A Criterion for Ferromagnetism in the Isotropic Quantum Heisenberg Model

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Abstract

A criterion for ferromagnetism is presented suggesting a line of proof to rigorously establish the phase transition. Spectral information will be required in certain invariant subspaces of the Hamiltonian, hopefully the relatively crude estimates needed will be not too difficult to establish.

We hope the criterion presented in this paper leads in short order to a rigorous proof of the ferromagnetic phase transition for the isotropic quantum Heisenberg model.

The model is constructed on a rectangular lattice, V. The Hamiltonian is taken as

$$H = -\sum_{i\sim j} \frac{1}{2}\Big(\vec{\sigma}_i \cdot \vec{\sigma}_j - 1\Big) = -\sum_{i\sim j} \Big(I_{ij} - 1\Big) \tag{1}$$

where I_{ij} interchanges the spins at nearest neighbor sites i and j. For the Hilbert space we introduce an o.n. basis with elements

$$\vec{i}_{\mathcal{S}} = \underset{i\in\mathcal{S}}{\otimes} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \underset{j\notin\mathcal{S}}{\otimes} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j \tag{2}$$

with each $\mathcal{S}$ a subset of the vertices. The r spin wave sector is spanned by the $\vec{i}_{\mathcal{S}}$ with $\#(\mathcal{S}) = r$, and is an invariant subspace of H.

In [1] operators $T^{r,s}$ were introduced mapping the r spin wave sector of the Hilbert space to the s spin wave sector. If

$$\vec{g} = \sum_{\mathcal{S}} g(\mathcal{S})\vec{i}_{\mathcal{S}} \tag{3}$$

with the cardinality of all sets in this sum equal r. Then

$$\vec{h} = T^{r,s}\vec{g} = \sum_{\mathcal{S}} h(\mathcal{S})\vec{i}_{\mathcal{S}} \tag{4}$$

where in this sum all sets are of cardinality s, and

$$h(\mathcal{S}) = \sum_{\mathcal{S}'\supset\mathcal{S}} g(\mathcal{S}') \ \text{ if } \ \#(\mathcal{S}) = s. \tag{5}$$

It was shown in [1] that these operators intertwine with H,

$$HT^{r,s} = T^{r,s}\ H, \tag{6}$$

(really a trivial property due to invariance under global rotation).

We will make some arbitrary choices in our present exposition, for simplicity. We let lattice size become infinite through a sequence of lattices all with, $v = |V|$, the number of their vertices, divisible by 10. We decompose the trace into contributions from the different spin wave sectors.

$$Tr\left(e^{-\beta H}\right)_V \equiv Tr(V,\beta) = \sum_{i=0}^{v} Tr(V,\beta,i) \tag{7}$$

where in $\mathrm{Tr}(V,\beta,i)$ only basis elements, $\vec{i}_{\mathcal{S}}$, with $\#(\mathcal{S}) = i$ are kept. We use the following criterion for permanent magnetism, derived from Section 2 of [2].

Magnetism Criterion: The system is ferromagnetic for $\beta \geq \beta_0$ if for some M_0, and $v \geq M_0$ one has

$$Tr(V,\beta,i) \leq 2\ Tr(V,\beta,i-\frac{v}{10})$$
$$\text{for} \quad i = \frac{4}{10}\ v+1, ..., \frac{v}{2} \tag{8}$$

when $\beta \geq \beta_0$ and $v \geq M_0$.

Our hope for an attack on the phase transition problem is to use the operators $T^{i,i-\frac{v}{10}}$ to study the relation between $\mathrm{Tr}(V,\beta,i)$ and $\mathrm{Tr}(V,\beta,i-\frac{v}{10})$. If we let K_i be the kernel of $T^{i,i-\frac{v}{10}}$ and R_i be the orthogonal complement of K_i in the i spin wave sector, and gets

$$Tr(V,\beta,i) = Tr^{(1)}(V,\beta,i) + Tr^{(2)}(V,\beta,i) \tag{9}$$

where $Tr^{(1)}$ is the portion of the trace from K_i and $Tr^{(2)}$ the portion from R_i. One has

$$Tr^{(2)}(V,\beta,i) = Tr(V,\beta,i-v/10) \tag{10}$$

and one is left with trying to prove

$$Tr^{(1)}(V,\beta,i) \leq Tr(V,\beta,i-v/10) \tag{11}$$

(for $\beta \geq \beta_0,\ v \geq M_0$). For the range of i needed, relation (10) is easily shown. We have ideas on how to develop a proof of (11).

The reduction of this paper, from proving the phase transition to studying (11), is truly trivial. But we believe it is new. If this has been known to someone, likely then for a long time, perhaps they've missed the boat in not pushing it through to a proof.

References

[1] P. Federbush, "A Polymer Expansion for the Quantum Heisenberg Wave Function", to be published in *J. Math. Phys.*, math-ph/0302067.

[2] P. Federbush, "For the Quantum Heisenberg Ferromagnet, Tao to the Proof of a Phase Transition", math-ph/0202044.