Convergence of Mayer Series via Cauchy–Kowalewski Majorant Methods with Application

Leonardo F. Guidi* & **Domingos H. U. Marchetti** †
Instituto de Física
Universidade de São Paulo
Caixa Postal 66318
05315 São Paulo, SP, Brasil

October 14, 2003

Abstract

We construct majorant functions $\Phi(\beta, z)$ for the Mayer series of pressure satisfying a non-linear differential equation of first order which can be solved by the method of characteristics. The domain $|z| < r(\beta)$ of convergence of Mayer series is given by the envelop of characteristics defined by the first crossing time of whole family. For non negative potentials we derive an explicit solution in terms of the Lambert W–function which is related to the exponential generating function of rooted trees T as $T(x) = -W(-x)$. For stable potentials the solution is majorized by a non negative potential solution. There are many choices in this case and we used this freedon to reexamine the ultraviolet problem of Yukawa potential. We also apply Cauchy–Kowalevsky theorem in order to discuss the analytic continuation of $\Phi(\beta, z)$ to the complex z–plane.

1 Introduction

One of the main investigation in equilibrium statistical mechanics concerning the determination of phase diagrams is to define regions of the parameter space in which a phase transition does not occur. According to Lee–Yang Theorem [YL, LY] theses regions are free of zeros of the partition function which, for lattice models with positive pair potentials, are located in the unit circle of the complex activity plane. Another way of determining whether the phase diagram is free of singularities is given by Mayer series. Although the theory based on these series does not describe the equation of state beyond the condensate point, it is able, however, to deal with continuum as well as lattice space for a larger class of admissible pair potentials.

*Supported by FAPESP under grant #98/10745 − 1. E-mail: *guidi@if.usp.br*.
†Partially supported by CNPq and FAPESP. E-mail: *marchett@if.usp.br*

The domain of convergence of Mayer series for classical gas with stable two–body summable potential can be established using various different techniques (see Ruelle [R], Brydges [B] and references therein). There are other cases, including attractive potential at short distances and Yukawa gas at inverse temperature β up to 4π, which require the use of finitely [GM] or infinitely [B] many interactions of Mayer series. The present investigation uses the flow equation of Ursell functions introduced by Brydges and Kennedy [BK, BK1] in order to reobtain previously known results on the convergence of Mayer series via partial differential equations (PDE).

The radius of convergence of a Mayer series is very often determined by a "non–Physical" singularity in the complex activity z–plane. In the present article we shall write a majorant series $\Phi(\beta, z)$ for the pressure $p(\beta, z)$ of a gas which satisfies a nonlinear partial differential equations of first order. The method of characteristics applied to the PDE allows to relate the "non–Physical" singularity with crossing of characteristics in the phase space $\Omega = \{(t, z) \in \mathbb{R}^2 : t, z \geq 0\}$ (t is eventually identified with β). As in Tonks' model of one–dimensional rods with hard core interaction, the singularity of $\Phi(\beta, z)$ is manifested by the W–Lambert function, an indication that this singularity is of combinatorial nature: the exponential generating function of rooted trees satisfies $T(x) = -W(-x)$.

As an attempt to extend our solutions beyond the disc of convergence of Mayer series, we shall also approach the PDE by Cauchy–Kowalevsky theorem developed by Nirenberg [N] for nonlinear equations.

This paper is organized as follows. Section 2 contains a review of Gaussian integrals, including their relation to the partition function of classical gases and the statements of our results. Section 3 gives an alternative derivation of Ursell integral equation of Brydges–Kennedy. The integral equation is the starting point for the Majorant differential equation presented in Section 4. This first order nonlinear partial differential equation is solved in Section 5 by the method of characteristics for non negative and stable potentials. Section 6 gives an application of the previous section to the ultraviolet problem of the two–dimension Yukawa gas. Finally, Section 8 applies Cauchy–Kowalevsky Theorem and Section concludes with open problems.

2 Gaussian Integrals and Summary of Results

In this section we introduce the model and describe our results. To fix notation, we need some facts on Gaussian integrals and their relation to the grand partition function (see references [FS, Br, BM], for reviews).

The state of a particle $\xi = (x, \sigma)$ consists of its position x and some internal degrees of freedom σ. *Examples:* 1. For one–component lattice gases, $\xi = x \in \mathbb{Z}^d$; 2. For a two–component continuous Coulomb gas, $\xi = (x, \pm e)$ where $x \in \mathbb{R}^d$ and e is the electrical charge; 3. For dipole lattice gas, $\xi = (x, \pm e_j)$ where $x \in \mathbb{Z}^d$ and e_j is the canonical unit vector.

The particles are constrained to move in a finite subset X of $\mathbb{Z}^d$ (or a compact region of $\mathbb{R}^d$). The lattice is preferred for pedagogical reason but we shall also consider particles in a continuum space. S is in general a compact space and we set $\Lambda = X \times S$. By the triple $(\mathbb{S}, \Sigma, d\rho)$ we mean the measure space of a single particle with $\mathbb{S} = \mathbb{R}^d \times S$, Σ a sigma algebra generated by the cylinder sets of $\mathbb{R}^d \times S$ and $d\rho(\xi) = d\mu(x)\, d\nu(\sigma)$ a product of positive measures ($d\mu(x) = \sum_{n\in\mathbb{Z}^d} \delta(x-n)\, d^dx$

is a counting or Lebesgue $d\mu(x) = d^d x$ measure times a probability measure $d\nu$ on S).

The grand canonical partition function is given by

$$\Xi_\Lambda(\beta, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} d^n\rho(\boldsymbol{\xi})\, \psi_n(\boldsymbol{\xi}) \tag{2.1}$$

where z is the particle activity, for each state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \Lambda^n$ of n–particles,

$$\psi_n(\boldsymbol{\xi}) = e^{-\beta U_n(\boldsymbol{\xi})} \tag{2.2}$$

is the associate Boltzmann factor with

$$U_n(\boldsymbol{\xi}) = \sum_{i<j} v(\xi_i, \xi_j) \tag{2.3}$$

the interaction energy and β is the inverse temperature. The two–body potential $v : \Lambda \times \Lambda \longrightarrow \mathbb{R}$ is a jointly measurable function and we use, sometimes, the abbreviation $v_{ij} := v(\xi_i, \xi_j)$. The finite volume pressure $p_\Lambda(\beta, z)$ of a classical gas is defined by

$$p_\Lambda(\beta, z) = \frac{1}{\beta\,|\Lambda|} \ln \Xi_\Lambda(\beta, z)\,. \tag{2.4}$$

Without loss, v is defined in the whole space $\mathbb{Z}^d$ (or $\mathbb{R}^d$) - the Green's function of some differential equation with free (insulating) boundary conditions - and it is assumed to be of the form

$$\beta v(\xi, \xi') = \omega\,(\sigma, \sigma', x - x') \tag{2.5}$$

for some real–valued measurable functions ω satisfying:

1. *Regularity at origin* $|\omega(\sigma, \sigma', 0)| \le B < \infty$;
2. *Symmetry* $\omega(\sigma, \sigma', -x) = \omega(\sigma, \sigma', x) = \omega(\sigma', \sigma, x)$;
3. *Summability*
$$\|\omega\| := \sup_{\sigma'} \int d\nu(\sigma)\, d\mu(x)\ |\omega(\sigma, \sigma', x)| < \infty\ ; \tag{2.6}$$
4. *Positivity*
$$\sum_{1\le i,j\le n} \overline{z}_i\, z_j\, \omega\,(\sigma, \sigma', x_i - x_j) \ge 0\,, \tag{2.7}$$
for any numbers $x_1, \ldots, x_n \in \mathbb{Z}^d$ ($\mathbb{R}^d$) and $z_1, \ldots, z_n \in \mathbb{C}$, $n = 1, 2, \ldots$.

Note these assumptions ensure that v is a stable potential, i.e., choosing $z_1 = \cdots = z_n = 1$ in (2.7),

$$\beta U_n(\boldsymbol{\xi}) \ge -\frac{1}{2} \sum_{1\le i\le n} |\omega(\sigma_i, \sigma_i, 0)| \ge -\frac{1}{2} B n \tag{2.8}$$

holds with $B \ge 0$ independent of n (see Section 3.2 of Ruelle [R], for further comments).

Since $\omega = \beta v$ is a positive definite symmetric function, it defines a Gaussian measure μ_ω with mean $\displaystyle\int d\mu_\omega(\phi)\phi(\xi) = 0$ and covariance $\int d\mu_\omega(\phi)\, \phi(\xi)\, \phi(\xi') = \beta v(\xi, \xi')$. Introducing the inner product

$$(\phi, \eta) = \int d\rho(\xi)\, \phi(\xi)\, \eta(\xi)$$

and the indicator function

$$\eta_n(\cdot) = \eta(\boldsymbol{\xi}; \cdot) = \sum_{j=1}^{n} \delta\left(\cdot - \xi_j\right) \tag{2.9}$$

of a n–state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, we have

$$\psi_n(\boldsymbol{\xi}) = \int d\mu_\omega\left(\phi\right)\, :e^{i(\phi,\eta_n)}:_\omega \tag{2.10}$$

where $::_\omega$ is the normal ordering with respect to covariance ω:

$$:e^{i\phi(\xi)}:_\omega = \exp\left\{\frac{\beta}{2} v\left(\xi, \xi\right)\right\}\, e^{i\phi(\xi)}\,. \tag{2.11}$$

Substituting (2.10) into (2.1) yields

$$\Xi_\Lambda(\beta, z) = \int d\mu_\omega\left(\phi\right)\, \exp\left\{V_0(\phi)\right\} \tag{2.12}$$

where

$$V_0(\phi) = z\int_\Lambda d\rho(\xi)\, :e^{i\phi(\xi)}:_\omega\,. \tag{2.13}$$

Example 2.1 *1.* $V_0(\phi) = z\sum_{x\in X} :e^{i\phi(x)}:_\omega$ *for one–component lattice gas;*

2. $V_0(\phi) = z\int_X d^dx :\cos\phi(x):_\omega$ *for standard Coulomb or Yukawa gas;*

3. $V_0(\phi) = z\sum_{x\in X}\int_{|\sigma|=1} d^d\sigma :\cos\left(\sigma\cdot\nabla\phi(x)\right):_\omega$ *for dipole lattice gas.*

In Example 2 we have respectively, $v = (-\Delta)^{-1}$ and $v = (-\Delta + 1)^{-1}$, the fundamental solution of Laplace and Yukawa equations. The potential of two dipoles σ and σ' in Example 3. is given by $v = (\sigma\cdot\nabla)(\sigma'\cdot\nabla')(-\Delta)^{-1}$, second difference of the fundamental solution of discrete Laplace equation. V_0 in Examples 2 and 3 contain cosine function instead exponential. This form expresses the pseudo–charge symmetry required for the existence of thermodynamic limit $\lim_{\Lambda\nearrow\mathbb{R}^d(\mathbb{Z}^d)} p_\Lambda = p$ of respective systems (see, e.g., [S]).

The potentials in Examples 2 and 3 do not fulfill all four conditions: Coulomb and dipole potentials do not satisfy summability whereas the Yukawa potential fails to be regular at origin. To circumvent these problems, a one parameter smooth decomposition of covariance $\omega(t)$ may be introduced so that $\int_t^T \left\|\dot{\omega}(s)\right\| ds < \infty$ holds for any finite interval $[t, T]$. This allows us to approach the ultraviolet problem of Yukawa gas by letting $t \to -\infty$ maintaining T fixed (e.g. $T = 0$) in Example 2, and the infrared problem of dipole gas in Example 3 as the limit $T \to \infty$ with $t = 0$ fixed. The application in the present work deals only with the former problem.

Equation (2.12) is the sine–Gordon field theory representation of a classical gas. In all examples $V_0(\phi)$ is a bounded perturbation to the Gaussian measure μ_ω satisfying $|V_0(\phi)| \leq z\,|\Lambda|$. Brydges and Kennedy [BK] have used the sine–Gordon representation to derive a PDE in the context of Mayer expansion. Although based in their work, our PDE approach in the present article is remarkably different.

It is convenient to write a flow

$$V_\Lambda\,(t, z; \phi) = \ln\left(\mu_{\omega(t)} * \exp\left\{V_0(\phi)\right\}\right) \tag{2.14}$$

with $V_\Lambda\,(0, z; \phi) = V_0(\phi)$ and $\mu_\omega * F$ denoting a convolution of F by a Gaussian measure μ_ω given, for the one–component lattice case, by the Wick theorem

$$\begin{aligned}(\mu_\omega * F)\,(\phi) &= \int d\mu_\omega(\zeta)\, F(\phi+\zeta) \\ &= \exp\left\{\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d}\omega\,(t; x-y)\,\frac{\partial^2}{\partial\phi(x)\partial\phi(y)}\right\}F(\phi)\,. \end{aligned}\tag{2.15}$$

To derive the standard (one scale) Mayer expansion, the covariance $\omega(t)$ is chosen to be any $\mathscr{C}^1$ parametrization such that $\omega(0) = 0$ and $\omega(1) = \beta v$ (e.g. $\omega(t) = t\beta v$). The parametrization involving multiscales, $\dot{\omega}(t)$ is required to be of the form (2.5) satisfying properties $1-4$ with $B = B(t) < \infty$ for all t. $\omega(t)$ is said to be admissible if fulfills these properties. Note that $V_0(\phi)$ also depends on t through the normal ordering $: :_{\omega(t)}$.

Equation (2.14) can be written as

$$e^{V_\Lambda(\beta,z,\phi)} = \Xi_\Lambda(\beta, z, e^{i\boldsymbol{\phi}}) \tag{2.16}$$

where $\Xi_\Lambda(\beta, z, \boldsymbol{\omega})$ generalizes the grand partition function by including an activity $\omega = \omega(\xi)$ which depends on the particle state:

$$\Xi_\Lambda(\beta, \boldsymbol{\omega}) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{\Lambda^n} d^n\rho(\boldsymbol{\xi})\,\omega_n(\boldsymbol{\xi})\,\psi_n(\boldsymbol{\xi}) \tag{2.17}$$

with $\omega_n(\boldsymbol{\xi}) = \prod_{j=1}^n \omega(\xi_j)$ and (2.1) is recovered if $\omega(\xi) = 1$.

We also have

$$\beta\,|p_\Lambda(\beta, z)| = \frac{1}{|\Lambda|}\,|V_\Lambda\,(\beta, z, 0)| \leq \sup_\phi \frac{1}{|\Lambda|}\,|V_\Lambda\,(\beta, z, \phi)| \tag{2.18}$$

and, as t goes to zero (the infinite temperature limit),

$$\lim_{t\to 0} V_\Lambda\,(t, z, 0) = V_0(0) = z\,|\Lambda| \quad .$$

The flow $\{V_\Lambda(t,z,\phi)/|\Lambda| : 0 \le t \le 1\}$ provides us an upper bound for the pressure of gases described above, starting from the pressure of an ideal gas.

The results presented in this article can be summarized as follows.

Theorem 2.2 *There exist a real analytic function $\Phi = \Phi(t,z)$, satisfying*

$$\Phi_t = \frac{1}{2}\Gamma\left(z\Phi_z\right)^2 + B\left(z\Phi_z - z\right) \tag{2.19}$$

with $\Phi(t_0,z) = z$ such that

$$\frac{1}{|\Lambda|}\left|V_\Lambda\left(t,z,0\right)\right| \le \Phi(t,z)$$

holds uniformly in Λ. Here, $\Gamma = \Gamma(t) = \left\|\dot{\omega}(t)\right\|$, $B = B(t)$ is defined by (2.8) with ω replaced by $\dot{\omega}$ and $V_\Lambda\left(t_0,z,\phi\right) = V_0(\phi)$ given by (2.13). The solution Φ has a convergent power series

$$\Phi(t,z) = z + \sum_{n=2}^{\infty} A_n(t)\ z^n$$

with $A_n(t) \ge 0$, in the open disc $D = \left\{|z| < (e\tau)^{-1}\right\}$ where

$$\tau(t_0,t) = \inf\left(\int_{t_0}^{t}\Gamma(s)\exp\left\{2\int_s^t B(s')\ ds'\right\}ds,\ \ \exp\left\{\int_{t_0}^{t}B(s')\ ds'\right\}\int_{t_0}^{t}\Gamma(s)ds\right)$$

and is bounded by

$$\Phi(t,z) \le \frac{-1}{\tau}\left(W\left(-z\tau\right) + \frac{1}{2}W^2\left(-z\tau\right)\right) \tag{2.20}$$

where $W(x)$ is the Lambert W–function defined as the inverse of $f(W) = We^W$.

Theorem 2.3 *Given $n \in \mathbb{N}$, the pressure of Yukawa gas, with the singular part of coefficients up to order n in the power series removed, exists and has a convergent Mayer series provided*

$$\beta < \beta_n \qquad \text{and} \qquad |z| < \frac{\beta_n - \beta}{\beta_n \beta e}$$

with $\beta_n = 8\pi/(1+n^{-1})$ the n–th threshold.

Theorems 2.2 and 2.3 will be proven in Sections 5 and 6, respectively, by majorizing the solution of (2.19) by an explicitly solution of (2.19) with $B = 0$ and Γ replaced by Γ/f for a conveniently chosen function f. Concerning the application of Cauchy–Kowalewski Theorem, we shall consider the equation for $\Theta = \Phi_z$, with Φ satisfying (2.19), given by (4.10). We have

Theorem 2.4 *For each $0 \le s < 1$, let $\Omega_s \subset \mathbb{C}$ be a family of open sets with $\Omega_{s'} \subset \Omega_s$ if $s' < s$ and let $\delta_0 > 0$ be so that $|B|_{\delta_0} = \sup_{|t|<\delta_0}|B(t)|$ and $|\Gamma|_{\delta_0}$ are finite. Then, there exist finite positive constants C and R such that $\Theta(t,z)$ is the unique solution of (4.10), continuously differentiable in t, for $|t| < \delta_0\left(1-s\right)\exp\left\{-2eC\delta_0\right\}$, with values in $H\left(\Omega_s\right)$ satisfying*

$$\left\|\Theta(t)\right\|_{H(\Omega_s)} \le 1 + \delta_0|B|_{\delta_0} + \frac{R}{2}\ .$$

Theorem 2.4 will be proven in Section 7 using Nirenberg's Cauchy–Kowalewski theorem [N] for nonlinear PDE.

3 Ursell Flow Equation

The representation of pressure (2.4) in power series of activity z (also named fugacity) is called Mayer series:

$$\beta p_\Lambda(\beta, z) = \sum_{n=1}^{\infty} b_{\Lambda,n}\, z^n \quad . \tag{3.1}$$

The Mayer coefficient $b_{\Lambda,n}$ has an expression analogous to (2.1),

$$|\Lambda|\; b_{\Lambda,n} = \frac{1}{n!}\int_{\Lambda^n} d^n\rho(\boldsymbol{\xi})\, \psi_n^c(\boldsymbol{\xi}) \tag{3.2}$$

where, for each state $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in (\mathbb{R}\times S)^n$ of n–particles, $\psi_n^c(\boldsymbol{\xi})$ is the Ursell functions of order n. In terms of Mayer graphs,

$$\psi_n^c(\boldsymbol{\xi}) = \sum_G \prod_{(i,j)\in G} \left(e^{-\beta v_{ij}} - 1\right) \tag{3.3}$$

where G runs over all connected graphs on $I_n = \{1, \ldots, n\}$ (G is connected if for any two vertex k, $l \in I_n$ there is a path of edges $(i,j) \in I_n \times I_n$ in G connecting k to l).

Since $|\sum_G 1| > c\,(n!)^{1+\delta}$ holds for some $c, \delta > 0$,[1] equation (3.3) is not useful to investigate whether (3.1) is a convergent series. We shall instead derive a flow equation for $\psi_n^c(\boldsymbol{\xi})$ that incorporate the cancellations required to reduce the cardinality of the sum in (3.3) to $O(n!)$.

Given $\boldsymbol{\xi} = (\xi_j)_{j\geq 1} \in (\mathbb{R}\times S)^\infty$, let $P_n\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ denote the projection of $\boldsymbol{\xi}$ into the subspace $(\mathbb{R}\times S)^n$ generated by its n first components. If $I \subset \mathbb{N}$ is a finite set $I = \{i_1, \ldots, i_n\}$, we set $P_I\boldsymbol{\xi} = (\xi_{i_1}, \ldots, \xi_{i_n})$. To each sequence $\boldsymbol{\xi}$ we associate a complex Boltzmann function

$$u(t, \boldsymbol{\xi}) = \exp\left\{-\beta \sum_{i<j} v(t, \xi_i, \xi_j) + i \sum_j \phi(\xi_j)\right\}$$

and define

$$u_n(t, \boldsymbol{\xi}) = u(t, P_n\boldsymbol{\xi}) = e^{-\beta U_n(\boldsymbol{\xi}) + i(\phi, \eta_n)}\,, \tag{3.4}$$

for $n \geq 2$ where $U_n(\boldsymbol{\xi}) = U(P_n\boldsymbol{\xi})$ is the interaction energy (2.3) and $\eta_n(\cdot) = \eta(P_n\boldsymbol{\xi}, \cdot)$ is the indicator function (2.9). Note that $u_n : \mathbb{R}_+ \times (\mathbb{R}\times S)^n \longrightarrow \mathbb{C}$ is symmetric with respect to the permutation π of the index set $\{1, \ldots, n\}$:

$$u(t, P_n\boldsymbol{\xi}) = u(t, M_\pi P_n\boldsymbol{\xi})$$

where $M_\pi : (\xi_1, \ldots, \xi_n) \longrightarrow (\xi_{\pi_1}, \ldots, \xi_{\pi_n})$ is the permutation matrix corresponding to π. Note also that $u_n(t, \boldsymbol{\xi})$ can be written as

$$u_n(t, \boldsymbol{\xi}) = \psi_n(\boldsymbol{\xi})\, e^{i(\phi,\eta_n)} = \mu_{\omega(t)} * : e^{i(\phi,\eta_n)} :_{\omega(t)}\,, \tag{3.5}$$

where ψ_n is the Boltzmann function (2.2), with $\omega = \beta v$ replaced by an admissible covariance $\omega(t)$.

[1] The number of Mayer graphs with n vertices is $2^{n(n-1)/2}$ and the number of connected Mayer graphs is of the same order. For this, note that a tree, the minimal connected graph, has exactly $n-1$ bonds, remaining $\dfrac{n^2}{2} - \dfrac{3n}{2} + 1$ bonds to be freely chosen to be connected.

Now, let $\boldsymbol{u}(t) = (u_n(t,\boldsymbol{\xi}))_{n\geq 0}$ denote the sequence of these functions with $u_0(t) = 1$ and $u_1(t,\boldsymbol{\xi}) = e^{i\phi(\xi_1)}$. At $t = 0$, $\omega(0) = 0$ (absence of interactions) and $\boldsymbol{u}(0) = \boldsymbol{u}^{(0)}$ is the sequence with $u_n^{(0)}(\boldsymbol{\xi}) = e^{i(\phi,\eta_n)}$ for $n \geq 1$ and $u_0^{(0)}(\boldsymbol{\xi}) = 1$. Using (3.5), (2.11) and (2.15), a simple computation yields

Proposition 3.1 *The sequence $\boldsymbol{u}(t) = (u_n(t,\boldsymbol{\xi}))_{n\geq 0}$ is the formal solution of the heat equation*

$$\boldsymbol{u}_t = \frac{1}{2}\Delta_{\dot{\omega}(t)}\boldsymbol{u} \tag{3.6}$$

with initial condition $\boldsymbol{u}(0) = \boldsymbol{u}^{(0)}$, where the action of Δ_D on $\boldsymbol{u}$ is a sequence $\Delta_D\boldsymbol{u}$ with

$$(\Delta_D u)_n = \sum_{1\leq i,j\leq n, i\neq j} D(\xi_i,\xi_j)\,\frac{\partial^2 u_n}{\partial\phi(\xi_i)\partial\phi(\xi_j)}\,. \tag{3.7}$$

Remark 3.2 *1. The derivatives in (3.7) are partial derivatives with respect to ϕ at the value ξ, even for continuous space. This has to be contrasted with the functional derivative in (2.15) whose correspondence to partial derivative can only be made for lattice space.*

2. We write $\Delta_D = \nabla\cdot D\nabla$ for the differential operator (3.7) to emphasize the order of operation.

Let $\mathfrak{S}$ denote the vector space over $\mathbb{C}$ of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$ of bounded Lebesgue measurable functions $\psi_n(\boldsymbol{\xi}) = \psi(P_n\boldsymbol{\xi})$ on $(\mathbb{R}\times\mathbb{S})^n$ which are symmetric with respect to the permutation π of the index set $\{1,\ldots,n\}$, with $\psi_0(\boldsymbol{\xi}) = \psi(P_\emptyset\boldsymbol{\xi}) \in \mathbb{C}$. We define a product operation $\circ : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathfrak{S}$ for any two sequences of symmetric functions $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ as a sequence

$$\boldsymbol{\psi}_1 \circ \boldsymbol{\psi}_2 = \boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_n \tag{3.8}$$

whose components are given by

$$\psi_n(\boldsymbol{\xi}) = \sum_{I\subseteq\{1,\ldots,n\}} \psi_1(P_I\boldsymbol{\xi})\,\psi_2(P_{I^c}\boldsymbol{\xi})$$

where $I^c = \{1,\ldots,n\}\setminus I$. Note that this operation is well defined and preserves the symmetry. $\mathfrak{S}$ endowed with the operation $\circ$ form a commutative algebra with the identity $\mathbf{1} = (\delta_n(\boldsymbol{\xi}))_{n\geq 0}$ given by

$$\delta(P_I\boldsymbol{\xi}) = \begin{cases} 1 & \text{if} \quad I = \emptyset \\ 0 & \text{if} \quad I \neq \emptyset \end{cases}\,.$$

Let $\mathfrak{S}_0$ and $\mathfrak{S}_1$ denote the subspace and the affine space of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$ such that $\psi_0(\boldsymbol{\xi}) = 0$ and $\psi_0(\boldsymbol{\xi}) = 1$, respectively. We define the algebraic exponential of sequences $\boldsymbol{\phi} = (\phi_n(\boldsymbol{\xi}))_{n\geq 0}$

$$\mathscr{E}xp : \mathfrak{S}_0 \longrightarrow \mathfrak{S}_1$$

as the following formal series

$$\mathscr{E}xp\,(\boldsymbol{\phi}) = \mathbf{1} + \boldsymbol{\phi} + \frac{1}{2}\boldsymbol{\phi}\circ\boldsymbol{\phi} + \cdots + \frac{1}{n!}\underbrace{\boldsymbol{\phi}\circ\cdots\circ\boldsymbol{\phi}}_{n-\text{times}} + \cdots$$

and the algebraic logarithmic of sequences $\boldsymbol{\psi} = (\psi_n(\boldsymbol{\xi}))_{n\geq 0}$

$$\mathscr{L}n : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_0$$

as ($\boldsymbol{\psi} \in \mathfrak{S}_0$)

$$\mathscr{L}n\,(\mathbf{1}+\boldsymbol{\psi}) = \boldsymbol{\psi} - \frac{1}{2}\boldsymbol{\psi}\circ\boldsymbol{\psi} + \cdots + \frac{(-1)^{n-1}}{n}\underbrace{\boldsymbol{\psi}\circ\cdots\circ\boldsymbol{\psi}}_{n-\text{times}} + \cdots .$$

Note that the following

$$\mathscr{L}n\,(\mathscr{E}xp\,(\boldsymbol{\phi})) = \boldsymbol{\phi} \qquad \text{and} \qquad \mathscr{E}xp\,(\mathscr{L}n\,(\mathbf{1}+\boldsymbol{\psi})) = \mathbf{1}+\boldsymbol{\psi}$$

holds for any $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathfrak{S}_0$, so $\mathscr{E}xp$ and $\mathscr{L}n$ are algebraic inverse function of each other. We thus have (for details, see Ruelle [R])

Proposition 3.3 *The sequence $\boldsymbol{\psi} = (\psi_n)_{n\geq 0}$of Boltzmann functions given by (2.2) $n \geq 2$ with $\psi_0 = \psi_1(\boldsymbol{\xi}) = 1$, is an algebraic exponential of the sequence $\boldsymbol{\psi}^c = (\psi_n^c)_{n\geq 0}$ of Ursell functions given by (3.3) with $\psi_0^c = 0$ and $\psi_1^c(\boldsymbol{\xi}) = 1$:*

$$\boldsymbol{\psi} = \mathscr{E}xp\,(\boldsymbol{\psi}^c)\ . \tag{3.9}$$

Proof. Let $\langle\cdot,\cdot\rangle : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{C}$ be the inner product

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int d^n\rho(\boldsymbol{\xi})\,\phi_n(\boldsymbol{\xi})\,\psi_n(\boldsymbol{\xi}) \tag{3.10}$$

and let $\boldsymbol{\chi} = (\chi_n(\boldsymbol{\xi}))_{n\geq 0}$ be the sequence in $\mathfrak{S}$ such that

$$\chi_n(\boldsymbol{\xi}) = \prod_{j=1}^{n}\chi_\Lambda(\xi_j)$$

where $\chi_\Lambda(\xi)$ denotes the characteristic function of the set Λ:

$$\chi_\Lambda(\xi) = \begin{cases} 1 & \text{if } \xi\in\Lambda \\ 0 & \text{otherwise} \end{cases} .$$

Using

$$\langle\boldsymbol{\chi},\boldsymbol{\phi}\circ\boldsymbol{\psi}\rangle = \langle\boldsymbol{\chi},\boldsymbol{\phi}\rangle\,\langle\boldsymbol{\chi},\boldsymbol{\psi}\rangle$$

($\boldsymbol{\phi} \longrightarrow \langle\boldsymbol{\chi},\boldsymbol{\phi}\rangle = \langle\boldsymbol{\chi},\boldsymbol{\phi}\rangle(z)$ defines an homeomorphism of $\mathfrak{S}$ into the algebra of formal power series) together with (3.9), (3.10) and (2.1), we have

$$\Xi_\Lambda(\beta,z) = \langle\boldsymbol{\chi},\boldsymbol{\psi}\rangle(z) = \langle\boldsymbol{\chi},\mathscr{E}xp\,(\boldsymbol{\psi}^c)\rangle(z) = \exp\{\langle\boldsymbol{\chi},\boldsymbol{\psi}^c\rangle(z)\}$$

which, in view of (3.1)–(3.3), proves the proposition.

□

Replacing $\boldsymbol{\psi}$ in (3.9) by the complex Boltzmann sequence $\boldsymbol{u}(t) = (u_n(t,\boldsymbol{\xi}))_{n\geq 0}$, leads to the complex Ursell sequence $\boldsymbol{v}(t) = (v_n(t,\boldsymbol{\xi}))_{n\geq 0}$ given by

$$\boldsymbol{v}(t) = \mathscr{L}n\,(\boldsymbol{u}(t))\ . \tag{3.11}$$

Note that, in view of (3.11), (3.5), (3.9) and the fact

$$\left(u \circ u\right)_n(t, \boldsymbol{\xi}) = \sum_{I \subseteq \{1,\ldots,n\}} \psi(t, P_I\boldsymbol{\xi})\, \psi(t, P_{I^c}\boldsymbol{\xi}) e^{i(\phi,\eta_n)} = \left(\psi \circ \psi\right)_n(t, \boldsymbol{\xi}) e^{i(\phi,\eta_n)} \ ,$$

we have $v_n(t,\boldsymbol{\xi}) = \psi_n^c(t,\boldsymbol{\xi})\, e^{i(\phi,\eta_n)}$. We shall use (3.11) together with (3.6) to derive a partial differential equation for the complex Ursell functions $v_n(t,\boldsymbol{\xi})$. A differential equation for the Ursell function $\psi_n^c(t,\boldsymbol{\xi})$, given by (3.3) with $\beta v\left(\xi_i, \xi_j\right)$ replaced by $\omega(t, \xi_i, \xi_j) \equiv \omega(t; \sigma_i, \sigma_j, x_i - x_j)$, is obtained setting $\phi = 0$ at the end of the computations.

Recalling Remark 3.2.2, we have

$$\begin{aligned}
\boldsymbol{v}_t &= \frac{1}{\boldsymbol{u}} \circ \boldsymbol{u}_t \\
&= \frac{1}{2}\mathscr{E}xp\left(-\boldsymbol{v}\right) \circ \Big(\nabla \cdot \dot{\omega} \nabla \mathscr{E}xp\left(\boldsymbol{v}\right)\Big) \\
&= \frac{1}{2}\Delta_{\dot{\omega}}\boldsymbol{v} + \frac{1}{2}\nabla\boldsymbol{v} \bullet \dot{\omega}\nabla\boldsymbol{v} \qquad (3.12)
\end{aligned}$$

where

$$\begin{aligned}
\left(\nabla\boldsymbol{v} \bullet D\nabla\boldsymbol{v}\right)_n &= \sum_{1\le i,j\le n, i\ne j} D(\xi_i,\xi_j)\left(\frac{\partial\boldsymbol{v}}{\partial\phi(\xi_i)} \circ \frac{\partial\boldsymbol{v}}{\partial\phi(\xi_j)}\right)_n \\
&= \sum_{I\subset\{1,\ldots,n\}} \nabla v\left(P_I\boldsymbol{\xi}\right) \cdot D\nabla v\left(P_{I^c}\boldsymbol{\xi}\right) \ ,
\end{aligned}$$

from which equation (6) of [BK1] is obtained

$$\frac{\partial}{\partial t}\psi_n^c(\boldsymbol{\xi}) = -U_n(\boldsymbol{\xi})\,\psi_n^c(\boldsymbol{\xi}) - \frac{1}{2}\sum_{I\subseteq\{1,\ldots,n\}} U\left(P_I\boldsymbol{\xi}; P_{I^c}\boldsymbol{\xi}\right)\, \psi^c(P_I\boldsymbol{\xi})\,\psi^c(P_{I^c}\boldsymbol{\xi}) \qquad (3.13)$$

with initial condition $\boldsymbol{\psi}_0^c = \left(\psi_{0,n}^c(\boldsymbol{\xi})\right)_{n\ge 0}$ given by

$$\psi_0^c\left(P_I\boldsymbol{\xi}\right) = \begin{cases} 1 & \text{if } |I| = 1 \\ 0 & \text{otherwise} \end{cases} \ . \qquad (3.14)$$

For brevity, we have omitted the time dependence in (3.13). Here, $U_n(t,\boldsymbol{\xi}) = U(t, P_n\boldsymbol{\xi})$ is the energy of the state $P_n\boldsymbol{\xi}$ due to interaction $\dot{\omega}\left(t, \xi_i, \xi_j\right)$ and

$$U\left(t, P_I\boldsymbol{\xi}; P_J\boldsymbol{\xi}\right) = \sum_{i\in I,\, j\in J} \dot{\omega}\left(t, \xi_i, \xi_j\right) \ . \qquad (3.15)$$

The minus sign in the r.h.s. of (3.13) comes from the derivatives of $e^{i\phi(\xi)}$ at $\phi = 0$.

Using the variation of constants formula, (3.13) is equivalent to the following integral equation

$$\psi_n^c(t,\boldsymbol{\xi}) = -\frac{1}{2}\int_0^t ds\, e^{-\displaystyle\int_s^t d\tau\, U_n(\tau,\boldsymbol{\xi})} \sum_{I\subseteq\{1,\ldots,n\}} U\left(s, P_I\boldsymbol{\xi}; P_{I^c}\boldsymbol{\xi}\right) \psi^c(s, P_I\boldsymbol{\xi})\, \psi^c(s, P_{I^c}\boldsymbol{\xi}) \ , \qquad (3.16)$$

for $n > 1$ with $\psi_0^c(t, \boldsymbol{\xi}) = 0$ and $\psi_1^c(t, \boldsymbol{\xi}) = 1$ for all t.

The second Ursell function ψ_2^c satisfies

$$\begin{aligned}\psi_2^c(t, \xi_1, \xi_2) &= -\int_0^t ds\, \dot{\omega}\,(s, \xi_1, \xi_2)\, e^{-\int_s^t d\tau\, \dot{\omega}\,(\tau, \xi_1, \xi_2)} \\ &= e^{-\int_0^t d\tau\, \dot{\omega}\,(\tau, \xi_1, \xi_2)} - 1 = e^{-\omega(t, \xi_1, \xi_2)} - 1 \end{aligned} \tag{3.17}$$

and agrees with (3.3) since at time $t = 0$, $\omega\,(0, \xi_1, \xi_2) = 0$. One can verify that there is a unique solution to the integral equation (3.16) formally given by the Ursell functions (3.3) with βv replaced by $\omega(t)$. These follows from (3.13) by induction in n. Equation (3.16), derived for the first time by Brydges–Kennedy [BK], is the starting point for the procedure of the next section.

4 Majorant Differential Equation

If $g(t, z)$ is an one–parameter family of holomorphic function on the open disc

$$D_R = \{z \in \mathbb{C} : |z| < R\}$$

it has a convergent power series

$$g(t, z) = \sum_{j=0}^{\infty} C_j(t)\, z^j$$

which we denote by $\boldsymbol{C} \cdot \boldsymbol{z}$ with $\boldsymbol{C} = (C_j)_{j \geq 0}$ and $\boldsymbol{z} = (z^j)_{j \geq 0}$.

Definition 4.1 *$g(t, z) = \boldsymbol{C}(t) \cdot \boldsymbol{z}$ is said to be a uniform majorant of a function $f(t, z) = \boldsymbol{c}(t) \cdot \boldsymbol{z}$ in $\mathscr{D} = [a, b] \times D_R$ if*

$$|c_j(t)| \leq C_j(t)$$

holds for every $j \in \mathbb{N}$ and $t \in [a, b]$. We write $f \trianglelefteq g$ for the majorant relation.

Note that a majorant g is a real analytic function and the majorant relation is preserved by differentiation in z and integration with respect to t.

Proof of Theorem 2.2. Part I. Our aim is to find an equation for a uniform majorant $\Phi(t, z) = \boldsymbol{A}(t) \cdot \boldsymbol{z}$ of the pressure (3.1). Let $\boldsymbol{a}(t) = (a_n(t))_{n \geq 1}$ be given by

$$a_n(t) = \sup_{\xi_1 \in \Lambda} \int d\rho(\xi_2) \cdots \int d\rho(\xi_n)\ |\psi_n^c(t, \boldsymbol{\xi})|$$

and define

$$\Gamma(t) := \left\| \dot{\omega}(t) \right\| , \qquad \gamma(s, t) := \int_s^t B(\tau)\, d\tau\,, \tag{4.1}$$

where B and $\|\cdot\|$ are given by (2.8) and (2.6). It follows from (3.16)–(3.15) and definitions (2.8) and (2.6), that

$$a_n(t) \leq \frac{1}{2}\int_0^t ds\, e^{n\gamma(s,t)}\Gamma(s)\sum_{k=1}^{n-1}\binom{n}{k} k\,(n-k)\,a_k(s)\,a_{n-k}(s) \tag{4.2}$$

holds for $n > 1$ with $a_1 = 1$. We thus have

$$|b_{\Lambda,n}| \leq \frac{1}{n!}a_n(t) \leq A_n(t) \tag{4.3}$$

uniformly in Λ where $b_{\Lambda,n}$ is the Mayer coefficients (3.2) with βv replaced by $\omega(t)$ and $\boldsymbol{A}(t) = (A_n(t))_{n\geq 1}$ is a sequence of continuous functions so that $(n!A_n(t))_{n\geq 1}$ satisfies (4.2) as an equality:

$$A_n(t) = \frac{1}{2}\int_0^t ds\, e^{n\gamma(s,t)}\Gamma(s)\sum_{k=1}^{n-1} k\, A_k(s)\,(n-k)\,A_{n-k}(s) \tag{4.4}$$

for $n > 1$ with $A_1 = 1$.

Writing

$$\Phi\,(t,z) := \sum_{n=1}^{\infty} A_n(t)\, z^n \tag{4.5}$$

equations (3.1), (4.3) and (4.4) yields

$$\beta p_\Lambda(\beta,z) \trianglelefteq \Phi\,(1,z) \tag{4.6}$$

uniformly in Λ where $\Phi\,(1,z)$ is the solution of

$$\Phi(t,z) = z + \frac{1}{2}\int_0^t ds\,\Gamma(s)\left(z\,\frac{\partial\Phi}{\partial z}\left(s, z\, e^{\gamma(s,t)}\right)\right)^2 \tag{4.7}$$

at time $t = 1$. Here, the following identities have been used

$$\begin{aligned}\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} kA_k\,(n-k)A_{n-k} &= \sum_{n-1=1}^{\infty}\sum_{k=1}^{n-1} kA_k z^k\,(n-k)A_{n-k}z^{n-k} \\ &= \sum_{k=1}^{\infty} kA_k z^k \sum_{n-1=k}^{\infty}(n-k)A_{n-k}z^{n-k} = \left(z\frac{\partial\Phi}{\partial z}\right)^2 .\end{aligned}$$

The partial derivative of Φ with respect to t and z are denoted by Φ_t and Φ_z. From (4.5) and (4.7), we have

$$\begin{aligned}\left(z\Phi_z\right)_t(s,ze^{\gamma}) &= \frac{\partial\gamma}{\partial t}\left(ze^{\gamma} + 2^2z^2e^{2\gamma}A_2(s) + 3^2z^3e^{3\gamma}A_3(s) + \cdots\right) \\ &= B(t)\; z\left(z\Phi_z\right)_z(s,ze^{\gamma})\ ,\end{aligned}$$

with $\gamma = \gamma(s,t)$, and

$$z\Phi_z = z + \int_0^t ds\, \Gamma(s)\, z\Phi_z \left(s, ze^{\gamma(s,t)}\right)\ z\left(z\Phi_z\right)_z \left(s, ze^{\gamma(s,t)}\right)\ .$$

Differentiating (4.7) with respect to t,

$$\Phi_t = \frac{1}{2}\Gamma(z\Phi_z)^2 + \int_0^t ds\, \Gamma(s)\ \ z\Phi_z \left(s, ze^{\gamma(s,t)}\right)\ \ (z\Phi_z)_t \left(s, ze^{\gamma(s,t)}\right)$$

thus yields

$$\Phi_t = \frac{1}{2}\Gamma(z\Phi_z)^2 + B\left(z\Phi_z - z\right)$$

with initial condition $\Phi(0,z) = z$. This establishes the majorant relation of Theorem 2.2 and equation (2.19).

□

It will be convenient to consider the density

$$\rho_\Lambda(\beta, z) = z\frac{\partial p_\Lambda}{\partial z}(\beta, z) \tag{4.8}$$

whose corresponding Mayer series has the same radius of convergence of the Mayer series of p_Λ. Writing $\Theta(t,z) = \Phi_z(t,z)$, we have

$$\rho_\Lambda(\beta, z) \trianglelefteq z\Theta(\beta, z) \tag{4.9}$$

with Θ satisfying the following quasi–linear first order PDE

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + B\left(z\Theta_z + \Theta - 1\right) \tag{4.10}$$

with initial condition $\Theta(0,z) = 1$.

In view of (4.5), (4.8) and (4.9), we seek for solutions of (4.10) of the form

$$\Theta(t,z) = 1 + \sum_{n=2}^{\infty} C_n(t) z^{n-1} \tag{4.11}$$

with $C_n = nA_n(t) \geq 0$ and this form is preserved by the equation.

Remark 4.2 *1. The convergence of Mayer series for an integrable stable potential ($\omega(t)$ so that $\|\omega(t)\| < \infty$ for $t \in [0,1]$ and $\omega(1) = \beta v$) can be established as in the case of non negative potential ($B \equiv 0$). Writing $w = e^{\gamma(0,t)} z$ and $\widetilde{A}_n = e^{-n\gamma(0,t)} A_n$, we have*

$$\Phi(t,z) = \sum_{n=1}^{\infty} \widetilde{A}_n(t)\ w^n := \widetilde{\Phi}(t,w)$$

with $\widetilde{A}_n$ and $\widetilde{\Phi}(t,w)$ satisfying $\widetilde{\Phi}_t = (\Gamma/2)\left(w\widetilde{\Phi}_w\right)^2$. The Mayer series of these systems converges for all (β, z) such that $\widetilde{\Phi}(1, e^{\gamma(0,1)}|z|)$ exists.

2. We also consider equation (4.10) satisfying initial condition $\Theta(t_0, z) = 1$ at time t_0 and let $t_0 \to -\infty$ at the end of computations to control the ultraviolet problem of Yukawa gas (more comments in Section 5 and Remarks at the end of [BK]).

5 The Method of Characteristics

In this section the convergence of Mayer series will be addressed from the point of view of (4.10). We seek for a subset of $\{(t,z) : t, z \geq 0\}$ in which there exists a unique classical solution of the initial value problem. We shall distinguish two cases, depending on whether the stability function $B(t)$ in (2.8) is or is not identically zero.

5.1 Non Negative Potentials

We begin by addressing equation (4.10) with $B(s) \equiv 0$,

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) \tag{5.1}$$

with $\Theta(0,z) = 1$, by the method of characteristics. This equation describes a density majorant $\rho_\Lambda(\beta,z) \trianglelefteq \Theta(\beta,z)$ of a gas with non negative (repulsive) potential $v \geq 0$.

It is convenient change the t variable by $\tau = \displaystyle\int_0^t \Gamma(s)\ ds$ and write $U(\tau) = \Theta(t(\tau), z(\tau))$. Equation

$$U' = t'\Theta_t + z'\Theta_z$$

together with (5.1) yield a pair of ordinary differential equations (ODE)

$$U' = zU^2 \tag{5.2}$$

$$z' = -z^2U \tag{5.3}$$

satisfying the initial conditions $U(0) = 1$ and $z(0) = z_0$. Dividing (5.2) by (5.3) gives

$$\frac{dU}{dz} = -\frac{U}{z}$$

which can be solved as a function of z

$$U(z) = \frac{z_0}{z}\ . \tag{5.4}$$

Inserting (5.4) into (5.3), leads to the following characteristic equation

$$z' = -z_0z$$

whose solution

$$z(\tau) = z_0 \exp\{-z_0\tau\}\ . \tag{5.5}$$

Now, $\Theta(t,z)$ is given by (5.4) with $z_0 = z_0(\tau,z)$ implicitly given by (5.5). If $W(x)$ denotes the Lambert W–function [K], defined as the inverse of $f(W) = We^W$ ($f(W(x)) = x$ and $W(f(w)) = w$), we have

$$z_0(\tau,z) = \frac{-1}{\tau}W(-z\tau)$$

and the solution of the initial value problem (5.1) is thus given by

$$\Theta(t,z) = \frac{-1}{z\tau}W(-z\tau)$$

provided

$$ez\tau = ez \int_0^t \Gamma(s)\ ds < 1\ . \tag{5.6}$$

Since $\Im(W(x)) \neq 0$ for $x < -1/e$ in the principal branch of the W –function (see e.g. [K]), this condition has to be imposed to ensure uniqueness. W has a branching point singularity at $-1/e+i0$ with $W(-1/e) = \Re(W(-1/e)) = -1$. At the singularity $z^* = z^*(t) = (z\tau)^{-1}$, $\Theta(t, z^*) = e$ and does not diverge.

Remark 5.1 *1. For non negative potentials we can actually find a pressure majorant $\Phi(\beta, z)$ explicitly. Differentiating $x = W(x)\exp(W(x))$ and solving for W' gives, after some manipulations,*

$$\frac{W}{x} = \left(W + \frac{W^2}{2}\right)'$$

and this, together with (4.8) and $W(0) = 0$, yields

$$\begin{aligned}\Phi(t, z) &= \frac{-1}{\tau}\int_0^{-z\tau} \frac{W(x)}{x} dx \\ &= \frac{-1}{\int_0^t \Gamma(s)\ ds}\left(W\left(-z\int_0^t \Gamma(s)\ ds\right) + \frac{1}{2}W^2\left(-z\int_0^t \Gamma(s)\ ds\right)\right)\ .\end{aligned}$$

2. *The interpretation of (5.6) is as follows. Let*

$$\mathscr{T}(r, z_0) = \frac{\ln r - \ln z_0}{r - z_0}$$

be the time necessary for two characteristics starting from r and z_0 to meet:

$$z = re^{-r\mathscr{T}} = z_0 e^{-z_0\mathscr{T}}\ .$$

$\mathscr{T}(r, z_0)$ is a monotone decreasing function of r with $\lim_{r\nearrow z_0} \mathscr{T}(r, z_0) = 1/z_0$. Note that, $\Theta(\mathscr{T}, z)$ assumes two different values $U(z) = r/z$ and $U_0(z) = z_0/z$ for $r < z_0$ which tends to e in the limit $r \nearrow z_0$. The minimum crossing time of two characteristics with $r \leq z_0$ is attained when both coincide and this occurs exactly at the branching singularity $-z\mathscr{T} = -1/e$ of $W(x)$ where uniqueness of (5.1) breaks down.

3. *From condition (5.6) and (4.6), the Mayer series (3.1) converges if*

$$|z| < \frac{1}{e\int_0^1 \Gamma(s)\ ds} \tag{5.7}$$

and this agrees with Theorem 2 of ([BK]) for $B(s) \equiv 0$. Ruelle's estimate (see Section 4.5 of [R]) replaces $\int_0^1 \Gamma(s)\ ds$ in (5.7) by $\|1 - \exp(-\beta v)\|$, with the norm (2.6). Choosing $\omega(t) = t\beta v$, we have $\int_0^1 \Gamma(s)\ ds = \beta\|v\| \geq \|1 - \exp(-\beta v)\|$ and (5.7) improves slightly Ruelle's bound.

4. *If $\omega(t) = t\beta v$, the estimate in Theorem 2 of ([BK]) for stable potentials ($B(s) \neq 0$) can be improved. According Remark 4.2.1, the Mayer series converges for all (β, z) so that*

$$e\,|z|\exp\left\{\int_0^1 B(s)\ ds\right\}\int_0^1 \Gamma(s)\ ds\ < 1 \tag{5.8}$$

and this gives

$$e\beta\,|z|\,e^{\beta B}\,\|v\| < 1 \tag{5.9}$$

compared with the condition of ([BK]) (see also (5.22) below)

$$e\beta\,|z|\,e^{\beta B}\frac{\sinh\beta B}{\beta B}\,\|v\| < 1$$

(note that $\sinh x/x \geq 1$*). Estimate (5.9) has been previously obtained by Brydges–Federbush [BF] (see eq. (11) there).*

5.2 Stable Potentials

We now turn to equation (4.10). Although we are not able to solve it explicitly, we shall obtain an upper solution $\overline{\Theta}(t, z)$ of (4.10) whose existence implies absolutely convergence of density majorant series $\Theta(t, z)$. $\overline{\Theta}(t, z)$ satisfies equation (5.1) for non–negative potential with Γ substituted by Γ/f for some properly chosen functions f. Our procedure removes the mystery behind this freedom and allows us to obtain various convergence conditions.

Proof of Theorem 2.2. Conclusion. Inspired on Remark 4.2.1, let us change the variable z to $w = f(t)z$, for a function $f(t) \geq 1$ such that $f(0) = 1$, and define $\Theta(t, z) = f(t)\Psi\left(t, f(t)z\right)$. Using the chain rule, we have

$$\begin{aligned}\Theta_t &= f\left(\Psi_t + \frac{f'}{f}\Psi + \frac{f'}{f}w\Psi_w\right)\\ z\Theta_z &= fw\Psi_w\end{aligned}$$

which, together with (4.10), yields

$$\Psi_t - \left(B - \frac{f'}{f} + \Gamma w\Psi\right)w\Psi_w = \left(B - \frac{f'}{f}\right)\Psi + \Gamma w\Psi^2 - \frac{B}{f}\ .$$

Choosing

$$f(t) = \exp\left\{\int_0^t B(s)\ ds\right\}$$

yields

$$\Psi_t - \Gamma w^2\Psi\Psi_w = \Gamma w\Psi^2 - \frac{B}{f} \leq \Gamma w\Psi^2 \tag{5.10}$$

by positivity of f and B. Note that Ψ satisfies (5.1) as an inequality.

As before, we define $\tau = \displaystyle\int_0^t \Gamma(s)\ ds$ and write

$$U(\tau) = \Psi(t(\tau), w(\tau)). \tag{5.11}$$

From $U' = \Psi_t t' + \Psi_w w'$and (5.10), we have

$$U' \leq wU^2 \tag{5.12}$$

$$w' = -w^2 U \tag{5.13}$$

with initial conditions $U(0) = 1$ and $w(0) = w_0$. Equations (5.12) and (5.13) can be written in terms of the variables $V(\tau) = wU$ and w as

$$\begin{aligned} V' &\leq 0 \\ \frac{w'}{w} &= -V \end{aligned} \tag{5.14}$$

with $V(0) = w_0$, which implies

$$V(\tau) \leq w_0 \tag{5.15}$$

and

$$w(\tau) \geq w_0 \exp\{-w_0 \tau\} \ . \tag{5.16}$$

Here, we need an inequality for $w_0 = w_0(\tau, z)$ so that (5.16) holds. Writing (5.16) as

$$-w_0\tau \ e^{-w_0\tau} \geq -w\tau \ ,$$

we have

$$w_0(\tau, w) \leq -\frac{1}{\tau} W(-w\tau) \ , \tag{5.17}$$

where W is the Lambert W–function, provided

$$w\tau \ = z \exp\left\{\int_0^t B(s') \ ds'\right\} \int_0^t \Gamma(s) \ ds \ < 1/e \ .$$

Plugging (5.17) into (5.15), together with (5.11), leads to

$$\Theta(t, z) = \frac{V(\tau(t))}{z} \leq \overline{\Theta}(t, z) \ ,$$

where

$$\overline{\Theta}(t, z) = \frac{-1}{z \displaystyle\int_0^t \Gamma(s) \ ds} W\left(-z \exp\left\{\int_0^t B(s) \ ds\right\} \int_0^t \Gamma(s) \ ds \right) \ , \tag{5.18}$$

recovering the statements of Remark 4.2.1.

We now go beyond inequality (5.16). For this we modify slightly the previous scaling by defining $\Theta(t, z) = \Psi(t, f(t)z)$. Using the chain rule, we now have

$$\Psi_t - \frac{\Gamma}{f} w^2 \Psi \Psi_w = \left(B - \frac{f'}{f}\right) w\Psi_w + B\Psi - B + \frac{\Gamma}{f} w\Psi^2 \ .$$

Choosing

$$f(t) = \exp\left\{2\int_0^t B(s)\ ds\right\} \tag{5.19}$$

and using the fact

$$\begin{aligned} -w\Psi_w + \Psi - 1 &= -\sum_{n=2}^{\infty}(n-1)C_n w^{n-1} + \sum_{n=1}^{\infty} C_n w^{n-1} - 1 \\ &= -\sum_{n=2}^{\infty}(n-2)C_n w^{n-1} \le 0 \end{aligned}$$

in view of (4.11) and $w = zf(t) \ge 0$, we have

$$\Psi_t - \frac{\Gamma}{f} w^2 \Psi \Psi_w = B\left(-w\Psi_w + \Psi - 1\right) + \frac{\Gamma}{f} w\Psi^2 \le \frac{\Gamma}{f} w\Psi^2 \ . \tag{5.20}$$

We define

$$\tau = \int_0^t \frac{\Gamma}{f}(s)\ ds \tag{5.21}$$

and repeat the steps from (5.11) to (5.18). Provided

$$w\tau \ = zf(t)\int_0^t \frac{\Gamma}{f}(s)\ ds \ = z\int_0^t e^{2\int_s^t B(s)\ ds}\Gamma(s)\ ds < 1/e \ , \tag{5.22}$$

the solution of the initial value problem (4.10) satisfies

$$\Theta(t,z) \le \frac{-1}{z\displaystyle\int_0^t \exp\left\{2\int_s^t B(\sigma)\ d\sigma\right\}\Gamma(s)\ ds} W\left(-z\int_0^t \exp\left\{2\int_s^t B(\sigma)\ d\sigma\right\}\Gamma(s)\ ds\right) \tag{5.23}$$

Equation (2.20) follows from (5.18), (5.23) and Remark 5.1.1 and this concludes the proof of Theorem 2.2.

□

Remark 5.2 *1. Working directly with the integral equation (3.16), Brydges–Kennedy [BK1] have established recursively the following*

$$a_n(t) \le \left(f(t)\tau(t)z\right)^{n-1} n^{n-2}$$

with f , τ and a_n given by (5.19) and (5.21) and (4.2). Hence, the density majorant of stable potentials satisfies

$$\begin{aligned} \Theta(t,z) &= \sum_{n=1}^{\infty} \frac{a_n(t)}{(n-1)!} z^{n-1} \\ &\le \frac{-1}{f\tau z}\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}\left(-f\tau z\right)^n = \frac{-1}{\tau w} W\left(-\tau w\right) \ , \end{aligned}$$

by the Taylor series of the Lambert W–function [K], and this is exactly (5.23).

2. *The Mayer series (3.1) converges if (t, z) is such that (5.22) holds and this agrees with Theorem* 2 *of ([BK]). Criterion (5.22) for convergence may be more advantageous than (5.8) when $\omega(t) = \beta v$ is a multiscale decomposition of a potential failing to satisfy either condition* 1. *or* 3. *of (2.5). As observed in [BK1] (see Remarks at the end of this work), a poor stability bound (large $B(t)$) at short distances ($t \ll 0$) may be overcome by the smallness of the interaction $\Gamma = \left\| \dot{w}(t) \right\|$ at these scales.*

6 An Application

We shall illustrate the usefulness of condition (5.22) in the ultraviolet problem of Yukawa gas for $\beta < 4\pi$. The procedure of previous subsection will be further extended in order to overcome the sequence of thresholds for $4\pi \le \beta < 8\pi$.

6.1 The Yukawa Gas

We here develop a Mayer series majorant $\Theta(\beta, z)$ for the density of the Yukawa gas in two space dimensions. Particles in this system have assigned two charges $\sigma \in \{-1, 1\}$ and they interact via Yukawa potential

$$\begin{aligned} v(\xi_1, \xi_2) &= \sigma_1 \sigma_2 \left(-\Delta + 1\right)^{-1} (x_1, x_2) \\ &= \frac{\sigma_1 \sigma_2}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{-ik\cdot(x_1 - x_2)}}{k^2 + 1} d^2 k \end{aligned} \tag{6.1}$$

where $-\Delta$ is the Laplace operator in $\mathbb{R}^2$. Potential (6.1) violates condition 1. of (2.5), since

$$|v(\xi, \xi)| = \frac{1}{4\pi} \lim_{R\to\infty} \int_0^R \frac{2\eta}{\eta^2 + 1} d\eta = \frac{\beta}{4\pi} \lim_{R\to\infty} \ln\left(R^2 + 1\right)$$

diverges logarithmically in the ultraviolet limit. We replace βv by

$$\begin{aligned} \omega(t, |x_1 - x_2|) &= \beta \sigma_1 \sigma_2 \left(-\Delta + \gamma(t)\right)^{-1} (x_1, x_2) \\ &= \frac{\beta \sigma_1 \sigma_2}{2\pi} K_0 \left(\sqrt{\gamma(t)}\, |x_1 - x_2|\right) \end{aligned} \tag{6.2}$$

where K_0 is the modified Bessel function of second kind (see e.g. [GJ] p. 126) and γ is a monotone decreasing scale function such that

$$\gamma(-\infty) = \infty \qquad \text{and} \qquad \gamma(0) = 1 \ ,$$

and decompose it into scales

$$\omega(t, r) = \int_{-\infty}^{t} \dot{\omega}(s, r)\, ds$$

where

$$\dot{\omega}(s, r) = \frac{\beta \sigma_1 \sigma_2}{4\pi} \frac{\gamma'(s)}{\sqrt{\gamma(s)}} r K_0' \left(\sqrt{\gamma(s)} r\right) \ . \tag{6.3}$$

By definitions (2.8) and (4.1), we have

Proposition 6.1 *For each* $-\infty < s < 0$, $\dot{\omega}(s, x-y)$ *is a positive kernel satisfying properties* $1-4$ *of (2.5) with*

$$B(s) = \frac{1}{2}\left|\dot{\omega}(s,0)\right| = \frac{\beta}{8\pi}\left(-\ln\ \gamma(s)\right)' \tag{6.4}$$

and

$$\Gamma(s) = \|\dot{\omega}(s,\cdot)\| = \beta\ \frac{-\gamma'(s)}{\gamma^2(s)}. \tag{6.5}$$

Proof. Properties $1-4$ of (2.5) follow immediately from (6.1), provided (6.4) and (6.5) are established. Applying Cauchy formula for the integral (6.1) (with 1 replaced by $\gamma(t)$) in the $x-y$ direction, gives

$$\omega(t,r) = \frac{\beta\sigma_1\sigma_2}{2\pi} g\left(\sqrt{\gamma(t)}r\right)$$

with

$$g(w) := \int_0^\infty \frac{e^{-w\sqrt{k^2+1}}}{\sqrt{k^2+1}} dk\ . \tag{6.6}$$

With $w = \sqrt{\gamma(s)}r$ we have by chain rule

$$(g\circ w)' = -w'\int_0^\infty e^{-w\sqrt{k^2+1}}dk\ = -\frac{w'}{w}\int_w^\infty \frac{e^{-\zeta}}{\sqrt{\zeta^2-w^2}}\zeta d\zeta \equiv -\frac{w'}{w}h(w) \tag{6.7}$$

and equation (6.3) can thus be written as

$$\dot{\omega}(s,r) = \frac{-\beta\sigma_1\sigma_2}{4\pi}\frac{\gamma'(s)}{\gamma(s)}h\left(\sqrt{\gamma(s)}\ r\right)\ , \tag{6.8}$$

which gives (6.4) by taking absolute value and $r = 0$. Note that $-\gamma'(s) \geq 0$, by assumption. Integrating (6.6) times w, yields

$$\int_0^\infty g(w)wdw = \int_0^\infty \frac{1}{(k^2+1)^{3/2}}dk = 1 \tag{6.9}$$

which implies

$$\begin{aligned} \|\dot{\omega}(s,\cdot)\| &= \frac{\beta}{2\pi}\frac{d}{ds}2\pi\int_0^\infty g\left(\sqrt{\gamma(s)}r\right)rdr \\ &= \beta\left(\frac{1}{\gamma(s)}\right)' \end{aligned}$$

and concludes the proof.

□

The majorant density $\Theta(t,z)$ of the Yukawa gas satisfies equation (4.10) with B and Γ given by (6.4) and (6.5), respectively, and initial data $\Theta(t_0,z) = 1$ where t_0 plays the role of a ultraviolet cutoff which will be removed by letting $t_0 \to -\infty$ at the end of calculations.

From Remark 5.2.2, the Mayer series converges if (5.22) holds with the necessary adjustments in the integral limits. We have

$$\begin{aligned}\lim_{t_0\to-\infty}\int_{t_0}^{0}\Gamma(t')e^{2\int_{t'}^{0}B(s)\;ds}dt' &= -\beta\lim_{t_0\to-\infty}\int_{t_0}^{0}\frac{\gamma'(t')}{\gamma^{2-\beta/4\pi}(t')}dt' \\ &= \frac{4\pi\beta}{4\pi-\beta}\lim_{t_0\to-\infty}\left\{1-\gamma^{\beta/4\pi-1}(t_0)\right\}=\frac{4\pi\beta}{4\pi-\beta}\end{aligned}$$

provided $\beta<4\pi$, independently of the scale function γ. The Mayer series of Yukawa gas converges if

$$\beta<4\pi \qquad \text{and} \qquad |z|<\frac{4\pi-\beta}{4\pi e\beta}\ .$$

6.2 The Yukawa Gas Beyond the First Threshold

The Mayer series for the Yukawa gas between the first and second thresholds, $4\pi\le\beta<6\pi$, diverges because the second Ursell function $\psi_2^c(t,\boldsymbol{\xi})$ diverges in the ultraviolet limit (see e.g. [BGN, BK] and references therein). We are going to show that the renormalized density majorant $\Theta^R(\beta,z)$, defined by $\Theta(\beta,z)$ with the singular part of $\psi_2^c(t,\boldsymbol{\xi})$ removed, exists and defines a convergent Mayer series provided $\beta<16\pi/3$. For $16\pi/3\le\beta<6\pi$, the upper bound A_3 of $\psi_3^c(t,\boldsymbol{\xi})$ also diverges in the ultraviolet limit. Although $\beta=16\pi/3$ is not a threshold for the Yukawa model, our procedure consider this value, and more generally all

$$\beta_n=8\pi\left(1+\frac{1}{n}\right)^{-1}=8\pi\frac{n}{n+1} \tag{6.10}$$

with $n\ge1$ even, as an actual singularity (see Remarks on pg. 47 of [BK]). We shall then explain how to extend the result to the n–th "threshold" and prove Theorem 2.3

The second Ursell function $\psi_2^c(t,\boldsymbol{\xi})$ is explicitly given by (3.17). Introducing the cutoff t_0 and the necessary adjustments in the integral limits together with (6.2), we have

$$\psi_2^c(t,\xi_1,\xi_2)=\exp\left\{\frac{-\beta\sigma_1\sigma_2}{2\pi}\ \left(K_0\left(\sqrt{\gamma(t)}r_{12}\right)-K_0\left(\sqrt{\gamma(t_0)}r_{12}\right)\right)\right\}-1 \tag{6.11}$$

where $r_{12}=|x_1-x_2|$.

To isolate the singularity of (6.11) we replace $a_2(t)=\sup_{\xi_1\in\Lambda}\int d\rho(\xi_2)\,|\psi_2^c(t,\xi_1,\xi_2)|$ by a refined version

$$a_2^{\pm}(t)=\sup_{x_1\in\Lambda}\int d\mu(x_2)\left|\frac{1}{2}\sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}:\\ \sigma_1\sigma_2=\pm}}\psi_2^c(t,\xi_1,\xi_2)\right|$$

which distinguishes whether the charge product $\sigma_1\sigma_2$ is positive or negative. Using the fact that

$K_0(x)$ is a monotone decreasing function together with $1-e^{-\delta K}\leq \delta K$ for $\delta K\geq 0$, we have

$$\begin{aligned} a_2^+(t) &\leq \beta \int_0^\infty r\left(K_0\left(\sqrt{\gamma(t)}r\right)dr - K_0\left(\sqrt{\gamma(t_0)}r\right)\right)dr \\ &\leq \beta\left(\frac{1}{\gamma(t)}-\frac{1}{\gamma(t_0)}\right) . \end{aligned}$$

For $\sigma_1\sigma_2=-1$, we have $-\dot{\omega}(s,r)\geq 0$, in view of (6.8), and from r.h.s. of the first line of (6.11)

$$\begin{aligned} a_2^-(t) &= \frac{\beta}{2}\int_0^\infty dw\ w\ h(w)\int_{t_0}^t \frac{-\gamma'(s)}{\gamma^2(s)}\exp\left\{\frac{\beta}{4\pi}\ln\left(\frac{\gamma(s)}{\gamma(t)}\right)h(w)\right\}\ ds \\ &= \frac{2\pi\beta}{4\pi-\beta}\int_0^\infty dw\ w\ h(w)\left(\frac{1}{\gamma(t)}-\frac{1}{\gamma(t_0)}\left(\frac{\gamma(t_0)}{\gamma(t)}\right)^{(\beta/4\pi)h(w)}\right) \\ &\leq \frac{4\pi\beta}{4\pi-\beta}\left(\frac{1}{\gamma(t)}-\frac{1}{\gamma(t_0)}\left(\frac{\gamma(t_0)}{\gamma(t)}\right)^{\beta/4\pi}\right) = A_2(t) . \end{aligned}$$

Note that $h(w)$ is a monotone decreasing function with $h(0)=1$ and

$$\int_0^\infty dw\ w\ h(w) = \int_0^\infty dw\ w^2\ g'(w) = 2\int_0^\infty dw\ w\ g(w) = 2$$

by (6.7), integration by parts and (6.9). As a consequence, for any $\beta>4\pi$ there exist $\varepsilon>0$ and a constant $C=C(\varepsilon)>0$ such that $\dfrac{\beta}{4\pi}+\varepsilon>1$ and

$$a_2^-(t)\geq C\left(\frac{1}{\gamma(t_0)}\left(\frac{\gamma(t_0)}{\gamma(t)}\right)^{\beta/4\pi+\varepsilon}-\frac{1}{\gamma(t)}\right)$$

diverges as $t_0\to-\infty$. In contradistinction, $a_2^+(t)$ remains bounded for any $\beta>0$.

Now we shall write the majorant equation (4.10) with the singularity of A_2 removed by a Lagrange multiplier. This can be done in several ways. We may, for example, define Θ^R such that $\Theta_z^R(t,0)=0$ for all $t\geq 0$,

$$\Theta_z^R(t,z) = 1+\sum_{n=3}^\infty nA_n(t)z^{n-1} ,$$

as the solution of

$$\Theta_t = \Gamma\left(z\Theta^2+z^2\Theta\Theta_z\right)+B\left(z\Theta_z+\Theta-1\right)-z\left(\Gamma+2B\Theta_z(t,0)\right)$$

with $\Theta(0,z)=1$. Note that $d\Theta_z^R(t,0)/dt=\Theta_{tz}^R(t,0)=0$ which is consequence of the regularity of Θ^R as stated in Theorem 2.4. Since the linear part of equation (4.10) is responsible to make A_2 singular, we may, instead, include in the Lagrange multiplier only terms proportional to B

$$\Theta_t = \Gamma\left(z\Theta^2+z^2\Theta\Theta_z\right)+B\left(z\Theta_z+\Theta-1\right)-\frac{1}{2}zB\Theta_z(t,0) \tag{6.12}$$

(with a quarter of intensity needed to make it vanishes). As in Subsection 5.2, defining $\Theta_2(t,z) = \Psi(t,w)$, where $w = f_2(t)z$ and

$$f_2(t) = \exp\left\{\frac{3}{2}\int_0^t B(s)\;ds\right\}\;, \tag{6.13}$$

and using (recall $C_n = nA_n$)

$$\begin{aligned} -\frac{1}{2}w\Psi_w + \Psi - 1 - \frac{1}{2}w\Psi_w(t,0) &= -\frac{1}{2}\sum_{n=2}^{\infty}(n-1)C_n w^{n-1} + \sum_{n=1}^{\infty}C_n w^{n-1} - 1 - \frac{1}{2}C_2 \\ &= -\frac{1}{2}\sum_{n=3}^{\infty}(n-3)C_n w^{n-1} \le 0 \end{aligned}$$

we have

$$\begin{aligned} \Psi_t - \frac{\Gamma}{f_2}w^2\Psi\Psi_w &= \left(B - \frac{f_2'}{f_2}\right)w\Psi_w + B\left(\Psi - 1 - \frac{1}{2}C_2\right) + \frac{\Gamma}{f_2}w\Psi^2 \\ &= B\left(-\frac{1}{2}w\Psi_w + \Psi - 1 - \frac{1}{2}C_2\right) + \frac{\Gamma}{f_2}w\Psi^2 \\ &\le \frac{\Gamma}{f_2}w\Psi^2\;. \end{aligned}$$

Repeating the steps from (5.11) to (5.18) with $\tau = \int_{t_0}^t(\Gamma/f_2)(s)\;ds$, provided

$$w\tau\;= zf_2(t)\int_{t_0}^t\frac{\Gamma}{f_2}(s)\;ds\;= z\int_{t_0}^t\exp\left\{\frac{3}{2}\int_s^t B(s')\;ds'\right\}\;\Gamma(s)\;ds < 1/e\;, \tag{6.14}$$

the solution of the initial value problem (6.12) Θ_2 satisfies

$$\Theta_2(t,z) \le \frac{-1}{\tau w}W\left(-\tau w\right)\;. \tag{6.15}$$

Substituting (6.4) and (6.5) into (6.14) with $t = 0$, we have

$$\begin{aligned} \lim_{t_0\to-\infty}\int_{t_0}^0\exp\left\{\frac{3}{2}\int_s^0 B(s')\;ds'\right\}\;\Gamma(t')\;dt' &= -\beta\lim_{t_0\to-\infty}\int_{t_0}^0\frac{\gamma'(t')}{\gamma^{2-3\beta/16\pi}(t')}dt' \\ &= \frac{16\pi\beta}{16\pi - 3\beta} \end{aligned}$$

provided $\beta < 16\pi/3$, independently of the scale function γ. The Mayer series of Yukawa gas, with the singular part of ψ_2^c "removed", converges if

$$\beta < \frac{16}{3}\pi \qquad \text{and} \qquad |z| < \frac{16\pi - 3\beta}{16\pi\beta e}\;.$$

The extension of the result to $\beta < \beta_n = 8\pi\left(1+n^{-1}\right)^{-1}$ immediately follows from the renormalized equation

$$\Theta_t = \Gamma\left(z\Theta^2 + z^2\Theta\Theta_z\right) + B\left(z\Theta_z + \Theta - 1\right) - B\sum_{j=1}^{n-1}\frac{1}{j!}\left(1-\frac{j}{n}\right)z^j\Theta_{\underbrace{z\cdots z}_{j-\text{times}}}(t,0) \tag{6.16}$$

with $\Theta(0,z)=1$. We define analogously $\Theta(t,z)=\Psi(t,w)$, where $w=f_n(t)z$ and

$$f_n(t) = \exp\left\{\frac{n+1}{n}\int_0^t B(s)\;ds\right\}\;, \tag{6.17}$$

and use

$$\left(1-\frac{f_n'}{f_n}\right)w\Psi_w + \Psi - 1 - \sum_{m=2}^{n}\left(1-\frac{m-1}{n}\right)C_m w^{m-1}$$

$$= -\frac{1}{n}\sum_{m=n}^{\infty}(m-n)C_m w^{m-1} \leq 0$$

in order to obtain

$$\Psi_t - \frac{\Gamma}{f_n}w^2\Psi\Psi_w \leq \frac{\Gamma}{f_n}w\Psi^2\;.$$

Denoting the solution of the initial value problem (6.16) by Θ_n, we have

$$\Theta_n(t,z) \leq \frac{-1}{\tau w}W\left(-\tau w\right) \tag{6.18}$$

provided

$$w\tau = zf_n(t)\int_{t_0}^{t}\frac{\Gamma}{f_n}(s)\;ds \;= z\int_{t_0}^{t}\exp\left\{\frac{n+1}{n}\int_s^t B(s')\;ds'\right\}\;\Gamma(s)\;ds < 1/e\;. \tag{6.19}$$

Repeating the calculations done before, the Mayer series of Yukawa gas, with the singular part of ψ_j^c for $j=2,\ldots,n$ "removed", converges if

$$\beta < \beta_n \qquad \text{and} \qquad |z| < \frac{\beta_n - \beta}{\beta_n\beta e}$$

with β_n as in (6.10). This concludes the proof of Theorem 2.3.

$\square$

Remark 6.2 *The convergence of Θ for $\beta < 16\pi/3$ with $O(z)$ term omitted was established directly from the integral equation (3.16) by [BK]. Our method is cleaner and does not require extra conditions for the radius of convergence (see Theorem* 4.3 *of [BK]).*

7 Cauchy–Kowalevsky Theorem

We shall address equation (4.10) by Nirenberg's Cauchy–Kowalevsky theorem [N] for nonlinear PDE. For this, we need some more notations and preliminaries.

The set of continuous function in $\overline{\Omega}$ and holomorphic in the open set $\Omega \subset \mathbb{C}$ will be denoted by $H(\Omega)$. Equipped with the sup norm $\|f\|_{H(\Omega)} = \sup_{z\in\overline{\Omega}} |f(z)|$, $H(\Omega)$ is a Banach space. Acting on $H(\Omega)$, the differential operator satisfies, by the Cauchy formula,

$$\left\|\frac{df}{dz}\right\|_{H(\Omega)} \leq \frac{1}{d}\,\|f\|_{H(\Omega')} \tag{7.1}$$

where $\Omega' \subset \Omega$ and $d = \operatorname{dist}(\Omega', \Omega\backslash\Omega')$. Let E_s, $0 \leq s < 1$, denote the one–parameter family of Banach spaces $H(\Omega_s)$ with $\Omega_s \subset \Omega_{s'}$ if $s < s'$ and set $\|f\|_s = \|f\|_{H(\Omega_s)}$. Note that $E_{s'} \subset E_s$, $\|f\|_s \leq \|f\|_{s'}$ and we choose the parametrization of Ω_s so that $\operatorname{dist}(\Omega_s, \Omega_{s'}\backslash\Omega_s) = d_0\,(s'-s)$ for some constant d_0. Hence, equation (7.1) reads

$$\left\|\frac{df}{dz}\right\|_{s} \leq \frac{1}{d_0\,(s'-s)}\,\|f\|_{s'}\ . \tag{7.2}$$

Finally, $\mathscr{C}\,(I, E_s)$ ($\mathscr{C}^1\,(I, E_s)$) will denote the set of continuous (continuously differentiable, resp.) functions $f : I \subset \mathbb{R} \longrightarrow E_s$ and we shall use the notation $|g|_\delta = \sup_{|t|<\delta} |g(t)|$ for a function $g : \mathbb{R} \longrightarrow \mathbb{R}$.

Proof of Theorem 2.4. We apply Theorem 1.1 of [N] to the problem at our hand. Our main concern here is to obtain concrete estimates on the sequence of approximate solutions of (4.10) in terms of their coefficients and parameters. So, we have to go into details but we shall quote equations of [N] whenever they lead to a lengthy or technical calculations. The proof of Theorem 2.4 uses multiscale analysis similar to KAM theory and is divided into several steps.

1st. Setup. For simplicity, let $\Omega_s = \{|z| < sR_1 + (1-s)R_0\}$ be a family of discs, centered at origin, for some $0 < R_0 < R_1 < \infty$ fixed. We are looking for a solution in $\mathscr{C}\,(I_s, E_s)$ of

$$\Theta(t) = 1 + \int_0^t B(\tau)\,d\tau + \int_0^t F(\Theta(\tau), \tau)\,d\tau \tag{7.3}$$

with F defined by the r.h.s. of (4.10),

$$F(\Theta(t), t) = \Gamma(t)\left(z\Theta^2(t) + z^2\Theta(t)\frac{d\Theta}{dz}(t)\right) + B(t)\left(z\frac{d\Theta}{dz}(t) - \Theta(t)\right)\,, \tag{7.4}$$

and a time interval $I_s = \{t \in \mathbb{R} : |t| < \delta\,(1-s)\}$ depending on the scale parameter $0 \leq s < 1$ and other constants involved in equation (4.10). The solution will be obtained as the limit $\theta_k \longrightarrow \Theta$ of a sequence $(\theta_k)_{k\geq 0}$ in $\mathscr{C}\,(I_s, E_s)$ defined recursively by

$$\theta^{k+1} = \theta^k + \vartheta \tag{7.5}$$

for $k \geq 1$ with

$$\theta^0 = 1 + \int_0^t B(\tau)\,d\tau\,,$$

$$\left\|\theta^k(t)\right\|_s \le R \qquad \text{for} \qquad |t| \le \delta_k\,(1-s)$$

and ϑ the solution of

$$\vartheta(t) = \int_0^t L_k\vartheta(\tau)\,d\tau\ + G_k(t) \tag{7.6}$$

where $L_k = L\left[\theta^k\right]$ is a linear operator in E_s obtained by linearizing the r. h. s. of (4.10) at θ^k:

$$L_k = \left(\Gamma z^2\theta^k + Bz\right)\frac{d}{dz} + \Gamma\left(2z\theta^k + z^2\theta^k\right) - B \tag{7.7}$$

and

$$G_k(t) = \int_0^t F(\theta^k(\tau),\tau)\,d\tau - \theta^k(t) + \theta^0(t). \tag{7.8}$$

Both θ^k and ϑ are defined in the Banach space $\mathscr{B}_k = \mathscr{B}_k(s_0)$ of functions $f(t) \in \mathscr{C}\left(I_{k,s}, E_s\right)$ for every $s < s_0 \le 1$, where $I_{k,s} = \{t \in \mathbb{R} : |t| \le \delta_k\,(s_0 - s)\}$, having finite norm

$$\|f\|_k := \sup_{(t,s)\in T_k}\left(\frac{\delta_k(s_0-s)}{|t|} - 1\right)\|f(t)\|_s \tag{7.9}$$

with the sup taken over the isosceles triangle in the (t,s) plane of base length $2\delta_k$ and height s_0:

$$T_k := \left\{|t| \le \delta_k\,(s_0 - s)\,, 0 \le s < s_0\right\}\,. \tag{7.10}$$

Note that $\mathscr{B}_k(s_0) \subseteq \mathscr{B}_k(1)$ for any $s_0 \le 1$ and, with exception of the fifth step, s_0 will be kept fixed equal to 1. The numbers $(\delta_k)_{k\ge 0}$ will be chosen to be a decreasing sequence with $\lim_{k\to\infty}\delta_k = \delta > 0$.

Note that, if $\theta_k \longrightarrow \Theta$ in $\mathscr{C}\left(I_s, E_s\right)$ and Θ is a solution of (7.3), then $\Theta \in \mathscr{C}^1\left(I_s, E_s\right)$ is a solution of (4.10) with $\Theta(0,z) = 1$. In addition, if $(\theta_k)_{k\ge 0}$ is defined by (7.5) - (7.8) and $\theta_k \longrightarrow \Theta$ in $\mathscr{C}\left(I_s, E_s\right)$ then $G_k(t) \longrightarrow 0$, by (7.3), and $\vartheta(t) = 0$ is the only solution of the limit equation $\vartheta(t) = \displaystyle\int_0^t L[\Theta]\vartheta(\tau)\,d\tau$ with $\vartheta(0) = 0$. This shows consistency of definition (7.5) with the limit $k \to \infty$.

2^{nd}. *Linear Cauchy–Kowalevsky theorem.* Let us assume that θ^j, $j \le k$, are determined with $\left\|\theta^j\right\|_j < \infty$ and set $\lambda_k = \|G_k\|_k$. By Theorem 1.2 of [N], there is a function $\vartheta \in \mathscr{C}\left(I_{s,k}, E_s\right)$ satisfying (7.6) such that

$$\|\vartheta\|_k \le 2\lambda_k\,. \tag{7.11}$$

This is obtained by Nirenberg's fixed point method. The method used here, an abstract Cauchy–Kowalevsky theorem introduced by Ovsjannikov (see e.g. [Tr]), shows directly the implication (7.15) of equation (7.11) that will be used in the next step.

From (7.7), (7.2) and the hypothesis on θ^k, there exist a finite constant C_k, depending on Γ, B, θ^k and $d_0 = R_1 - R_0$, such that

$$\|L_k\vartheta\|_s \le \frac{C_k}{(s-s')}\,\|\vartheta\|_{s'}\ .$$

If $(\vartheta_n)_{n\ge 0}$ is a sequence defined by

$$\vartheta_n(t) = \int_0^t L_k\vartheta_{n-1}(\tau)\,d\tau\ + G_k(t) \tag{7.12}$$

with $\vartheta_0(t) = G_k(t)$, we claim that

$$\|\vartheta_n(t) - \vartheta_{n-1}(t)\|_s \le \eta_k \left(\frac{eC_k\,|t|}{1-s}\right)^n \tag{7.13}$$

holds for any $|t| < \delta_{k+1}(1-s) := (2eC_k)^{-1}(1-s)$, $0 \le s < 1$ and $n \ge 0$ with $\vartheta_{-1} = 0$ and

$$\eta_k = \frac{\lambda_k}{\delta_k/\delta_{k+1} - 1}. \tag{7.14}$$

Note that, in view of (7.9), we have for $n = 0$

$$\|\vartheta_0(t)\|_s = \|G_k(t)\|_s \le \frac{\lambda_k}{(\delta_k(1-s)/\,|t| - 1)} \le \eta_k\,,$$

provided $|t| < \delta_{k+1}(1-s)$. Now, assuming (7.13) holds for $n = r-1$, we have

$$\begin{aligned}
\|\vartheta_r(t) - \vartheta_{r-1}(t)\|_s &\le \frac{C_k}{\varepsilon}\int_0^{|t|} \|\vartheta_r(\tau) - \vartheta_{r-1}(\tau)\|_{s+\varepsilon}\, d\tau \\
&\le \eta_k \frac{C_k}{\varepsilon}\left(\frac{eC_k}{1-s-\varepsilon}\right)^{r-1}\int_0^{|t|} \tau^{r-1}\, d\tau \le \eta_k \left(\frac{eC_k\,|t|}{1-s}\right)^r
\end{aligned}$$

provided one chooses $\varepsilon = (1-s)/r$. This completes the induction and establishes (7.13).

It thus follows from (7.13) that $(\vartheta_n)_{n\ge 0}$ is a Cauchy sequence in $\mathscr{C}(I_{k+1,s}, E_s)$ which converges to the solution ϑ of (7.6) and satisfies

$$\|\vartheta(t)\|_s \le \sum_{n=0}^{\infty} \|\vartheta_n(t) - \vartheta_{n-1}(t)\|_s \le 2\eta_k \tag{7.15}$$

for $|t| < \delta_{k+1}(1-s)$.

Inserting (7.15) in (7.5), we have

$$\left\|\theta^{k+1}(t)\right\|_s \le \left\|\theta^k(t)\right\|_s + 2\eta_k$$

and hence, by recursion,

$$\left\|\theta^{k+1}(t)\right\|_s \le \theta^0(t) + 2\sum_{j=0}^{k} \eta_j. \tag{7.16}$$

Provided

$$2\sum_{j=0}^{k} \eta_j \le \frac{R}{2} \tag{7.17}$$

can be established, $\|\theta^j(t)\|_s \le \theta^0(t) + R/2$ holds for $t \in I_{j,s}$ and $j \le k+1$, so $F\left(\theta^{k+1}(t), t\right)$ is well defined and furnishes through (7.8) the initial step $\vartheta_0(t) = G_{k+1}(t)$ for equation (7.12) with k replaced by $k+1$. Note that, under the condition (7.17), $C_k = C\left(d_0, |\Gamma|_\delta, |B|_\delta, R\right)$ and consequently, $\delta_{k+1} = (2eC_k)^{-1}$ does not depend on k and we are free to choose the sequence $(\delta_k)_{k\ge 0}$ as we wish provided $\delta_{k+1} < \delta_1 \le (2eC)^{-1}$.

3^{rd}. *Estimate on* $(\lambda_k)_{k\geq 0}$. From (7.5), (7.6) and (7.8), we have

$$
\begin{aligned}
G_{k+1}(t) &= \int_0^t F\left(\theta^{k+1}(\tau),\tau\right)\, d\tau - \theta^{k+1} - \vartheta \\
&= \int_0^t \left\{F\left(\theta^{k+1}(\tau),\tau\right) - F\left(\theta^{k}(\tau),\tau\right) - L_k\vartheta(\tau)\right\}\, d\tau \\
&= \int_0^t \Gamma(\tau)\left(z\vartheta^2(\tau) + z^2\vartheta(\tau)\frac{d\vartheta}{dz}(\tau)\right) d\tau
\end{aligned}
$$

where in the last line (7.4), and the expansion of $F(\theta^{k+1}) = F\left(\theta^k + \vartheta\right)$ around θ^k, have been used. Using estimate (7.2), there is a finite constant C', depending on Γ and $d_0 = R_1 - R_0$, such that

$$
\left\|\Gamma(\tau)\left(z\left(\vartheta\right)^2(\tau) + z^2\vartheta(\tau)\frac{d\vartheta}{dz}(\tau)\right)\right\|_s \leq \frac{C'}{s'-s}\left\|\vartheta(\tau)\right\|_{s'}^2 ,
$$

which yields

$$
\left\|G_{k+1}(t)\right\|_s \leq C'\int_0^t \frac{1}{s(\tau)-s}\left\|\vartheta(\tau)\right\|_{s(\tau)}^2 d\tau\,.
$$

Choosing $s(\tau) = (1-\tau/\delta_k + s)/2$, which satisfies $s(\tau) - s = (1-\tau/\delta_k - s)/2 > 0$ for $(\tau,s)\in T_k$, together with (7.11), we have

$$
\begin{aligned}
\left\|G_{k+1}(t)\right\|_s &\leq 8C'\lambda_k^2\delta_k\int_0^{|t|}\frac{\tau^2}{\left(\delta_k(1-s)-\tau\right)^3}d\tau \\
&\leq 8C'\frac{\lambda_k^2}{1-s}\frac{|t|}{\left(\delta_k(1-s)\,/\,|t|-1\right)^2}
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
\lambda_{k+1} &= \left\|G_{k+1}\right\|_{k+1} \\
&\leq 8C'\lambda_k^2\sup_{(t,s)\in T_{k+1}}\left\{\frac{|t|\left(\delta_{k+1}(1-s)/\,|t|-1\right)}{(1-s)\left(\delta_k(1-s)/\,|t|-1\right)^2}\right\} \\
&\leq 8C'\lambda_k^2\frac{\delta_{k+1}}{\delta_k/\delta_{k+1}-1}\,, \qquad (7.18)
\end{aligned}
$$

with

$$
\lambda_0 = \left\|\int_0^t F(\theta^0(\tau),\tau)\, d\tau\right\|_0 \leq \frac{\delta_0 d_0}{4}\left(1+\frac{R_0}{d_0}\right)^2\left(1+\delta_0\left|B\right|_{\delta_0}\right)^2\left|\Gamma\right|_{\delta_0} := M\,.
$$

Let $\delta_{k+1} = \delta_k\left(1-c^{k+1}\right)$ be chosen for some positive constants δ_0 and $c<1$ and note that

$$
\delta = \lim_{k\to\infty}\delta_k = \delta_0\prod_{j\geq 0}\left(1-c^{j+1}\right) \leq \delta_0\exp\left(\frac{-c}{1-c}\right) \qquad (7.19)
$$

and

$$\delta_{k+1}\frac{\delta_{k+1}/\delta_{k+2}-1}{\delta_k/\delta_{k+1}-1} \le \delta_1\frac{c}{1+c} \le \frac{\delta_0(1-c)}{2}\,.$$

This equation, together with (7.14) and (7.18), yields

$$\begin{aligned} \eta_{k+1} &\le 4\delta_0(1-c)C'\eta_k^2 \\ &\le r_0\,\eta_k \end{aligned} \tag{7.20}$$

with $\eta_0 = \lambda_0(1-c)/c$, and $\eta_k \to 0$ exponentially fast provided

$$r_0 := 4\delta_0(1-c)C'\eta_0 < 1. \tag{7.21}$$

4^{th}. *Convergence of θ^k to Θ.* Substituting (7.20) in (7.17), leads

$$4\sum_{j=0}^{k}\eta_j \le 4\lambda_0\frac{1-c}{c\,(1-r_0)} \le \frac{4M(1-c)}{c\,(1-r_0)} := R\,.$$

Hence, $\theta^k(t)$ is defined as a function of $\mathscr{C}\left(I_{s,k},E_s\right)$ for each $k\ge 0$ and $0\le s<1$, satisfying, in view of (7.16),

$$\left\|\theta^k(t)\right\|_s \le \theta^0(t) + \frac{R}{2}$$

for $|t| < \delta_k(1-s)$. Furthermore, from (7.5) and (7.11),

$$\left\|\theta^{k+1}(t)-\theta^k(t)\right\|_s \le \|\vartheta(t)\|_s \le \frac{2\lambda_k}{(\delta_k(1-s)/\left|t\right|-1)}$$

holds for $|t| < \delta(1-s) < \delta_k(1-s)$ and, consequently,

$$\left\|\theta^{k+1}(t)-\theta^k(t)\right\|_\infty = \sup_{(t,s)\in T_\infty}\left(\delta(1-s)/\left|t\right|-1\right)\left\|\theta^{k+1}(t)-\theta^k(t)\right\|_s \le 2\lambda_k\,.$$

Since $\lambda_j = \eta_j\left(\delta_j/\delta_{j+1}-1\right) \le \left(cr_0\right)^j/2$, $\lambda_j \to 0$ exponentially fast and $\left(\theta^k(t)\right)_{k\ge 0}$ is a Cauchy sequence which converges to $\Theta(t)$ in the Banach space $\mathscr{B} = \bigcap_{k\ge 0}\mathscr{B}_k$.

Note that $\theta^k(t)$ converges to a solution $\Theta(t)$ of (7.3) since, for $|t| < \delta\left(1-s'\right) < \delta\left(1-s\right)$ we have

$$\begin{aligned} &\left\|\Theta(0)+\int_0^t F(\Theta(\tau),\tau)\,d\tau - \Theta(t)\right\|_{s'} \\ \le\ & \int_0^t \left\|F(\Theta(\tau),\tau)\ - F(\theta^k(\tau),\tau)\right\|_{s'} d\tau + \left\|\Theta(t)-\theta^k(t)\right\|_{s'} + \|\vartheta(t)\|_{s'} \\ \le\ & \frac{C}{s-s'}\int_0^t \left\|\Theta(\tau)-\theta^k(\tau)\right\|_s d\tau + \frac{C'}{s-s'}\int_0^t \left\|\Theta(\tau)-\theta^k(\tau)\right\|_s^2 d\tau + \left\|\Theta(t)-\theta^k(t)\right\|_{s'} + \|\vartheta(t)\|_{s'}\ , \end{aligned}$$

and all these terms go to zeros as $k\to\infty$.

5^{th}. *Uniqueness.* Suppose that $\Theta'(t)$ is also a solution of (7.3). Then $W(t) = \Theta'(t) - \Theta(t)$ satisfies

$$W(t) = \int_0^t \left(F(\Theta'(\tau), \tau) \; - F(\Theta(\tau), \tau)\right) d\tau \, .$$

For a fixed $0 \le s_0 < 1$, let $\mathscr{B}_0$ be the subspace of $\mathscr{B}$ with $1-s$ replaced by $s_0 - s$ in all places (see definitions in the first step). For $|t| < \delta\,(s_0 - s)$, arguing as in the third step, we have

$$\begin{aligned} \|W(t)\|_s &\le (C + C'R) \int_0^{|t|} \frac{\|W(\tau)\|_s}{s(\tau) - s} d\tau \\ &\le 4\delta\,(C + C'R) \frac{\|W(t)\|_{s_0}}{\delta\,(s_0 - s) \,/\, |t| - 1} \end{aligned}$$

so that

$$\|W(t)\|_{s_0} \le 4\delta\,(C + C'R) \, \|W(t)\|_{s_0} < \|W(t)\|_{s_0} \ ,$$

and hence $\|W(t)\|_{s_0} = 0$, provided

$$4\delta\,(C + C'R) < 1 \tag{7.22}$$

holds. Thus $\|W(t)\|_s = 0$ for $|t| < \delta\,(s_0 - s)$ and since this is true for any s_0 we conclude $W(t) \equiv 0$.

6^{th}. *Summary.* We now choose the free remaining constants. Since the initial condition $\theta^0(t) = 1 + \displaystyle\int_0^t B(\tau)\, d\tau$ is an entire function, we can take $R_0 = 0$ and $d_0 = R_1$ as large as we wish. We also take δ_0 so large that $|B|_{\delta_0} = \sup_{|t|<\delta_0} |B(t)|$ remains finite. Now, the sequence $(\delta_k)_{k\ge 0}$ will be completely defined by choosing $c < 1$ so close to 1 that $\delta_1 = \delta_0(1-c) \le (2eC)^{-1}$ and $r_0 < 1$, given by (7.21), hold. Note that C and C' grows linearly with d_0 and δ_0 (see second and third steps). As far as existence is concerned, these are all we need to fix. Finally, we observe that $\Theta(t, z)$ exists for all $|t| < \delta\,(1-s)$ and, in view of (7.19), δ is exponentially small in $1/(1-c) \ge 2eC/\delta_0$ and the uniqueness condition (7.22) holds.

□

8 Concluding Remarks

Cauchy–Kowalewski method establishes a family of open regions $\mathscr{R}_s = \{|t| < \delta(1-s), \ z \in \Omega_s\}$ of $\mathbb{R} \times \mathbb{C}$, parametrized by $0 \le s < 1$, where (4.10) has a unique real analytic solution $\Theta\,(t, z)$. In Section 7, Ω_s was considered to be discs centered at the origin but it can be centered at any point z_i. A solution $\Theta\,(t, z)$ of the original initial value problem may be the initial data for equation (4.10) with respect to $(t_1, z_1) \in \mathscr{R}_s$, for some $s \in [0, 1)$, and new family

$$\mathscr{R}^1_s = \{|t - t_1| < \delta_1(1-s), \ \ |z - z_1| < sR_1 + (1-s)R_0\} \ \ , \quad 0 \le s < 1$$

can be obtained by applying the Cauchy–Kowalewski theorem again. Repeating this procedure, we find a sequence of points $(t_n, z_n) \in \mathscr{R}^{n-1}_s$, for some $s \in [0, 1)$, and families of open regions $\mathscr{R}^n_s$. The union of all families $\{\mathscr{R}^n_s\}$, over all sequences, gives the maximum open region $\mathscr{R}^*$ in which a unique holomorphic solution $\Theta\,(t, z)$ of the initial value problem (4.10) exists. Note that $\Theta\,(t, z)$ has a continuous extension to the closure $\overline{\mathscr{R}^*}$.

According to the analysis in Section 5, for an increasing sequence of positive numbers, $\{(t_n, z_n)\}$ with $t_m < t_k$ and $z_m < z_k$ if $m < k$, there is a point (t^*, z^*) at the boundary of $\bigcup_{n,s} \mathscr{R}_s^n$ corresponding the branching singularity of the Lambert W–function, which can certainly be avoided by picking another sequence which goes around it.

The analytic continuation of Mayer series in the Tonks' gas of hard spheres [T], whose singularity is also due the branching point of Lambert W–function, has Physical meaning. Whether analytic continuation $\widetilde{\Theta}\,(t, z)$ of the pressure majorant $\Theta\,(t, z)$ remains a "pressure majorant" for $(t, z) \in \mathscr{R}^*$ in the complement of convergence domain, is a question to be investigate. It is not clear to us the precise meaning of this notion but we believe that Cauchy–Kowalewski theorem has to be applied to equation (3.12), written in terms of the complex variables $\zeta_j = ze^{i\phi(\xi_j)}$.

The convergence of Mayer series of a dipole gas is another open problem which can be addressed by the techniques employed in this paper. For this case, a majorant differential equation must incorporate, at least, the symmetry expressed by the cosine in Example 2.1.3, called θ–symmetry in [PPNM].

Acknowledgment

We are grateful to Prof. Paulo D. Cordaro who has pointed us out some instructive references [N, Tr] concerning abstract Cauchy–Kowalewski theorem.

References

[Be] G. Benfatto. "An iterated Mayer Expansion for the Yukawa Gas", J. Statist. Phys. **41**, 671–684 (1985)

[BGN] G. Benfatto, G. Gallavotti and F. Nicoló. "On the Massive Sine Gordon Equation in the First Few Regions of Collapse, Commun. Math. Phys. **83**, 387 (1982)

[B] D. C. Brydges. "A Short Course on Cluster Expansion", in *Critical Phenomena, Random Systems, Gauge Theories*, K. Ostervaldand R. Stora ed. (1986)

[Br] D. C. Brydges. "Functional Integrals and their Applications", notes for *Troisième Cycle de la Physique en Suisse Romande*, Lausanne (1992)

[BF] David Brydges and Paul Federbush. "A new Form of the Mayer Expansion in Classical Statistical Mechanics", J. Math. Phys. **19**, 2064 - 2067 (1978)

[BK] D. C. Brydges and T. Kennedy. "Mayer Expansions and the Hamilton-Jacobi Equation", Journ. Stat. Phys. **48**, 19 - 49 (1987)

[BK1] D. C. Brydges and T. Kennedy. "Convergence of Multiscale Mayer Series", in *$VIII^{th}$ International Congress in Mathematical Physics*, Marselle, World Scientific (1987)

[BM] David C. Brydges and Ph. A. Martin. "Coulomb Systems at Low Density: A Review", Journ. Stat. Phys. **96**, 1163 - 1330 (1999)

[FS] Jürg Fröhlich and Thomas Spencer. "Phase Diagram and Critical Properties of (Classical) Coulomb Systems", Erice Lectures (1980)

[GJ] James Glimm and Arthur Jaffe. "Quantum Physics: A Functional Integral Point of View", Springer-Verlag New York Inc. (1981)

[GM] Markus Göpfert and Gerhard Mack. "Iterated Mayer Expansion for Classical Gases at Low Temperatures", Commun. Math. Phys. **81**, 97 - 126 (1981)

[I] John Imbrie. "Iterated Mayer Expansions and their Application to Coulomb Gases" in *Scaling and self-similarity in physics*, Progr. Phys., **7**, 163–179. Birkhuser, Boston (1983)

[K] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth. "On the Lambert W Function", Adv. Comput. Math. **5**, 329-359 (1996)

[LY] T. D. Lee and C. N. Yang. "Statistical Theory of Equation of States and Phase Transitions. II Lattice Gas and Ising Model", Phys. Rev. **87**, 404-409 (1952)

[N] L. Nirenberg. "An Abstract Form of the Nonlinear Cauchy-Kowalewski Theorem", J. Differential Geometry **6**, 561-576 (1972)

[PPNM] Aldo Procacci, Emmanuel Pereira, Armando G. M. Neves and Domingos H. U. Marchetti. "Coulomb Interaction Symmetries and the Mayer Series in the Two–Dimensional Dipole Gas", Journ. Stat. Phys. **87**, 877-889 (1997)

[R] David Ruelle. "Statistical Mechanics: Rigorous Results", Addison-Wesley (1989)

[S] Barry Simon. "The Statistical Mechanics of Lattice Gases", Princeton University Press (1993)

[T] L. Tonks. "The Complete Equation of State of One, Two and Three Dimensional Gases of Hard Elastic Spheres", Phys. Rev. **50**, 955 (1936)

[Tr] François Treves. "Basic Linear Differential Equations", Academic Press (1975)

[YL] C. N. Yang and T. D. Lee. "Statistical Theory of Equation of States and Phase Transitions. I Theory of Condensation", Phys. Rev. **87**, 404-409 (1952)