij}'' := \frac{\partial^2}{\partial x_i \partial x_j}\Phi_{h,\Lambda}$ etc. and define

$$Z_i \;:=\; h^{1/2}\,\tfrac{\partial}{\partial x_i} \,+\, \tfrac{1}{h^{1/2}}\,\Phi_i' \;, \qquad Z_i^* \;:=\; -\,h^{1/2}\,\tfrac{\partial}{\partial x_i} \,+\, \tfrac{1}{h^{1/2}}\,\Phi_i' \;, \tag{1}$$

for each $i \in \Lambda$. Then the local expressions for d_h and its formal adjoint, δ_h, read

$$d_h \;=\; \sum_{i\in\Lambda} Z_i \otimes a_i^\dagger \;, \qquad \delta_h \;=\; \sum_{i\in\Lambda} Z_i^* \otimes a_i \;.$$

We are using the fermionic calculus. That is $a_i^\dagger$ denotes exterior multiplication with the i^{th} coordinate differential and a_i interior multiplication with the i^{th} basis vector field. They fulfill the canonical anti-commutation relations (CAR)

$$\{a_i\,,\,a_j^\dagger\} \;=\; \delta_{ij}\,\mathbb{1}\;, \qquad \{a_i\,,\,a_j\} \;=\; \{a_i^\dagger\,,\,a_j^\dagger\} \;=\; 0\;,$$

for $i, j \in \Lambda$. In contrast the operators Z_i and Z_i^* fulfill the following canonical commutation relations (CCR),

$$[\,Z_i, Z_j^*\,] \;=\; 2\,\Phi_{ij}''\;, \qquad [\,Z_i, Z_j\,] \;=\; [\,Z_i^*, Z_j^*\,] \;=\; 0\;,$$

for $i, j \in \Lambda$. Using the CAR and CCR as well as the complex property for d_h and δ_h, i.e. $d_h^2 = \delta_h^2 = 0$, one easily obtains the following representation for the Witten-Laplacian $\Delta_{h,\Lambda}$, which is for the present defined on $\Omega_0 := \bigoplus_{k=0}^{|\Lambda|} \Omega_0^k$,

$$\Delta_{h,\Lambda} \;:=\; (\,d_h + \delta_h\,)^2 \;=\; \Delta_{h,\Lambda}^{(0)} \otimes \mathbb{1} \,+\, 2\sum_{i,j\in\Lambda} \Phi_{ij}'' \otimes a_i^\dagger a_j \;. \tag{2}$$

Here $\Delta_{h,\Lambda}^{(0)}$ denotes the Witten-Laplacian restricted to zero-forms,

$$\Delta_{h,\Lambda}^{(0)} \;=\; \sum_{i\in\Lambda} Z_i^* Z_i \;=\; -h\,\Delta \,+\, \tfrac{1}{h}\,|\nabla\Phi_{h,\Lambda}|^2 \,-\, \Delta\Phi_{h,\Lambda}\;.$$

Accordingly $\Delta_{h,\Lambda}^{(k)}$ denotes the restriction of $\Delta_{h,\Lambda}$ to k-forms, for $0 \leqslant k \leqslant |\Lambda|$. Frequently we also use the notation

$$d\Gamma(\Phi_{h,\Lambda}''(x)) \;:=\; \sum_{i,j\in\Lambda} \Phi_{ij}''(x) \otimes a_i^\dagger a_j$$

for the second quantization of $\Phi_{h,\Lambda}''$ at $x \in \mathbb{R}^\Lambda$, and $d\Gamma^{(k)}(\Phi_{h,\Lambda}''(x))$ for its restriction to k-forms.

Since $\delta_h \subset d_h^*$ is densely defined d_h is closable. A limiting argument shows that we again have $\bar{d}_h \circ \bar{d}_h = 0$. Therefore the sequence

$$0 \xrightarrow{\bar{d}_h^{(-1)} \equiv 0} \mathscr{D}_0 \xrightarrow{\bar{d}_h^{(0)}} \mathscr{D}_1 \xrightarrow{\bar{d}_h^{(1)}} \cdots \xrightarrow{\bar{d}_h^{(|\Lambda|-1)}} \mathscr{D}_{|\Lambda|} \xrightarrow{\bar{d}_h^{(|\Lambda|)}} 0 \,, \tag{3}$$

where $\mathscr{D}_k := \mathcal{D}(\bar{d}_h^{(k)}) \subset \mathscr{H}^k$, for $0 \leqslant k \leqslant |\Lambda|$, defines a complex of closed operators between the Hilbert spaces

$$\mathscr{H}^k \;\equiv\; \mathscr{H}^k(\Lambda) \;:=\; L^2(\mathbb{R}^\Lambda) \otimes \bigwedge\nolimits^{(k)} \mathbb{C}^\Lambda \,, \qquad 0 \leqslant k \leqslant |\Lambda| \,.$$

Here, as well as below, the symbol $\mathcal{D}(T)$ denotes the domain of an operator T. The complex (3) is an example of a Hilbert complex in the sense of [7]. We also use the following abbreviations:

$$R_k \;:=\; \operatorname{Ran} \bar{d}_h^{(k-1)} \,, \qquad R_k^* \;:=\; \operatorname{Ran} d_h^{(k)*} \,, \qquad 0 \leqslant k \leqslant |\Lambda| \,.$$

Here we note that $d_h^{(|\Lambda|)*} \equiv 0$ In the proof of our main theorem the Hodge decomposition associated to the complex (3) will play a crucial rôle. We will formulate all the necessary ingredients in the next three propositions. For related discussions we refer to [2, 7, 23].

Proposition 1.1 *(i) Each power of $d_h + \delta_h$ is essentially selfadjoint on Ω_0. In particular the Witten-Laplacian $\Delta_{h,\Lambda}$ is essentially self-adjoint on Ω_0 and so are its restrictions $\Delta_{h,\Lambda}^{(k)}$ on Ω_0^k, for $0 \leqslant k \leqslant |\Lambda|$. The symbols $\Delta_{h,\Lambda}$ resp. $\Delta_{h,\Lambda}^{(k)}$ will in the following always denote the respective unique selfadjoint extensions acting in $\mathscr{H}$ resp. $\mathscr{H}^k$.*
(ii) We have $\mathcal{D}(\Delta_{h,\Lambda}) = \mathcal{D}(\bar{d}_h d_h^*) \cap \mathcal{D}(d_h^* \bar{d}_h)$ and

$$\Delta_{h,\Lambda} \;=\; \bar{d}_h d_h^* + d_h^* \bar{d}_h \;.$$

(iii) The following weak Hodge decompositions *are valid:*

$$\mathscr{H}^k \;=\; \ker \Delta_{h,\Lambda}^{(k)} \oplus \overline{R_k} \oplus \overline{R_k^*} \,, \qquad 1 \leqslant k \leqslant |\Lambda| \tag{4}$$

$$\mathscr{H}^0 \;=\; \ker \Delta_{h,\Lambda}^{(0)} \oplus \overline{R_0^*} \tag{5}$$

Proof: Part (i) is due to a well-known result by Chernoff [8].

In order to prove part (ii) we first recall that a theorem of von Neumann implies that both $\bar{d}_h d_h^*$ and $d_h^* \bar{d}_h$ are selfadjoint and in particular closed. Since we have $\langle\, \bar{d}_h d_h^* \phi \,,\, d_h^* \bar{d}_h \psi \,\rangle = 0$, for all $\phi, \psi \in \mathcal{D}(\bar{d}_h d_h^*) \cap \mathcal{D}(d_h^* \bar{d}_h)$, an easy argument shows that $\bar{d}_h d_h^* + d_h^* \bar{d}_h$ is closed. Since it is also symmetric and extends $(d_h + \delta_h)^2$, it is equal to $\Delta_{h,\Lambda}$ because of part (i).

Finally, we note that $\Delta_{h,\Lambda}^{(k)}$ is identical to the unique selfadjoint operator associated to the closed form $\mathsf{q}_k(\phi,\psi) := \langle\, \bar d_h^{(k)}\phi\,,\, \bar d_h^{(k)}\psi\,\rangle + \langle\, d_h^{(k-1)*}\phi\,,\, d_h^{(k-1)*}\psi\,\rangle$, for $\phi,\psi \in \mathcal{D}(\bar d_h) \cap \mathcal{D}(d_h^*)$. Here $d_h^{(-1)*} \equiv 0$. Part (iii) now follows from [2, Proposition A.1]. ∎

Proposition 1.2 *Let $l \in \{1,\ldots,|\Lambda|\}$ and suppose that $\Delta_{h,\Lambda}^{(k)}$ is strictly positive, for $1 \leqslant k \leqslant l$. Then we have:*
(i) For $1 \leqslant k \leqslant l$, the subspaces R_k and R_k^* are closed and the following Hodge decompositions *are valid:*

$$\begin{aligned} \mathscr{H}^0 &= \mathbb{C}\,\exp(-\tfrac{1}{h}\,\Phi_{h,\Lambda}) \oplus R_0^*\,, && (6)\\ \mathscr{H}^k &= R_k \oplus R_k^*\,, \qquad 1 \leqslant k \leqslant l\,. && (7) \end{aligned}$$

(ii) For $0 \leqslant k \leqslant l$, there exist unitary operators $U_k : R_k^* \to R_{k+1}$ having the property

$$\Delta_{h,\Lambda}^{(k)}\restriction_{R_k^*} = U_k^*\, \Delta_{h,\Lambda}^{(k+1)}\restriction_{R_{k+1}} U_k\,.$$

Consequently

$$\sigma(\Delta_{h,\Lambda}^{(k)}\restriction_{R_k^*}) = \sigma(\Delta_{h,\Lambda}^{(k+1)}\restriction_{R_{k+1}})\,, \qquad 0 \leqslant k \leqslant l\,.$$

Proof: We note that $\ker \Delta_{h,\Lambda}^{(0)} = \ker d_h^{(0)*}\bar d_h^{(0)} = \ker \bar d_h^{(0)} = \mathbb{C}\,\exp(-\frac{1}{h}\,\Phi_{h,\Lambda})$. The remaining assertions follow directly from [23, Theorem 1.3] together with the weak Hodge decompositions (4) and (5). ∎

From now on we assume that $\Phi_{h,\Lambda}$ fulfills Hypothesis 1 below.

Convention: *We call a constant uniform, if it neither depends on $\Lambda \in \mathfrak{L}$ nor $h \in (0,h_0]$, for some Λ-independent $h_0 \in (0,1]$.*

Hypothesis 1 *(i) For each $h \in (0,1]$ and $\Lambda \in \mathfrak{L}$, the phase function $\Phi_{h,\Lambda} \in C^\infty(\mathbb{R}^\Lambda)$ has some critical point $x_0 = x_0(h,\Lambda) \in \mathbb{R}^\Lambda$ such that*

$$|\,\Phi_{ij}''(x) - \Phi_{ij}''(x_0)\,| \leqslant \sum_{k\in\Lambda} b_{ijk}(h,\Lambda)\,|\Phi_k'(x)|\,, \tag{8}$$

for all $i,j \in \Lambda$ and $x \in \mathbb{R}^\Lambda$. Here, for $\Lambda \in \mathfrak{L}$ and $i,j,k \in \Lambda$, the $b_{ijk}(h,\Lambda)$ are non-negative numbers such that

$$\max_{i\in\Lambda} \sum_{k\in\Lambda} b_{iik}(h,\Lambda) \leqslant B\,, \qquad \max_{i\in\Lambda} \sum_{j,k\in\Lambda} b_{ijk}(h,\Lambda) \leqslant B'\,, \tag{9}$$

where $0 < B, B' < \infty$ are uniform constants.

(ii) Moreover we assume that there exist uniform constants $0 < \underline{m} \leqslant \overline{m} < \infty$ *such that*

$$\underline{m}\,\mathbb{1} \;\leqslant\; 2\,\Phi''_{h,\Lambda}(x_0) \;\leqslant\; \overline{m}\,\mathbb{1}\;, \tag{10}$$

for all $h \in (0,1]$ *and* $\Lambda \in \mathfrak{L}$.

This hypothesis is taken over from [3] in order to apply the estimation techniques developed there in our Section 2. It implies that $\Phi_{h,\Lambda}$ has only one critical point, namely its non-degenerate minimum at x_0, but is in general not convex [3].

Proposition 1.3 *Assume that Hypothesis 1 holds. Then, for each* $l \in \mathbb{N}$, *there is some* $h_0 \in (0,1]$ *such that* $\Delta^{(k)}_{h,\Lambda}$ *is strictly positive, for* $1 \leqslant k \leqslant l$ *and* $h \in (0, h_0]$.

PROOF: This is an immediate consequence of Lemma 2.1 below. ■

If the spaces R_k resp. R^*_k are closed, for $0 \leqslant k \leqslant l$, we will often refer to them as the subspaces of closed resp. non-closed k-forms. This is justified by the fact that in this situation the first l Betti numbers of our Hilbert complex vanish except for the zeroth which equals one. Since $\bar{d}_h$ and d^*_h both commute with $\Delta_{h,\Lambda}$ the subspaces of closed and non-closed k-forms are in fact spectral subspaces of $\Delta^{(k)}_{h,\Lambda}$.

Before we formulate the main result of this subsection we introduce some further abbreviating notation: If T is a selfadjoint operator in some Hilbert space and if $E_J(T)$ denotes its spectral projection corresponding to some interval J, we write

$$D_J(T) \;:=\; \dim \operatorname{Ran} E_J(T)\;. \tag{11}$$

Theorem 1.4 *Assume that Hypothesis 1 holds. Let* $n \in \mathbb{N}$ *and suppose that*

$$(n+1)\,\underline{m} \;-\; n\,\overline{m} \;=:\; \triangle_n \;>\; 0\;.$$

Then, for each $0 < \eta < \frac{2}{5}\triangle_n$, *there exists some* $h_0 = h_0(n,\eta) > 0$ *such that, for all* $h \in (0,h_0]$, $\Lambda \in \mathfrak{L}$, *and* $0 \leqslant k \leqslant \min\{n, |\Lambda|\}$, *the spectrum of* $\Delta^{(k)}_{h,\Lambda}$ *below* $(n+1)\,\underline{m} \,-\, \eta$ *is purely discrete and has a band structure: We set*

$$I^\eta_l \;:=\; [\,l\,\underline{m} - \eta, l\,\overline{m} + \eta\,]\;.$$

Then we have, for $k \leqslant l \leqslant n$,

$$D_{I^\eta_l}(\Delta^{(k)}_{h,\Lambda}) \;=\; \binom{|\Lambda| + l - k - 1}{l-k}\binom{|\Lambda|}{k}\;. \tag{12}$$

All eigenforms of $\Delta^{(k)}_{h,\Lambda}$ corresponding to the eigenvalues lying in I^η_k are closed, for $0 \leqslant k \leqslant n$. For $k+1 \leqslant l \leqslant \min\{|\Lambda|-1, n\}$, the number of eigenvalues among those counted in (12) corresponding to non-closed eigenforms equals

$$D_{I^\eta_l}(\Delta^{(k)}_{h,\Lambda}\restriction_{R^*_k}) \;=\; \sum_{\alpha=0}^{l-1-k} (-1)^{l-k-1-\alpha} \binom{|\Lambda|+\alpha-1}{\alpha} \binom{|\Lambda|}{l-\alpha} \,. \tag{13}$$

If $n \geqslant |\Lambda|$ and $|\Lambda| \leqslant l \leqslant n$, the last formula has to be replaced by

$$D_{I^\eta_l}(\Delta^{(k)}_{h,\Lambda}\restriction_{R^*_k}) \;=\; \sum_{\alpha=0}^{|\Lambda|-1-k} (-1)^{|\Lambda|-k-1-\alpha} \binom{|\Lambda|+\alpha+l-1}{\alpha+l-|\Lambda|} \binom{|\Lambda|}{|\Lambda|-\alpha} \,. \tag{14}$$

For $k=0$ the formulas (13) and (14) both simplify to

$$D_{I^\eta_l}(\Delta^{(0)}_{h,\Lambda}\restriction_{R^*_0}) \;=\; \binom{|\Lambda|+l-1}{l} \,.$$

Moreover, for $0 \leqslant kklg \min\{n, |\Lambda|\}$ there is no spectrum of $\Delta^{(k)}_{h,\Lambda}$ contained in the set (δ_{0k} is Kronecker's symbol)

$$\big(-\infty\,,\, k\,\underline{m} - (1-\delta_{0k})\,\eta\big) \cup \bigcup_{l=k}^{n} \big(\,l\,\overline{m} + \eta\,,\, (l+1)\,\underline{m} - \eta\,\big) \,.$$

We emphasize that the width and the separation of the bands of eigenvalues described in Theorem 1.4 is uniform in the dimension $|\Lambda|$. Note also that the number $\binom{|\Lambda|+l-1}{l}$ is just the degeneracy of the l^{th} energy level of an isotropic $|\Lambda|$-dimensional harmonic oscillator.

The first step in the proof of Theorem 1.4 is to show that, for small $h \in (0,1]$, the Witten-Laplacian $\Delta_{h,\Lambda}$ is a small perturbation of some simpler comparison operator $A_{h,\Lambda}$ in the quadratic form sense uniformly in $\Lambda \in \mathfrak{L}$. For $\Delta^{(1)}_{h,\Lambda}$, this is already done in [3]. The generalization to the full Witten-Laplacian is straight forward. The numbers $b_{ijk}(h,\Lambda)$ occuring in Hypothesis 1 are used to compensate for summations over Λ in the derivation of this relative bound. We present all this in detail in Section 2 below. In Section 3 we then prove Theorem 1.4 by combining the spectral information given by $A_{h,\Lambda}$ with the Hodge decomposition.

1.2 Application to Schrödinger operators with strictly convex potentials

Now we describe an application of Theorem 1.4 to the semi-classical analysis of Schrödinger operators $P_{h,\Lambda} := -h^2\Delta + V_\Lambda$, $h \in (0,1]$, acting in $\mathscr{H}^0$ uniformly in $\Lambda \in \mathfrak{L}$. The potentials $V_\Lambda \in C^\infty(\mathbb{R}^\Lambda)$ are small perturbations of harmonic oscillator potentials in a sense made explicite in Hypothesis 2 below. In particular they are strictly convex. More precisely there are probably Λ-dependent constants $0 < c < c' < \infty$, and $E \in \mathbb{R}$, such that $c\,x^2 \leqslant V_\Lambda - E \leqslant c'\,x^2$. So $P_{h,\Lambda}$ can be realized as a selfadjoint operator on $\mathcal{D}(P_{h,\Lambda}) = \mathcal{D}(\Delta) \cap \mathcal{D}(x^2)$. The space of test functions $C_0^\infty(\mathbb{R}^\Lambda)$ is a domain of essential selfadjointness [27, Theorem X.28]. Furthermore, since $V_\Lambda(x) \to \infty$, as $|x| \to \infty$, we know that each $P_{h,\Lambda}$ has purely discrete spetrum. This is for example a consequence of a well-known theorem of Persson [24] (see also [28, Theorem XIII.67]). The connection between this family of Schrödinger operators and the Witten-Laplacian considered in the last subsection is based on the following well-known observation. Under our hypothesis on V_Λ the usual Perron-Frobenius argument [28, Theorem XIII.47] implies that the lowest eigenvalue, $\mu_0(h,\Lambda)$, of $P_{h,\Lambda}$ is simple and that the corresponding eigenfunction, ϕ_0, can be chosen strictly positive. Hence it can be expressed as $\phi_0 = \exp(-\frac{1}{h}\,\Phi_{h,\Lambda})$. As suggested by our notation we use the phase $\Phi_{h,\Lambda}$ of the ground state eigenfunction to construct a Witten deformation as in the preceeding subsection. The relation of the resulting Witten-Laplacian to the Schrödinger operator under investigation is obtained through the eigenvalue equation for $\exp(-\frac{1}{h}\,\Phi_{h,\Lambda})$. Since the latter function is strictly positive we get

$$\begin{aligned} V_\Lambda - \mu_0(h,\Lambda) &= |\nabla\Phi_{h,\Lambda}|^2 - h\,\Delta\Phi_{h,\Lambda} \\ \Longrightarrow \qquad P_{h,\Lambda} - \mu_0(h,\Lambda) &= h\,\Delta^{(0)}_{h,\Lambda}\,. \end{aligned} \tag{15}$$

Consequently apart from the factor h the spectrum of $\Delta^{(0)}_{h,\Lambda}$ is identical to the spectrum of the "renormalized" Schrödinger operator $P_{h,\Lambda} - \mu_0(h,\Lambda)$. Therefore we would also obtain a band structure of the low-lying spectrum of $P_{h,\Lambda}$, for small values of $h \in (0,1]$, uniformly in $\Lambda \in \mathfrak{L}$, if we were able to verify Hypothesis 1, for the phase $\Phi_{h,\Lambda}$ of the ground state eigenfunction of $P_{h,\Lambda}$. This is done in Section 6 below. The numbers $b_{ijk}(h,\Lambda)$ occuring in Hypothesis 1 will essentially be given as products of exponentials of minus the distance between the points $i,j,k \in \Lambda$ in some metric we now introduce. For each $\Lambda \in \mathfrak{L}$, we assume that a metric $d_\Lambda : \Lambda \times \Lambda \to [0,\infty)$ is given such that

$$\max_{k\in\Lambda} \sum_{l\in\Lambda} e^{-d_\Lambda(k,l)} \;\leqslant\; C_d \;<\; \infty\,, \tag{16}$$

for some uniform constant C_d. This is the only condition on d_Λ which is otherwise arbitrary. In fact we only use the properties of a semi-metric when dealing with d_Λ in this article so that the distance between distinct points may also be zero.

The main example we have in mind is when the sets Λ are given as finite subsets of $\mathbb{Z}^\nu$ for some $\nu \in \mathbb{N}$ and the metrics are given as, e.g., the ℓ^1-distance on $\mathbb{Z}^\nu$ supplemented by some boundary conditions. For example we could chose periodic boundary conditions, i.e. succesively increasing cubes contained in $\mathbb{Z}^\nu$ could be interpreted as disrete tori. In this case, for different $\Lambda \Subset \mathbb{Z}^\nu$, the metrics d_Λ would actually give different distances for corresponding pairs of points.

For all $o \in \Lambda$, we denote the $\Lambda \times \Lambda$-diagonal matrix with entries $e^{-\varkappa\, d_\Lambda(o,j)}\,\delta_{ij}$, $i,j \in \Lambda$, $\varkappa \in \mathbb{R}$, by $\Theta^{\varkappa}_{\Lambda,o}$. Furthermore, for $p \in [1,\infty]$, let $\ell^p_{o,\varkappa} \equiv \ell^p_{o,\varkappa}(\Lambda)$ be the vector space $\mathbb{R}^\Lambda$ equipped with the norm

$$\| v \|_{\ell^p_{o,\varkappa}} \;:=\; \| \Theta^{\varkappa}_{\Lambda,o} v \|_{\ell^p(\Lambda)}\,, \qquad v \in \mathbb{R}^\Lambda\,.$$

We note that we preserve the notation $|\cdot|$ for the standard Euklidean norm on $\mathbb{R}^\Lambda$.

Using the notation introduced above we assume the following

Hypothesis 2 *For $\Lambda \in \mathfrak{L}$, the potential V_Λ can be expressed as*

$$V_\Lambda(x) \;=\; (D_\Lambda x)^2 \,+\, W_\Lambda(x)\,, \qquad x \in \mathbb{R}^\Lambda\,,$$

where D_Λ is some constant real diagonal $\Lambda \times \Lambda$-matrix and $W_\Lambda \in C^\infty(\mathbb{R}^\Lambda)$. There exist two uniform real numbers, $\lambda_{\min}, \lambda_{\max}$, such that

$$0 \;<\; \lambda_{\min} \;\leqslant\; D_\Lambda \;\leqslant\; \lambda_{\max} \;<\; \infty\,, \qquad \Lambda \in \mathfrak{L}\,.$$

Moreover, there is some uniform constant γ such that

$$\| W''_\Lambda(x) \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;=\; \| \Theta^{\varkappa}_{\Lambda,o}\, W''_\Lambda(x)\, \Theta^{-\varkappa}_{\Lambda,o} \|_{\mathscr{L}(\ell^\infty(\Lambda))} \;\leqslant\; 2\,\gamma \;\leqslant\; 2\,\lambda^2_{\min}\,, \tag{17}$$

for $\varkappa = 0,1,2$ and for all $\Lambda \in \mathfrak{L}$, $o \in \Lambda$, $x \in \mathbb{R}^\Lambda$. Additionally γ fulfills

$$C_\gamma \;:=\; \frac{\gamma}{\lambda_{\min} + \sqrt{\lambda^2_{\min} - \gamma}} \;<\; \frac{\lambda_{\min}}{C_d}\,. \tag{18}$$

Furthermore the third derivative of W_Λ obeys the following bound,

$$\| W^{(3)}_\Lambda(x) \|_{\mathscr{L}(\ell^\infty_{o,1},\, \mathscr{L}(\ell^\infty_{o,1}, \ell^\infty_{o,2}))} \;\leqslant\; \Gamma \;<\; \infty\,, \tag{19}$$

for each $\Lambda \in \mathfrak{L}$, $o \in \Lambda$, $x \in \mathbb{R}^\Lambda$, where Γ is some uniform constant.

We remark that (17) and (19) are implied by

$$\max_{k\in\Lambda} \sum_{l\in\Lambda} e^{2\,d_\Lambda(k,l)}\, |W''_{kl}(x)| \;\leqslant\; 2\,\gamma\,, \qquad \max_{j\in\Lambda} \sum_{k,l\in\Lambda} e^{d_\Lambda(j,k)+d_\Lambda(j,l)}\, |W^{(3)}_{jkl}(x)| \;\leqslant\; \Gamma\,,$$

for each $\Lambda \in \mathfrak{L}$, $x \in \mathbb{R}^\Lambda$, which is easily verified using the triangle inequality for d_Λ. Under Hypothesis 2 the potentials V_Λ model a system of weakly coupled harmonic oscillators. The conditions (17) and (19) allow for an unbounded w.r.t. Λ interaction range but impose an exponential decay on the interaction strength relative to $d_\Lambda(i,j)$, $i,j \in \Lambda$. Hypothesis 2 implies the following two theorems. Theorem 1.5 is due to [34], and Theorem 1.6 is obtained in [33] under the assumption $D_\Lambda = \mathbb{1}$.

Theorem 1.5 *Under Hypothesis 2 we have* $\Phi_{h,\Lambda}(x) = \frac{1}{2}\, x \cdot D_\Lambda x + \Psi_{h,\Lambda}(x)$ *where* $\Psi_{h,\Lambda}$ *obeys the bound*

$$\sup_{x \in \mathbb{R}^\Lambda} \| \, \Psi''_{h,\Lambda}(x) \, \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;\leqslant\; C_\gamma \;,$$

for $\varkappa = 0, 1, 2$, *and for all* $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, $o \in \Lambda$.

Theorem 1.6 *Under Hypothesis 2 there exists some uniform constant* $C^{(3)} > 0$ *such that*

$$\sup_{x \in \mathbb{R}^\Lambda} \| \Psi^{(3)}_{h,\Lambda}(x) \|_{\mathscr{L}(\ell^\infty_{o,1}, \mathscr{L}(\ell^\infty_{o,1}, \ell^\infty_{o,2}))} \;\leqslant\; C^{(3)} \;,$$

for all $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, *and* $o \in \Lambda$.

Theorems 1.5 and 1.6 are both derived by applying so-called maximum principles developped in [22, 33, 34]. To make this article self-contained we give detailed proofs of both theorems in Section 5. These proofs require an asymptotic analysis of the logarithm of the first eigenfunction of the operator $-\Delta + x^2 + t\, U$, where $t \in [0,1]$, and U is some compactly supported smooth function. This analysis has to be uniform in $t \in [0,1]$. It is carried out in [33] without taking the parameter t explicitly into account. Since one has to fill in some more details in that paper we felt that it might be worthwhile to work out everything carefully. This is done in Section 4.

Before stating the next theorem we remark further that in the present situation we know à priori that the associated Witten-Laplacian has a compact resolvent: As a consequence of Theorem 1.5 and (17) with $\varkappa = 0$, we obtain the estimate

$$0 \;<\; \lambda_{\min} \,-\, C_\gamma \;\leqslant\; \sup_{x \in \mathbb{R}^\Lambda} \| \, \Phi''_{h,\Lambda}(x) \, \|_{\mathscr{L}(\ell^\infty(\Lambda))} \;\leqslant\; \lambda_{\max} \,+\, C_\gamma \;. \tag{20}$$

Since $\Phi''_{h,\Lambda}(x)$ is symmetric we have the same bounds also in the norm on $\mathscr{L}(\ell^1(\Lambda))$, and, by the Riesz-Thorin interpolation theorem, also in the norm on $\mathscr{L}(\ell^2(\Lambda))$. For $1 \leqslant k \leqslant |\Lambda|$, the second quantization $d\Gamma^{(k)}(\Phi''_{h,\Lambda})$ thus defines a bounded operator on $\mathscr{H}^k$, satisfying $\|d\Gamma^{(k)}(\Phi''_{h,\Lambda})\| \;\leqslant\; k\,(\lambda_{\max} \,+\, C_\gamma)$. The representation (2) and the fact that $\Delta^{(0)}_{h,\Lambda} \otimes \mathbb{1}$ has a compact resolvent for trivial reasons imply that $\Delta_{h,\Lambda}$ is self-adjoint on $\mathcal{D}(P_{h,\Lambda}) \otimes \bigwedge \mathbb{C}^\Lambda$ and that

$$(\Delta_{h,\Lambda} + 1)^{-1} \;=\; (\Delta^{(0)}_{h,\Lambda} \otimes \mathbb{1} + 1)^{-1} - (\Delta^{(0)}_{h,\Lambda} \otimes \mathbb{1} + 1)^{-1}\, 2d\Gamma(\Phi''_{h,\Lambda})\, (\Delta_{h,\Lambda} + 1)^{-1}$$

is compact. We also note that the lower bound in (20) with the ℓ^∞-norm replaced by the ℓ^2-norm shows that $\Phi_{h,\Lambda}$ is strictly convex. Actually, this is already a direct consequence of Hypothesis 2 by a classical result of Brascamp and Lieb [6]. Consequently $\Delta^{(k)}_{h,\Lambda}$ is strictly positive, for $1 \leqslant k \leqslant |\Lambda|$, which is evident from (2). Therefore the Hodge decompositons (6) and (7) are valid without any restrictions on h.

Our main result on the family of Schrödinger operators $P_{h,\Lambda}$ is

Theorem 1.7 *Assume that Hypothesis 2 holds and set*

$$\underline{m} \;:=\; \lambda_{\min} - C_\gamma \,, \qquad \overline{m} \;:=\; \lambda_{\max} + C_\gamma \,.$$

Let $n \in \mathbb{N}$ and suppose that

$$(n+1)\,\underline{m} \;-\; n\,\overline{m} \;=:\; \triangle_n \;>\; 0\,.$$

Then, for each $0 < \eta < \frac{2}{5}\triangle_n$, there exists some $h_0 = h_0(n,\eta) > 0$ such that, for all $h \in (0, h_0]$, $\Lambda \in \mathfrak{L}$, and $0 \leqslant k \leqslant n$, we have

$$D_{\mu_0(h,\Lambda)+h\,[\,k\,\underline{m}\,-\,\eta\,,\,k\,\overline{m}\,+\,\eta\,]}(P_{h,\Lambda}) \;=\; \binom{|\Lambda| + k - 1}{k}\,.$$

Furthermore there is no spectrum of $P_{h,\Lambda}$ contained in the set

$$\mu_0(h,\Lambda) \,+\, h\Big\{\,(\,0\,,\,\underline{m}\,-\,\eta\,) \cup \bigcup_{k=1}^{n} \big(\,k\,\overline{m}\,+\,\eta\,,\,(k+1)\,\underline{m}\,-\,\eta\,\big)\,\Big\}\,.$$

Again we point out that the crucial fact in Theorem 1.7 is the uniformity in Λ, since, for fixed dimension, everything is well-known.

As already indicated above, the proof of Theorem 1.7 given in Section 6 consists of verifying the conditions of Hypothesis 1 and then applying Theorem 1.4 in view of (15). As ingredients we need Theorem 1.5 and 1.6.

Remark on perturbation theory.

We note that our methods also give Λ-independent results in the perturbative case. By this we mean the case where $h = 1$ is fixed and γ tends to zero in Hypothesis 2. Here it is essential that the unperturbed Schrödinger operator, $-\Delta + (D_\Lambda x)^2$, is a sum of non-interacting harmonic oscillators.

Theorem 1.8 *Let $n \in \mathbb{N}$ and suppose that*

$$(n+1)\,\lambda_{\max} \;-\; n\,\lambda_{\min} \;=:\; \triangle_n \;>\; 0\;, \qquad \eta \in (0\,,\, \tfrac{2}{5}\triangle_n)\;.$$

Then there exists some $\gamma_0 = \gamma_0(n,\eta) > 0$ such that, for all W_Λ such that Hypothesis 2 is fulfilled, for some $\gamma \in (0, \gamma_0]$,

$$D_{\mu_0(1,\Lambda)+[\,k\,\lambda_{\min}-\eta\,,\,k\,\lambda_{\max}+\eta\,]}(P_{1,\Lambda}) \;=\; \binom{|\Lambda|+k-1}{k}\,,$$
$$\sigma(P_{1,\Lambda}) \cap \Big\{\mu_0(1,\Lambda) \;+\; \big(\,k\,\lambda_{\max} + (1-\delta_{0k})\,\eta\;,\;(k+1)\,\lambda_{\min} - \eta\,\big)\Big\} \;=\; \varnothing\;,$$

holds true, for $0 \leqslant k \leqslant n$.

Proof: We construct the Witten-Laplacian $\Delta_{1,\Lambda}$ associated to $P_{1,\Lambda}$ and define a selfadjoint comparison operator $\mathscr{A}_\Lambda$ on $\mathcal{D}(P_{1,\Lambda}) \otimes \bigwedge \mathbb{C}^\Lambda$ by

$$\mathscr{A}_\Lambda \;:=\; \Delta^{(0)}_{1,\Lambda} \otimes \mathbb{1} \;+\; 2\,d\Gamma(D_\Lambda)\;.$$

From Theorem 1.5 we get $\Psi''_{1,\Lambda} \leqslant C_\gamma \mathbb{1} \leqslant \frac{C_\gamma}{\lambda_{\min}}\,D_\Lambda$. We recall that $C_\gamma = \mathcal{O}(\gamma)$, as $\gamma \to 0$. With this we readily derive the following quadratic form bound,

$$\pm\big(\Delta_{1,\Lambda} \;-\; \mathscr{A}_\Lambda\big) \;=\; \pm 2\,d\Gamma(\Psi''_{1,\Lambda}) \;\leqslant\; \tfrac{C_\gamma}{\lambda_{\min}}\,\mathscr{A}_\Lambda\;.$$

Using this form bound we can proceed as in the proofs of Corollary 2.2 resp. Theorem 1.4 given in Section 2 resp. Section 3 to get the assertion. ■

2 Uniform semi-classical asymptotics of $\Delta_{h,\Lambda}$

First we show that, in the quadratic form sense and for sufficiently small $h \in (0,1]$, the Witten-Laplacian $\Delta_{h,\Lambda}$ is a small perturbation of

$$A_{h,\Lambda} \;:=\; \Delta^{(0)}_{h,\Lambda} \otimes \mathbb{1} \;+\; 2\,d\Gamma(\Phi''_{h,\Lambda}(x_0))\;.$$

Since $d\Gamma(\Phi''_{h,\Lambda}(x_0))$ is bounded, the operator $A_{h,\Lambda}$ is apparently selfadjoint on $\mathcal{D}(\Delta^{(0)}_{h,\Lambda}) \otimes \bigwedge \mathbb{C}^\Lambda$ and essentially selfadjoint on Ω_0. The form bound derived in Lemma 2.1 enables us to compare the dimensions of spectral subspaces corresponding to $\Delta^{(k)}_{h,\Lambda}$ and $A^{(k)}_{h,\Lambda}$, the restriction of $A_{h,\Lambda}$ to $\mathscr{H}^k$, for small $h \in (0,1]$. This is done in a succeeding corollary.

In the following $N := d\Gamma(\mathbb{1}) = \sum_{i\in\Lambda} a_i^\dagger a_i$ denotes the number operator on the exterior algebra (or the fermionic Fock space) $\bigwedge \mathbb{C}^\Lambda$. The form domain of $\Delta_{h,\Lambda}$ is denoted by $\mathcal{Q}(\Delta_{h,\Lambda}) = \mathcal{D}(\bar{d}_h) \cap \mathcal{D}(d_h^*)$.

Lemma 2.1 ([3]) *Assume that Hypothesis 1 holds and let $h' \in (0,1]$ be strictly smaller than the uniform constant B appearing in (9). Then there exists some uniform constant $C = C(h') > 0$ such that, for all $h \in (0, h']$ and $\Lambda \in \mathfrak{L}$, we have*

$$\begin{aligned} \pm\big(\, \Delta_{h,\Lambda} - A_{h,\Lambda} \,\big) \;&=\; \pm 2\, d\Gamma(\Phi''_{h,\Lambda}(\cdot) - \Phi''_{h,\Lambda}(x_0)) \\ &\leqslant\; C\, h^{1/2} \left(\Delta^{(0)}_{h,\Lambda} \otimes N \,+\, 2\, d\Gamma(\Phi''_{h,\Lambda}(x_0)) \right) , \end{aligned} \tag{21}$$

on $\mathcal{Q}(\Delta_{h,\Lambda})$ in the sense of quadratic forms. In particular we have

$$\pm\big(\, \Delta^{(k)}_{h,\Lambda} - A^{(k)}_{h,\Lambda} \,\big) \;\leqslant\; C\, k\, h^{1/2}\, A^{(k)}_{h,\Lambda} \,, \qquad \textit{for } 1 \leqslant k \leqslant |\Lambda| \,. \tag{22}$$

Proof: We note that, as a domain of essential selfadjointness, Ω_0 is also a form core for both $\Delta_{h,\Lambda}$ and $A_{h,\Lambda}$. To begin with we have, for $v \in \Omega_0$,

$$\begin{aligned} &\Big| \Big\langle v \,, \sum_{i,j \in \Lambda} \big(\Phi''_{ij} - \Phi''_{ij}(x_0) \big) \otimes a_i^\dagger a_j\, v \Big\rangle \Big| \\ &\quad\leqslant\; \int\limits_{\mathbb{R}^\Lambda} \sum_{i,j \in \Lambda} \big| \Phi''_{ij}(x) - \Phi''_{ij}(x_0) \big|\, \| a_i v(x) \|\, \| a_j v(x) \|\, dx \\ &\quad\leqslant\; \int\limits_{\mathbb{R}^\Lambda} \sum_{i,j \in \Lambda} \big| \Phi''_{ij}(x) - \Phi''_{ij}(x_0) \big|\, \| a_i v(x) \|^2\, dx \,, \end{aligned} \tag{23}$$

by Cauchy-Schwarz. Now (8) and (9) imply, for each $i \in \Lambda$ and $x \in \mathbb{R}^\Lambda$,

$$\begin{aligned} \sum_{j \in \Lambda} \big| \Phi''_{ij}(x) - \Phi''_{ij}(x_0) \big| \;&\leqslant\; \sum_{j,k \in \Lambda} b_{ijk}(h,\Lambda)\, |\Phi'_k(x)| \\ &\leqslant\; \frac{1}{2} \Big(B'\, h^{1/2} \,+\, h^{-1/2} \sum_{j,k \in \Lambda} b_{ijk}(h,\Lambda)\, |\Phi'_k(x)|^2 \Big) \end{aligned} \tag{24}$$

Furthermore we have, for each $k \in \Lambda$, in the sense of quadratic forms in $L^2(\mathbb{R}^\Lambda)$,

$$\begin{aligned} |\Phi'_k(x)|^2 \;&\leqslant\; |\Phi'_k(x)|^2 \,-\, h\, \Phi''_{kk}(x) \,+\, h\, \Phi''_{kk}(x_0) \,+\, h\, \big|\Phi''_{kk}(x) - \Phi''_{kk}(x_0)\big| \\ &\leqslant\; h\, Z_k^* Z_k \,+\, h\, \Phi''_{kk}(x_0) \,+\, h \sum_{l \in \Lambda} b_{kkl}(h,\Lambda)\, |\Phi'_l(x)| \\ &\leqslant\; h\, Z_k^* Z_k \,+\, \frac{h\, \overline{m}}{2} \,+\, \frac{h}{4}\, B \,+\, h \sum_{l \in \Lambda} b_{kkl}(h,\Lambda)\, |\Phi'_l(x)|^2 \,, \end{aligned}$$

where we used the notation introduced in (1). This is equivalent to

$$\sum_{l \in \Lambda} \big(\, \delta_{kl} \,-\, h\, b_{kkl}(h,\Lambda) \,\big)\, |\Phi'_l(x)|^2 \;\leqslant\; h\, Z_k^* Z_k \,+\, h\, C' \,, \tag{25}$$

with $C' := \overline{m}/2 + B/4$. Since the $\Lambda \times \Lambda$-matrix Ξ with entries $\Xi_{kl} = b_{kkl}(h, \Lambda)$ is bounded by B in the norm of $\mathscr{L}(\ell^\infty(\Lambda))$ by (9), for $h \leqslant h' < B$, the inverse of the $\Lambda \times \Lambda$-matrix $\mathbb{1} - h\,\Xi$ is given by a $\mathscr{L}(\ell^\infty(\Lambda))$-norm-convergent Neumann series. Furthermore each entry of $(\mathbb{1} - h\,\Xi)^{-1}$ is strictly positive, so that we can apply $(\mathbb{1} - h\,\Xi)^{-1}$ formally to (25), giving

$$\begin{aligned} |\Phi'_k(x)|^2 \;&\leqslant\; \sum_{l \in \Lambda} (\mathbb{1} - h\,\Xi)^{-1}_{kl} \big(h\, Z_l^* Z_l \,+\, h\, C' \big) \\ &\leqslant\; \|\, (\mathbb{1} - h\,\Xi)^{-1} \,\|_{\mathscr{L}(\ell^\infty(\Lambda))} \Big(\sum_{l \in \Lambda} h\, Z_l^* Z_l \,+\, h\, C' \Big) \\ &\leqslant\; \frac{1}{1 - h\,B} \Big(h\, \Delta^{(0)}_{h,\Lambda} \,+\, h\, C' \Big) \,, \end{aligned} \tag{26}$$

for $h \in (0, h']$. Combining (23), (24) and (26), we arrive at

$$\begin{aligned} &\Big| \Big\langle v \,,\, \sum_{i,j \in \Lambda} \big(\Phi''_{ij} - \Phi''_{ij}(x_0) \big) \otimes a_i^\dagger a_j \, v \Big\rangle \Big| \\ &\leqslant\; \frac{B'}{2} \Big\langle v \,,\, \sum_{i \in \Lambda} \Big(h^{1/2}\, \mathbb{1} \,+\, \frac{h^{-1/2}}{1 - h\,B} \Big(h\, \Delta^{(0)}_{h,\Lambda} \,+\, h\, C' \Big) \otimes a_i^\dagger a_i \, v \Big\rangle \\ &\leqslant\; C'' \Big\langle v \,,\, \Big(h^{1/2}\, \Delta^{(0)}_{h,\Lambda} \otimes N \,+\, h^{1/2}\, \mathbb{1} \otimes N \Big) \, v \Big\rangle \,, \end{aligned}$$

for $v \in \Omega_0$, where $C'' > 0$ is uniform in $h \in (0, h']$ and $\Lambda \in \mathfrak{L}$. Now the claim follows since $\Phi''_{h,\Lambda}(x_0) \geqslant \underline{m}/2$, thus $\frac{2}{\underline{m}}\, d\Gamma(\Phi''_{h,\Lambda}(x_0)) \geqslant N$. ■

In the next corollary we again use the notation introduced in (11) and the symbol $\varrho(T)$ for the resolvent set of an operator T.

Corollary 2.2 *Assume that Hypothesis 1 holds and let $l \in \{1, \ldots, |\Lambda|\}$. Let $a, b \geqslant 0$ and $0 < \varepsilon < 1$ be such that $a + 2\,\varepsilon < b$,*

$$(a - \varepsilon, a + \varepsilon) \subset \varrho(A^{(l)}_{h,\Lambda}), \quad \textit{and} \quad (b - \varepsilon, b + \varepsilon) \subset \varrho(A^{(l)}_{h,\Lambda}) \,. \tag{27}$$

Furthermore let J denote one of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. Then there exists some $h_0 \in (0, 1]$ such that

$$D_J(\Delta^{(l)}_{h,\Lambda}) \;=\; D_J(A^{(l)}_{h,\Lambda}) \,, \tag{28}$$

for all $h \in (0, h_0]$ and $\Lambda \in \mathfrak{L}$. Moreover, if $k \in \{1, \ldots, |\Lambda|\}$ and if (27) holds, for all $1 \leqslant l \leqslant k$, then (28) holds simultaneously for all $1 \leqslant l \leqslant k$, $h \in (0, h_0]$ and $\Lambda \in \mathfrak{L}$.

Proof: Let $k \in \{1, \ldots, |\Lambda|\}$ and (27) be fulfilled, for all $1 \leqslant l \leqslant k$. We show that, for $1 \leqslant l \leqslant k$, and sufficiently small $h \in (0,1]$, we get $\|E_J(\Delta^{(l)}_{h,\Lambda}) - E_J(A^{(l)}_{h,\Lambda})\| < 1$. Clearly this implies (28).

First we pick some smooth function $\chi \in C_0^\infty([a-\varepsilon, b+\varepsilon], [0,1])$, such that $\chi \equiv 1$ on $[a+\varepsilon, b-\varepsilon]$, and

$$\big|\, \chi - \mathbb{1}_J \,\big|_\infty \;\leqslant\; \frac{1}{2}\,. \tag{29}$$

Furthermore we may assume that there is some J- and ε-independent $K > 0$ such that

$$\int\limits_{\mathbb{R}} |\partial^\nu \chi| \;\leqslant\; K\,\varepsilon^{-(\nu-1)}\,, \qquad \nu = 1,2,3\,. \tag{30}$$

Assumption (27) implies

$$E_J(A^{(l)}_{h,\Lambda}) \;=\; \chi(A^{(l)}_{h,\Lambda})\,, \qquad 1 \leqslant l \leqslant k\,. \tag{31}$$

Due to (29) and (31) we have the estimate

$$\|\, E_J(\Delta^{(l)}_{h,\Lambda}) - E_J(A^{(l)}_{h,\Lambda}) \,\| \;\leqslant\; \frac{1}{2} \;+\; \|\, \chi(\Delta^{(l)}_{h,\Lambda}) - \chi(A^{(l)}_{h,\Lambda}) \,\|\,, \tag{32}$$

for $1 \leqslant l \leqslant k$. So it suffices to show that the norm on the right side of the last equation is less than $\frac{1}{2}$, for small h. To this end we employ the following formula due to Amrein et al. [1, Theorem 6.1.4(b)] valid for any selfadjoint operator T in some Hilbert space and $\nu \in \mathbb{N}$:

$$\begin{aligned} \chi(T) \;=\; & \sum_{\varkappa=0}^{\nu-1} \frac{1}{\pi\varkappa!} \int\limits_{\mathbb{R}} \chi^{(\varkappa)}(\lambda)\, \Im\big[i^\varkappa (T-\lambda-i)^{-1}\big]\, d\lambda \\ & + \int\limits_0^1 \frac{t^{\nu-1}}{\pi(\nu-1)!} \int\limits_{\mathbb{R}} \chi^{(\nu)}(\lambda)\, \Im\big[i^\nu (T-\lambda-it)^{-1}\big]\, d\lambda\, dt\,. \end{aligned} \tag{33}$$

To make use of this formula we need the estimate

$$\begin{aligned} & \big\|\, (\Delta^{(l)}_{h,\Lambda} - \lambda \pm it)^{-1} (\Delta^{(l)}_{h,\Lambda} - A^{(l)}_{h,\Lambda}) (A^{(l)}_{h,\Lambda} - \lambda \pm it)^{-1} \,\big\| \\ \leqslant\; & \big\|\, (\Delta^{(l)}_{h,\Lambda} - \lambda \pm it)^{-1} (\Delta^{(l)}_{h,\Lambda} + 1)^{1/2} \,\big\| \\ & \cdot \big\|\, (\Delta^{(l)}_{h,\Lambda} + 1)^{-1/2} (A^{(l)}_{h,\Lambda} + 1)^{1/2} \,\big\| \\ & \cdot \big\|\, (A^{(l)}_{h,\Lambda} + 1)^{-1/2} (\Delta^{(l)}_{h,\Lambda} - A^{(l)}_{h,\Lambda}) (A^{(l)}_{h,\Lambda} + 1)^{-1/2} \,\big\| \\ & \cdot \big\|\, (A^{(l)}_{h,\Lambda} + 1)^{1/2} (A^{(l)}_{h,\Lambda} - \lambda \pm it)^{-1} \,\big\|\,, \end{aligned}$$

for $t > 0$. The norms in the second and fifth line of the last equation are less or equal to some uniform constant times $\sqrt{b+1}/t$, for $\lambda \in \operatorname{supp}\chi$. The norm in the fourth line is less or equal to $Ch^{1/2}l \leqslant Ch^{1/2}k$, for sufficiently small h, because of Lemma 2.1. In order to bound the norm in the third line we consider the polar decomposition $(\Delta^{(l)}_{h,\Lambda}+1)^{-1/2}(A^{(l)}_{h,\Lambda}+1)^{1/2} = UG^{1/2}$, where U is a partial isometry and

$$\begin{aligned} 0 \;\leqslant\; G \;:=&\; (A^{(l)}_{h,\Lambda}+1)^{1/2}\,(\Delta^{(l)}_{h,\Lambda}+1)^{-1}\,(A^{(l)}_{h,\Lambda}+1)^{1/2} \\ =&\; \Big(\mathbb{1} \,-\, (A^{(l)}_{h,\Lambda}+1)^{-1/2}\,(A^{(l)}_{h,\Lambda}-\Delta^{(l)}_{h,\Lambda})\,(A^{(l)}_{h,\Lambda}+1)^{-1/2}\Big)^{-1}\,. \end{aligned}$$

The norm of G is less or equal to $(1 - C\,k\,h^{1/2})^{-1}$, for $1 \leqslant l \leqslant k$, and for sufficiently small h satisfying $C\,k\,h^{1/2} < 1$. Collecting all estimates and using formula (33) with $\nu = 3$, the second resolvent equation and (30) we see that there is some $h_0 \in (0,1]$ such that

$$\|\,\chi(\Delta^{(l)}_{h,\Lambda}) - \chi(A^{(l)}_{h,\Lambda})\,\| \;\leqslant\; \text{const}\;\frac{h^{1/2}\,k\,(b+1)(b-a+1)}{\varepsilon^2}\,, \tag{34}$$

for $h \in (0,h_0]$ and $0 < \varepsilon < 1$. The constant appearing in (34) is uniform. In view of (32) this implies the assertion. ■

3 Proof of Theorem 1.4

We prove the theorem by induction on $n \in \mathbb{N}$. To this end we fix $\Lambda \in \mathfrak{L}$. We observe, however, that the restrictions on the semi-classical parameter $h \in (0,1]$ below are uniform in Λ. We start by proving the theorem for $n = 1$, then for $1 \leqslant n < |\Lambda|$, and finally for $n \geqslant |\Lambda|$. In all steps we use the following simple facts: Suppose that the spectrum of $\Delta^{(0)}_{h,\Lambda}$ below or equal to some $E > 0$ is purely discrete and consists of the eigenvalues $\mu_0 \leqslant \ldots \leqslant \mu_L$, for some $L \in \mathbb{N}_0$. Moreover suppose that $\Delta^{(0)}_{h,\Lambda}$ has no spectrum contained in (E,E'), where $E+\overline{m} < E'+\underline{m}$. Let ϕ_j be the eigenfunction corresponding to μ_j, for $0 \leqslant j \leqslant L$. We recall that $\mu_0 = 0$ is simple with corresponding eigenfunction $\phi_0 = \exp(-\frac{1}{h}\Phi_{h,\Lambda})$. Accordingly, let m_j denote the j^{th} eigenvalue (again counted with multiplicity) of $2\,\Phi''_{h,\Lambda}(x_0)$, and let v_j denote the corresponding eigenvector, for $1 \leqslant j \leqslant |\Lambda|$. By Hypothesis 1(ii) we know that $\underline{m} \leqslant m_j \leqslant \overline{m}$, for $1 \leqslant j \leqslant |\Lambda|$, uniformly in $h \in (0,1]$ and $\Lambda \in \mathfrak{L}$. Then the spectrum of

$$A^{(k)}_{h,\Lambda} \;=\; \Delta^{(0)}_{h,\Lambda} \otimes \mathbb{1} \;+\; 2\,d\Gamma^{(k)}(\Phi''_{h,\Lambda}(x_0))$$

strictly below $E' + \underline{m}$ is given by

$$\begin{aligned}
&\sigma(A^{(k)}_{h,\Lambda}) \cap (-\infty, E' + \underline{m}) \\
&= \Big\{ \mu_j + \sum_{\alpha=1}^{k} m_{i_\alpha} : 0 \leqslant j \leqslant L\,,\ 1 \leqslant i_1 < \cdots < i_k \leqslant |\Lambda| \Big\}\,.
\end{aligned}$$

The eigenvalue $\mu_j + \sum_{\alpha=1}^{k} m_{i_\alpha}$ corresponds to the eigenfunction $\phi_j \otimes v_{i_1} \wedge \cdots \wedge v_{i_k}$. In particular

$$A^{(k)}_{h,\Lambda} \ \geqslant\ k\,\underline{m}\,, \qquad 1 \leqslant k \leqslant |\Lambda|\,. \tag{35}$$

$\boxed{\textbf{(1)}\ n=1}$ Let $0 < \eta < \frac{2}{5}\left(2\,\underline{m} - \overline{m}\right)$. We set $\varepsilon := \eta/2$. Due to Propositions 1.2 and 1.3 there is some $h_0 \in (0,1]$ such that the Hodge decompositions

$$\mathscr{H}^0 \ =\ \mathbb{C}\,\exp(-\tfrac{1}{h}\Phi_{h,\Lambda}) \oplus R_0^*\,, \qquad \mathscr{H}^k \ =\ R_{k-1} \oplus R_k^*\,, \qquad k = 1,2\,, \tag{36}$$

and unitary equivalences

$$\Delta^{(k)}_{h,\Lambda}\restriction_{R_k^*} =\ U_k^*\,\Delta^{(k+1)}_{h,\Lambda}\restriction_{R_{k+1}}\,U_k\,, \qquad k = 0,1\,, \tag{37}$$

are valid, for all $h \in (0, h_0]$. First we observe that (35) together with Corollary 2.2 implies that there is a $h_1 \in (0, h_0]$ such that

$$\sigma(\Delta^{(1)}_{h,\Lambda}) \cap (-\infty, \underline{m} - \varepsilon\,] \ =\ \varnothing\,,$$

for all $h \in (0, h_1]$. Since (36) and (37) imply $\sigma(\Delta^{(0)}_{h,\Lambda}) \setminus \{0\} \subset \sigma(\Delta^{(1)}_{h,\Lambda})$, it follows that

$$\sigma(\Delta^{(0)}_{h,\Lambda}) \cap (-\infty, \underline{m} - \varepsilon\,] \ =\ \{0\}\,,$$

for all $h \in (0, h_1]$. By assumption we have $\overline{m} + 5\,\varepsilon < 2\,\underline{m}$. Hence $A^{(1)}_{h,\Lambda} = \Delta^{(0)}_{h,\Lambda} + 2\Phi''_{h,\Lambda}(x_0)$ has exactly $|\Lambda|$ eigenvalues in $[\,\underline{m}, \overline{m}\,]$, namely those of $2\,\Phi''_{h,\Lambda}(x_0)$. Besides, $A^{(1)}_{h,\Lambda}$ has empty spectrum in $(\overline{m}, 2\,\underline{m} - \varepsilon)$. Now Corollary 2.2 implies that there is a $0 < h_2 \leqslant h_1$, such that

$$\begin{aligned}
D_{[\,\underline{m}-\varepsilon, \overline{m}+\varepsilon\,]}(\Delta^{(1)}_{h,\Lambda}) \ &=\ D_{[\,\underline{m}-\varepsilon, \overline{m}+\varepsilon\,]}(A^{(1)}_{h,\Lambda}) \ =\ |\Lambda|\,, \\
D_{(\,\overline{m}+\varepsilon, 2\,\underline{m}-2\,\varepsilon\,)}(\Delta^{(1)}_{h,\Lambda}) \ &=\ D_{(\,\overline{m}+\varepsilon, 2\,\underline{m}-2\,\varepsilon\,)}(A^{(1)}_{h,\Lambda}) \ =\ 0\,,
\end{aligned} \tag{38}$$

for $h \in (0, h_2]$. It follows from $A^{(2)}_{h,\Lambda} \geqslant 2\,\underline{m}$ and Corollary 2.2 that

$$\sigma(\Delta^{(2)}_{h,\Lambda}) \cap (-\infty, 2\,\underline{m} - \varepsilon) \ =\ \varnothing\,,$$

for $h \in (0, h_3]$, where $0 < h_3 \leqslant h_2$. Since $\Delta^{(1)}_{h,\Lambda} \restriction_{R_1^*}$ is unitarily equivalent to $\Delta^{(2)}_{h,\Lambda} \restriction_{R_2}$ all the eigenforms of $\Delta^{(1)}_{h,\Lambda}$ corresponding to its eigenvalues in the interval $[\,\underline{m} - \varepsilon, \overline{m} + \varepsilon\,]$ are closed, for $h \in (0, h_3]$. Thus the unitary equivalence of $\Delta^{(0)}_{h,\Lambda} \restriction_{R_0^*}$ and $\Delta^{(1)}_{h,\Lambda} \restriction_{R_1}$ implies

$$\sigma(\Delta^{(0)}_{h,\Lambda}) \cap (-\infty, 2\,\underline{m}\, - 2\,\varepsilon\,) \;=\; \{0\} \cup \sigma(\Delta^{(1)}_{h,\Lambda}) \cap (-\infty, 2\,\underline{m}\, - 2\,\varepsilon\,)$$

with equal multiplicities of all eigenvalues, for $h \in (0, h_3]$. This together with (38) proves the theorem for $n = 1$.

$\boxed{\textbf{(2)}\ 1 \leqslant n < |\Lambda|}$ Suppose $n \in \{2, \ldots, |\Lambda| - 1\}$, and assume that the theorem is true for all $1 \leqslant k \leqslant n - 1$.

Let $0 < \eta' < \frac{2}{5}\big((n+1)\underline{m} - n\,\overline{m}\big)$ and set $\varepsilon := \eta'/2$. We pick some $h_0 \in (0, 1]$ such that the assertions of the theorem for $1 \leqslant k \leqslant n - 1$ hold with $\eta = \varepsilon$, and such that the decompositions

$$\mathscr{H}^0 \;=\; \mathbb{C}\, \exp(-\tfrac{1}{h}\Phi_{h,\Lambda}) \oplus R_0^* \;, \qquad \mathscr{H}^k \;=\; R_k \oplus R_k^* \;, \qquad 1 \leqslant k \leqslant (n+1) \;, \tag{39}$$

and unitary equivalences

$$\Delta^{(k)}_{h,\Lambda} \restriction_{R_k^*} = \; U_k^*\, \Delta^{(k+1)}_{h,\Lambda} \restriction_{R_{k+1}}\, U_k \;, \qquad 0 \leqslant k \leqslant n \;, \tag{40}$$

are valid, for all $h \in (0, h_0]$. Then we know that, for $1 \leqslant k \leqslant n$ and $h \in (0, h_0]$, $A^{(k)}_{h,\Lambda}$ has exactly $\binom{|\Lambda|+n-k-1}{n-k}\binom{|\Lambda|}{k}$ eigenvalues in the interval $[\,n\,\underline{m} - \varepsilon, n\,\overline{m} + \varepsilon\,]$, namely those corresponding to the eigenvectors

$$\phi_j \otimes v_{i_1} \wedge \cdots \wedge v_{i_k} \;,$$

where ϕ_j is some eigenvector of $\Delta^{(0)}_{h,\Lambda}$ with corresponding eigenvalue

$$\mu_j \in [\,(n-k)\,\underline{m} - \varepsilon\,,\,(n-k)\,\overline{m} + \varepsilon\,]\;.$$

Moreover,

$$\begin{aligned} \sigma(A^{(k)}_{h,\Lambda}) \cap \big(\,(n-1)\,\overline{m} + \varepsilon, n\,\underline{m} - \varepsilon\big) \;&=\; \varnothing\;, \\ \sigma(A^{(k)}_{h,\Lambda}) \cap \big(\,n\,\overline{m} + \varepsilon, (n+1)\,\underline{m} - \varepsilon\big) \;&=\; \varnothing\;. \end{aligned}$$

Since $(n+1)\underline{m} - n\,\overline{m} > 5\,\varepsilon$ we can apply Corollary 2.2 and find some $h_1 \in (0, h_0]$, such that, for all $h \in (0, h_1]$ and $1 \leqslant k \leqslant n$,

$$\begin{aligned} D_{[\,n\,\underline{m}-2\,\varepsilon\,,\,n\,\overline{m}+2\,\varepsilon\,]}(\Delta^{(k)}_{h,\Lambda}) \;&=\; \binom{|\Lambda| + n - k - 1}{n-k}\binom{|\Lambda|}{k}\;, \qquad (41) \\ D_{(\,(n-1)\,\overline{m}+2\,\varepsilon\,,\,n\,\underline{m}-2\,\varepsilon\,)}(\Delta^{(k)}_{h,\Lambda}) \;&=\; 0\;, \\ D_{(\,n\,\overline{m}+2\,\varepsilon\,,\,(n+1)\,\underline{m}-2\,\varepsilon\,)}(\Delta^{(k)}_{h,\Lambda}) \;&=\; 0\;. \end{aligned}$$

Moreover we know that $A_{h,\Lambda}^{(n+1)} \geqslant (n+1)\,\underline{m} > n\,\overline{m} + 5\,\varepsilon$, i.e. by Corollary 2.2 we find some $h_2 \in (0, h_1]$ such that

$$\sigma(\Delta_{h,\Lambda}^{(n+1)}) \cap (-\infty, (n+1)\,\underline{m} - \varepsilon\,] \;=\; \varnothing\;,$$

for $h \in (0, h_2]$. This together with (40) shows that all $\binom{|\Lambda|}{n}$ eigenvalues of $\Delta_{h,\Lambda}^{(n)}$ lying in $[\,n\,\underline{m} - 2\,\varepsilon, n\,\overline{m} + 2\,\varepsilon\,]$ correspond to closed eigenforms. The decompositions (39) and unitary equivalences (40) together with (41) now lead recursively to the formulas

$$D_{[\,n\,\underline{m}-2\,\varepsilon\,,\,n\,\overline{m}+2\,\varepsilon\,]}(\Delta_{h,\Lambda}^{(k)}\restriction_{R_k^*}) \;=\; \sum_{\alpha=0}^{n-1-k} (-1)^{n-k-1-\alpha}\binom{|\Lambda|+\alpha-1}{\alpha}\binom{|\Lambda|}{n-\alpha}\;,$$

for $0 \leqslant k < n$. These remarks together with (103) prove the theorem for $1 \leqslant n < |\Lambda|$.

$\boxed{\textbf{(3)}\ n \geqslant |\Lambda|}$ Suppose $n \geqslant |\Lambda|$ and assume that Theorem 1.7 holds for all $1 \leqslant k \leqslant n-1$. Let $0 < \eta' < \frac{2}{5}\big((n+1)\,\underline{m} - n\,\overline{m}\big)$. We set $\varepsilon := \eta'/2$ and pick some $h_0 \in (0,1]$ such that the assertions of the theorem for $1 \leqslant k \leqslant n-1$ hold with $\eta = \varepsilon$, and such that the decompositions

$$\mathscr{H}^0 \;=\; \mathbb{C}\,\exp(-\tfrac{1}{h}\Phi_{h,\Lambda}) \oplus R_0^*\;, \qquad \mathscr{H}^k \;=\; R_k \oplus R_k^*\;, \qquad 1 \leqslant k \leqslant |\Lambda|\;,$$

and unitary equivalences

$$\Delta_{h,\Lambda}^{(k)}\restriction_{R_k^*} \;=\; U_k^*\,\Delta_{h,\Lambda}^{(k+1)}\restriction_{R_{k+1}}\;U_k\;, \qquad 1 \leqslant k \leqslant |\Lambda|\;,$$

are valid, for all $h \in (0, h_0]$. Exactly as in the preceeding step we see that we can find some $h_1 \in (0, h_0]$, such that, for all $h \in (0, h_1]$ and $1 \leqslant k \leqslant n$, there are precisely $\binom{|\Lambda|+n-k-1}{n-k}\binom{|\Lambda|}{k}$ eigenvalues of $\Delta_{h,\Lambda}^{(k)}$ in the interval $[\,n\,\underline{m}-2\,\varepsilon, n\,\overline{m}+2\,\varepsilon\,]$ and no eigenvalues in $\big(\,(n-1)\,\overline{m}+2\,\varepsilon, n\,\underline{m}-2\,\varepsilon\big) \cup \big(\,n\,\overline{m}+2\,\varepsilon, (n+1)\,\underline{m}-2\,\varepsilon\big)$. Trivially all eigenforms of $\Delta_{h,\Lambda}^{(|\Lambda|)}$ are closed so that we now obtain the formulas

$$\begin{aligned} &D_{[\,n\,\underline{m}-2\,\varepsilon\,,\,n\,\overline{m}+2\,\varepsilon\,]}(\Delta_{h,\Lambda}^{(k)}\restriction_{R_k^*}) \\ &\quad= \sum_{\alpha=0}^{|\Lambda|-1-k} (-1)^{|\Lambda|-k-1-\alpha}\binom{|\Lambda|+\alpha+(n-|\Lambda|)-1}{\alpha+(n-|\Lambda|)}\binom{|\Lambda|}{|\Lambda|-\alpha}\;, \end{aligned}$$

for $0 \leqslant k \leqslant |\Lambda|$. Together with (104) this finishes the proof of Theorem 1.4. ■

4 Harmonic oscillators perturbed by compactly supported potentials

In this section we work out [33, Section 1]. In particular we verify that everything works uniformly in the parameter t introduced below and give a proof of Lemma 4.6 which was left to the reader in [33]. Our goal is to obtain the following proposition. Its proof is carried out throughout the whole section.

Proposition 4.1 *Let $n \in \mathbb{N}$ and $U \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let μ_t denote the first eigenvalue and $\exp(-\Phi_t)$ the first normalized strictly positive eigenfunction of the unique selfadjoint extension of $-\Delta + x^2 + t\,U$ starting from $C_0^\infty(\mathbb{R}^n)$, for $t \in [0,1]$. Then it follows that $\Phi_t(x) = x^2/2 + \Psi_t(x)$, where, for each multi-index $\beta \in \mathbb{N}_0^n$, we have*

$$\partial^\beta \Psi_t(x) \;\longrightarrow\; 0\,, \qquad \textit{as } |x| \to \infty\,,$$

uniformly in $t \in [0,1]$.

We start with a well-known result, which we state in a slightly more general form for later reference:

Let $\{V_s\}_{s\geqslant 0}$ be a family of smooth potential defined on $\mathbb{R}^n$ such that

$$\lim_{|x|\to\infty} V_s(x) \quad = \quad \infty\,, \qquad \text{for } s \geqslant 0\,, \tag{42}$$

$$\text{and} \qquad \sup_{x\in K} |\,\partial^\beta (V_s - V)\,| \quad\longrightarrow\quad 0\,, \qquad \text{as } s \to s_0\,, \tag{43}$$

for each compact set $K \Subset \mathbb{R}^n$ and each multi-index $\beta \in \mathbb{N}_0^n$. We consider the family of Schrödinger operators P_s obtained as the closures of $-\Delta + V_s$, which are defined and essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$, for $s \geqslant 0$. Due to (42) each P_s has purely discrete spectrum. We assume furthermore that P_s converges to P_0 in the norm resolvent sense, as $s \to 0$. In the following lemma we study uniform convergence properties of the unique normalized strictly positive ground state eigenfunction of P_s, which we denote by u_s, for $s \geqslant 0$. The corresponding lowest eigenvalue of P_s is denoted by e_s, for $s \geqslant 0$. Finally, E_s stands for the orthogonal projection onto the space spanned by u_s, for $s \geqslant 0$.

Lemma 4.2 *We have*

$$\sup_{x\in K} |\,\partial^\beta (u_s - u_0)\,| \quad\longrightarrow\quad 0\,, \qquad \textit{as } s \to 0\,, \tag{44}$$

$$\sup_{x\in K} |\,\partial^\beta (\ln u_s - \ln u_0)\,| \quad\longrightarrow\quad 0\,, \qquad \textit{as } s \to 0\,, \tag{45}$$

for each compact set $K \Subset \mathbb{R}^n$ and each multi-index $\beta \in \mathbb{N}_0^n$.

PROOF: By norm resolvent convergence we get $e_s \to e_0$, and $\| E_0 - E_s \| \to 0$, as $s \to 0$ [26, Theorem VIII.23]. Since u_s is strictly positive and normalized, for $s \geqslant 0$, this implies

$$1 \geqslant \langle u_s , u_0 \rangle = \| \langle u_s , u_0 \rangle u_0 \| = \| E_0 u_s \| \geqslant 1 - \| E_0 - E_s \| \to 1 ,$$

as $s \to 0$. Hence $\| u_s - u_0 \|^2 = 2 - 2 \langle u_s , u_0 \rangle \to 0$, as $s \to 0$. Assertion (44) follows from these remarks and (42) by considering the identity

$$\Delta(u_s - u_0) = (e_0 - V_0)(u_0 - u_s) + (e_0 - e_s) u_s + (V_s - V_0) u_s ,$$

using recursively the regularity theorem for the Laplacian and applying the Sobolev imbedding theorem as in [17, §3]. Furthermore, we know that u_0^{-1} is bounded on each compact set $K \Subset \mathbb{R}^n$. So the family $(u_s^{-1})_{s_1 \geqslant s \geqslant 0}$ is uniformly bounded on K, for some sufficiently small $s_1 > 0$, because of (44) with $\beta = 0$. Therefore assertion (45) follows easily from (44). ■

Now we come back to the proof of Proposition 4.1.

First we note that μ_t is continuous in t on $[0, 1]$. Hence it is possible to pick some $\rho_0 > \sup_{t \in [0,1]} \sqrt{\mu_t}$ such that $\operatorname{supp} U \Subset \{|x| < \rho_0\}$. We define $S := \{|x| = \rho_0\}$ and $\Omega := \{|x| \geqslant \rho_0\}$. Then $\exp(-\Phi_t)$ is a solution of the Dirichlet problem

$$\begin{aligned} (-\Delta + x^2 - \mu_t) u &= 0 \quad \text{on } \overset{\circ}{\Omega} , \\ u &\equiv \exp(-\Phi_t)\restriction_S \quad \text{on } S . \end{aligned} \tag{46}$$

Therefore we will analyze the homogenous Dirichlet problem with inhomogenous boundary conditions

$$\begin{aligned} (-\Delta + x^2 - \mu_t) u &= 0 \quad \text{on } \overset{\circ}{\Omega} , \\ u &\equiv \tilde{v}_t \quad \text{on } S , \end{aligned} \tag{47}$$

where $\tilde{v}_t \in C^\infty(S)$. We are only interested in square integrable solutions of (47). First we transform (47) into an inhomogenous Dirichlet problem with homogenous boundary conditions. To this end we extend $\tilde{v}_t$ to a function $\hat{v}_t \in C^\infty(\Omega)$ such that $\hat{v}_t \equiv 0$ on $\{|x| \geqslant r_0\}$, for some $r_0 > \rho_0$. For example, we may set $\hat{v}_t(x) = \tilde{v}_t(x/|x|)\, \chi(|x|)$, for some appropriate smooth cut-off function χ. Then we define $v_t := -(-\Delta + x^2 - \mu_t)\hat{v}_t$, so that $v_t \in C^\infty(\Omega) \cap L^2(\Omega)$. We denote by P the Dirichlet realization of $-\Delta + x^2$ on Ω. This is identical to the Friedrichs extension of $-\Delta + x^2$ starting from $C_0^\infty(\overset{\circ}{\Omega})$ and its domain $\mathcal{D}(P)$ is contained in $\overset{\circ}{H}{}^1(\Omega)$, the closure of $C_0^\infty(\overset{\circ}{\Omega})$ with respect to the norm on $H^1(\mathbb{R}^n)$

(see, e.g., [24]). Its spectrum is purely discrete and contained in $[\rho_0, \infty)$ [24]. Therefore $P - \mu_t$ is continuously invertible and $w_t := (P - \mu_t)^{-1} v \in \overset{\circ}{H}{}^1(\Omega)$. We readily verify that the standard elliptic regularity theorems (e.g. in the form presented in [25, §5.9]) apply and show that $w_t \in C^\infty(\Omega)$, $w_t \equiv 0$ on S. In summary we see that w_t solves the problem

$$\begin{aligned} (-\Delta + x^2 - \mu_t)\, w_t &= v_t \quad \text{on } \overset{\circ}{\Omega}\,, \\ w_t &\equiv 0 \quad \text{on } S\,. \end{aligned}$$

By construction $u_t := w_t + \hat{v}_t \in C^\infty(\Omega)$ is then a solution of the original problem (47). Using weighted L^2-estimates, regularity theorems for the Laplace operator, and the Sobolev imbedding theorem as in [12, §6] and [17, §3] we verify that u_t decays exponentially together with all its derivatives:

Lemma 4.3 *Let* $\alpha := \min_{t\in[0,1]} \sqrt{(\rho_0^2 - \mu_t)/2}$ *and suppose that* $\sup_{t\in[0,1]} \max_{y\in S} |\tilde{v}_t(y)| < \infty$. *Then, for each multi-index* $\beta \in \mathbb{N}_0^n$, *and* $\varepsilon > 0$, *there exists some* t-*independent constant* $c_{\beta,\varepsilon} > 0$ *such that*

$$\sup_{t\in[0,1]} |\partial_x^\beta u_t(x)| \leqslant c_{\beta,\varepsilon}\, e^{-(\alpha-\varepsilon)\,|x|}\,, \qquad x \in \Omega\,.$$

This implies

Lemma 4.4 *(i) Any square-integrable solution* $\tilde{u}_t$ *of (47) is identical to* $u_t := w_t + \hat{v}_t$. *In particular the solution* u_t *of (47) does not depend on the choice of* $\hat{v}_t$.
(ii) We have $\min\{\,0\,, \min_S \tilde{v}_t\,\} \leqslant u_t \leqslant \max\{\,0\,, \max_S \tilde{v}_t\,\}$. *In particular* $\tilde{v}_t \geqslant 0$ *implies* $u_t \geqslant 0$.

Proof: (i) We have $(-\Delta + x^2 - \mu_t)(u_t - \tilde{u}_t) = 0$ on $\overset{\circ}{\Omega}$ and $u_t - \tilde{u}_t \equiv 0$ on S. Furthermore we find, for each $\varepsilon > 0$, some $R > \rho_0$ such that $|u_t|, |\tilde{u}_t| \leqslant \varepsilon$ on $\{|x| \geqslant R\}$. The classical maximum principle (see, e.g. [25, §5.11. Corollary 7]) implies that $|u_t - \tilde{u}_t| \leqslant 2\varepsilon$ on $\{\rho_0 \leqslant |x| \leqslant R\}$. Since $\varepsilon > 0$ is arbitrarily small and R arbitrarily large this gives the assertion
(ii) For each $\varepsilon > 0$, we find some $R > 0$ such that $|u_t| \leqslant \varepsilon$ on $\{|x| \geqslant R\}$. The maximum principle again implies $\min\{-\varepsilon, \min_S \tilde{v}_t\} \leqslant u_t \leqslant \max\{\varepsilon, \max_S \tilde{v}_t\}$ on $\{|x| \geqslant R\}$. ■

Direct calculation shows that if $K(\tilde{v}_t) := u_t$ is a solution of (47) with boundary condition $\tilde{v}_t$ then $u_t \circ R = K(\tilde{v}_t \circ R)$, for any rotation matrix $R \in \mathrm{SO}(n, \mathbb{R})$.

Denoting the expressions for u_t and $\tilde v_t$ in polar coordinates $x = (r, \theta)$ again by the same symbols this implies $XK(\tilde v_t) = K(X\tilde v_t)$, for each vector field X on S^{n-1} that is the infinitesimal generator of a rotation. In particular

$$\partial_\theta^\alpha K(\tilde v_t) \;=\; K(\partial_\theta^\alpha \tilde v_t)\ , \quad \text{for each multi-index } \alpha\ . \tag{48}$$

Now we focus our attention on radial symmetric solutions. That is we suppose that $\tilde v_t \equiv 1$ in (47). Due to (48) the corresponding solution u_t° depends only on the absolut value of $x \in \Omega$ and can therefore be written in the form $u_t^\circ(x) = u_t^\circ(|x|)$, for $x \in \Omega$. After passing to polar coordinates the radial Dirichlet problem reads

$$\Big(-\partial_r^2 - \frac{n-1}{r}\,\partial_r + r^2 - \mu_t \Big)\, u^\circ \;=\; 0 \quad \text{on } [\rho_0, \infty)\ , \qquad u^\circ(\rho_0) \;=\; 1\ , \tag{49}$$

and we consider solutions which are in $L^2([\rho_0,\infty), r^{n-1}dr)$. A further substitution $f(r) := r^{\frac{n-1}{2}} u^\circ(r)$, for $r \in [\rho_0, \infty)$, turns the Dirichlet problem above into

$$\Big(-\partial_r^2 - \frac{(n-1)(n-3)}{4r^2} + r^2 - \mu_t \Big)\, f \;=\; 0 \quad \text{on } [\rho_0, \infty)\ , \quad f(\rho_0) \;=\; \rho_0^{\frac{n-1}{2}}\ , \tag{50}$$

where $f \in L^2([\rho_0, \infty), dr)$. In the following we will construct an asymptotic solution of (50). More precisely we have

Lemma 4.5 *For $t \in [0,1]$, there is a smooth function $\varphi_t : [\rho_0, \infty) \to \mathbb{R}$, such that*

$$\Big(-\partial_r^2 - \frac{(n-1)(n-3)}{4r^2} + r^2 - \mu_t \Big) \exp(-\varphi_t) \;=\; \widetilde R_t \exp(-\varphi_t)\ , \tag{51}$$

where, for each $\beta \in \mathbb{N}_0$,

$$\partial_r^\beta \widetilde R_t \;\sim\; 0 \qquad \textit{as} \qquad r \to \infty\ , \ \textit{uniformly in}\, t\ (u.i.t)\ . \tag{52}$$

There is a sequence of continuous functions $(a_\nu)_{\nu\in\mathbb{N}_0}$, $a_\nu \in C([0,1],\mathbb{R})$, for $\nu \in \mathbb{N}_0$, such that, for each $\beta \in \mathbb{N}_0$,

$$\partial_r^{\beta+1}\varphi_t(r) \;\sim\; r\,\delta_{\beta,0} + \sum_{\nu=0}^{\infty} a_\nu(t)\,(-1)^\beta\,\frac{(\beta+2\nu)!}{(2\nu)!}\, r^{-1-2\nu-\beta}\ , \quad (r \to \infty,\ u.i.t)\ . \tag{53}$$

Equation (52) means that, for all $\beta \in \mathbb{N}_0$ and $N \in \mathbb{N}$, there exist two t-independent constants $C_{\beta,N}, R_{\beta,N} > 0$, such that

$$\big|\, \partial_r^\beta \widetilde R_t(r) \,\big| \;\leqslant\; \frac{C_{\beta,N}}{r^N}\ , \qquad r \geqslant R_{\beta,N},\ t \in [0,1]\ .$$

In particular,

$$\forall\,\beta \in \mathbb{N}_0 : \ \ \partial_r^\beta \widetilde{R}_t(r) \ \longrightarrow\ 0\ , \qquad \text{as } r \to \infty, \text{ u.i.}\, t\ . \tag{54}$$

Correspondingly (53) says that, for all $\beta \in \mathbb{N}_0$ and $N \in \mathbb{N}$, there are two t-independent constants $D_{\beta,N}, S_{\beta,N} > 0$, such that

$$\Big|\, \partial_r^{\beta+1}\varphi_t(r) \ -\ r\,\delta_{\beta,0} \ -\ \sum_{\nu=0}^{N} a_\nu(t)\,(-1)^\beta\,\frac{(\beta+2\nu)!}{(2\nu)!}\, r^{-1-2\nu-\beta}\,\Big| \ \leqslant\ \frac{D_{\beta,N}}{r^{2N+3+\beta}}\ ,$$

for all $r \geqslant S_{\beta,N}$ and $t \in [0,1]$. In particular

$$\begin{aligned} \varphi_t'(r) - r \ &=\ \mathcal{O}(\tfrac{1}{r})\ , \qquad\qquad \varphi_t''(r) - 1 \ =\ \mathcal{O}(\tfrac{1}{r^2})\ , \\ \partial^\beta \varphi_t(r) \ &=\ \mathcal{O}(\tfrac{1}{r^{\beta+1}})\ , \qquad \beta \geqslant 3\ , \end{aligned} \tag{55}$$

where the $\mathcal{O}$-symbols are all unifom in $t \in [0,1]$.

Proof of Lemma 4.5 (Sketch): First we plug the formal power series $\rho_t(r) = r + \sum\limits_{\nu=0}^{\infty} a_\nu(t)\, r^{-1-2\nu}$ into the Riccati equation

$$\rho_t' \ -\ \rho_t^2 \ -\ \frac{(n-1)(n-3)}{4r^2} \ +\ r^2 \ -\ \mu_t \ =\ 0$$

and convince ourselfs that the coefficients $a_\nu(t)$ are determined by recursive formulas. Moreover each coefficient $a_\nu(t)$ is a polynomial in μ_t and therefore continuous in $t \in [0,1]$. Now we carry through a Borel procedure: We pick some $\chi \in C^\infty(\mathbb{R},[0,1])$ such that $\chi \equiv 0$ on $(-\infty,1]$ and $\chi \equiv 1$ on $[2,\infty)$ and define

$$\varphi_t'(r) \ =\ r \ +\ \sum_{\nu=0}^{\infty} a_\nu(t)\,\chi_{\nu,t}(r)\, r^{-1-2\nu}\ , \quad \varphi_t(r) \ =\ \int_{\rho_0}^{r} \varphi_t'(s)\,ds\ ,$$

for $r \in [\rho_0,\infty)$, where $\chi_{\nu,t}(r) := \chi(r\cdot 2^{-A_\nu(t)})$ and $A_\nu(t) := 1+2\nu+\max\limits_{0\leqslant\varkappa\leqslant\nu} |a_\varkappa(t)|$. By construction we have

$$\big|\, r^N\, a_\nu(t)\,\chi_{\nu,t}(r)\, r^{-1-2\nu}\,\big| \ \leqslant\ \frac{|a_\nu(t)|}{2^{(1+2\nu-N)\,A_\nu(t)}} \qquad (\nu \in \mathbb{N}_0)\ .$$

Similar estimates show that all involved series converge geometrically. Besides, the supports of $\partial^\beta\chi_{\nu,t}$ are disjoint for different ν and $\beta \in \mathbb{N}$. Using these remarks and the continuity of $t \mapsto A_\nu(t)$ on $[0,1]$, we verify (51) with

$$\widetilde{R}_t \ :=\ \varphi_t'' \ -\ \varphi_t'^2 \ -\ \frac{(n-1)(n-3)}{4r^2} \ +\ r^2 \ -\ \mu_t$$

and (53) by direct calcuation. ■

Next we claim that the exact solution of (50) can be written in the form $f_t = e^{-\varphi_t - R_t}$, where $\partial^\alpha R_t \sim 0$ uniformly in $t \in [0,1]$, for each $\alpha \in \mathbb{N}$. Plugging this expression for f_t into the differential equation in (50) we see that R_t' is necessarily a solution of

$$R_t'' - R_t'^2 - 2\,\varphi_t' R_t' + \widetilde{R}_t = 0 \qquad \text{on } [\rho_0, \infty) \, .$$

Lemma 4.6 *For $\xi > \rho_0$ big enough, there is a smooth solution m_t of*

$$m_t' = 2\,\varphi_t' m_t + m_t^2 - \tilde{R}_t \qquad \text{on } [\,\xi, \infty) \, , \tag{56}$$

that is asymptotically zero together with all its derivatives, $\partial^\alpha m_t \sim 0$, uniformly in $t \in [0,1]$, for $\alpha \in \mathbb{N}_0$. In particular, for each $\alpha \in \mathbb{N}_0$, $\partial^\alpha m_t(r)$ converges to zero as $r \to \infty$, uniformly in $t \in [0,1]$.

Proof: We proceed along the lines of [37, Section 14]. For $\eta \geqslant \rho_0$, let $\mathscr{X}_\eta = \{ f \in C([\eta, \infty)) : \| f \|_\infty \leqslant \frac{1}{2} \}$. This is a closed subset of the Banach space $\langle\, C_b^\infty([\,\eta, \infty)), \| \cdot \|_\infty \,\rangle$. We define a mapping $\mathscr{P}_\eta^{(t)}$ on $\mathscr{X}_\eta$ by

$$\mathscr{P}_\eta^{(t)} f(r) = - \int_r^\infty e^{-(s^2 - r^2)} \Big(- \widetilde{R}_t(s) + \big(2\,\varphi_t'(s) - 2\,s\big)\, f(s) + f^2(s) \Big)\, ds \, , \tag{57}$$

for $r \in [\,\eta, \infty)$, $f \in \mathscr{X}_\eta$. We want to show that $\mathscr{P}_\xi^{(t)}$ possesses a fixpoint $m_t \in \mathscr{X}_\xi$ which is asymptotically zero uniformly in $t \in [0,1]$, if we choose the t-independent parameter ξ large enough. Since $\mathscr{P}_\xi^{(t)} f \in C^{k+1}([\,\xi, \infty))$ if $f \in C^k([\,\xi, \infty))$, for all $k \in \mathbb{N}_0$, any fixpoint of $\mathscr{P}_\xi^{(t)}$ is automatically smooth. Moreover, differentiating the fixpoint equation shows that any fixpoint of $\mathscr{P}_\xi^{(t)}$ solves (56).

To begin with we observe that, since $\frac{1}{r}\,(2\,\varphi_t'(r) - 2r) \to 0$, as $r \to \infty$, u.i.t, there is some $\tilde{\xi} \in [\max\{2, \rho_0\}\,, \infty)$ such that

$$\Big| \frac{1}{r} \big(2\,\varphi_t'(r) - 2\,r\big) \Big| \leqslant \frac{1}{2} \, , \tag{58}$$

for all $r \geqslant \tilde{\xi}$ and $t \in [0,1]$. In particular, we then have

$$\Big| \frac{1}{r} \big(2\,\varphi_t'(r) - 2\,r\big)\,(z_1 - z_2) + \frac{1}{r} \big(z_1^2 - z_2^2\big) \Big| \leqslant |z_1 - z_2| \, , \tag{59}$$

for all $r \geqslant \tilde{\xi} \geqslant 2$, $t \in [0,1]$, and $z_1, z_2 \in \mathbb{R}$ with $|z_1|, |z_2| < 1/2$. Furthermore, by (54), for each $\omega \in \mathbb{N}$, there exist $\xi_\omega \in [\,\tilde{\xi}, \infty)$ and $0 < c_\omega < \infty$ such that

$$\Big| \frac{\widetilde{R}_t(r)}{r} \Big| \;\leqslant\; c_\omega\, r^{-\omega} \;\leqslant\; \frac{1}{2}\,, \tag{60}$$

for all $r \geqslant \xi_\omega$ and $t \in [0,1]$. We may suppose that the sequence $(\xi_\omega)_{\omega\in\mathbb{N}}$ is increasing. We stress that due to (55) and (54) we can choose each ξ_ω and c_ω independend from t. In our argumentation below we will also need the following observation: For some $\chi \in C_b([1,\infty)$ such that $|\chi(r)| \leqslant c\, r^{-\omega}$ with some fixed $\omega \in \mathbb{N}$, we have

$$\Big| \int\limits_r^\infty e^{-(s^2-r^2)}\, s\, \chi(s)\, ds \Big| \;\leqslant\; \frac{c}{2}\, r^{-\omega}\,, \qquad r \in [1,\infty)\,. \tag{61}$$

First we note that it follows from (58), (60) and (61) that $\mathscr{P}_{\xi_\omega}$ actually maps $\mathscr{X}_{\xi_\omega}$ into itself, for each $\omega \in \mathbb{N}$. Furthermore, using (59) and (61) it is easily seen that each $\mathscr{P}_{\xi_\omega}$ is a contraction with Lipschitz constant $\frac{1}{2}$ on $\mathscr{X}_{\xi_\omega}$.

Now we consider, for $\omega \in \mathbb{N}$, the iteration sequence (in the following we drop the parameter t in the notation)

$$m_0^{(\omega)} \;\equiv\; 0\,, \qquad m_{\nu+1}^{(\omega)} \;=\; \mathscr{P}_{\xi_\omega} m_\nu^{(\omega)}\,, \qquad \nu \in \mathbb{N}_0\,,$$

which converges uniformly to the unique fixpoint $m^{(\omega)}$ of $\mathscr{P}_{\xi_\omega}$ in $\mathscr{X}_{\xi_\omega}$. We will show the following two inequalities by induction on $\nu \in \mathbb{N}_0$:

$$|\, m_{\nu+1}^{(\omega)} \,-\, m_\nu^{(\omega)} \,| \;\leqslant\; \frac{1}{2^{\nu+1}}\, c_\omega\, r^{-\omega}\,, \qquad |\, m_{\nu+1}^{(\omega)} \,| \;\leqslant\; c_\omega\, r^{-\omega} \;\leqslant\; \frac{1}{2}\,, \tag{62}$$

for all $\nu \in \mathbb{N}_0$ and $r \in [\,\xi_\omega, \infty)$. Clearly this implies

$$|\, m^{(\omega)}(r) \,| \;\leqslant\; c_\omega\, r^{-\omega}\,, \qquad r \in [\,\xi_\omega, \infty)\,, \tag{63}$$

for $\omega \in \mathbb{N}$. Evidently, $m^{(\tilde{\omega})}\!\restriction_{[\,\xi_\omega,\infty)} = m^{(\omega)}$, for all $\omega, \tilde{\omega} \in \mathbb{N}$, $\omega \geqslant \tilde{\omega}$. So, setting $\xi := \xi_1$, the lemma is proved if (62) is true. The case $\nu = 0$ follows immediatly by definition of $m_0^{(\omega)}$ and by (57), (60) and (61). So let $\nu \in \mathbb{N}$ and suppose that (62) holds true, for all $0 \leqslant \varkappa \leqslant \nu - 1$. In particular, we then have $|m_{\nu-1}^{(\omega)}|, |m_\nu^{(\omega)}| \leqslant 1/2$. Using the estimate (59) we get

$$\Big| \mathscr{P}_{\xi_\omega} m_\nu^{(\omega)}(r) \,-\, \mathscr{P}_{\xi_\omega} m_{\nu-1}^{(\omega)}(r) \Big| \;\leqslant\; \int\limits_r^\infty e^{-(s^2-r^2)}\, s\, \big|\, m_\nu^{(\omega)}(s) \,-\, m_{\nu-1}^{(\omega)}(s) \,\big|\, ds\,,$$

which is by the induction hypothesis less than or equal to

$$\int_r^\infty e^{-(s^2-r^2)}\, s\, \frac{1}{2^\nu}\, c_\omega\, s^{-\omega}\, ds \;\leqslant\; \frac{1}{2^{\nu+1}}\, c_\omega\, r^{-\omega} \;,$$

for all $r \in [\,\xi_\omega, \infty)$. Here we again used (61). This gives the first inequality in (62). The second follows from

$$|\, m^{(\omega)}_{\nu+1}(r)\,| \;\leqslant\; \sum_{\varkappa=0}^{\nu} |m^{(\omega)}_{\varkappa+1} - m^{(\omega)}_{\varkappa}| \;\leqslant\; \frac{c_\omega}{2}\, r^{-\omega} \sum_{\varkappa=0}^{\nu} \frac{1}{2^\varkappa} \;\leqslant\; c_\omega\, r^{-\omega} \;,$$

for all $r \in [\,\xi_\omega, \infty)$.

So far we have $m_t \sim 0$, uniformly in $t \in [0,1]$. Differentiating equation (56) succesively we also get the asymptotics $\partial_r^\alpha m_t \sim 0$, for $\alpha \in \mathbb{N}_0$, uniformly in $t \in [0,1]$ by induction. ■

Now we return to our discussion of the exact solution f_t of (50). Having derived the previous lemma we define $R_t(r) \;:=\; \int_\xi^r m_t(s)\, ds \;-\; \varphi_t(\xi) \;-\; \ln(f_t(\xi))$, for $r \in [\xi, \infty)$, so that $f_t(\xi) = e^{-\varphi_t(\xi)-R_t(\xi)}$. We conclude that $f_t \equiv e^{-\varphi_t - R_t}$ on $[\,\xi, \infty)$ using maximum principles as in Corollary 4.4. Writing $f_t = e^{-\psi_t}$ we thus have

$$\partial_r^{\beta+1}\psi_t(r) \;\sim\; r\, \delta_{\beta,0} \,+\, \sum_{\nu=0}^{\infty} a_\nu(t)\, (-1)^\beta\, \frac{(\beta+2\nu)!}{(2\nu)!}\, r^{-1-2\nu-\beta} \;, \quad r \to \infty, \text{ u.i.}\, t \;, \tag{64}$$

for all $\beta \in \mathbb{N}_0$. The expression for $u_t^\circ = u_t^\circ(t)$ in Cartesian coordinates is then given by $u_t^\circ(x) = e^{-g_t(|x|)}$ where $g_t(r) := \psi_t(r) + \frac{n-1}{2}\ln(r)$, for $r \in [\xi, \infty)$. We have $\partial_r g_t(r) = r + \mathcal{O}(\frac{1}{r})$ and

$$\begin{aligned} \partial_{x_i} g_t(|x|) \;&=\; x_i \,+\, \mathcal{O}(\tfrac{1}{|x|}) \;, \qquad \partial_{x_i}\partial_{x_j} g_t(|x|) \;=\; \delta_{ij} \,+\, \mathcal{O}(\tfrac{1}{|x|^2}) \;, \\ \partial_x^\alpha g_t(|x|) \;&=\; \mathcal{O}(\tfrac{1}{|x|^{|\alpha|}}) \;, \end{aligned} \tag{65}$$

for $i, j \in \Lambda$ and each multiindex α with $|\alpha| \geqslant 3$. Due to (55) and Lemma 4.6 these $\mathcal{O}$-symbols are all uniform in t.

We come back to our original problem (46) and abbreviate $\tilde v_t := \exp(-\Phi_t)\restriction_S$. Of coures $\tilde v_t$ is in general not radially symmetric. By Lemma 4.2 we know, however, that the functions $t \mapsto \min_S \tilde v_t$ resp. $t \mapsto \max_S \tilde v_t$ are continuous on $[0,1]$ and strictly positive. Hence $\underline{v} := \inf_{t\in[0,1]} \min_S \tilde v_t > 0$ and $\overline{v} := \sup_{t\in[0,1]} \max_S \tilde v_t < \infty$ are attained and we can use them to compare solutions of (46) with radially symmetric solutions. Recall that the L^2-solution of (46) is given by $u_t = \exp(-\Phi_t)$. Corollary 4.4 implies

$$\underline{v}\, e^{-g_t} \;\leqslant\; u_t \;\leqslant\; \overline{v}\, e^{-g_t} \qquad \text{on } \Omega\,,$$

for each $t \in [0,1]$. Hence there is a smooth function k_t on Ω such that

$$\ln(\underline{v}) \;\leqslant\; k_t \;\leqslant\; \ln(\overline{v}) \qquad \text{and} \qquad u_t \;=\; e^{-g_t+k_t}\,, \qquad t \in [0,1]\,. \tag{66}$$

Moreover, we have $K(\partial_{\theta_i}\tilde{v}_t) = \partial_{\theta_i} u_t = e^{-g_t+k_t}\,\partial_{\theta_i} k_t$. This together with Corollary 4.4 gives

$$-\infty \;<\; e^{-g_t} \inf_{\tau\in[0,1]} \min_S \partial_{\theta_i}\tilde{v}_\tau \;\leqslant\; e^{-g_t+k_t}\,\partial_{\theta_i} k_t \;\leqslant\; e^{-g_t} \sup_{\tau\in[0,1]} \max_S \partial_{\theta_i}\tilde{v}_\tau \;<\; \infty\,,$$

and therefore

$$-\infty \;<\; \overline{v}^{-1} \inf_{\tau\in[0,1]} \min_S \partial_{\theta_i}\tilde{v}_\tau \;\leqslant\; \partial_{\theta_i} k_t \;\leqslant\; \underline{v}^{-1} \sup_{\tau\in[0,1]} \max_S \partial_{\theta_i}\tilde{v}_\tau \;<\; \infty\,. \tag{67}$$

Proceeding on in this way we get by induction

$$\sup_\Omega |\partial_\theta^\alpha k_t| \;\leqslant\; C(\alpha) \;<\; \infty \tag{68}$$

uniformly in $t \in [0,1]$, for each multi-index α.

To obtain Proposition 4.1 some estimates on the radial derivatives of k_t are still missing. The differential equation (46) for $u_t = u_t^\circ\, e^{k_t}$ written in polar coordinates implies, for each $\alpha \in \mathbb{N}_0^{n-1}$,

$$\Big(-\partial_r^2 - \frac{n-1}{r}\,\partial_r + r^2 - \mu_t \Big)\, u_t^\circ(r)\,\partial_\theta^\alpha e^{k_t} \;=\; \frac{u_t^\circ(r)}{r^2}\,\partial_\theta^\alpha \Delta_\theta e^{k_t}\,.$$

Since u_t° solves (49) we obtain

$$\Big(\partial_r + \frac{2\,\partial_r u_t^\circ}{u_t^\circ} + \frac{n-1}{r} \Big)\,\partial_r\partial_\theta^\alpha e^{k_t} \;=\; \frac{1}{r^2}\,\partial_\theta^\alpha \Delta_\theta e^{k_t}\,.$$

Due to (68) the right side of the last equation is $\mathcal{O}(\frac{1}{r^2})$ uniformly in $t \in [0,1]$. Since $\frac{2\,\partial_r u_t^\circ}{u_t^\circ} + \frac{n-1}{r} = -2\psi_t'$ we thus get

$$\partial_r\big(e^{-2\,\psi_t}\,\partial_r\,\partial_\theta^\alpha e^{k_t} \big) \;=\; e^{-2\,\psi_t}\mathcal{O}(\tfrac{1}{r^2})\,. \tag{69}$$

Using Lemma 4.3 we easily see that $e^{-2\psi_t}\,\partial_r\,\partial_\theta^\alpha e^{k_t} \longrightarrow 0$, as $r\to\infty$. Integrating (69) we hence obtain

$$\partial_r\partial_\theta^\alpha e^{k_t} \;=\; \int\limits_r^\infty e^{2\,(\psi_t(r)\,-\,\psi_t(s)\,)}\mathcal{O}(\tfrac{1}{s^2})\,ds\;.$$

Since we have the asymptotic expansion (64) for ψ_t, and since $r^2-s^2\leqslant 2r(r-s)$, for $r\leqslant s$, we have $2\,(\psi_t(r)\,-\,\psi_t(s)\,)\leqslant(2r-c)(r-s)$, for some t-independent constant $c>0$. This yields $\partial_r\partial_\theta^\alpha e^{k_t} \;=\; \mathcal{O}(\frac{1}{r^3})$ u.i.t. Differentiating (69) succesively with respect to r and using (64) we can repeat the previous argument and get by induction

$$\partial_r^\beta\partial_\theta^\alpha\,e^{k_t} \;=\; \mathcal{O}(\tfrac{1}{r^{2+\beta}})\,,\qquad \text{for }\beta\in\mathbb{N}\,,\qquad \text{u.i.}\,t\,.$$

Differentiating $k_t=\ln(e^{k_t})$ we derive from this again by induction

$$\partial_r^\beta\partial_\theta^\alpha\,k_t \;=\; \mathcal{O}(\tfrac{1}{r^{2+\beta}})\,,\qquad \text{for }\beta\in\mathbb{N}\,,\qquad \text{u.i.}\,t\,.$$

In Cartesian coordinates this reads

$$\partial_x^\alpha k_t \;=\; \mathcal{O}(\tfrac{1}{|x|^{|\alpha|}})\,,\qquad \text{for }\alpha\in\mathbb{N}_0^n,\ \alpha\neq 0\,,\qquad \text{u.i.}\,t\,. \tag{70}$$

Since $\Phi_t=g_t-k_t$, Proposition 4.1 finally follows from (65) and (70).

5 Maximum principles and the phase of the ground state eigenfunction of $P_{h,\Lambda}$

In this section we prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5: We fix $h\in(0,1]$ and $\Lambda\in\mathfrak{L}$. To begin with we pick some $\rho\in C^\infty(\mathbb{R},[0,1])$ such that $\rho\equiv 1$ on $(-\infty,1]$ and $\rho\equiv 0$ on $[2,\infty)$. We set $\chi_\varepsilon(x)=\rho(\varepsilon\ln|x|)$, for $\varepsilon>0$, $x\in\mathbb{R}^\Lambda\setminus\{0\}$, and $\chi_\varepsilon(0)=1$, for $\varepsilon>0$, such that $\chi_\varepsilon\equiv 1$ on $\{|x|\leqslant e^{1/\varepsilon}\}$ and $\chi_\varepsilon\equiv 0$ on $\{|x|\geqslant e^{2/\varepsilon}\}$. Then $\partial_x^\alpha\chi_\varepsilon=\mathcal{O}(\varepsilon/|x|^{|\alpha|})$, for each multi-index α.

Furthermore we define, for $\varepsilon>0$ and $t\in[0,1]$,

$$\begin{aligned} V_\varepsilon &\;=\; \lambda_{\min}^2\,(1-\chi_\varepsilon)\,x^2\;+\;\chi_\varepsilon\,V_\Lambda\;,\\ V_{\varepsilon,t} &\;=\; \lambda_{\min}^2\,(1-t)\,x^2+t\,V_\varepsilon \;=\; \lambda_{\min}^2\,x^2+t\,\chi_\varepsilon\big(V_\Lambda-\lambda_{\min}^2\,x^2\big)\;. \end{aligned} \tag{71}$$

Then we observe that

$$\tfrac{1}{2}\,V_{\varepsilon,t}'' \;=\; D_{\varepsilon,t}^2\;+\;Y_{\varepsilon,t} \tag{72}$$

holds with

$$D_{\varepsilon,t}^2 \;:=\; t\left(\chi_\varepsilon D_\Lambda^2 \;+\; \lambda_{\min}^2(1-\chi_\varepsilon)\,\mathbb{1}\right) \;+\; (1-t)\,\lambda_{\min}^2\,\mathbb{1} \;\longrightarrow\; \lambda_{\min}^2\,\mathbb{1}\;, \tag{73}$$

as $|x| \to \infty$. Due to (17) and the properties of χ_ε, the smooth matrix-valued function $Y_{\varepsilon,t}$ satisfies

$$\sup_{x\in\mathbb{R}^\lambda} \|\, Y_{\varepsilon,t}(x)\,\|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;\leqslant\; t\,\gamma \;+\; t\,\mathcal{O}_\Lambda(\varepsilon)\;. \tag{74}$$

Here the Λ-index on the $\mathcal{O}$-symbol indicates that the latter might depend on Λ. This causes no problem since our results in the limits $t \to 1$ and $\varepsilon \to 0$ below will again be uniform in Λ.

For later use we note that

$$\left\|\,(-h^2\Delta + V_{\varepsilon,t} + i)^{-1} - (-h^2\Delta + V_{\varepsilon,t_0} + i)^{-1}\,\right\| \;\longrightarrow\; 0 \quad \text{as } t \to t_0\;, \tag{75}$$

for each $t_0 \in [0,1]$, and

$$\left\|\,(-h^2\Delta + V_\varepsilon + i)^{-1} - (-h^2\Delta + V_\Lambda + i)^{-1}\,\right\| \;\longrightarrow\; 0 \quad \text{as } \varepsilon \to 0\;. \tag{76}$$

Here (76) is true since we can write

$$\begin{aligned}&(-h^2\Delta + V_\varepsilon + i)^{-1} - (-h^2\Delta + V_\Lambda + i)^{-1}\\ &\quad=\; (-h^2\Delta + V_\Lambda + i)^{-1}\,(1-\chi_\varepsilon)\left\{(V_\Lambda - \lambda_{\min}^2 x^2)(-h^2\Delta + V_\varepsilon + i)^{-1}\right\}\,,\end{aligned}$$

where the operator in the curly brackets is bounded uniformly in $\varepsilon > 0$, $(1-\chi_\varepsilon)$ converges weakly to zero, as $\varepsilon \to 0$, and $(-h^2\Delta + V + i)^{-1}$ is compact.

Let $\exp(-\frac{1}{h}\,\Phi_{\varepsilon,t})$ denote the normalized strictly positive eigenfunction corresponding to the lowest eigenvalue $\mu_{\varepsilon,t}$ of $-\,h^2\,\Delta + V_{\varepsilon,t}$, for $\varepsilon > 0$, $t \in [0,1]$. Of course both $\Phi_{\varepsilon,t}$ and $\mu_{\varepsilon,t}$ also depend on h and Λ. Then, after a scaling, Proposition 4.1 and (73) imply

$$\begin{aligned}&& \Phi''_{\varepsilon,t}(x) \;-\; \lambda_{\min}\,\mathbb{1} \;&\to\; 0 \qquad \text{as } |x| \to \infty\;,\\ \Rightarrow\quad & \Psi''_{\varepsilon,t}(x) \;:=\; & \Phi''_{\varepsilon,t}(x) \;-\; D_{\varepsilon,t}(x) \;&\to\; 0 \qquad \text{as } |x| \to \infty\;,\end{aligned} \tag{77}$$

for each $\varepsilon > 0$, $t \in [0,1]$. So the supremum in the next definition is actually attained,

$$m_\varepsilon(t) \;:=\; \sup_{x\in\mathbb{R}^\Lambda} \|\, \Psi''_{\varepsilon,t}(x)\,\|_{\mathscr{L}(\ell^\infty_{o,\varkappa})}\;, \qquad \varepsilon > 0\;,\; t \in [0,1]\;. \tag{78}$$

We indicate briefly why m_ε is uniformly continuous on $[0,1]$ as a function of t. In order to show this it is of course sufficient to prove that m_ε is continuous at $t_0 \in [0,1]$. Due to (71) and (75) the assumptions of Proposition 4.1 and Lemma 4.2 are fulfilled by $V_{\varepsilon,t}$ after a scaling, for each fixed $\varepsilon > 0$. So let $\eta > 0$. By Proposition 4.1 there is some $R = R(\varepsilon, \eta, \Lambda) > 0$ such that

$$\sup_{t\in[0,1]} \sup_{|x|>R} \| \, \Psi''_{\varepsilon,t}(x) \, \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \; < \; \frac{\eta}{2} \, .$$

Having chosen such an $R > 0$ it suffices to observe that due to Lemma 4.2 there is some $\delta = \delta(R) > 0$ such that

$$\sup_{t\in[0,1]\cap(t_0-\delta\,,\,t_0+\delta)} \sup_{|x|\leqslant R} \| \, \Psi''_{\varepsilon,t}(x) \; - \; \Psi''_{\varepsilon,t_0}(x) \, \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \; < \; \eta \, .$$

This shows that m_ε is uniformly continuous on $[0,1]$.

Proceeding on in the proof of the lemma we note that the eigenvalue equation for $\exp(-\frac{1}{h}\,\Phi_{\varepsilon,t})$ yields

$$V_{\varepsilon,t} \; - \; \mu_{\varepsilon,t} \;\; = \;\; (\nabla\Phi_{\varepsilon,t})^2 \; - \; h \; \Delta\Phi_{\varepsilon,t} \; .$$

Differentiating this identity twice we get

$$\begin{aligned} &\langle \, v \, , V''_{\varepsilon,t} \, w \, \rangle \; - \; 2 \, \nabla\Phi_{\varepsilon,t} \cdot \nabla \langle \, v \, , D_{\varepsilon,t} \, w \, \rangle \; + \; h \, \Delta \langle \, v \, , D_{\varepsilon,t} \, w \, \rangle \\ &\quad = \;\; 2 \, \nabla\Phi_{\varepsilon,t} \cdot \nabla \langle \, v \, , \Psi''_{\varepsilon,t} \, w \, \rangle \; - \; h \, \Delta \langle \, v \, , \Psi''_{\varepsilon,t} \, w \, \rangle \; + \; 2 \, \langle \, v \, , (\Phi''_{\varepsilon,t})^2 \, w \, \rangle \; , \end{aligned} \tag{79}$$

for $w \in \ell^\infty_{o,\varkappa}$, $v \in \ell^1_{o,-\varkappa}$. In view of (72) and (77) and since $\Psi''_{\varepsilon,t}$ is symmetric (79) is equivalent to

$$\begin{aligned} &\langle \, v \, , Y_{\varepsilon,t} \, w \, \rangle \; - \; \nabla\Phi_{\varepsilon,t} \cdot \nabla \langle \, v \, , D_{\varepsilon,t} \, w \, \rangle \; + \; \tfrac{h}{2} \, \Delta \langle \, v \, , D_{\varepsilon,t} \, w \, \rangle \\ &\quad = \;\; \nabla\Phi_{\varepsilon,t} \cdot \nabla \langle \, v \, , \Psi''_{\varepsilon,t} \, w \, \rangle \; - \; \tfrac{h}{2} \, \Delta \langle \, v \, , \Psi''_{\varepsilon,t} \, w \, \rangle \\ &\qquad + \langle \, D_{\varepsilon,t} v \, , \Psi''_{\varepsilon,t} \, w \, \rangle \; + \; \langle \, \Psi''_{\varepsilon,t} \, v \, , D_{\varepsilon,t} \, w \, \rangle \; + \; \langle \, v \, , (\Psi''_{\varepsilon,t})^2 \, w \, \rangle \, , \end{aligned} \tag{80}$$

for $w \in \ell^\infty_{o,\varkappa}$, $v \in \ell^1_{o,-\varkappa}$. We observe that, by our choice of χ_ε, we have

$$\nabla D_{\varepsilon,t} = t \, \mathcal{O}_\Lambda(\varepsilon/|x|) \; , \qquad \Delta D_{\varepsilon,t} = t \, \mathcal{O}_\Lambda(\varepsilon/|x|^2) \; . \tag{81}$$

Writing

$$\nabla\Phi_{\varepsilon,t}(x) \;\; = \;\; \nabla\Phi_{\varepsilon,t}(0) \; + \; \int_0^1 \Phi''_{\varepsilon,t}(\tau \, x) \, x \, d\tau$$

and observing that $\|x\|_{\ell^\infty_{o,\varkappa}} \leqslant |x|$ and

$$\lambda_{\min} \, \mathbb{1} \;\; \leqslant \;\; D_{\varepsilon,t}(x) \;\; \leqslant \;\; \lambda_{\max} \, \mathbb{1} \; , \qquad x \in \mathbb{R}^\Lambda \; , \; \varepsilon > 0 \; , \; t \in [0,1] \; , \tag{82}$$

we get

$$|\nabla\Phi_{\varepsilon,t}(x)| \;\leqslant\; |\nabla\Phi_{\varepsilon,t}(0)| \,+\, \lambda_{\max}\,|x| \,+\, m_\varepsilon(t)\,|x|\,. \tag{83}$$

Here $|\nabla\Phi_{\varepsilon,t}(0)| \leqslant c$, for some $c < \infty$, uniformly in $t \in [0,1]$ and $\varepsilon > 0$ because of (75), (76) and Lemma 4.2. Due to (74), (81), and (83), we have the following upper bound on the left hand side of (80), for $\|v\|_{\ell^1_{o,-\varkappa}} = \|w\|_{\ell^\infty_{o,\varkappa}} = 1$,

$$\begin{aligned}
&\langle\, v\,, Y_{\varepsilon,t}\, w\,\rangle \,-\, \nabla\Phi_{\varepsilon,t}\cdot\nabla\langle\, v\,, D_{\varepsilon,t}\, w\,\rangle \,+\, \tfrac{h}{2}\,\Delta\langle\, v\,, D_{\varepsilon,t}\, w\,\rangle \\
&\quad\leqslant\; \|\, Y_{\varepsilon,t} \,+\, \nabla\Phi_{\varepsilon,t}\cdot\nabla D_{\varepsilon,t} \,+\, \tfrac{h}{2}\,\Delta D_{\varepsilon,t}\,\|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \\
&\quad\leqslant\; t\,\big(\gamma + \mathcal{O}_\Lambda(\varepsilon)\big) \,+\, t\,\big(\,c \,+\, \lambda_{\max}|x| \,+\, m_\varepsilon(t)\,|x|\,\big)\,\mathcal{O}_\Lambda(\tfrac{\varepsilon}{|x|}) \,+\, t\,h\,\mathcal{O}_\Lambda(\tfrac{\varepsilon}{|x|^2})\,.
\end{aligned}\tag{84}$$

Let $x_{\varepsilon,t} \in \mathbb{R}^\Lambda$ denote the point where the supremum (78) it attained. We find $r \in \ell^\infty_{o,\varkappa}$, $s \in \ell^1_{o,-\varkappa}$ with $\|r\|_{\ell^\infty_{o,\varkappa}} = \|s\|_{\ell^1_{o,-\varkappa}} = 1$, such that $\langle\, s\,, \Psi''_\varepsilon(x_{\varepsilon,t})\, r\,\rangle = m_\varepsilon(t)$. In particular $\langle\, s\,, \Psi''_{\varepsilon,t}(\cdot)\, r\,\rangle \in C^\infty(\mathbb{R}^\Lambda, \mathbb{R})$ has a local maximum at $x_{\varepsilon,t}$, that is

$$\nabla\Phi_{\varepsilon,t}(x_{\varepsilon,t})\cdot\nabla\langle\, s\,, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \;=\; 0\,, \qquad -\tfrac{h}{2}\,\Delta\langle\, s\,, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \;\geqslant\; 0\,. \tag{85}$$

From (80), (84) and (85) we obtain

$$\begin{aligned}
&t\,\big(\gamma + \mathcal{O}_\Lambda(\varepsilon)\big) \,+\, t\, m_\varepsilon(t)\,\mathcal{O}_\Lambda(\varepsilon) \\
&\quad\geqslant\; \langle\, D_{\varepsilon,t}(x_{\varepsilon,t}) s\,, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \,+\, \langle\, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, s\,, D_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \\
&\qquad + \langle\, s\,, (\Psi''_{\varepsilon,t}(x_{\varepsilon,t}))^2\, r\,\rangle\,.
\end{aligned}\tag{86}$$

Since

$$\begin{aligned}
\langle\, s\,, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \;&=\; \langle\, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, s\,, r\,\rangle \;=\; \|s\|_{\ell^1_{o,-\varkappa}}\,\|\Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\|_{\ell^\infty_{o,\varkappa}} \\
&=\; \|\Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, s\|_{\ell^1_{o,-\varkappa}}\,\|r\|_{\ell^\infty_{o,\varkappa}} \;=\; \|\Psi''_{\varepsilon,t}(x_{\varepsilon,t})\|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;=\; m_\varepsilon(t)\,,
\end{aligned}$$

we see that $s_j(\Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r)_j \geqslant 0$ and $(\Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, s)_j r_j \geqslant 0$, for each $j \in \Lambda$. Since $D_{\varepsilon,t}$ is diagonal, this together with (82) implies

$$\langle\, D_{\varepsilon,t}(x_{\varepsilon,t}) s\,, \Psi''_{\varepsilon,t}(x_{\varepsilon,t})\, r\,\rangle \;\geqslant\; \lambda_{\min}\, m_\varepsilon(t)\,.$$

Altogether the previous estimate, its analogue for the second term in the second line of (86), and (86) itself, lead to

$$t\,(\gamma \,+\, \mathcal{O}_\Lambda(\varepsilon)) \;\geqslant\; \big(\,2\,\lambda_{\min} \,-\, t\,\mathcal{O}_\Lambda(\varepsilon)\,\big)\, m_\varepsilon(t) \,-\, m_\varepsilon(t)^2\,.$$

Now we assume that $m_\varepsilon(t) \leqslant \lambda_{\min}$. Solving the last inequality for $m_\varepsilon(t)$ shows that, for sufficiently small $\varepsilon > 0$, the only possibility for this to be fulfilled is

$$m_\varepsilon(t) \;\leqslant\; \lambda_{\min} \,-\, (\lambda^2_{\min} - t\,\gamma)^{1/2} \,+\, t\,\mathcal{O}_\Lambda(\varepsilon) \,+\, t^{1/2}\,\mathcal{O}_\Lambda(\varepsilon^{1/2})\,. \tag{87}$$

From this we obtain the theorem as follows. Let $0 < \eta < (\lambda_{\min}^2 - \gamma)^{1/2}$. By (87) we find some $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leqslant \varepsilon_0$ and $t \in [0,1]$, we have

$$m_\varepsilon(t) \leqslant \lambda_{\min} \quad \Longrightarrow \quad m_\varepsilon(t) \leqslant \lambda_{\min} - (\lambda_{\min}^2 - \gamma)^{1/2} + \eta \;=\; C_\gamma + \eta \;<\; \lambda_{\min} \,.$$

The uniform continuity of m_ε and $m_\varepsilon(0) = 0$ show that, for all $0 < \varepsilon \leqslant \varepsilon_0$, there is some $t_0 \in (0,1]$ such that $m_\varepsilon(t_0) \leqslant \lambda_{\min}$. Then we get after a simple bootstrap argument

$$\sup_{x \in \mathbb{R}^\Lambda} \| \, \Psi''_{\varepsilon,1}(x) \, \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;=\; m_\varepsilon(1) \;\leqslant\; C_\gamma + \eta \,, \tag{88}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. Due to (76) we are again allowed to apply Lemma 4.2. Thus we see that, for all $R > 0$, there is some $0 < \varepsilon_1 \leqslant \varepsilon_0$ such that, for all $0 < \varepsilon \leqslant \varepsilon_1$, we have

$$\sup_{|x| \leqslant R} \|\Psi''(x)\|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} \;\leqslant\; \sup_{|x| \leqslant R} \| \, \Psi''_{\varepsilon,1}(x) \, \|_{\mathscr{L}(\ell^\infty_{o,\varkappa})} + \eta \;\leqslant\; C_\gamma \,+\, 2\,\eta \,.$$

Letting R tend to infinity gives the result since $\eta > 0$ was arbitrarily small. ■

Proof of Theorem 1.6: The parameter t is no longer needed. That is we set $t = 1$ in the definition of the objects introduced in the preceeding proof and drop this from the notation. Differentiating (79) once more gives, for $\varepsilon > 0$, $u \in \ell^\infty_{o,1}$, $v \in \ell^1_{o,-2}$, $w \in \ell^\infty_{o,1}$

$$\begin{aligned}
&\Big\langle v \,, \big(V^{(3)}_\varepsilon u - 2\,\nabla\Phi_\varepsilon \cdot \nabla D'_\varepsilon u\big) w \Big\rangle \\
&\qquad -2\,\nabla(\Phi'_\varepsilon u) \cdot \nabla\langle v \,, D_\varepsilon w\rangle \;+\; h\,\Delta\langle v \,, (D'_\varepsilon u) w\rangle \\
&= \;\; 2\,\nabla\Phi_{\varepsilon,t} \cdot \nabla\langle v \,, (\Psi^{(3)}_\varepsilon u) w\rangle \;+\; 2\,\nabla(\Phi'_\varepsilon u) \cdot \nabla\langle v \,, \Psi''_\varepsilon w\rangle \\
&\qquad -h\,\Delta\langle v \,, (\Psi^{(3)}_\varepsilon u) w\rangle \;+\; 2\,\langle v \,, (\Phi^{(3)}_\varepsilon u) \Phi''_\varepsilon w\rangle \;+\; 2\,\langle v \,, \Phi''_\varepsilon (\Phi^{(3)}_\varepsilon u) w\rangle \,.
\end{aligned}$$

Since $\lambda_{\max}$ is an upper bound for the eigenvalues of $D_\varepsilon(x)$, uniformly in $x \in \mathbb{R}^\Lambda$ and $\varepsilon > 0$, Recalling $\Phi''_\varepsilon = D_\varepsilon + \Psi''_\varepsilon$ we see that this is equivalent to

$$\begin{aligned}
&\nabla\Phi_\varepsilon \cdot \nabla\langle v \,, (\Psi^{(3)}_\varepsilon u) w\rangle \;+\; 3\lambda_{\max}\,\langle v \,, (\Psi^{(3)}_\varepsilon u) w\rangle \\
&\qquad + \langle v \,, \big(D_\varepsilon - \lambda_{\max}\,\mathbb{1} + \Psi''_\varepsilon\big)\,(\Psi^{(3)}_\varepsilon u) w\rangle \\
&\qquad + \langle v \,, (\Psi^{(3)}_\varepsilon u)\,\big(D_\varepsilon - \lambda_{\max}\,\mathbb{1} + \Psi''_\varepsilon\big) w\rangle \\
&\qquad + \langle v \,, \big(\Psi^{(3)}_\varepsilon \big(D_\varepsilon - \lambda_{\max}\,\mathbb{1} + \Psi''_\varepsilon\big) u\big) w\rangle \\
&\qquad - \tfrac{h}{2}\,\Delta\langle v \,, (\Psi^{(3)}_\varepsilon u) w\rangle \\
&= \;\; \Big\langle v \,, \big(\tfrac{1}{2}\,V^{(3)}_\varepsilon u - \nabla\Phi_\varepsilon \cdot \nabla D'_\varepsilon u\big) w \Big\rangle \;+\; \tfrac{h}{2}\,\Delta\langle v \,, (D'_\varepsilon u) w\rangle \\
&\qquad - \langle v \,, \Phi''_\varepsilon (D'_\varepsilon u) w\rangle \;-\; \langle v \,, (D'_\varepsilon u)\Phi''_\varepsilon w\rangle \;-\; \langle v \,, (D'_\varepsilon \Phi''_\varepsilon u) w\rangle \,.
\end{aligned} \tag{89}$$

First we estimate the right hand side of (89) from above assuming that u, v, and w are normalized,

$$\|u\|_{\ell^\infty_{o,1}} \;=\; \|v\|_{\ell^1_{o,-2}} \;=\; \|w\|_{\ell^\infty_{o,1}} \;=\; 1\,. \tag{90}$$

The proof of the previous theorem, in particular (82), (83) and the remark following it, as well as (88), shows that, for all $x \in \mathbb{R}^\Lambda$,

$$|\nabla\Phi_\varepsilon(x)| \;\leqslant\; c + \big(\lambda_{\max} + C_\gamma + \mathcal{O}_\Lambda(\varepsilon)\big)\,|x|\,, \tag{91}$$

$$\|\Psi''_\varepsilon(x)\|_{\mathscr{L}(\ell^\infty_{o,1})} \;\leqslant\; C_\gamma + \mathcal{O}_\Lambda(\varepsilon) \;=\; \lambda_{\min} - (\lambda^2_{\min} - \gamma)^{1/2} + \mathcal{O}_\Lambda(\varepsilon)\,, \tag{92}$$

$$\|\Phi''_\varepsilon(x)\|_{\mathscr{L}(\ell^\infty_{o,1})} \;\leqslant\; \lambda_{\max} + C_\gamma + \mathcal{O}_\Lambda(\varepsilon)\,. \tag{93}$$

We recall that

$$\partial^\beta_x D_\varepsilon \;=\; \mathcal{O}_\Lambda(\varepsilon/|x|^{|\beta|})\,, \tag{94}$$

for each multi-index $\beta \in \mathbb{N}^\Lambda_0$. A straight forward calculation using (94) and (19) yields

$$\|\,V^{(3)}_\varepsilon\,\|_{\mathscr{L}(\ell^\infty_{o,1},\mathscr{L}(\ell^\infty_{o,1},\ell^\infty_{o,2}))} \;\leqslant\; \Gamma + \mathcal{O}_\Lambda(\varepsilon)\,. \tag{95}$$

Combining (91), (93), (94), and (95), we find that, under the assumption (90), the right hand side of (89) is bounded from above by $\Gamma/2 + \mathcal{O}_\Lambda(\varepsilon)$.

Now we consider the left hand side of (89). Proposition 4.1 again implies that $\Psi^{(3)}_\varepsilon(x) \to 0$, as $|x| \to \infty$. Proceeding as in the last theorem we pick some $x_\varepsilon \in \mathbb{R}^\Lambda$ where $\|\Psi^{(3)}_\varepsilon(\cdot)\|_{\mathscr{L}(\ell^\infty_{o,1},\mathscr{L}(\ell^\infty_{o,1},\ell^\infty_{o,2}))}$ attains its maximum. We fix u, v, and w, such that

$$\langle\, v\,, (\Psi^{(3)}_\varepsilon(x_\varepsilon)u)w\,\rangle \;=\; \|\Psi^{(3)}_\varepsilon(x_\varepsilon)\|_{\mathscr{L}(\ell^\infty_{o,1},\mathscr{L}(\ell^\infty_{o,1},\ell^\infty_{o,2}))} \;=:\; M_\varepsilon\,, \tag{96}$$

in addition to (90). So we have

$$\nabla\Phi_\varepsilon(x_\varepsilon)\cdot\nabla\langle v\,, (\Psi^{(3)}_\varepsilon(x_\varepsilon)u)w\rangle \;=\; 0\,, \qquad -\tfrac{h}{2}\,\Delta\langle v\,, (\Psi^{(3)}_\varepsilon(x_\varepsilon)u)w\rangle \;\geqslant\; 0\,. \tag{97}$$

Furthermore, (82) implies

$$\|D_\varepsilon(x_\varepsilon) - \lambda_{\max}\,\mathbb{1}\|_{\mathscr{L}(\ell^\infty_{o,1})} \;\leqslant\; \lambda_{\max} - \lambda_{\min}\,. \tag{98}$$

If we combine (90), (92), (96), (97), and (98), with our upper bound on the right hand side of (89) we arrive at

$$3\lambda_{\max}\,M_\varepsilon - 3\big(\lambda_{\min} - (\lambda^2_{\min} - \gamma)^{1/2} + \mathcal{O}_\Lambda(\varepsilon) + \lambda_{\max} - \lambda_{\min}\big)M_\varepsilon \;\leqslant\; \tfrac{\Gamma}{2} + \mathcal{O}_\Lambda(\varepsilon)\,.$$

This gives

$$M_\varepsilon \;\leqslant\; \frac{1}{6}\,\frac{\Gamma + \mathcal{O}_\Lambda(\varepsilon)}{(\lambda^2_{\min} - \gamma)^{1/2} - \mathcal{O}_\Lambda(\varepsilon)}\,,$$

for $\varepsilon > 0$ small enough. Now the theorem follows from arguments similar to those at the end of the proof of the preceeding theorem. ■

6 Weighted estimates on the phase of the ground state eigenfunction of $P_{h,\Lambda}$

In the following proposition we apply the estimates on the second and third derivatives of $\Phi_{h,\Lambda}$ obtained from Theorem 1.5 and 1.6 to show that $\Phi_{h,\Lambda}$ fulfills Hypothesis 1.

Proposition 6.1 *The phase $\Phi_{h,\Lambda}$ of the ground state eigenfunction of $P_{h,\Lambda}$ is strictly convex. More precisely we have*

$$0 \;<\; \lambda_{\min} - C_\gamma \;\leqslant\; \Phi''_{h,\Lambda}(x) \;\leqslant\; \lambda_{\max} + C_\gamma \,, \tag{99}$$

uniformly in $h > 0$, $\Lambda \in \mathfrak{L}$, $x \in \mathbb{R}^\Lambda$. Let $x_0 = x_0(h) \in \mathbb{R}^\Lambda$ denote the point where $\Phi_{h,\Lambda}$ attains its unique minimum. Then there is some uniform constant $K > 0$ such that

$$|\,\Phi''_{ij}(x) - \Phi''_{ij}(x_0)\,| \;\leqslant\; K \sum_{k\in\Lambda} e^{-d_\Lambda(i,j)-d_\Lambda(j,k)}\,|\Phi'_k(x)| \tag{100}$$

for all $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, $i,j \in \Lambda$ and $x \in \mathbb{R}^\Lambda$.

Proof of Theorem 1.7: The inequalities (99) show in particular that condition (10) holds with $\underline{m} := 2(\lambda_{\min} - C_\gamma)$ and $\overline{m} := 2(\lambda_{\max} + C_\gamma)$. Furthermore we readily verify that (8) and (9) hold by (100) if we set $b_{ijk}(h,\Lambda) := K\,e^{-d_\Lambda(i,j)-d_\Lambda(j,k)}$, for $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, $i,j,k \in \Lambda$, and $B := K\,C_d$, $B' := K^2\,C_d^2$. Theorem 1.7 is then a direct consequence of Theorem 1.4 because of (15). ■

Proof of Proposition 6.1: The validity of (99) has been observed earlier. See for the remarks below (20).

In order to prove (100) we we first observe that, for $h \in (0,1]$, $\Lambda \in \mathfrak{L}$, and $x \in \mathbb{R}^\Lambda$, we have the following estimate on the second partial derivatives of $\Psi_{h,\Lambda} = \Phi_{h,\Lambda} - \frac{1}{2}\,x \cdot D_\Lambda x$ with respect to $k, l \in \Lambda$,

$$\begin{aligned} |\,\Psi''_{kl}(x)\,| \;&=\; |\langle\, e_k\,,\, \Psi''_{h,\Lambda}(x)\,e_l\,\rangle| \\ &\leqslant\; \|\,\Psi''_{h,\Lambda}(x)\,\|_{\mathscr{L}(\ell^\infty_{k,2})}\,\|e_k\|_{\ell^1_{k,-2}}\,\|e_l\|_{\ell^\infty_{k,2}} \\ &=\; e^{-2\,d_\Lambda(k,l)}\,\|\,\Psi''_{h,\Lambda}(x)\,\|_{\mathscr{L}(\ell^\infty_{k,2})} \;\leqslant\; C_\gamma\,e^{-2\,d_\Lambda(k,l)}\,. \end{aligned} \tag{101}$$

In the second and third step we have observed that the dual space of $\ell^\infty_{k,2}$ is $\ell^1_{k,-2}$ and $\|e_k\|_{\ell^1_{k,-2}} = 1$. The last inequality is a direct consequece of (17) with $\varkappa = 2$ together with Theorem 1.5.

In a similar fashion (17) with $\varkappa = 1$ and (19) together with Theorem 1.6 show that

$$\begin{aligned} |\,\Psi^{(3)}_{ijk}(x)\,| &= |\langle\, e_i\,,\,(\Psi^{(3)}_{h,\Lambda}(x) e_j)\,e_k\,\rangle| \\ &\leqslant \|\,\Psi^{(3)}_{h,\Lambda}(x)\,\|_{\mathscr{L}(\ell^\infty_{i,1},\,\mathscr{L}(\ell^\infty_{i,1},\ell^\infty_{i,2}))}\,\|e_i\|_{\ell^1_{i,-2}}\,\|e_j\|_{\ell^\infty_{i,1}}\,\|e_k\|_{\ell^\infty_{i,1}} \\ &\leqslant C^{(3)}\,\|e_i\|_{\ell^1_{i,-2}}\,\|e_j\|_{\ell^\infty_{i,1}}\,\|e_k\|_{\ell^\infty_{i,1}} \\ &= C^{(3)}\,e^{-d_\Lambda(i,j)-d_\Lambda(i,k)}\;. \end{aligned} \tag{102}$$

Since x_0 is the unique global minimum of $\Phi_{h,\Lambda}$, we can write

$$\begin{aligned} \Phi'_k(x) \;=\; \Phi'_k(x) - \Phi'_k(x_0) \;&=\; \int_0^1 \nabla\Phi'_k(t\hat{x}+x_0)\cdot\hat{x}\,dt \\ &=\; \lambda_k\,\hat{x}_k \;+\; \int_0^1 \sum_{l\in\Lambda} \Psi''_{kl}(t\hat{x}+x_0)\,\hat{x}_l\,dt\;, \end{aligned}$$

where $\hat{x} := x - x_0$ and λ_k denotes the k^{th} diagonal entry of D_Λ, $k \in \Lambda$. Therefore the estimate (101) yields

$$|\,\Phi'_k(x)\,| \;\geqslant\; \lambda_k\,|\hat{x}_k| \;-\; C_\gamma \sum_{l\in\Lambda} e^{-2\,d_\Lambda(k,l)}\,|\hat{x}_l|\;.$$

This implies, for each $o \in \Lambda$, $x \in \mathbb{R}^\Lambda$,

$$\begin{aligned} &\sum_{k\in\Lambda} e^{-d_\Lambda(o,k)}\,|\,\Phi'_k(x)\,| \\ &\quad\geqslant\; \lambda_{\min}\sum_{k\in\Lambda} e^{-d_\Lambda(o,k)}\,|\hat{x}_k| \;-\; C_\gamma \sum_{k,l\in\Lambda} e^{-(\,d_\Lambda(o,k)+d_\Lambda(k,l)\,)-d_\Lambda(k,l)}\,|\hat{x}_l| \\ &\quad\geqslant\; \lambda_{\min}\sum_{k\in\Lambda} e^{-d_\Lambda(o,k)}\,|\hat{x}_k| \;-\; C_\gamma\,C_d \sum_{l\in\Lambda} e^{-d_\Lambda(o,l)}\,|\hat{x}_l|\;. \end{aligned}$$

Thus we get

$$\sum_{k\in\Lambda} e^{-d_\Lambda(o,k)}\,|\hat{x}_k| \;\leqslant\; \frac{1}{\lambda_{\min} - C_\gamma\,C_d} \sum_{k\in\Lambda} e^{-d_\Lambda(o,k)}\,|\,\Phi'_k(x)\,|\;,$$

for all $o \in \Lambda$ and $x \in \mathbb{R}^\Lambda$. To derive this inequality we used (18) for the first time. Finally this together with (102) implies, for fixed $i, j \in \Lambda$, $x \in \mathbb{R}^\Lambda$, and suitable $\xi_{x,k} = \xi_{x,k}(h) \in \mathbb{R}^\Lambda$, $k \in \Lambda$, the desired estimate,

$$
\begin{aligned}
|\, \Phi''_{ij}(x) - \Phi''_{ij}(x_0)\, | \;&\leqslant\; \sum_{k\in\Lambda} |\Psi^{(3)}_{ijk}(\xi_{x,k})|\, |\hat{x}_k| \\
&\leqslant\; C^{(3)} \sum_{k\in\Lambda} e^{-d_\Lambda(i,j)-d_\Lambda(j,k)}\, |\hat{x}_k| \\
&\leqslant\; \frac{C^{(3)}}{\lambda_{\min} - C_\gamma\, C_d} \sum_{k\in\Lambda} e^{-d_\Lambda(i,j)-d_\Lambda(j,k)}\, |\, \Phi'_k(x)\, |\,.
\end{aligned}
$$

■

Remark. Using the symmetry of $\Psi^{(3)}_{ijk}$ in i, j and k we obtain

$$
|\, \Psi^{(3)}_{ijk}(x)\, | \;\leqslant\; C^{(3)}\, e^{-\frac{2}{3}\,(\, d_\Lambda(i,j)+d_\Lambda(j,k)+d_\Lambda(k,i)\,)}
$$

from (102). Exactly as in the preceeding proof we derive from this the more symmetric estimate

$$
|\, \Phi''_{ij}(x) - \Phi''_{ij}(x_0)\, | \;\leqslant\; \text{const} \sum_{k\in\Lambda} e^{-\frac{2}{3}\,(\, d_\Lambda(i,j)+d_\Lambda(j,k)+d_\Lambda(k,i)\,)}\, |\Phi'_k(x)|
$$

under the assumption $\max\limits_{j\in\Lambda} \sum\limits_{k\in\Lambda} e^{-\frac{2}{3}\, d_\Lambda(k,j)} \leqslant C_d$. This, however, would not give any substantial improvement.

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A Some elementary combinatorial identities

Lemma A.1 *For $k, l \in \mathbb{N}$, $r \in \mathbb{N}_0$, the following formulas are true:*

$$
\sum_{\alpha=0}^{k} (-1)^{k-\alpha} \binom{l+\alpha-1}{\alpha} \binom{l}{k-\alpha+1} \;=\; \binom{l+k}{k+1} \tag{103}
$$

$$
\sum_{\alpha=0}^{l-1} (-1)^{l-1-\alpha} \binom{l+\alpha+r-1}{\alpha+r} \binom{l}{l-\alpha} \;=\; \binom{2\,l+r-1}{l+r} \tag{104}
$$

PROOF: We recall the elementary identity $(1-x)^{-l} = \sum_{s=0}^{\infty} \binom{l+s-1}{s} x^s$, for $l \in \mathbb{N}$. Then we observe that the left side of (103) is just the coefficient of x^k in the Cauchy product of $(1-x)^{-l}$ with $\frac{1}{x} - \frac{(1-x)^l}{x}$. On the other hand the right side of (103) is the coefficient of x^k in the power series $\frac{(1-x)^{-l}}{x} - \frac{1}{x}$. The second identity (104) follows in the same manner by replacing $(1-x)^{-l}$ in the above argument by the expression $\frac{(1-x)^{-l}}{x^r} - \sum_{s=0}^{r-1} \binom{l+s-1}{s} \frac{x^s}{x^r}$. ■

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