Randomly forced CGL equation: stationary measures and the inviscid limit

Sergei Kuksin[1], Armen Shirikyan[2]

Abstract

We study a complex Ginzburg–Landau (CGL) equation perturbed by a random force which is white in time and smooth in the space variable x. Assuming that $\dim x \le 4$, we prove that this equation has a unique solution and discuss its asymptotic in time properties. Next we consider the case when the random force is proportional to the square root of the viscosity and study the behaviour of stationary solutions as the viscosity goes to zero. We show that, under this limit, a subsequence of solutions in question converges to a nontrivial stationary process formed by global strong solutions of the nonlinear Schrödinger equation.

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Contents

[1]Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland, UK; Steklov Institute of Mathematics, 8 Gubkina St., 117966 Moscow, Russia; e-mail: `S.B.Kuksin@ma.hw.ac.uk`

[2]Laboratoire de Mathématiques, Université de Paris–Sud XI, Bâtiment 425, 91405 Orsay Cedex, France; e-mail: `Armen.Shirikyan@math.u-psud.fr`

0 Introduction

In this work, we consider the randomly forced complex Ginzburg–Landau (CGL) equation

$$\dot{u} - (\nu + i)\Delta u + i\lambda |u|^2 u = \eta(t,x), \quad u = u(t,x), \tag{0.1}$$

where $0 < \nu \le 1$, $\lambda > 0$, $\dim x \le 4$, and either $x \in D \Subset \mathbb{R}^n$, and then the equation is supplemented with the Dirichlet boundary condition, or $x \in \mathbb{R}^n$, and then the odd-periodic boundary conditions are imposed (see relation (1.7) below). The force $\eta(t,x)$ is a random field white in time and sufficiently smooth in x:

$$\eta(t,x) = \frac{\partial}{\partial t} \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \quad \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty. \tag{0.2}$$

Here $\{e_j\}$ are normalised (real) eigenfunctions of the Dirichlet Laplacian with eigenvalues $\{\alpha_j\}$, and $\{\beta_j(t), t \ge 0\}$ are independent standard complex-valued Wiener processes.

In the first section of the paper, we show that Eq. (0.1) has a unique solution defined for $t \ge 0$ and equal to a given initial function at $t = 0$ (see Proposition 1.2). For this solution, we derive various a priori estimates and next use them to prove that Eq. (0.1) has stationary in time solutions (see Proposition 1.5). We note that in [Kuk99] similar results are obtained for the CGL equation without linear dispersion:

$$\dot{u} - \nu\Delta u + i\lambda |u|^2 u = \eta(t,x). \tag{0.3}$$

Proof in [Kuk99] differs essentially from the argument used in this work, since it is based on rather different a priori estimates. (The main estimates used in the study of (0.1) do not hold for (0.3), and vice versa.)

The last result of the section, Theorem 1.6, states that, if all coefficients b_j are non-zero, then Eq. (0.1) (whose solutions are interpreted as Markov processes in a suitable functional space) has a unique stationary measure $\hat{\mu}$, and any solution $u(t,x)$ of (0.1) converges to $\hat{\mu}$ in distribution:

$$\mathcal{D}\big(u(t,\cdot)\big) \rightharpoonup \hat{\mu} \quad \text{as } t \to +\infty. \tag{0.4}$$

Section 2 contains our main results. There we study the CGL equation perturbed by a random force of order $\sqrt{\nu}$:

$$\dot{u} - (\nu + i)\Delta u + i\lambda |u|^2 u = \sqrt{\nu}\,\eta(t,x), \tag{0.5}$$

where $\eta(t,x)$ is defined in (0.2). Let $u_\nu(t)$, $t \ge 0$, be a stationary (in time) solution of (0.5). In Theorem 2.4, we show that every sequence $\hat{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that the random processes $u_{\nu_j}(t)$ converge in distribution (as $j \to \infty$) to a stationary process $v(t) = v(t,\cdot)$ with values in the Sobolev space H^2. The process v possesses the following properties:

(i) The trajectories of v are solutions of the NLS equation

$$\dot{v} - i\Delta v + i\lambda|v|^2 v = 0. \tag{0.6}$$

(ii) The trajectories $v(t,\cdot)$ have two integrals of motion:

$$\frac{1}{2}\int_D |v(t,x)|^2 dx \equiv E_0, \qquad \int_D \Big(\frac{1}{2}|\nabla v(t,x)|^2 + \frac{\lambda}{4}|v(t,x)|^4\Big)\, dx \equiv E_1,$$

where E_0 and E_1 are random constants.

(iii) The mean values of the L^2, H^1, and H^2 norms of v satisfy the relations

$$\begin{gathered}\tfrac{\alpha_1 B_0^2}{\alpha_1 B_1 + 2\lambda M B_0} \le \mathbb{E}\|v(0)\|^2 \le \tfrac{B_0}{\alpha_1},\\ \mathbb{E}\|\nabla v(0)\|^2 = B_0,\\ \mathbb{E}\big\{\|\Delta v(0)\|^2 + \tfrac{\alpha_1\lambda}{4}\|v(0)\|_{L^4}^4\big\} \le \tfrac{\alpha_1 B_1 + 2\lambda M B_0}{\alpha_1}.\end{gathered}$$

Here $\|\cdot\|$ is the L^2 norm, and we set

$$B_0 = \sum_{j=1}^{\infty} b_j^2, \quad B_1 = \sum_{j=1}^{\infty} \alpha_j b_j^2. \tag{0.7}$$

The process v is the inviscid limit for the solutions u_{ν_j}. Due to the results above, this limit is a nontrivial stationary process whose trajectories are H^2-solutions of the NLS equation (0.6).

If either $n \le 3$, and the equations are supplemented with the odd-periodic boundary conditions, or $n \le 2$, and the Dirichlet boundary condition is imposed, then the NLS equation (0.6) defines a group of continuous transformations of the Sobolev space H^2 (see [Bou99, BGT03] and [BG80, SS99], respectively). The measure $\mu = \mathcal{D}(v(0))$ is invariant for this group of transformations. For $n \le 2$, the NLS equation under the odd-periodic boundary condition has invariant Gibbs measures (see [Bou99]). Presumably, the measure μ is different from them.

Let us now assume that $b_j \ne 0$ for all $j \ge 1$, and let $u_\nu(t)$, $t \ge 0$, be a solution of (0.5) with the initial condition $u_\nu(0) = u_0$, where $u_0(x)$ belongs to the Sobolev space H^1. Then due to (0.4) and the above results, we have

$$\lim_{j\to\infty}\lim_{T\to\infty} \mathcal{D}\big(u_{\nu_j}(T)\big) = \mu.$$

In Theorem 2.7, we show that the following stronger statement is true:

$$\lim_{j\to\infty}\lim_{T\to+\infty} \mathcal{D}\big(u_{\nu_j}(T+\cdot)\big) = \mathcal{D}\big(v(\cdot)\big),$$

where the convergence holds in the space of probability measures on $C(\mathbb{R}_+, L^2)$. Thus, the measures μ and $\mathcal{D}(v(\cdot))$ corresponding to various sequences $\nu_j \to 0$ as above describe the distributions of solutions for Eq. (0.5) when $t \gg 1$ and $\nu \ll 1$.

The CGL equation (0.5) is an example of a damped/driven Hamiltonian PDE (now the Hamiltonian PDE is the NLS equation (0.6)). The approach to study its small-viscosity solutions exploited in this work applies to any equation of this kind, provided that the underlying Hamiltonian PDE has two (or more) "good" integrals of motion. In [Kuk03], this approach is used to study small-viscosity solutions of the 2D Navier–Stokes system, interpreted as a damped/driven Euler equation.

Notation. For a random variable ξ, we denote by $\mathcal{D}(\xi)$ its distribution. All metric spaces are endowed with the Borel σ-algebras, and the measures on these spaces are assumed to be probability Borel measures. We deal with complex random fields $u(t,x)$ and often interpret them as random processes $u(t)=u(t,\cdot)$ in suitable functional spaces.

1 A priori estimates and stationary solutions for the CGL equation

1.1 Local well-posedness

Let us consider a randomly forced complex Ginzburg–Landau equation in a bounded domain $D\subset\mathbb{R}^n$, $n\le 4$, with a smooth boundary ∂D:

$$\dot u-(\nu+i)\Delta u+i\lambda|u|^2u=\eta(t,x), \tag{1.1}$$

$$u\big|_{\partial D}=0. \tag{1.2}$$

Here $\nu\in(0,1]$ and $\lambda>0$ are constants and $\eta(t,x)$ is a random process of the form

$$\eta(t,x)=\frac{\partial}{\partial t}\zeta(t,x),\quad \zeta(t,x)=\sum_{j=1}^{\infty}b_j\beta_j(t)e_j(x), \tag{1.3}$$

where $e_j(x)$, $j\ge 1$, are the normalised eigenfunctions of the Laplace operator $-\Delta$ in D with the Dirichlet boundary condition that correspond to eigenvalues $\alpha_1<\alpha_2\le\alpha_3\le\cdots$, $\{\beta_j(t)=\beta_{1j}(t)+i\beta_{2j}(t),t\in\mathbb{R}_+\}$ is a sequence of independent complex-valued standard Brownian motions defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and $b_j\ge 0$ are some constants satisfying the condition

$$B_1=\sum_{j=1}^{\infty}\alpha_j b_j^2<\infty. \tag{1.4}$$

We denote by $(H,\|\cdot\|)$ the L^2-space of complex-valued functions on D. In what follows, we view it as a real Hilbert space with the scalar product

$$(u,v)=\mathrm{Re}\int_D u(x)\bar v(x)\,dx.$$

Clearly, we have

$$(u, iu) = 0 \quad \text{for any } u \in H. \tag{1.5}$$

Setting $e_{-j} = ie_j$ for $j \geq 1$, we see that the functions e_j, $j \in \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, form an orthonormal basis in H.

For any $s \in \mathbb{R}$, let H^s be the domain of the operator $(-\Delta_D)^{s/2}$, where Δ_D is the Laplace operator acting on complex-valued functions in D supplemented with the Dirichlet boundary condition. In particular, H^1 is the space of functions that belong to the Sobolev space $H^1(D, \mathbb{C})$ and satisfy the boundary condition (1.2), and H^2 is the intersection of $H^2(D, \mathbb{C})$ and H^1. We provide H^1 and H^2 with the norms

$$\|u\|_1 = \left(\int_D \left|\nabla u(x)\right|^2 dx\right)^{\frac{1}{2}}, \quad \|u\|_2 = \left(\int_D \left|\Delta u(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Since $n \leq 4$, the space H^1 is continuously embedded into $L^4 = L^4(D, \mathbb{C})$.

Often the Dirichlet problem (1.1), (1.2) will be supplemented with the initial condition

$$u(0, x) = u_0(x), \tag{1.6}$$

where u_0 is an H^1-valued random variable independent of ζ.

We shall also study equation (1.1) under the odd-periodic boundary conditions:

$$x \in \mathbb{R}^n, \quad u(x_1, \dots, x_j, \dots, x_n) = u(\dots, x_j + 2\pi, \dots) = -u(\dots, -x_j, \dots), \tag{1.7}$$

where $j = 1, \dots, n$. Relations (1.7) imply the Dirichlet boundary conditions for the cube of half-periods, so properties of the problem (1.1), (1.7) are similar to those of (1.1), (1.2) (in fact, in many respects the former is easier than the latter). Accordingly, below we prove our results for the problem (1.1), (1.2) and only briefly discuss their (obvious) reformulations for (1.1), (1.7).

The initial-value problem (1.1), (1.2), (1.6) is well posed. To formulate the corresponding result, we shall need some notations.

For any Banach space X and any $T > 0$, we denote by $C(0, T; X)$ the space of continuous functions $f : [0, T] \to X$ endowed with the norm

$$\|f\|_{C(0,T;X)} = \sup_{t \in [0,T]} \|f(t)\|_X .$$

For $1 \leq p < \infty$, $L^p(0, T; X)$ denotes the space of Bochner-measurable functions $f(t)$ such that

$$\|f\|_{L^p(0,T;X)} := \left(\int_0^T \|f(t)\|_X^p \, dt\right)^{\frac{1}{p}} < \infty.$$

Proposition 1.1. *For any random initial function u_0 independent of ζ there is an a.s. positive random constant T and a random process $u(t) = u(t,x)$, $0 \le t \le T$, adapted to the filtration generated by u_0 and ζ such that the following assertions hold:*

(i) *Almost every realization of $u(t,x)$ belongs to the space*

$$\mathcal{S}(T) := L^2(0,T;H^2) \cap C(0,T;H^1).$$

(ii) *The random process $u(t,x)$ satisfies Eq.* (1.1) *and the initial condition* (1.6) *in the sense that, with probability* 1, *we have*

$$u(t) = u_0 + \int_0^t \left((\nu+i)\Delta u - i\lambda|u|^2 u\right) ds + \zeta(t), \quad 0 \le t \le T, \tag{1.8}$$

where the left- and right-hand sides are regarded as elements of H.

(iii) *If $u'(t,x)$, $0 \le t \le T'$, is another random process satisfying (i) and (ii), then, with probability* 1, *we have $u(t) = u'(t)$ for $0 \le t \le T \wedge T'$.*

Scheme of the proof. The uniqueness (in the sense of (iii)) can be easily established with the help of (1.8) and the Gronwall inequality. To prove the existence, we set

$$z(t,x) = \int_0^t e^{(\nu+i)(t-s)\Delta_D} d\zeta(t,x),$$

and seek a solution of the form

$$u(t,x) = z(t,x) + v(t,x).$$

Substituting this formula into (1.1), (1.2), (1.6), we obtain for v the following problem:

$$\dot v - (\nu+i)\Delta v + i\lambda|z+v|^2(z+v) = 0,$$
$$v\big|_{\partial D} = 0, \quad v(0,x) = u_0(x).$$

Its local in time solution can be constructed as a fixed point of the operator $\mathcal{F}$,

$$\mathcal{F}(w)(t) = e^{(\nu+i)t\Delta_D}u_0 - i\lambda \int_0^t e^{(\nu+i)(t-s)\Delta_D}\left(|z+w|^2(z+w)\right) ds,$$

restricted to an appropriate subset of $\mathcal{S}(T)$ with a sufficiently small $T > 0$. $\square$

In what follows, we shall need *maximal* solutions of the problem (1.1), (1.2). More precisely, for any $\omega \in \Omega$, we can extend the solution $u(t,x)$ constructed in Proposition 1.1 to a random time interval $[0, T(u_0))$ with the following property:

$$\text{if } T(u_0) < +\infty, \text{ then } \|u(t)\|_1 \to +\infty \text{ as } t \to T(u_0)^-. \tag{1.9}$$

We claim that $T(u_0)$ is a Markov time with respect to the filtration of σ-algebras generated by the initial condition u_0 and the process ζ. Indeed, let us fix an arbitrary integer $R \geq 1$ and consider the truncated equation

$$\dot{u} - (\nu + i)\Delta u + i\lambda\chi_R\big(\|u\|_1\big)|u|^2 u = \eta(t,x), \tag{1.10}$$

where $\chi_R \in C_0^\infty(\mathbb{R})$ is a function such that $0 \leq \chi_R \leq 1$ and $\chi_R(r) = 1$ for $|r| \leq R$. Repeating the scheme used in the proof of Proposition 1.1, it is not difficult to establish the following assertion: for any $N > 0$ and a.e. $\omega \in \Omega$, there is a non-increasing function $\theta(r) > 0$, $r \geq 0$, such that, for any $t_0 \in [0, N]$ and $u_0 \in H^1$, the problem (1.10), (1.2) has a unique solution $u(t,x)$ defined on the interval $[t_0, t_0 + \theta(\|u_0\|_1)]$ and satisfying the initial condition $u(t_0) = u_0$. Therefore, if the H^1-norm of a solution of (1.10), (1.2) remains bounded on any compact interval, then it can be extended to the half-line $t \geq 0$. Using the fact that the nonlinear term in (1.10) vanishes for functions with large H^1-norm, we conclude that, for any H^1-valued random variable u_0, the solution of the problem (1.10), (1.2), (1.6) is defined for all $t \geq 0$. Let us denote this solution by $u_R(t,x)$ and introduce the Markov time

$$T_R(u_0) = \min\big\{t \geq 0 : \|u_R(t)\|_1 \geq R\big\}$$

(here and below, the minimum of an empty set is $+\infty$). It is a matter of direct verification to show that $T(u_0) = \sup_R T_R(u_0)$. Hence, we conclude that $T(u_0)$ also is a Markov time.

1.2 A priori estimates and global existence

Our next goal is to show that any maximal solution of the problem (1.1), (1.2) is defined for all $t \geq 0$. This assertion will be proved under the additional assumption that the mean value of the energy is finite at the initial instant. Namely, let us define the following continuous functionals on H^1:

$$\mathcal{H}_0(u) := \frac{1}{2}\|u\|^2 = \frac{1}{2}\int_D |u(x)|^2 dx,$$
$$\mathcal{H}_1(u) := \int_D \Big(\frac{1}{2}\,|\nabla u(x)|^2 + \frac{\lambda}{4}\,|u(x)|^4\Big)\,dx.$$

The functional $\mathcal{H}_1$ is the Hamiltonian (i.e. "the energy") of the NLS equation (0.6), see [Kuk00], and $\mathcal{H}_0$ is called its "total number of particles". Both functionals are integrals of motion for Eq. (0.6).

Proposition 1.2. *Let u_0 be an H^1-valued random variable independent of $\zeta(t)$, $t \geq 0$, such that $\mathbb{E}\mathcal{H}_1(u_0) < \infty$. Suppose that*

$$M := \sup_{x\in D}\sum_{j=1}^{\infty} b_j^2 e_j^2(x) < \infty. \tag{1.11}$$

Then the following assertions hold for the maximal solution $u(t)$ of the problem (1.1), (1.2), (1.6):

(i) *For almost every ω we have $T(u_0) = +\infty$.*

(ii) *The random processes $\mathcal{H}_0(u(t))$ and $\mathcal{H}_1(u(t))$ possess stochastic differentials, which have the form*

$$d\mathcal{H}_0(u(t)) = \Big(-\nu \|\nabla u(t)\|^2 + B_0\Big)\, dt + \big(u(t), d\zeta(t)\big), \tag{1.12}$$

$$\begin{aligned} d\mathcal{H}_1(u(t)) = \Big(&-\nu\left[\|\Delta u\|^2 + 2\lambda\big(|u|^2, |\nabla u|^2\big) + \lambda\big(u^2, (\nabla u)^2\big)\right] + B_1 \\ &+ 2\lambda \sum_{j=1}^{\infty} b_j^2\big(|u|^2, e_j^2\big)\Big)\, dt + \big(-\Delta u + \lambda|u|^2 u, d\zeta\big), \end{aligned} \tag{1.13}$$

where the constants B_0 and B_1 are defined in (0.7).

(iii) *For any $t \geq 0$, we have*

$$\mathbb{E}\mathcal{H}_0(u(t)) + \nu \int_0^t \mathbb{E}\|u(s)\|_1^2\, ds = \mathbb{E}\mathcal{H}_0(u_0) + B_0 t, \tag{1.14}$$

$$\begin{aligned} &\mathbb{E}\mathcal{H}_1(u(t)) + \int_0^t \mathbb{E}\big\{\nu\|\Delta u\|^2 + \nu\lambda\big(|u|^2, |\nabla u|^2\big) - 2\lambda M\|u\|^2\big\}\, ds \\ &\quad \leq \mathbb{E}\mathcal{H}_1(u_0) + B_1 t \\ &\quad\quad \leq \mathbb{E}\mathcal{H}_1(u(t)) + \nu \int_0^t \mathbb{E}\big\{\|\Delta u\|^2 + 3\lambda\big(|u|^2, |\nabla u|^2\big)\big\} ds. \end{aligned} \tag{1.15}$$

The proof of this proposition is based on the assertion below, which is a variant of Itô's lemma for randomly forced PDE's (cf. [Shi02, Theorem 1.3]). For any $R > 0$, let us introduce the Markov time

$$\tau_R := \min\big\{t : 0 \leq t < T(u_0), \|u(t)\|_1 \geq R\big\}.$$

Using (1.9) we see that, if $T(u_0) < \infty$, then the minimum is taken over a non-empty set. Hence,

$$\tau_R \leq T(u_0) \quad \text{for all } R > 0. \tag{1.16}$$

Lemma 1.3. *Let $\mathcal{H} : H^1 \to \mathbb{C}$ be a twice continuously $\mathbb{R}$-differentiable functional such that $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{H}''$ are uniformly continuous on bounded subsets of H^1, and $\mathcal{H}'$ satisfies the inequality*

$$\big|\mathcal{H}'(u; v)\big| \leq K(\|u\|_1)\|u\|_2\|v\|, \quad u \in H^2, \quad v \in L^2, \tag{1.17}$$

where $K(r) \geq 0$ is a continuous function defined for $r \geq 0$. Let $u(t)$ be a maximal solution of the problem (1.1), (1.2). *Then, for any $R > 0$, the process $\mathcal{H}(u(t \wedge \tau_R))$ possesses a stochastic differential, which has the form*

$$\begin{aligned} d\mathcal{H}(u(t \wedge \tau_R)) = I_{\{\tau_R > t\}} \Big(&\mathcal{H}'\big(u; (\nu + i)\Delta u - i\lambda|u|^2 u\big) + \frac{1}{2} \sum_{j \in \mathbb{Z}_0} b_j^2 \mathcal{H}''(u; e_j)\Big)\, dt \\ &+ I_{\{\tau_R > t\}} \sum_{j=1}^{\infty} b_j \mathcal{H}'(u; e_j)\, d\beta_j(t), \end{aligned} \tag{1.18}$$

where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, $\mathcal{H}'(u;\cdot)$ denotes the continuous extension of the derivative of $\mathcal{H}$ to the space L^2, and $I_\Gamma = I_\Gamma(\omega)$ is the indicator function of a set Γ.

A proof of Lemma 1.3 can be obtained by standard arguments, based on applying Itô's lemma (see [DZ92, Theorem 4.17]) to the process $\mathcal{H}(u_\varepsilon(t \wedge \tau_R))$, where $u_\varepsilon(t) = (1-\varepsilon\Delta_D)^{-1}u(t)$, $\varepsilon > 0$, and passing to the limit as $\varepsilon \to 0$. (Note that the process $u_\varepsilon(t)$ possesses a stochastic differential in the space H^1.) We shall not give the details.

Proof of Proposition 1.2. The main idea of the proof is rather standard. Namely, we apply Itô's formula (1.18) to the processes $\mathcal{H}_0(u(t\wedge\tau_R))$ and $\mathcal{H}_1(u(t\wedge\tau_R))$, integrate with respect to t, and take the expectation. This results in some a priori estimates for $\mathbb{E}\mathcal{H}_0(u(t\wedge\tau_R))$ and $\mathbb{E}\mathcal{H}_1(u(t\wedge\tau_R))$ that are uniform in R. We next note that, if $\tau_R(\omega) \le t_0$ for all $R > 0$ and $\omega \in \Gamma$, where $t_0 > 0$ is a constant and $\Gamma \in \mathcal{F}$, then $\|u(t_0 \wedge \tau_R)\|_1 \ge R$ for $\omega \in \Gamma$. Hence,

$$\mathbb{E}\mathcal{H}_1\big(u(t_0\wedge\tau_R)\big) \ge \mathbb{E}\big\{ I_\Gamma \mathcal{H}_1\big(u(t_0\wedge\tau_R)\big)\big\} \ge \tfrac{1}{2}\,\mathbb{P}(\Gamma)\,R^2.$$

Since $\mathbb{E}\mathcal{H}_1(u(t_0\wedge\tau_R))$ is bounded uniformly with respect to $R>0$, we conclude that $\mathbb{P}(\Gamma) = 0$. Thus, for a.e. ω we have

$$\tau_R \to +\infty \quad \text{as} \quad R \to +\infty.$$

Now (1.16) implies the assertion (i).

We shall confine ourselves to a formal derivation of relations (1.12) – (1.15). It is a matter of direct verification to show that the functionals $\mathcal{H}_0(u)$ and $\mathcal{H}_1(u)$ are infinitely $\mathbb{R}$-differentiable on H^1, and their first two derivatives have the form

$$\begin{aligned} \mathcal{H}_0'(u;v) &= (u,v),\\ \mathcal{H}_0''(u;v) &= \|v\|^2,\\ \mathcal{H}_1'(u;v) &= \big(-\Delta u + \lambda|u|^2u, v\big),\\ \mathcal{H}_1''(u;v) &= \|\nabla v\|^2 + 2\lambda\big(|u|^2,|v|^2\big) + \lambda(u^2,v^2). \end{aligned} \tag{1.19}$$

Applying Itô's formula to $\mathcal{H}_0(u)$, we derive

$$\begin{aligned} d\mathcal{H}_0(u(t)) &= \mathcal{H}_0'(u;du) + \frac{1}{2}\sum_{j\in\mathbb{Z}_0} b_j^2 \mathcal{H}_0''(u;e_j)\,dt\\ &= \big(u,(\nu+i)\Delta u - i\lambda|u|^2u\big)\,dt + (u,d\zeta) + B_0dt, \end{aligned}$$

whence follows (1.12).

In a similar way, application of Itô's formula to the functional $\mathcal{H}_1(u)$ results in the relation

$$\begin{aligned} d\mathcal{H}_1(u(t)) &= \mathcal{H}_1'(u;du) + \frac{1}{2}\sum_{j\in\mathbb{Z}_0} b_j^2\mathcal{H}_1''(u;e_j)\,dt\\ &= \big(-\Delta u + \lambda|u|^2u, (\nu+i)\Delta u - i\lambda|u|^2u\big)\,dt + \big(-\Delta u + \lambda|u|^2u, d\zeta\big)\\ &\quad + B_1dt + 2\lambda\sum_{j\in\mathbb{Z}_0} b_j^2\big(|u|^2,e_j^2\big)\,dt, \end{aligned} \tag{1.20}$$

where we used that

$$\mathcal{H}_1''(u;e_j)+\mathcal{H}_1''(u;e_{-j})=4\lambda(|u|^2,e_j^2),\quad j\ge 1.$$

Due to (1.5), the first term in the right-hand side of (1.20) equals

$$(-\Delta u+\lambda|u|^2u,\nu\Delta u)\,dt=-\nu\Bigl(\|\Delta u\|^2+2\lambda(|u|^2,|\nabla u|^2)+\lambda(u^2,(\nabla u)^2)\Bigr)\,dt.$$

It follows that (1.20) implies (1.13).

We now establish inequalities (1.14) and (1.15). Integrating relation (1.12) with respect to t and taking the expectation, we obtain (1.14). To prove (1.15), we note that

$$\bigl|\bigl(u^2,(\nabla u)^2\bigr)\bigr|\le\bigl(|u|^2,|\nabla u|^2\bigr),\qquad 0\le\sum_{j=1}^\infty b_j^2\bigl(|u|^2,e_j^2\bigr)\le M\,\|u\|^2.$$

Integrating relation (1.13) with respect to t, taking the expectation, and using the above inequalities, we derive (1.15). □

In what follows, we shall also need an estimate for a higher moment of the H^1-norm of solutions for (1.1), (1.2). Let us set

$$P:=\sum_{j=1}^\infty b_j^2\|e_j\|_{L^4}^2. \tag{1.21}$$

In view of the Sobolev embedding theorem (e.g., see Theorem 2.1 in [VF88, Chapter I]), if $n\le 4$, then

$$\|e_j\|_{L^4}^2\le C\,\|e_j\|_1^2=C\alpha_j,$$

where $C>0$ is a constant not depending on j. Hence, if condition (1.4) is satisfied, then $P<\infty$.

Proposition 1.4. *Suppose that the conditions of Proposition 1.2 are satisfied. Let $u(t,x)$ be a solution of the problem (1.1), (1.2), (1.6) with an initial condition $u_0(x)$ such that $\mathbb{E}\mathcal{H}_1^2(u_0)<\infty$. Then there are positive constants c and C, depending only on the domain D, such that, for $t\ge 0$, we have*

$$\mathbb{E}\mathcal{H}_1^2(u(t))\le e^{-c\nu t}\mathbb{E}\mathcal{H}_1^2(u_0)+C\nu^{-2}B_1^2+C\nu^{-4}\lambda^2(M^4+P^4). \tag{1.22}$$

Proof. We shall give a formal derivation of (1.22). Applying Itô's formula to the process $\mathcal{H}_1^2(u(t))$, we derive [1]

$$d\mathcal{H}_1^2(u(t))=2\,\mathcal{H}_1(u(t))\,d\mathcal{H}_1(u(t))+\sum_{j\in\mathbb{Z}_0}b_j^2\bigl(\mathcal{H}_1'(u(t);e_j)\bigr)^2dt.$$

[1]Note that the functional $\mathcal{H}_1^2(u)$ satisfies the conditions of Lemma 1.3, and therefore application of Itô's formula is justified.

Integrating this relation with respect to t, using formula (1.13), taking the expectation, and repeating the arguments in the proof of Proposition 1.2, we can show that

$$\mathbb{E}\mathcal{H}_1^2(u(t)) + \mathbb{E}\int_0^t \mathcal{F}_\nu\big(u(s)\big)\,ds \le \mathbb{E}\mathcal{H}_1^2(u_0), \tag{1.23}$$

where we set

$$\begin{aligned}\mathcal{F}_\nu(u) = 2\nu\,\mathcal{H}_1(u)\big(\|\Delta u\|^2 + \lambda\big(|u|^2, |\nabla u|^2\big)\big)\\ - 2\mathcal{H}_1(u)\big(B_1 + 2\lambda M\|u\|^2\big) - \sum_{j\in\mathbb{Z}_0} b_j^2\big(\mathcal{H}_1'(u;e_j)\big)^2.\end{aligned} \tag{1.24}$$

It follows from (1.19) that

$$\begin{aligned}\sum_{j\in\mathbb{Z}_0} b_j^2\big|\mathcal{H}_1'(u;e_j)\big|^2 &\le 2\sum_{j\in\mathbb{Z}_0} b_j^2\Big(\alpha_j^2u_j^2 + \lambda^2(|u|^2u, e_j)^2\Big)\\ &\le 2B_1\|u\|_1^2 + 2\lambda^2P\|u\|_{L^4}^6,\end{aligned}$$

where we set $\alpha_{-j} = \alpha_j$ for $j \ge 1$. Furthermore, we have

$$\begin{gathered}\|u\|_1^2 \le 2\,\mathcal{H}_1(u), \qquad \|u\|_{L^4}^6 \le 8\lambda^{-\frac{3}{2}}\mathcal{H}_1(u)^{\frac{3}{2}},\\ \|u\|^2 \le \big(\mathrm{vol}(D)\big)^{\frac{1}{2}}\|u\|_{L^4}^2 \le 2\lambda^{-\frac{1}{2}}\big(\mathrm{vol}(D)\big)^{\frac{1}{2}}\mathcal{H}_1(u)^{\frac{1}{2}},\end{gathered}$$

where $\mathrm{vol}(D)$ is the volume of D. Hence, for any $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ not depending on ν such that

$$\begin{aligned}2\mathcal{H}_1(u)\big(B_1 + 2\lambda M\|u\|^2\big) &\le 2B_1\mathcal{H}_1(u) + 8\lambda^{\frac{1}{2}}M\,\big(\mathrm{vol}(D)\big)^{\frac{1}{2}}\mathcal{H}_1(u)^{\frac{3}{2}}\\ &\le \varepsilon\nu\mathcal{H}_1^2(u) + C_\varepsilon\big(\nu^{-1}B_1^2 + \nu^{-3}\lambda^2M^4\big),\\ 2B_1\|u\|_1^2 &\le 4B_1\mathcal{H}_1(u) \le \varepsilon\nu\mathcal{H}_1^2(u) + C_\varepsilon\nu^{-1}B_1^2,\\ 2\lambda^2P\|u\|_{L^4}^6 &\le 16\lambda^{\frac{1}{2}}P\,\mathcal{H}_1(u)^{\frac{3}{2}} \le \varepsilon\nu\mathcal{H}_1^2(u) + C_\varepsilon\nu^{-3}\lambda^2P^4.\end{aligned}$$

Now note that [2]

$$\big\|u\big\|_{L^4(D)}^4 \le 4\alpha_1^{-1}\big(|u|^2, |\nabla u|^2\big), \tag{1.25}$$

Hence, there is $c > 0$ such that

$$\|\Delta u\|^2 + \lambda\big(|u|^2, |\nabla u|^2\big) \ge c\mathcal{H}_1(u).$$

Substituting the above inequalities into (1.24) and choosing $\varepsilon = \frac{c}{3}$, we derive

$$\mathcal{F}_\nu(u) \ge c\nu\,\mathcal{H}_1^2(u) - C_0\nu^{-1}B_1^2 - C_0\nu^{-3}\lambda^2(M^4 + P^4),$$

[2] This inequality follows by applying the Poincaré inequality to the function $|u|^2$.

where $C_0 > 0$ is a constant depending only on the domain D. Combining this with (1.23), we obtain

$$\mathbb{E}\mathcal{H}_1^2(u(t))+c\nu\int_0^t \mathbb{E}\mathcal{H}_1^2(u(s))\,ds \le \mathbb{E}\mathcal{H}_1^2(u_0)+C_0\nu^{-1}tB_1^2+C_0\nu^{-3}\lambda^2 t(M^4+P^4).$$

The required estimate (1.22) with $C = C_0/c$ follows now from the Gronwall inequality. □

Next we shall derive some corollaries on stationary solutions of problem (1.1), (1.2). We first note that such a solution exists as soon as conditions (1.4) and (1.11) are satisfied. Indeed, let us denote by $u(t)$ the solution of (1.1), (1.2), (1.6) with $u_0 \equiv 0$. The a priori estimates (1.14) and (1.15), combined with the Gronwall inequality, imply that

$$\nu \sup_{t\ge 0}\int_t^{t+1} \mathbb{E}\|\Delta u(s)\|^2 ds \le \text{const}\,. \tag{1.26}$$

Following the classical Bogolyubov–Krylov argument, we set

$$\bar{\mu}_t = \frac{1}{t}\int_0^t \mu_s\, ds,$$

where μ_s is the distribution of $u(s)$. It follows from inequality (1.26) that

$$\int_H \|v\|_2^2\,\bar{\mu}_t(dv) \le t^{-1}\int_0^t \mathbb{E}\|\Delta u(s)\|^2 ds \le \text{const}\,\nu^{-1}.$$

Combining this with the Chebyshev inequality, we conclude that the family of measures $\{\mathcal{D}(\bar{u}(t)), t \ge 0\}$ is tight in H^1. The existence of a stationary measure can now be obtained by a standard argument (see [DZ96, Section 3.1]).

The following result, which is a consequence of (1.14), (1.15), and (1.22), concerns *all* stationary measures in H^1.

Proposition 1.5. *Suppose that conditions* (1.4) *and* (1.11) *are satisfied, and let $u(t,x)$ be an H^1-valued stationary solution of the problem* (1.1), (1.2). *Then*

$$\mathbb{E}\|\nabla u\|^2 = \frac{B_0}{\nu}, \tag{1.27}$$

$$\mathbb{E}\big\{\|\Delta u\|^2 + \lambda\big(|u|^2, |\nabla u|^2\big)\big\} \le \frac{B_1}{\nu} + \frac{2\lambda M B_0}{\nu^2\alpha_1}, \tag{1.28}$$

$$\mathbb{E}\mathcal{H}_1^2(u) \le \frac{CB_1^2}{\nu^2} + \frac{C\lambda^2(M^4+P^4)}{\nu^4}. \tag{1.29}$$

Proof. Inequality (1.22) implies that, for any H^1-valued stationary solution of the problem (1.1), (1.2), we have

$$\mathbb{E}\mathcal{H}_1^2(u(t)) < \infty \quad \text{for all} \quad t \ge 0. \tag{1.30}$$

Indeed, let μ be a stationary measure, $u(t)$ be a stationary solution with distribution μ, and let $\chi_R(s)\ge 0$ be a smooth function equal to 1 for $s\le R$ and to 0 for $s\ge R+1$. Then, for any $t\ge 0$, we have

$$\int_{H^1}\mathcal{H}_1^2(v)\chi_R(\|v\|_1)\mu(dv)=\int_{H^1}\mathbb{E}\{\mathcal{H}_1^2(u(t,v))\chi_R(\|u(t,v)\|_1)\}\mu(dv),$$

where $u(t,v)$ denotes the solution of (1.1), (1.2), starting from $v\in H^1$. Using (1.22), we can pass to the limit in the right-hand side of this relation as $t\to+\infty$. This results in

$$\mathbb{E}\mathcal{H}_1^2(u(t))\chi_R(\|u(t)\|_1)=\int_{H^1}\mathcal{H}_1^2(v)\chi_R(\|v\|_1)\mu(dv)\le\text{const}\,.$$

Application of Fatou's lemma gives (1.30). An accurate proof of (1.30) can be obtained by repeating the arguments used in the proof of Theorem 2.2 in [Shi02].

Now Propositions 1.2 and 1.4 imply that relation (1.14) and inequalities (1.15) and (1.22) hold for any H^1-valued stationary solution $u(t)$. Relation (1.27) follows immediately from (1.14) and the fact that the expectation $\mathbb{E}\mathcal{H}_0(u(t))$ does not depend on t. Similarly, to prove (1.28), it suffices to note that $\|u\|^2\le\alpha_1^{-1}\|u\|_1^2$ and to use inequality (1.15). Finally, passing to the limit in (1.22) as $t\to+\infty$, we obtain (1.29). □

We conclude this subsection by formulating (without proof) a result on uniqueness of a stationary measure for problem (1.1), (1.2). Its proof is carried out by the methods used in [KS00, KS01, KS02] and will be given elsewhere. [3]

Theorem 1.6. *Suppose that conditions* (1.4) *and* (1.11) *are satisfied and that*

$$b_j\ne 0\quad\textit{for all}\quad j\ge 1.\tag{1.31}$$

Then for any $\nu>0$ the problem (1.1), (1.2) *has a unique stationary measure μ_ν. Moreover, for any $\nu>0$ there are positive constants C and γ such that, for any Lipschitz functional $f: H^1\to\mathbb{R}$ with* $\sup|f|\le 1$ *and* $\mathrm{Lip}(f)\le 1$, *and any random variable u_0 satisfying the condition $\mathbb{E}\mathcal{H}_1(u_0)<\infty$, we have*

$$\big|\mathbb{E}f(u(t;u_0))-(f,\mu_\nu)\big|\le C\big(1+\mathbb{E}\mathcal{H}_1(u_0)\big)\,e^{-\gamma t},\quad t\ge 0,\tag{1.32}$$

where $u(t;u_0)$ is the solution of (1.1), (1.2), (1.6) *and (f,μ_ν) is the mean value of f with respect to μ_ν.*

All the results of this section remain valid for the odd-periodic boundary conditions (1.7). In this case, the trigonometric functions $\left(\frac{2}{\pi}\right)^{\frac{n}{2}}\sin(k_1x_1)\cdots\sin(k_nx_n)$ form an eigenbasis in $L^2(K)$, where $K=(0,\pi)^n$ is the cube of half-periods, and condition (1.11) is implied by (1.4).

[3] The papers [KS00, KS01] deal with a large class of PDE's (including the CGL equation) perturbed by a random kick force, while [KS02] concerns the 2D Navier–Stokes system perturbed by a white noise in time. The results obtained in those papers do not imply the uniqueness of a stationary measure for the CGL equation (2.1), however, the main ideas remain valid (see also [EMS01, BKL02, Mat02, Hai02]).

2 The inviscid limit

2.1 Tightness of stationary measures

We now consider Eq. (1.1) with the right-hand side replaced by $\eta_\nu = \sqrt{\nu}\,\eta$:

$$\dot{u} - (\nu + i)\Delta u + i\lambda|u|^2 u = \sqrt{\nu}\,\eta(t,x). \tag{2.1}$$

Let $u_\nu(t,x)$, $0 < \nu \le 1$, be stationary solutions for the problem (2.1), (1.2) that are defined for $t \ge 0$. We denote by μ_ν (respectively, $\boldsymbol{\mu}_\nu$) their distribution in the space H (respectively, $L^2((0,1)\times D;\mathbb{C})$).

For a Banach space X and $1 \le p < \infty$, let $W^{1,p}(0,T;X)$ be the space of absolutely continuous functions $f\colon [0,T] \to X$ such that

$$\|f\|^p_{W^{1,p}(0,T;X)} := \|f\|^p_{L^p(0,T;X)} + \|f'\|^p_{L^p(0,T;X)} < \infty.$$

For any $\alpha \in (0,1)$, let us define $W^{\alpha,p}(0,T;X)$ as the space of functions $f \in L^p(0,T;X)$ such that

$$\|f\|^p_{W^{\alpha,p}(0,T;X)} = \|f\|^p_{L^p(0,T;X)} + \int_0^T\!\!\int_0^T \frac{\|f(t)-f(s)\|_X^p}{|t-s|^{1+\alpha p}}\,dsdt < \infty.$$

Finally, for $\gamma \in (0,1)$, let us denote by $C^\gamma(0,1;X)$ the space of functions $f:[0,1] \to X$ that are Hölder continuous with the exponent γ.

We fix arbitrary $\alpha \in (\frac14, \frac12)$ and $\varepsilon > 0$, and introduce the following spaces endowed with the natural norms:

$$\begin{aligned}
\mathcal{X} &= L^2(0,1;H^2) \cap \Big(W^{1,\frac43}(0,1;L^{\frac43}) + W^{\alpha,4}(0,1;H^1)\Big) = \mathcal{X}_1 + \mathcal{X}_2,\\
\mathcal{Y} &= L^2(0,1;H^{2-\varepsilon}) \cap C(0,1;H^{-\frac{n}{4}-\varepsilon}),
\end{aligned}$$

where

$$\mathcal{X}_1 = L^2(0,1;H^2) \cap W^{1,\frac43}(0,1;L^{\frac43}), \qquad \mathcal{X}_2 = L^2(0,1;H^2) \cap W^{\alpha,4}(0,1;H^1).$$

Let us note that we have a compact embedding $\mathcal{X} \subset \mathcal{Y}$. Indeed, it follows from Theorems 5.1 and 5.2 in [Lio69, Chapter 1] that the spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are compactly embedded in $L^2(0,1;H^{2-\varepsilon})$. Furthermore, we have continuous embeddings [4]

$$\mathcal{X}_1 \subset C^{\frac14}(0,1;L^{\frac43}), \quad \mathcal{X}_2 \subset C^{\alpha-\frac14}(0,1;H^1). \tag{2.2}$$

It remains to note that $H^1 \subset L^{\frac43} \subset H^{-\frac{n}{4}}$, and therefore the spaces in (2.2) are compactly embedded in $C(0,1;H^{-\frac{n}{4}-\varepsilon})$.

Theorem 2.1. *Suppose that conditions* (1.4) *and* (1.11) *are satisfied. Then the following assertions hold:*

[4] The first embedding in (2.2) is obvious, and the other is a well-known Sobolev-type embedding theorem; see [VF88, Theorem I.2.1] or [Kry95, Lemma II.2.4].

(i) *The family $\{\mu_\nu\}$ is tight in $H^{2-\varepsilon}$.*

(ii) *There is a constant $C>0$ not depending on ν such that*

$$\mathbb{E}\left\|u_\nu\right\|_{\mathcal{X}} \le C \quad \textit{for} \quad 0<\nu\le 1. \tag{2.3}$$

(iii) *The family $\{\boldsymbol{\mu}_\nu\}$ is tight in the space $\mathcal{Y}$.*

Proof. It suffices to establish (2.3), because assertions (i) and (iii) are straightforward consequences of the Prokhorov theorem, inequalities (1.28) and (2.3), and the compact embeddings $H^2\subset H^{2-\varepsilon}$ and $\mathcal{X}\subset\mathcal{Y}$.

In what follows, we denote by C_i inessential positive constants that do not depend on ν. To prove (2.3), we first note that, by (1.28),

$$\mathbb{E}\left\|u_\nu\right\|^2_{L^2(0,1;H^2)} \le C_1. \tag{2.4}$$

Using (1.8), we represent u_ν in the form

$$u_\nu(t) = v_\nu(t) - w_\nu(t) + \zeta_\nu(t), \tag{2.5}$$

where $\zeta_\nu(t)=\sqrt{\nu}\,\zeta(t)$ and

$$v_\nu(t) = u_\nu(0) + (\nu+i)\int_0^t \Delta u_\nu(s)\,ds,$$

$$w_\nu(t) = i\lambda\int_0^t |u_\nu(s)|^2 u_\nu(s)\,ds.$$

Applying (1.28) again and taking into account inequality (1.25), we obtain

$$\mathbb{E}\left\|v_\nu\right\|^2_{W^{1,2}(0,1;L^2)} \le C_3, \tag{2.6}$$

$$\mathbb{E}\left\|w_\nu\right\|^{4/3}_{W^{1,4/3}(0,1;L^{4/3})} \le C_4\lambda^{4/3}. \tag{2.7}$$

Furthermore, well-known properties of the Brownian motion imply that

$$\mathbb{E}\int_0^1 \|\zeta\|_1^4 dt = \mathbb{E}\int_0^1 \Bigl(\sum_{j=1}^\infty \alpha_j b_j^2|\beta_j(t)|^2\Bigr)^2 dt$$

$$\le B_1\sum_{j=1}^\infty \alpha_j b_j^2\int_0^1 \mathbb{E}|\beta_j(t)|^4 dt \le C_5 B_1^2, \tag{2.8}$$

$$\mathbb{E}\int_0^1\int_0^1 \frac{\|\zeta(t)-\zeta(s)\|_1^4}{|t-s|^{1+4\alpha}}\,dsdt \le B_1\sum_{j=1}^\infty \alpha_j b_j^2\int_0^1\int_0^1 \frac{\mathbb{E}\left|\beta_j(t)-\beta_j(s)\right|^4}{|t-s|^{1+4\alpha}}\,dsdt$$

$$\le C_6 B_1^2. \tag{2.9}$$

Combining (2.4)–(2.9) and using the embedding $W^{1,2}(0,1;L^2)\subset W^{1,\frac43}(0,1;L^{\frac43})$, we obtain (2.3). □

As we explained at the end of Section 1.2, in the case of the odd-periodic boundary condition relation (1.4) implies (1.11). So for the problem (1.1), (1.7) the assertions of Theorem 2.1 hold under assumption (1.4).

2.2 Limiting process and stationary solutions for the NLS equation

Let us fix an arbitrary sequence $\hat{\nu}_j \to 0$. The following result is a consequence of assertion (iii) of Theorem 2.1, weak compactness of bounded closed subsets of a reflexive Banach space, and the Skorokhod embedding theorem (cf. [Kuk03]).

Theorem 2.2. *Suppose that conditions* (1.4) *and* (1.11) *are satisfied. Then there is a subsequence* $\{\nu_j\} \subset \{\hat{\nu}_j\}$, *a new probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *and random fields* $v_j(t,x)$ *and* $v(t,x)$, *defining* $\mathcal{Y}$*-valued random variables, such that the following assertions hold:*

(i) *The distribution of* v_j *coincides with* $\boldsymbol{\mu}_{\nu_j}$ *for any* $j \geq 1$. *In particular, a.e. realization of* v_j *belongs to* $\mathcal{X}$.

(ii) *The sequence* $\{v_j\}$ *converges to* v *in the space* $\mathcal{Y}$ *for a.e.* $\omega \in \Omega$.

(iii) *The random field* v *defines a stationary random process* $t \mapsto v(t,\cdot) \in H^2$. *Moreover, for any* $t \in [0,1]$ *we have*

$$\mathbb{E}\|\nabla v(t)\|^2 = B_0, \tag{2.10}$$

$$\mathbb{E}\left\{\|\Delta v(t)\|^2 + \tfrac{\alpha_1 \lambda}{4}\|v(t)\|_{L^4}^4\right\} \leq \tfrac{\alpha_1 B_1 + 2\lambda M B_0}{\alpha_1}. \tag{2.11}$$

(iv) *Every realization of* $v(t,x)$ *belongs to the space*

$$L^2(0,1;H^2) \cap W^{1,\frac{4}{3}}(0,1;L^{\frac{4}{3}}) \tag{2.12}$$

and satisfies the NLS equation

$$\dot{v} - i\Delta v + i\lambda |v|^2 v = 0, \tag{2.13}$$

in the sense that

$$v(t) = v(0) + i\int_0^t (\Delta v - \lambda |v|^2 v)\, ds, \tag{2.14}$$

where $0 \leq t \leq 1$. *Moreover, the functions* $\mathcal{H}_0(v(t))$ *and* $\mathcal{H}_1(v(t))$ *are constant in* $t \in [0,1]$.

Proof. By assertion (iii) of Theorem 2.1, the family of measures $\boldsymbol{\mu}_{\hat{\nu}_j}$ is tight in $\mathcal{Y}$. Therefore, the Skorokhod embedding theorem implies the existence of random variables v_j and v, possessing properties (i) and (ii). The fact that $v(t)$ is stationary as a random process valued in $H^{-\frac{n}{4}-\varepsilon}$ follows from the stationarity of $v_j(t)$.

To prove the remaining assertions, we denote by $\mathfrak{H}$ the Banach space of functions $h \in L^2(0,1;H^2)$ with finite norm

$$\|h\|_{\mathfrak{H}} = \Big(\int_0^1 \int_D |\Delta h(t,x)|^2 dx dt\Big)^{\frac{1}{2}} + \Big(\int_0^1 \int_D |h(t,x)|^4 dx dt\Big)^{\frac{1}{4}}.$$

It is clear that $\mathfrak{H}$ is uniformly convex and therefore reflexive (e.g., see Problems III.25 and V.15 in [RS80]). Hence, the space $L^2(\Omega, \mathfrak{H})$ is also reflexive, and any its closed bounded subset is compact in the weak topology (see Chapter IV in [RS80]).

Regarding the functions $v_j(t, x; \omega)$ as elements of the space $L^2(\Omega, \mathfrak{H})$ we note that, in view of inequality (1.28) (with B_0 and B_1 replaced by νB_0 and νB_1, respectively) and (1.25), their norms are uniformly bounded. Therefore, the sequence $\{v_j\}$ is relatively compact in the weak topology, and we can assume (passing to a subsequence if necessary) that it converges weakly in $L^2(\Omega, \mathfrak{H})$. The limiting function, which belongs to $L^2(\Omega, \mathfrak{H})$, must coincide with $v(t, x; \omega)$ as an element of $L^2(\Omega, L^2(0, 1; H^{2-\varepsilon}))$. Since the distribution of $v(t, \cdot)$ does not depend on t, we conclude from (1.28) that $v(t, x)$ satisfies inequality (2.11) for any $t \in [0, 1]$ and belongs to the space $\mathfrak{H}$ for a.e. ω. In particular, the distribution of $v(t)$ is concentrated on H^2.

Since $v_j(t, x)$ is a weak solution of (2.1) with $\nu = \nu_j$, than for a.e. ω we have

$$v_j(t) = v_j(0) + \int_0^t \left((\nu_j + i)\Delta v_j - i\lambda |v_j|^2 v_j\right) ds + \sqrt{\nu_j}\, \zeta_j(t), \quad 0 \le t \le 1, \tag{2.15}$$

where $\zeta_j(t)$ is a process distributes as $\zeta(t)$. It is easy to check that the sequence $\{\sqrt{\nu_j}\, \zeta_j\} \subset C(0, 1; L^2)$ goes to zero in probability. Therefore, passing to a subsequence, we can assume that, for a.e. ω,

$$\sqrt{\nu_j}\, \zeta_j \to 0 \quad \text{in } C(0, 1; L^2) \text{ as } j \to \infty. \tag{2.16}$$

Let us fix an arbitrary $\omega \in \Omega$ such that $v_j \to v$ in $\mathcal{Y}$, $v \in \mathfrak{H}$, and (2.16) holds. Then we can pass to the limit in relation (2.15) as $j \to \infty$ regarding its left- and right-hand sides as elements of $C(0, 1; H^{-\frac{n}{4}-\varepsilon})$. This results in (2.14), which shows that $v(t, x)$ belongs to the space (2.12) and satisfies Eq. (2.13) almost surely. Redefining v to be zero on the corresponding negligible subset of Ω, we achieve that every realisation of v belongs to space (2.12) and satisfies (2.13).

It remains to show that $\mathcal{H}_0(v(t))$ and $\mathcal{H}_1(v(t))$ are constant in time and that relation (2.10) holds. To prove the first assertion, let us fix an arbitrary ω for which $v(t, x)$ belongs to space (2.12). Then the real-valued functions $\mathcal{H}_0(v(t))$ and $\mathcal{H}_1(v(t))$ are absolutely continuous, and their derivatives are given by the formulas

$$\frac{d}{dt}\mathcal{H}_0(v(t)) = \mathrm{Re}(v, \dot{v}),$$
$$\frac{d}{dt}\mathcal{H}_1(v(t)) = \mathrm{Re}(-\Delta v + \lambda |v|^2 v, \dot{v}),$$

valid for a.e. $t \in [0, 1]$. Replacing $\dot{v}$ with the expression found from Eq. (2.13), we easily show that the right-hand sides are zero for a.e. t.

Finally, to establish (2.10), we need the following auxiliary assertion (cf. proof of Lemma 3.2 in [Kuk03]).

Lemma 2.3. *Let $\xi_j \geq 0$ be a sequence of random variables that converge to ξ almost surely. Suppose that*

$$\mathbb{E}\xi_j^2 \leq C, \tag{2.17}$$

where $C > 0$ is a constant not depending to j. Then

$$\mathbb{E}\xi_j \to \mathbb{E}\xi \quad as \quad j \to \infty. \tag{2.18}$$

To prove the lemma, it suffices to note that, in view of inequality (2.17), the random variables ξ_j are uniformly integrable, and therefore convergence (2.18) holds. We now set

$$\xi_j = \int_0^1 \|\nabla v_j(t,\cdot)\|^2 dt, \quad \xi = \int_0^1 \|\nabla v(t,\cdot)\|^2 dt.$$

By construction, for a.e. ω, we have $v_j \to v$ in $L^2(0,1;H^{2-\varepsilon})$ as $j \to \infty$, so the sequence ξ_j converges to ξ almost surely. Furthermore, it follows from inequality (1.29) (with B_1, P, and M replaced by νB_1, νP, and νM, respectively) that condition (2.17) is satisfied. Therefore, by (2.18),

$$\lim_{j\to\infty} \mathbb{E}\xi_j = \lim_{j\to\infty} \mathbb{E}\|\nabla v_j(t,\cdot)\|^2 = B_0 = \mathbb{E}\xi = \int_0^1 \mathbb{E}\|\nabla v(t,\cdot)\|^2 dt.$$

The required relation (2.10) follows now from the fact that the function $\mathbb{E}\|\nabla v(t,\cdot)\|^2$ does not depend on t. $\square$

Clearly, for any integer $N > 0$ there exists a subsequence $\{\nu_j\}$ depending on N such that the assertions of Theorem 2.2 hold with the segment $[0,1]$ replaced by $[0,N]$. Applying the diagonal process, we arrive at the following result:

Theorem 2.4. *Under the conditions of Theorem 2.2, there exists a subsequence $\{\nu_j\} \subset \{\hat{\nu}_j\}$, a new probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and random fields $v_j(t,x)$ and $v(t,x)$ defined for $t \geq 0$ and $x \in D$ such that the following assertions hold:*

- **(i)** *The random fields v_j are distributed as u_{ν_j}.*
- **(ii)** *For any $T > 0$, the sequence of random processes $\{v_j(t,\cdot), 0 \leq t \leq T\}$ converge almost surely to $\{v(t,\cdot), 0 \leq t \leq T\}$ in the norm of the space $L^2(0,T;H^{2-\varepsilon}) \cap C(0,T;H^{-\frac{n}{4}-\varepsilon})$.*
- **(iii)** *The random field $v(t,x)$ defines a stationary process in H^2, which satisfies (2.10) and (2.11).*
- **(iv)** *Every realization of $v(t,x)$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}_+;H^2) \cap W^{1,\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}_+;L^{\frac{4}{3}})$ and satisfies the NLS equation (2.13). Moreover, the random functions $\mathcal{H}_0(v(t))$ and $\mathcal{H}_1(v(t))$ are t-independent.*

Inequalities (2.10) and (2.11) imply the following two-sided estimate for the L^2-norm of the limiting process.

Corollary 2.5. *Let $v(t,x)$ be a stationary solution of Eq.* (2.13) *constructed in Theorem 2.4. Then for any $t \geq 0$ we have*

$$\frac{\alpha_1 B_0^2}{\alpha_1 B_1 + 2\lambda M B_0} \leq \mathbb{E}\|v(t,\cdot)\|^2 \leq \frac{B_0}{\alpha_1}. \tag{2.19}$$

Proof. It follows from (2.10) that

$$\mathbb{E}\|v\|^2 \leq \alpha_1^{-1}\mathbb{E}\|\nabla v\|^2 = \frac{B_0}{\alpha_1}.$$

On the other hand, since $\|\nabla v\|^2 \leq \|v\|\,\|\Delta v\|$, we conclude that

$$\left(\mathbb{E}\|\nabla v\|^2\right)^2 \leq \mathbb{E}\|v\|^2\,\mathbb{E}\|\Delta v\|^2.$$

Combining this with (2.10) and (2.11), we derive the left-hand inequality in (2.19). ☐

For the odd-periodic boundary conditions, the assertions of Theorem 2.4 and Corollary 2.5 hold under assumption (1.4).

2.3 The inviscid limit and the NLS equation

The NLS equation

$$\dot{u} - i\Delta u + i\lambda|u|^2u = 0, \quad u(t) \in H^1, \tag{2.20}$$

is a Hamiltonian PDE with the Hamiltonian $\mathcal{H}_1$ (see [Kuk00]). Moreover, if $n \leq 2$, or $n \leq 3$ and the odd-periodic conditions (1.7) are imposed, then for any $u_0 \in H^2$ Eq. (2.20) has a unique solution $u \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1)$ that is equal to u_0 at $t = 0$. This solution continuously depends on u_0 and $\mathcal{H}_0$ and $\mathcal{H}_1$ are its integrals of motions; see [BG80, SS99, Bou99, BGT03]. Accordingly, Eq. (2.20) defines a continuous dynamical system in H^2 with two integrals of motions.

Let us denote by μ the distribution of $v(0,\cdot)$. Theorem 2.2 implies the following assertion:

Corollary 2.6. *Suppose that either $n \leq 2$ or $n \leq 3$ and the odd-periodic conditions* (1.7) *are imposed. Then any sequence $\hat{\nu}_j \to 0$ has a subsequence $\nu_j \to 0$ such that the stationary measures μ_{ν_j} weakly converge, as $j \to \infty$, to a measure μ which is invariant for the dynamical system in H^2. This measure satisfies the relations* (*cf.* (2.10), (2.11), *and* (2.19))

$$\frac{\alpha_1 B_0^2}{\alpha_1 B_1 + 2\lambda M B_0} \leq \int_{H^2} \|v\|^2 \mu(dv) \leq \frac{B_0}{\alpha_1},$$

$$\int_{H^2} \|\nabla v\|^2 \mu(dv) = B_0,$$

$$\int_{H^2} \{\|\Delta v\|^2 + \tfrac{\alpha_1\lambda}{4}\|v\|_{L^4}^4\}\mu(dv) \leq \frac{\alpha_1 B_1 + 2\lambda M B_0}{\alpha_1}.$$

Let us assume that, in (1.3), all coefficients b_j are nonzero. In this case Theorem 1.6 implies that, for any $\nu \in (0,1]$ Eq. (2.1) has a unique stationary measure μ_ν, and

$$\mathcal{D}\big(u_\nu(t)\big) \rightharpoonup \mu_\nu \quad \text{as} \quad t \to \infty, \tag{2.21}$$

where $u_\nu(t)$ is an arbitrary solution of (2.1) defined for $t \geq 0$ and satisfying the condition $\mathbb{E}\|u_\nu(0)\|^2 < \infty$. The convergence (2.21) is exponentially fast if the space of measures is provided with the Kantorovich distance; see [KS02]. Moreover, if $U_\nu(t)$, $t \geq 0$, is a stationary solution of (2.1), then we have the weak convergence of measures in the space $C(\mathbb{R}_+, H)$:

$$\mathcal{D}\big(u_\nu(T+\cdot)\big) \rightharpoonup \mathcal{D}\big(U_\nu(\cdot)\big) \quad \text{as} \quad T \to \infty. \tag{2.22}$$

To see this, let us note that, if $v(t)$ and $v_T(t)$, $T \geq 0$, are weak solutions of (2.1) defined for $t \geq 0$ and satisfying the condition

$$\mathcal{D}\big(v_T(0)\big) \rightharpoonup \mathcal{D}\big(v(0)\big) \quad \text{as} \quad T \to \infty, \tag{2.23}$$

then we have

$$\mathcal{D}\big(v_T(\cdot)\big) \rightharpoonup \mathcal{D}\big(v(\cdot)\big) \quad \text{as} \quad T \to \infty, \tag{2.24}$$

where convergences in (2.23) and (2.24) hold in the space of probability measures on H and $C(\mathbb{R}_+, H)$, respectively. Applying this result to the weak solutions $v(t) = U_\nu(t)$ and $v_T(t) = u_\nu(T+t)$ (which satisfies (2.23) in view of (2.21)), we arrive at (2.22).

Theorem 2.4 combined with (2.22) implies the following result:

Theorem 2.7. *Suppose that $b_j \neq 0$ for all $j \geq 1$. Let us fix a vector $u_0 \in H^1$ and denote by $u_\nu(t)$, $t \geq 0$, a solution of (2.1) equal to u_0 at $t = 0$. Then, under the assumption of Theorem 2.2, there is a subsequence $\{\nu_j\}$ such that*

$$\lim_{j\to\infty} \lim_{T\to\infty} \mathcal{D}\big(u_{\nu_j}(T+\cdot)\big) = \mathcal{D}\big(v(\cdot)\big).$$

Here the limits are understood in the sense of weak convergence of Borel measures in the space $L^2_{\mathrm{loc}}(\mathbb{R}_+, H) \cap C(\mathbb{R}_+, H^{-\frac{n}{4}-\varepsilon})$, and $v(t)$ is a stationary solution of the NLS equation (2.13) such that (2.10) and (2.11) hold, and $\mathcal{H}_0$ and $\mathcal{H}_1$ are integrals of motion for a.e. realisation of v.

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