A combinatorial proof of tree decay of semi–invariants

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Abstract

We consider finite range Gibbs fields and provide a purely combinatorial proof of the exponential tree decay of semi–invariants, supposing that the logarithm of the partition function can be expressed as a sum of suitable local functions of the boundary conditions. This hypothesis holds for completely analytical Gibbs fields; in this context the tree decay of semi–invariants has been proven via analyticity arguments. However the combinatorial proof given here can be applied also to the more complicated case of disordered systems in the so called Griffiths' phase when analyticity arguments fail.

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1. Introduction

In this note we present a purely combinatorial proof of the tree decay of semi–invariants, also called truncated correlations, Ursell functions, or cumulants, for a finite range Gibbsian field under the condition that the logarithm of the partition function can be expressed as the sum of suitable local functions of the boundary condition.

Let $Z_\Lambda(\tau)$ be the partition function in the finite volume $\Lambda \subset \mathbb{Z}^d$ with boundary condition τ outside Λ; we assume that

$$\log Z_\Lambda(\tau) \;=\; \sum_{X \subset \mathbb{Z}^d:\, X \cap \Lambda \neq \emptyset} \phi_{X,\Lambda}(\tau) \tag{1.1}$$

where the "effective potentials" $\phi_{X,\Lambda}$ are such that:

(i) given $X \subset \mathbb{Z}^d$, the functions $\phi_{X,\Lambda}$ are constant w.r.t. Λ for the Λ's with a given intersection with X;

(ii) have a suitable decay property with the size of X, uniformly in Λ.

The expression (1.1) can be obtained via cluster expansion in the weak coupling (high temperature and/or small activity) region but it holds in more general situations. It can also be obtained in the framework of Dobrushin–Shlosman complete analyticity as well as in the framework of the so–called *scale–adapted cluster expansion*, see [3], provided the volume Λ is a disjoint union of cubes whose side length equals the scale of the expansion. We refer to [3] for a more exhaustive discussion; here we only say that scale–adapted cluster expansions have been introduced in [17, 18] in order to perturbatively treat the whole uniqueness region of lattice spin systems, arbitrarily close to the coexistence line. Moreover, as we shall see in [5], a variant of (1.1) holds in the context of disordered lattice systems, also in the delicate situation of Griffiths' singularity that makes necessary the use of a *graded cluster expansion*, see [3].

In the framework of the renormalization group maps one often encounters an expression like (1.1) for the renormalized partition function. In that case the family $\{\phi_{X,\Lambda},\, X \subset \mathbb{Z}^d\}$ represents the "finite–volume renormalized potential". Both in the case of disordered systems and of renormalization group maps the decay properties of $\phi_{X,\Lambda}$ are weaker than the corresponding ones of the case of weakly coupled short range Gibbs fields

The tree decay of semi–invariants is often deduced from analyticity properties of the pressure, see [8–11, 19]; however, there are physically interesting situations in which these analyticity properties do not hold but nevertheless we expect the exponential decay of semi–invariants. The main example is given by the already quoted case of a disordered lattice spin system, like a spin glass or a ferromagnetic system subject to a random field, in presence of the so–called "Griffiths' singularity". Consider, for example, a random coupling Ising spin system in $\mathbb{Z}^d$ described by the formal Hamiltonian:

$$H(\sigma) = \sum_{x,y:|x-y|=1} J_{x,y}\sigma_x\sigma_y - h\sum_x \sigma_x \tag{1.2}$$

where $\sigma_x \in \{-1,+1\}$, $h \in \mathbb{R}$ is fixed and $J_{x,y}$ are i.i.d. Gaussian random variables with mean zero and variance one. At high temperature we expect an exponential tree decay of semi–invariants with a deterministic rate (this has actually been proved long time ago in [12]) but we do not expect analyticity of thermodynamic functions. This behavior is a consequence of the fact that, even though in average the system is weakly coupled nonetheless, with a positive probability, arbitrarily large regions with strong ferromagnetic couplings can appear inducing, locally, long–range order, as a consequence of the unboundedness of the random couplings.

The starting point of our combinatorial computation can be illustrated in the simple case of the semi–invariant of order two namely, the covariance between two local functions, see [1]. For instance, consider a lattice spin system with finite state space and finite range interaction, say $r \in [0,\infty)$, whose Hamiltonian, in a finite box Λ for a configuration σ_Λ in Λ and a boundary condition τ_{Λ^c} is denoted by $H_\Lambda(\sigma_\Lambda\tau_{\Lambda^c})$. More detailed and precise definitions will be given later on; here we only say that $H_\Lambda(\sigma_\Lambda\tau_{\Lambda^c})$ contains the self–interaction of σ_Λ in Λ and the mutual interaction between σ_Λ and τ_{Λ^c}. The Gibbs measure is $\mu^\tau_\Lambda(\sigma_\Lambda) = \exp\{H_\Lambda(\sigma_\Lambda\tau_{\Lambda^c})\}/Z_\Lambda(\tau_{\Lambda^c})$ where $Z_\Lambda(\tau_{\Lambda^c}) = \sum_{\sigma_\Lambda}\exp\{H_\Lambda(\sigma_\Lambda\tau_{\Lambda^c})\}$. Notice that we have included the inverse temperature in H_Λ and changed the usual convention on the sign in the exponent.

Let f, g be local functions with supports $\Lambda_f, \Lambda_g \subset \Lambda$ such that $\mathrm{dist}(\Lambda_f,\Lambda_g) > r$. We may write

$$\begin{aligned}\mu^\tau_\Lambda(f;g) &= \mu^\tau_\Lambda(fg) - \mu^\tau_\Lambda(f)\,\mu^\tau_\Lambda(g)\\ &= \sum_{\sigma_{\Lambda_f},\sigma_{\Lambda_g}} f(\sigma_{\Lambda_f})g(\sigma_{\Lambda_g})\,e^{H_{\Lambda_f}(\sigma_{\Lambda_f}\tau_{\Lambda^c})}e^{H_{\Lambda_g}(\sigma_{\Lambda_g}\tau_{\Lambda^c})}\\ &\qquad\times\Big(\frac{Z_{\Lambda\setminus(\Lambda_f\cup\Lambda_g)}(\sigma_{\Lambda_f}\sigma_{\Lambda_g}\tau_{\Lambda^c})}{Z_\Lambda(\tau_{\Lambda^c})} - \frac{Z_{\Lambda\setminus\Lambda_f}(\sigma_{\Lambda_f}\tau_{\Lambda^c})Z_{\Lambda\setminus\Lambda_g}(\sigma_{\Lambda_g}\tau_{\Lambda^c})}{Z^2_\Lambda(\tau_{\Lambda^c})}\Big)\end{aligned} \tag{1.3}$$

It is clear that the exponential decay of $\mu^\tau_\Lambda(f;g)$ with $\mathrm{dist}(\Lambda_f,\Lambda_g)$ easily follows from the analogous property of the quantity

$$\sup_{\sigma_{\Lambda_f},\sigma_{\Lambda_g},\tau_{\Lambda^c}}\left|\frac{Z_{\Lambda\setminus(\Lambda_f\cup\Lambda_g)}(\sigma_{\Lambda_f}\sigma_{\Lambda_g}\tau_{\Lambda^c})Z_\Lambda(\tau_{\Lambda^c})}{Z_{\Lambda\setminus\Lambda_f}(\sigma_{\Lambda_f}\tau_{\Lambda^c})Z_{\Lambda\setminus\Lambda_g}(\sigma_{\Lambda_g}\tau_{\Lambda^c})} - 1\right| \tag{1.4}$$

This, in turn, is easily seen to follow from (1.1) and suitable decay properties of $\phi_{X,\Lambda}$, see (2.11) below. Indeed by plugging (1.1) into (1.4) and using (i) above we easily see that, in the resulting expression, $\phi_{X,\Lambda}$ cancels out unless X intersects both Λ_f and Λ_g.

The case of a generic semi–invariant of order n is much more subtle and some more efforts are required to disclose the cancellation mechanism. The crucial point in our proof is the combinatorial result in Lemma 3.1 which generalizes (1.3) and expresses the semi–invariant in terms of ratios of partition functions.

The paper is organized as follows. In Section 2 we give the notation and a theorem stating our main result, with some comments and exempla. The proof of the theorem is finally given in Section 3.

2. Notation and result

In this Section we recall the general framework of Gibbs states for lattice systems, state our main results, and discuss some possible applications.

2.1. The lattice

For $a, b \in \mathbb{R}$ we set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ we set $|x| := \sup_{k=i,\cdots,d} |x_i|$. The spatial structure is modeled by the d–dimensional cubic lattice $\mathbb{L} := \mathbb{Z}^d$. We shall denote by $x, y, \cdots$ the points in $\mathbb{L}$, called *sites*, and by $\Lambda, V, X, \ldots$ the subsets of $\mathbb{L}$. We use $\Lambda^c := \mathbb{L} \setminus \Lambda$ to denote the complement of Λ. For Λ a finite subset of $\mathbb{L}$, we use $\Lambda \subset\subset \mathbb{L}$ to indicate that Λ is finite, $|\Lambda|$ denotes the cardinality of Λ. We consider $\mathbb{L}$ endowed with the distance $\mathrm{d}(x, y) = |x - y|$. As usual for $X, Y \subset \mathbb{L}$ we set $\mathrm{d}(X, Y) := \inf\{\mathrm{d}(x, y),\ x \in X,\ y \in Y\}$, $\mathrm{diam}(X) := \sup\{\mathrm{d}(x, x'),\ x, x' \in X\}$.

For $x \in \mathbb{L}$ and m a positive integer we let $Q_m(x) := \{y \in \mathbb{L} : x_i \le y_i \le x_i + (m - 1)\,,\ i = 1, \ldots, d\}$ be the cube of side m with x the site with smallest coordinates. We denote by $\mathbb{F} := \{X \subset\subset \mathbb{L}\}$ the collection of all finite subsets of $\mathbb{L}$. Let L be a positive integer, we denote by $\mathbb{F}_L$ the collection of sets in $\mathbb{F}$ which can be written as the disjoint union of cubes of side L, more precisely $X \in \mathbb{F}_L$ iff there exist $x_1, \ldots, x_k \in \mathbb{L}$ such that $X = \bigcup_{h=1}^{k} Q_L(Lx_h)$.

Let $\mathbb{E} := \big\{\{x, y\}, x, y \in \mathbb{L} : \mathrm{d}(x, y) = 1\big\}$ be the collection of *edges* in $\mathbb{L}$. Note that, according to our definitions, the edges can be also diagonal. We say that two edges $e, e' \in \mathbb{E}$ are connected iff $e \cap e' \neq \emptyset$. A subset $(V, E) \subset (\mathbb{L}, \mathbb{E})$ is said to be connected iff for each pair $x, y \in V$, $x \neq y$, there exists in E a path of connected edges joining them. For $X \subset\subset \mathbb{L}$ we then set

$$\mathbb{T}(X) := \inf\{|E|\,,\ (V, E) \subset (\mathbb{L}, \mathbb{E}) \text{ connected} :\ V \supset X\} \tag{2.1}$$

and remark that the infimum is attained (not necessary uniquely) for a graph $(V_X, E_X) \subset (\mathbb{L}, \mathbb{E})$ which is a tree, i.e. a connected and loop–free graph. We agree that $\mathbb{T}(X) = 0$ if $|X| = 1$ and note that for $x, y \in \mathbb{L}$ we have $\mathbb{T}(\{x, y\}) = \mathrm{d}(x, y)$.

2.2. The configuration space

The *single spin space* is given by a finite set $\mathcal{S}_0 \subset \mathbb{R}$ which we consider endowed with its discrete σ–algebra $\mathcal{F}_0$. The configuration space in $\Lambda \subset \mathbb{L}$ is $\mathcal{S}_\Lambda := \mathcal{S}_0^\Lambda$ equipped with the product σ–algebra $\mathcal{F}_\Lambda = \mathcal{F}_0^\Lambda$; we denote $\mathcal{S}_0^\mathbb{L}$ and $\mathcal{F}_0^\mathbb{L}$ simply by $\mathcal{S}$ and $\mathcal{F}$. Elements of $\mathcal{S}$, called *configurations*, are denoted by $\sigma, \tau, \ldots$. In other words a configuration $\sigma \in \mathcal{S}$ is a function $\sigma : \mathbb{L} \to \mathcal{S}_0$; for $\Lambda \subset \mathbb{L}$ we denote by σ_Λ the restriction of σ to Λ. Let $\Lambda_1, \Lambda_2 \subset \mathbb{L}$ be disjoint subsets of $\mathbb{L}$; if $\sigma_i \in \mathcal{S}_{\Lambda_i}$, $i = 1, 2$, we denote by $\sigma_1\sigma_2$ the configuration in $\mathcal{S}_{\Lambda_1 \cup \Lambda_2}$ given by $\sigma_1\sigma_2(x) := \sum_{i=1}^2 \mathbb{I}_{\{x \in \Lambda_i\}} \sigma_i(x)$ for any $x \in \Lambda_1 \cup \Lambda_2$.

A measurable function $f : \mathcal{S} \to \mathbb{R}$ is called a *local function* iff there exists $\Lambda \in \mathbb{F}$ such that $f \in \mathcal{F}_\Lambda$, namely f is $\mathcal{F}_\Lambda$–measurable for some $\Lambda \in \mathbb{F}$. For f a local function we shall denote by $\mathrm{supp}(f)$, the so called *support* of f, the smallest $\Lambda \subset\subset \mathbb{L}$ such that $f \in \mathcal{F}_\Lambda$. If $f \in \mathcal{F}_\Lambda$ we shall sometimes abuse the notation by writing $f(\sigma_\Lambda)$ instead of $f(\sigma)$. For $f \in \mathcal{F}$ we let $\|f\|_\infty := \sup_{\sigma \in \mathcal{S}} |f(\sigma)|$ be the sup norm of f.

2.3. The Gibbs state

A *potential* U is a collection of local functions $U_X : \mathcal{S} \to \mathbb{R}$, $\mathcal{F}_X$–measurable, labeled by finite subsets of $\mathbb{L}$, namely $U := \{U_X \in \mathcal{F}_X,\ X \in \mathbb{F}\}$. We shall consider only *finite range* potential namely, potentials U for which there exists an integer r, called *range* such that $U_X = 0$ if $\mathrm{diam}(X) > r$. We remark that we do not require the potential U to be translationally invariant.

For $\Lambda \subset\subset \mathbb{L}$ and $\sigma \in \mathcal{S}$ we define the *Hamiltonian* as

$$H_\Lambda(\sigma) := \sum_{X \cap \Lambda \neq \emptyset} U_X(\sigma) \tag{2.2}$$

In this paper we shall consider only finite volume Gibbs measures defined as follows: let $\tau \in \mathcal{S}$, the finite volume Gibbs measure μ_Λ^τ, with boundary condition τ, is the probability measure on $\mathcal{S}_\Lambda$ given by

$$\mu_\Lambda^\tau(\sigma) := \frac{1}{Z_\Lambda(\tau)}\, e^{H_\Lambda(\sigma\tau_{\Lambda^c})} \tag{2.3}$$

where $\sigma \in \mathcal{S}_\Lambda$ and $Z_\Lambda(\tau)$, called the *partition function*, is the normalization constant given by

$$Z_\Lambda(\tau) := \sum_{\sigma \in \mathcal{S}_\Lambda} e^{H_\Lambda(\sigma\tau_{\Lambda^c})} \tag{2.4}$$

we remark that, since the potential U has range r, we have $Z_\Lambda \in \mathcal{F}_{\{x \in \Lambda^c : \mathrm{d}(x,\Lambda) \le r\}}$.

For $V \subset \Lambda \subset\subset \mathbb{L}$ we shall denote by $\mu^\tau_{\Lambda,V}$ the projection (marginal) of μ^τ_Λ to $\mathcal{S}_V$ namely, the probability measure on $\mathcal{S}_V$ given by $\mu^\tau_{\Lambda,V}(A) = \mu^\tau_\Lambda(A)$, $A \in \mathcal{F}_V$.

2.4. Semi–invariants

Let $\Lambda \in \mathbb{F}$, $n \geq 2$ an integer, f_i, with $i = 1, \ldots, n$, local functions with $\Lambda_i := \mathrm{supp}(f_i) \subset \Lambda$, $t_i \in \mathbb{R}$, with $i = 1, \ldots, n$, and $\tau \in \mathcal{S}$; we define

$$Z_\Lambda\big(\tau; t_1, \ldots, t_n\big) := \mu^\tau_\Lambda\Big(\exp\Big\{ \sum_{i=1}^n t_i f_i \Big\}\Big) \tag{2.5}$$

The semi–invariant of $f_1, \ldots, f_n$ w.r.t. the finite volume Gibbs measure μ^τ_Λ is then defined by

$$\mu^\tau_\Lambda\big(f_1; \cdots; f_n\big) := \frac{\partial^n \log Z_\Lambda\big(\tau; t_1, \ldots, t_n\big)}{\partial t_1 \cdots \partial t_n}\bigg|_{t_1 = \cdots = t_n = 0} \tag{2.6}$$

note that for $n = 2$ we have $\mu^\tau_\Lambda\big(f_1; f_2\big) = \mu^\tau_\Lambda\big(f_1\, f_2\big) - \mu^\tau_\Lambda\big(f_1\big)\mu^\tau_\Lambda\big(f_2\big)$ namely, the covariance between f_1 and f_2.

It is possible to express the semi–invariant in terms of the moments of $f_1, \ldots, f_n$. For notation compactness let us set $N := \{1, \ldots, n\}$ and denote by $\mathcal{D}^\ell_N$ the collection of the partitions of N into ℓ atoms namely,

$$\mathcal{D}^\ell_N := \Big\{ \underline{D} \equiv \big\{D_1, \ldots, D_\ell\big\} : D_i \subset N,\ D_i \neq \emptyset,\ D_i \cap D_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^\ell D_i = N \Big\} \tag{2.7}$$

We then have, see e.g. [21, II, §12.8]

$$\mu^\tau_\Lambda\big(f_1; \cdots; f_n\big) = \sum_{\ell=1}^n (-1)^{\ell-1}(\ell-1)! \sum_{\underline{D} \in \mathcal{D}^\ell_N} \prod_{k=1}^\ell \mu^\tau_\Lambda\Big(\prod_{i \in D_k} f_i \Big) \tag{2.8}$$

2.5. Tree decay of semi–invariants

We may now state our main result. Let $f_1, \ldots, f_n$ be local functions, $n \geq 2$. Given a positive integer L, by enlarging $\Lambda_i := \mathrm{supp}(f_i)$, we may (and do) assume that $\Lambda_i \in \mathbb{F}_L$; we shall further assume that for $i \neq j \in N$ we have $\mathrm{d}(\Lambda_i, \Lambda_j) > r$. We stress that the supports Λ_i can be arbitrarily large, possibly diverging with Λ.

Let us denote by $(\mathbb{V}_N, \mathbb{E}_N)$ the graph obtained from $(\mathbb{L}, \mathbb{E})$ by contracting each Λ_i, $i \in N$, to a single point, in other words we define $\mathbb{V}_N := \{x : x \in \mathbb{L} \setminus \bigcup_{i=1}^n \Lambda_i\} \cup \bigcup_{i=1}^n \{\Lambda_i\}$, $\mathbb{E}_N := \{\{v, v'\},\ v, v' \in \mathbb{V}_N\ :\ \mathrm{d}(v, v') = 1\}$, and

$$\mathcal{T}\big(f_1; \ldots; f_n\big) := \inf\Big\{ |E|,\ (V, E) \subset (\mathbb{V}_N, \mathbb{E}_N) \text{ connected} :\ V \supset \bigcup_{i=1}^n \{\Lambda_i\} \Big\} \tag{2.9}$$

Theorem 2.1. *Let $L \in \mathbb{N}$, assume that for each $\Lambda \in \mathbb{F}_L$ and $\tau \in \mathcal{S}$ we have the expansion*

$$\log Z_\Lambda(\tau) = \sum_{X \cap \Lambda \neq \emptyset} \phi_{X,\Lambda}(\tau) \tag{2.10}$$

for some local functions $\phi_{X,\Lambda} \in \mathcal{F}_{\Lambda^c}$, $X \in \mathbb{F}$, such that given $\Lambda, \Lambda' \subset\subset \mathbb{L}$, we have that $X \cap \Lambda = X \cap \Lambda'$ implies $\phi_{X,\Lambda} = \phi_{X,\Lambda'}$. If there exist reals $a, b \geq 0$ and $C < \infty$ such that for any $\Lambda \in \mathbb{F}_L$

$$\sup_{x \in \mathbb{L}} \sum_{X \ni x} \exp\{a\mathbb{T}(X) + b \operatorname{diam}(X)\} \, \|\phi_{X,\Lambda}\|_\infty \leq C \tag{2.11}$$

then for each $n \geq 2$ there exists a real $K_n = K_n(C; |\Lambda_1|, \dots, |\Lambda_n|)$ such that

$$\big|\mu_\Lambda^\tau(f_1; \dots; f_n)\big| \leq K_n \, \exp\Big\{ - \Big[a + \frac{b}{n-1}\Big] \, \mathcal{T}(f_1; \dots; f_n)\Big\} \prod_{i=1}^n \mu_\Lambda^\tau(|f_i|) \tag{2.12}$$

for any $\Lambda \in \mathbb{F}_L$ and $\tau \in \mathcal{S}$.
Furthermore, if (2.11) is satisfied with $a > 0$ and $\Lambda_1, \dots, \Lambda_n$ are such that for some $\delta \in (0,1)$

$$\lambda := \sup_{i \in N} \sum_{j \neq i} \big(|\Lambda_i| \wedge |\Lambda_j|\big) \, \exp\Big\{ - \frac{1}{2} \, a \, \delta \, \mathrm{d}(\Lambda_i, \Lambda_j)\Big\} \leq \frac{1}{6\, e\, (1 + 18\, C\, e)} \tag{2.13}$$

then

$$\big|\mu_\Lambda^\tau(f_1; \dots; f_n)\big| \leq \exp\Big\{ - a\,(1-\delta)\, \mathcal{T}(f_1; \dots; f_n)\Big\} \prod_{i=1}^n \mu_\Lambda^\tau(|f_i|) \tag{2.14}$$

for any $\Lambda \in \mathbb{F}_L$, $\tau \in \mathcal{S}$, and $n \geq 2$.

Note that the hypotheses (2.10) and (2.11) with $a + b > 0$ imply [8, Condition IVa] which is one of the Dobrushin–Shlosman complete analyticity conditions. Indeed by setting

$$g(x, \Lambda, \tau) := \sum_{X \ni x} \frac{1}{|X|} \, \phi_{X,\Lambda}(\tau) \tag{2.15}$$

for all $\tau \in \mathcal{S}$, $\Lambda \subset\subset \mathbb{L}$, and $x \in \Lambda$, we have that (i) e (ii) of [8, Condition IVa] hold.

Remark 2.2. Instead of (2.10) we can assume an expansion of the form

$$\log Z_\Lambda(\tau) = \sum_{X \cap \Lambda \neq \emptyset} \big[\psi_{X,\Lambda}(\tau) + \phi_{X,\Lambda}(\tau)\big] \tag{2.16}$$

where $\phi_{X,\Lambda}$ satisfies the bound (2.11) whereas $\psi_{X,\Lambda}$ satisfy the same measurability condition namely, that $\psi_{X,\Lambda} \in \mathcal{F}_{\Lambda^c}$ and $X \cap \Lambda = X \cap \Lambda'$ implies $\psi_{X,\Lambda} = \psi_{X,\Lambda'}$, $X \in \mathbb{F}$, and $\psi_{X,\Lambda}$ are of finite range, i.e. for some integer $\bar{r}$ we have $\psi_{X,\Lambda} = 0$ for $\operatorname{diam}(X) > \bar{r}$. Then the thesis of Theorem 2.1 still holds provided $\mathrm{d}(\Lambda_i, \Lambda_j) > \bar{r}$, $i \neq j \in N$. Note that no bound on the norm of the family $\{\psi_{X,\Lambda}, \, X \in \mathbb{F}\}$ is required.

Addenda:

- One may wonder how we can bound the semi–invariant of n functions in terms of their L^1 (rather than L^n) norm. This is possible because f_i have disjoint supports.
- By the methods in [17,18], it is possible to prove the following converse to Theorem 2.1. If the bound (2.12) holds for $n = 2$ then there are an integer $L' > 0$ and a real $a' > 0$ such that (2.10) and (2.11) hold for any $\Lambda \in \mathbb{F}_{L'}$.
- If there exists a unique infinite volume Gibbs state μ, as it is typically the case under conditions implying the validity of (2.10)–(2.11), then the bounds (2.12) and (2.14) holds also for μ.
- If the supports Λ_i are at distance large enough (depending on $|\Lambda_i|$, a and C), then the condition (2.13) is satisfied. Note also that *one* of the functions f_i might have arbitrarily large support.
- In [22, Corollary II.12.8] it is shown how, in a general setting, it is possible to deduce some decay of semi–invariants from suitable decay properties of covariances.

2.6. Exempla

In order to clarify how (2.10) and (2.11) can be shown to hold assuming a convergent cluster expansion, we discuss the standard Ising model at high temperature; much more general models can be analyzed along the same lines. The single spin configuration space is $\mathcal{S}_0 = \{-1, +1\}$ and the potential U is then given by

$$U_X(\sigma) := \begin{cases} J\sigma(x)\sigma(y) & \text{if } X = \{x, y\} \text{ and } |x - y|_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $J \in \mathbb{R}$ and $|x|_2$ is the Euclidean norm of $x \in \mathbb{Z}^d$. The partition function (2.4) can be written as

$$Z_\Lambda(\tau) = 2^{|\Lambda|} \Bigg[1 + \sum_{n \geq 1} \sum_{\substack{\gamma_1, \ldots, \gamma_n \in \Gamma_\Lambda : \\ \widetilde{\gamma}_i \cap \widetilde{\gamma}_j = \emptyset \quad 1 \leq i < j \leq n}} \prod_{k=1}^{n} \zeta_{\gamma_k}(\tau) \Bigg]$$

where Γ_Λ is the set of polymers intersecting Λ; a polymer $\gamma \in \Gamma_\Lambda$ is a connected set of bonds: for some $k \geq 1$, $\gamma = \{b_1, \ldots, b_k\}$ with $b_i = \{x_i, y_i\}$, $|x_i - y_i|_2 = 1$, $b_i \cap \Lambda \neq \emptyset$. We have also set $\widetilde{\gamma} := \cup_{b \in \gamma} b$ and

$$\zeta_\gamma(\tau) := \frac{1}{2^{|\Lambda|}} \sum_{\sigma \in \{-1,+1\}^\Lambda} \prod_{b \in \gamma} \left[e^{U_b(\sigma \tau_{\Lambda^c})} - 1 \right]$$

note that for each $\gamma \in \Gamma_\Lambda$ we have $\zeta_\gamma \in \mathcal{F}_{\widetilde{\gamma} \cap \Lambda^c}$. For $|J|$ small enough it is possible to show, see e.g. [13, §20.4] or [22, §V.7], that

$$\log Z_\Lambda = \sum_{n=1}^{\infty} \sum_{\gamma_1, \ldots, \gamma_n \in \Gamma_\Lambda} \varphi_T(\gamma_1, \ldots, \gamma_n) \prod_{k=1}^{n} \zeta_{\gamma_k} \tag{2.17}$$

where φ_T is a combinatorial factor, see e.g. [13, Eq. (20.2.8)] or [22, Eq. (V.7.9)], vanishing whenever $\{\gamma_1, \dots, \gamma_n\}$ can be split into two subsets with every polymer of the first one not intersecting any polymer of the second one. From (2.17) we get (2.10) with $\phi_{X,\Lambda}$ given by

$$\phi_{X,\Lambda} = \sum_{n=1}^{\infty} \sum_{\substack{\gamma_1, \dots, \gamma_n \in \Gamma_\Lambda: \\ \bigcup_{i=1}^n \tilde{\gamma}_i = X}} \varphi_T(\gamma_1, \dots, \gamma_n) \prod_{k=1}^{n} \zeta_{\gamma_k}$$

Finally, by standard estimates, see e.g. [13, §20.4] or [22, §V.7], we get that the bound (2.11) holds for some $a > 0$.

Without entering into the details, we discuss here some models to which Theorem 2.1 might be applied on the basis of a convergent cluster expansion.

– *High temperature / low activity expansions.*
  The convergence of the cluster expansion for any $\Lambda \in \mathbb{F}$ and the tree decay of the semi–invariants for $f_i(\sigma) = \sigma(x_i)$, $x_i \in \mathbb{L}$ is a classical topic in equilibrium statistical mechanics, see e.g. [13, §20.4], [22, Theorem V.7.13], and [9–11]. However we are not aware of any reference where the case of local functions f_i with arbitrary support is discussed in detail.

– *Strong Mixing (SM) potentials.*
  The tree decay of the semi–invariants uniform in the boundary configuration is one, called condition IIc, of the equivalent conditions of the Dobrushin–Shlosman's completely analytical interactions [7, 8]. It is stated in a somewhat different form than the one given here: there is no restriction on $\mathrm{d}(\Lambda_i, \Lambda_j)$, but the supports Λ_i are required to have $\mathrm{diam}(\Lambda_i) \leq r$. We mention that the equivalence of the tree decay of the semi–invariants with the other conditions is proven, via a very elegant analytical function argument, under the additional assumption that the potential U is in the same connected component (among the interactions satisfying the conditions) of the zero potential, see [7, Comment 2.1].

  In the original Dobrushin–Shlosman's setting the exponential decay (2.12) is supposed to hold for all $\Lambda \in \mathbb{F}$; however, as discussed in [15], there are examples in which it holds only for $\Lambda \in \mathbb{F}_L$ with L large enough. This has lead to the so–called restricted completely analytical (or Strong Mixing) scenario, see [15,16,20], in which one considers only the "regular" volumes $\Lambda \in \mathbb{F}_L$. The usual argument to get the tree decay of the semi–invariants for SM potentials is the following. Consider a rescaled system whose new single spin variables are the old spin configurations in the blocks $Q_L(Lx)$, $x \in \mathbb{Z}^d$; we can then apply Dobrushin–Shlosman's results [7, 8] to this rescaled system and get all their equivalent mixing and analyticity properties of the Gibbs state for every $\Lambda \in \mathbb{F}_L$.

Theorem 2.1 allows a direct proof of the tree decay of semi–invariants for SM potentials (without the hypotheses that U is in the same connected component of the zero potential) according to the following route. SM potentials satisfy the *finite size* condition introduced in [17, 18] which yields a convergent cluster expansion for which (2.10) and (2.11) hold for some $a > 0$ and some integer L. As a matter of fact in [17, 18] it is considered only the case when Λ is a torus, but it is not too difficult, see [2, 4] for some details, to extend it to any $\Lambda \in \mathbb{F}_L$ and $\tau \in \mathcal{S}$. Then Theorem 2.1 yields the tree decay of the semi–invariants in the sense given by (2.14).

- *Continuous systems. High temperature / low activity expansions.*
  We have described only lattice models, but it is possible to extend Theorem 2.1 to continuous models. For the infinite volume state, absolute integrability of the Ursell functions is proven in [19, Thm. 4.4.8]. For a positive pairwise interaction, the convergence of the cluster expansion uniform in the boundary condition is proven in [23], see also [1] for the exponential decay of the covariance between local functions.

- *Disordered systems in the Griffiths' phase. High temperature / low activity expansions.*
  The convergence of an appropriate multi–scale cluster expansion in such a situation has been obtained in [12] where the tree decay of the semi–invariants is proven in detail only for $f_i(\sigma) = \sigma(x_i)$. We are in a situation like the one described in Remark 2.2 with the additional complication that, depending on the disorder configuration, the functions $\psi_{X,\Lambda}$ can have arbitrary large supports. One then obtains some probability estimates on the disorder which lead to a tree decay in a set of full measure.

- *Disordered systems in the Griffiths' phase. Small perturbation of SM potentials.*
  The convergence of an appropriate multi–scale cluster expansion in such a situation will be proven in [5]. We stress that, due to the presence of arbitrary large regions of strong interaction, the bound (2.11) holds with $b > 0$ but $a = 0$. We refer to [3] for a more detailed discussion.

3. Proof of the tree decay

The usual proofs of tree decay of the semi–invariants from the convergence of the cluster expansion, see e.g. [13, §20.4] or [22, §V.7], are based on the expansion of the perturbed partition function (2.5) and then in the estimates of the derivatives in (2.6). If one is willing to consider functions f_i with arbitrary supports Λ_i there are some difficulties related in the need of cluster expand the perturbed measure also inside Λ_i where the interaction is not necessary weak. The combinatorial proof we present here is instead based on the identity (2.8) and will involve (2.10) and (2.11), which abstract the convergence of a cluster expansion, only outside the supports Λ_i namely, for the unperturbed system. For simplicity we have required that the supports of the functions f_i are at a distance greater than the range of the potential.

Let us start by a purely combinatorial lemma which reduces the estimate of the semi–invariant to ratios of partition functions. For $\Lambda \in \mathbb{F}_L$ and $I \subset N = \{1, \dots, n\}$ we set $\Lambda_I := \bigcup_{i \in I} \Lambda_i \subset \Lambda$ and $V_I := \Lambda \setminus \Lambda_I$; note that since we have assumed $\Lambda_i \in F_L$ we have also $V_I \in F_L$. For $\sigma \in \mathcal{S}$ let also $R_I = R_I(\sigma)$ be defined by $R_\emptyset = R_{\{i\}} = 1$ and for $|I| \geq 2$ by

$$\log R_I := \sum_{J \subset I} (-1)^{|I|-|J|} \log Z_{V_J} \tag{3.1}$$

note that $R_I \in \mathcal{F}_{V_I^c}$ and $V_I^c = \Lambda^c \cup \Lambda_I$. We point out the analogy between definition (3.1) and the combinatorial set up of Kotecký–Preiss [14, Eq. (3)]. We set finally $\varrho_I := R_I - 1$ and define $\varrho^{(\Lambda)} \in \mathcal{F}_{\Lambda_N \cup \Lambda^c}$ as

$$\varrho^{(\Lambda)}(\sigma) := \sum_{k \geq 1} \sum_{\underline{I} \in \mathcal{I}_k} \prod_{I \in \underline{I}} \varrho_I(\sigma) \tag{3.2}$$

in which

$$\mathcal{I}_k := \left\{ \underline{I} \equiv \{I_1, \dots, I_k\} \, : \, I_h \subset N \, , \; h \neq h' \Rightarrow I_h \neq I_{h'} \, , \; \bigcup_{h=1}^{k} I_h = N \, , \; \underline{I} \text{ is connected} \right\} \tag{3.3}$$

where $\underline{I}$ connected means that for each pair I, I' in $\underline{I}$ there exists a sequence $J_q \in \underline{I}$, $q = 0, \dots, m$, such that $I = J_0$, $J_m = I'$, and $J_{q-1} \cap J_q \neq \emptyset$, $q = 1, \dots, m$. We note that for $k > 2^n$ we have $\mathcal{I}_k = \emptyset$.

Lemma 3.1. *If* $\mathrm{d}(\Lambda_i, \Lambda_j) > r$ *for any* $i \neq j \in N$ *then for each* $\tau \in \mathcal{S}_{\Lambda^c}$ *we have*

$$\mu_\Lambda^\tau\big(f_1; \cdots; f_n\big) = \sum_{\sigma \in \mathcal{S}_{\Lambda_N}} \prod_{i \in N} \left[\mu^\tau_{\Lambda;\Lambda_i}(\sigma_{\Lambda_i}) f_i(\sigma_{\Lambda_i})\right] \varrho^{(\Lambda)}(\sigma\tau) \tag{3.4}$$

In particular

$$\Big|\mu_\Lambda^\tau\big(f_1; \cdots; f_n\big)\Big| \leq \big\|\varrho^{(\Lambda)}\big\|_\infty \prod_{i=1}^{n} \mu_\Lambda^\tau\big(|f_i|\big) \tag{3.5}$$

Proof. For $\Lambda \in \mathbb{F}$, $F \in \mathcal{F}_{\Lambda_F}$, with $\Lambda_F \subset \Lambda$, and $\tau \in \mathcal{S}_{\Lambda^c}$, by using the definition (2.3) of the Gibbs state, we get

$$\mu_\Lambda^\tau(F) = \sum_{\sigma \in \mathcal{S}_{\Lambda_F}} \frac{Z_{\Lambda \setminus \Lambda_F}(\sigma\tau)}{Z_\Lambda(\tau)} \exp\Big\{ \sum_{\substack{X \cap \Lambda_F \neq \emptyset \\ X \cap \Lambda \subset \Lambda_F}} U_X(\sigma\tau) \Big\} F(\sigma)$$

By using (2.8), the hypotheses $\mathrm{d}(\Lambda_i,\Lambda_j)>r$, and that $\underline{D}$ is a partition of N, we thus find

$$\begin{aligned}\mu^{\tau}_{\Lambda}(f_1;\cdots;f_n) = & \sum_{\sigma\in\mathcal{S}_{\Lambda_N}}\prod_{i=1}^{n}\left[\frac{Z_{V_{\{i\}}}(\sigma\tau)}{Z_\Lambda(\tau)}\exp\Big\{\sum_{\substack{X\cap\Lambda_i\neq\emptyset\\ X\cap\Lambda\subset\Lambda_i}}U_X(\sigma\tau)\Big\}f_i(\sigma)\right]\\ & \times\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!\sum_{\underline{D}\in\mathcal{D}^{\ell}_N}\prod_{k=1}^{\ell}\frac{Z_{V_{D_k}}(\sigma\tau)\big[Z_\Lambda(\tau)\big]^{|D_k|-1}}{\prod_{i\in D_k}Z_{V_{\{i\}}}(\sigma\tau)}\end{aligned}\tag{3.6}$$

We therefore need to show that

$$\varrho^{(\Lambda)}=\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!\sum_{\underline{D}\in\mathcal{D}^{\ell}_N}\prod_{k=1}^{\ell}\frac{Z_{V_{D_k}}Z_\Lambda^{|D_k|-1}}{\prod_{i\in D_k}Z_{V_{\{i\}}}}\tag{3.7}$$

Let $I\subset N$, $|I|\geq 2$, by (3.1) we have the following chain of identities

$$\begin{aligned}\sum_{J\subset I}\log R_J &= \sum_{\substack{J\subset I\\ |J|\geq 2}}\sum_{K\subset J}(-1)^{|J|-|K|}\log Z_{V_K}=\sum_{K\subset I}\sum_{\substack{J:\,I\supset J\supset K\\ |J|\geq 2}}(-1)^{|J|-|K|}\log Z_{V_K}\\ &= \sum_{k=0}^{|I|}\sum_{\substack{K\subset I\\ |K|=k}}\log Z_{V_K}\sum_{j=2\vee k}^{|I|}(-1)^{j-k}\binom{|I|-k}{j-k}\\ &= \log Z_{V_I}+(|I|-1)\log Z_\Lambda-\sum_{i\in I}\log Z_{V_{\{i\}}}\end{aligned}\tag{3.8}$$

Therefore, given $\underline{D}\in\mathcal{D}^{\ell}_N$ and $k\in\{1,\ldots,\ell\}$ we have

$$\prod_{J\subset D_k}(1+\varrho_J)=\prod_{J\subset D_k}R_J=\frac{Z_{V_{D_k}}Z_\Lambda^{|D_k|-1}}{\prod_{i\in D_k}Z_{V_{\{i\}}}}\tag{3.9}$$

Hence, formula (3.7) follows from (3.9) and the following identity

$$\varrho^{(\Lambda)}=\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!\sum_{\underline{D}\in\mathcal{D}^{\ell}_N}\prod_{k=1}^{\ell}\prod_{J\subset D_k}(1+\varrho_J)\tag{3.10}$$

where $\varrho_\emptyset=0$ (we also have $\varrho_{\{i\}}=0$ but this will not be used in the proof of (3.10)).

To prove (3.10), we define

$$\widetilde{\mathcal{I}}_k:=\Big\{\underline{I}\equiv\{I_1,\ldots,I_k\}\ :\ I_h\subset N\,,\ h\neq h'\Rightarrow I_h\neq I_{h'}\Big\}$$

by expanding the products on the right hand side of (3.10) we get that it is equal to

$$\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!\big|\mathcal{D}^{\ell}_N\big|+\sum_{k\geq 1}\sum_{\underline{I}\in\widetilde{\mathcal{I}}_k}a(\underline{I})\prod_{I\in\underline{I}}\varrho_I\tag{3.11}$$

for appropriate coefficients $a(\underline{I})$ which can be computed as follows. Let $d_n^\ell := \left|\mathcal{D}_N^\ell\right|$ be the number of partitions into ℓ atoms of N; we understand that $d_n^\ell = 0$ if $\ell \geq n+1$. Given $\underline{I} \in \widetilde{\mathcal{I}}_k$ let us decompose it into maximal connected components $\mathcal{C}_1, \dots, \mathcal{C}_h$ namely, $\underline{I} = \bigcup_{m=1}^h \mathcal{C}_m$ where each $\mathcal{C}_m$ is connected and for any pair $I \in \mathcal{C}_m$, $J \in \mathcal{C}_{m'}$ with $m \neq m'$ we have $I \cap J = \emptyset$; let also $\widetilde{\mathcal{C}}_m := \bigcup_{I \in \mathcal{C}_m} I \subset N$ and $c_m := \left|\widetilde{\mathcal{C}}_m\right|$. Then

$$
\begin{aligned}
a(\underline{I}) &= a(\mathcal{C}_1, \dots, \mathcal{C}_h) \\
&= \sum_{\ell=1}^n (-1)^{\ell-1}(\ell-1)! \left| \left\{ \underline{D} \in \mathcal{D}_N^\ell \,:\, \forall m = 1, \dots, h \;\; \exists j \in \{1, \dots, \ell\} : \, D_j \supset \widetilde{\mathcal{C}}_m \right\} \right| \\
&= \sum_{\ell=1}^n (-1)^{\ell-1}(\ell-1)!\, d_{n+h-(c_1+\cdots+c_h)}^\ell
\end{aligned}
$$

We note that the recursion relation $d_1^1 = 1$ and $d_{i+1}^\ell = d_i^{\ell-1} + \ell\, d_i^\ell$, with $i, \ell = 1, 2, \dots$, holds. Such relation implies that the first term in (3.11) vanishes (recall that $n \geq 2$). Moreover, for the same reason, we have that $a(\underline{I}) = 1$ if $h = 1$ and $c_1 = n$ namely, if $\underline{I} \in \mathcal{I}_k$, and $a(\underline{I}) = 0$ otherwise. Recalling (3.2), we have thus completed the proof of (3.10). □

The next Lemma states that each ϱ_I has an exponential decay with the tree intersecting each Λ_i, $i \in I$. Given $I = \{h_1, \dots, h_{|I|}\} \subset N$, we define

$$
T(I) := \inf \left\{ \mathbb{T}\big(\{x_1, \dots, x_{|I|}\}\big),\, x_j \in \Lambda_{h_j} \text{ with } j = 1, \dots, |I| \right\} \tag{3.12}
$$

and note that $T(N) \geq \mathcal{T}(f_1; \dots; f_n)$.

Lemma 3.2. *Let $L \in \mathbb{N}$, assume that for each $\Lambda \in \mathbb{F}_L$ and $\tau \in \mathcal{S}$ we have the expansion (2.10) as in Theorem 2.1 and the bound (2.11) holds. Set $\theta_\emptyset := \theta_{\{i\}} := 0$ and*

$$
\theta_I := C\, 2^{|I|} \inf_{i \in I} |\Lambda_i| \, \exp\left\{ -\left[a + \frac{b}{|I|-1} \right] T(I) \right\} \tag{3.13}
$$

for $|I| \geq 2$. Then, recalling ϱ_I has been defined below (3.1), for any $I \subset N$ we have

$$
\left\| \varrho_I \right\|_\infty \leq \theta_I \, e^{\theta_I} \tag{3.14}
$$

Proof. By plugging (2.10) into definition (3.1) and understanding $\phi_{X,V} = 0$ whenever

$X \cap V = \emptyset$, we get

$$\begin{aligned}
\log R_I &= (-1)^{|I|} \sum_{j=0}^{|I|} (-1)^j \sum_{\substack{J \subset I \\ |J|=j}} \sum_{\substack{X \subset\subset \mathbb{L} \\ X \cap V_J \neq \emptyset}} \phi_{X,V_J} = (-1)^{|I|} \sum_{j=0}^{|I|} (-1)^j \sum_{\substack{X \subset\subset \mathbb{L}: \\ X \cap \Lambda \neq \emptyset}} \sum_{\substack{J \subset I \\ |J|=j}} \phi_{X,V_J} \\
&= (-1)^{|I|} \sum_{K \subset I} \sum_{\substack{X \cap \Lambda \neq \emptyset,\, X \cap \Lambda_{I \setminus K} = \emptyset \\ X \cap \Lambda_k \neq \emptyset \, \forall k \in K}} \sum_{j=0}^{|I|} (-1)^j \sum_{\substack{J \subset I \\ |J|=j}} \phi_{X,V_J} \\
&= (-1)^{|I|} \sum_{K \subset I} \sum_{\substack{X \cap \Lambda \neq \emptyset,\, X \cap \Lambda_{I \setminus K} = \emptyset \\ X \cap \Lambda_k \neq \emptyset \, \forall k \in K}} \sum_{j=0}^{|I|} (-1)^j \sum_{H \subset K} \sum_{\substack{J \subset I \\ |J|=j,\, J \cap K = H}} \phi_{X,V_J}
\end{aligned}$$

We now note that the hypotheses on $\phi_{X,V}$ imply that $\phi_{X,V} = \phi_{X,V'}$ if $X \cap V^{\mathrm{c}} = X \cap (V')^{\mathrm{c}}$. Hence

$$\begin{aligned}
\log R_I &= (-1)^{|I|} \sum_{K \subset I} \sum_{\substack{X \cap \Lambda \neq \emptyset,\, X \cap \Lambda_{I \setminus K} = \emptyset \\ X \cap \Lambda_k \neq \emptyset \, \forall k \in K}} \sum_{H \subset K} \phi_{X,V_H} \sum_{j=0}^{|I|} (-1)^j \sum_{\substack{J \subset I: \\ |J|=j,\, J \cap K = H}} 1 \\
&= (-1)^{|I|} \sum_{K \subset I} \sum_{\substack{X \cap \Lambda \neq \emptyset,\, X \cap \Lambda_{I \setminus K} = \emptyset \\ X \cap \Lambda_k \neq \emptyset \, \forall k \in K}} \sum_{H \subset K} \phi_{X,V_H} \sum_{j=|H|}^{|I|-|K|+|H|} (-1)^j \binom{|I|-|K|}{j-|H|} \\
&= \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda_i \neq \emptyset \, \forall i \in I}} \sum_{H \subset I} (-1)^{|I|-|H|} \phi_{X,V_H}
\end{aligned}$$

where we used that the sum on j on the second line equals $(-1+1)^{|I|-|K|} = 0$ for $K \subset I$ and $K \neq I$.

Now, by using the bound (2.11) and the remark that if $X \cap \Lambda_i \neq \emptyset$ for all $i \in I$ we have $T(I) \leq (|I|-1)\, \mathrm{diam}(X)$ and $T(I) \leq \mathbb{T}(X)$, we get

$$\big\| \log R_I \big\|_\infty \leq C\, 2^{|I|} \inf_{i \in I} |\Lambda_i| \, \exp\Big\{ -\Big(a + \frac{b}{|I|-1}\Big) T(I) \Big\} \tag{3.15}$$

which, by using the inequality $|e^u - 1| \leq e^{|u|}\, |u|$, implies the bound (3.14). $\square$

We remark that it is not difficult to check that Lemma 3.2 holds also under the condition in Remark 2.2.

We can now prove the first part of Theorem 2.1.

Proof of the bound (2.12). Recalling $\mathcal{I}_k$ has been defined in (3.3), it is easy to show that for each $k \geq 1$ and $\underline{I} \in \mathcal{I}_k$ we have

$$\sum_{I \in \underline{I}} \Big(a + \frac{b}{|I|-1}\Big)\, T(I) \geq \Big(a + \frac{b}{n-1}\Big)\, \mathcal{T}(f_1; \ldots; f_n) \tag{3.16}$$

Hence (2.12) follows from Lemmata 3.1 and 3.2 provided we define K_n as

$$K_n(C; |\Lambda_1|, \ldots, |\Lambda_n|) := \sum_{k\geq 1} \sum_{\underline{I}\in\mathcal{I}_k} \prod_{I\in\underline{I}} \Big[C\, 2^{|I|} \inf_{i\in I} |\Lambda_i| \, \exp\big\{ C\, 2^{|I|} \inf_{i\in I} |\Lambda_i| \big\} \Big] \tag{3.17}$$

which is finite since it is the sum of a finite number of terms. □

The constant K_n in (3.17) is highly non–optimal. If $a > 0$ we can improve the estimates and get (2.14).

Proposition 3.3. *Assume condition (2.13) holds for some $a > 0$, $\delta \in (0, 1)$. Recalling θ_I has been defined in (3.13), set*

$$\bar{\theta}_I := \exp\big\{ a\,(1-\delta)\, T(I) \big\}\, \theta_I \tag{3.18}$$

then, recalling $\mathcal{I}_k$ has been defined in (3.3),

$$\sum_{k\geq 1} \sum_{\underline{I}\in\mathcal{I}_k} \prod_{I\in\underline{I}} \bar{\theta}_I\, e^{\theta_I} \leq 1 \tag{3.19}$$

To prove Proposition 3.3 we start by a general result on trees which gives a lower bound on the number of edges in term of a path connecting all the vertices.

Lemma 3.4. *Let $I \subset N$ with $2 \leq |I| = k + 1$ be given by $I = \{i_0, i_1, \ldots, i_k\}$. Let us denote by $\Pi_0(k)$ the set of permutations π of $\{0, 1, \ldots, k\}$ such that $\pi(0) = 0$. Recalling $T(I)$ has been defined in (3.12), we then have*

$$T(I) \geq \frac{1}{2} \inf_{\pi\in\Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}\big(\Lambda_{i_{\pi(l-1)}}, \Lambda_{i_{\pi(l)}}\big) \tag{3.20}$$

Proof. Let $X = \{\bar{x}_{i_0}, \bar{x}_{i_1}, \ldots, \bar{x}_{i_k}\}$, $\bar{x}_{i_h} \in \Lambda_{i_h}$ be a minimizer for (3.12) and, for such X, let $T_X = (V_X, E_X) \subset (\mathbb{L}, \mathbb{E})$ with $V_X \supset X$ be a tree in which the infimum in (2.1) is attained. The Lemma is implied by

$$|E_X| \geq \frac{1}{2} \inf_{\pi\in\Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}\big(\bar{x}_{i_{\pi(l-1)}}, \bar{x}_{i_{\pi(l)}}\big) \tag{3.21}$$

which is proven as follows. By induction on the number of edges in T_X it is easy to prove that there exists a path (see Fig. 1) $\{\ell_0, \ldots, \ell_{M-1}\}$, with $\ell_m \in E_X$ for all $m = 0, \ldots, M-1$, satisfying the following properties: $\ell_{m-1} \cap \ell_m \neq \emptyset$ for all $m = 1, \ldots, M-1$, $\bar{x}_{i_0} \in \ell_0$, for each $v \in V_X$ there exists $m \in \{0, \ldots, M-1\}$ such that $\ell_m \ni v$, and each edge $e \in E_X$ appears in the path at most twice. Recalling that $\mathrm{d}(x, y) = \mathbb{T}(\{x, y\})$, the bound (3.21) follows. □

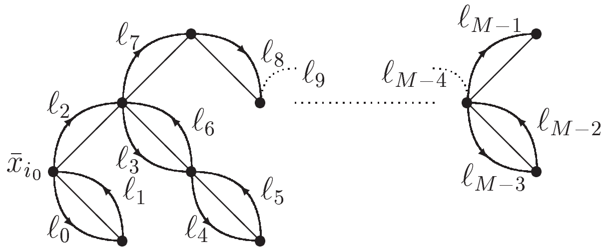

Figure 1: The path $\ell = \{\ell_0, \dots, \ell_{M-1}\}$ introduced in the proof of Lemma 3.4

Lemma 3.5. *Assume condition (2.13) is satisfied then, recalling $\bar{\theta}_I$ has been defined in (3.18),*

$$\widetilde{K} := \sup_{i \in N} \sum_{I \subset N,\, I \ni i} (3\,e)^{|I|}\, \bar{\theta}_I \le \frac{1}{3} \tag{3.22}$$

Proof. By using Lemma 3.4 and recalling that $\theta_{\{i\}} = 0$ for all $i \in N$, we get

$$\begin{aligned}
\sup_{i \in N} \sum_{I \subset N,\, I \ni i} (3\,e)^{|I|}\, \bar{\theta}_I \; &\le \; 6\,C\,e \sup_{i \in N} \sum_{k \ge 1} \sum_{\substack{I \subset N \setminus \{i\} \\ |I| = k}} (6\,e)^k \Big(\inf_{j \in I} |\Lambda_j| \Big)\, e^{-a\,\delta\, T(I)} \\
&\le \; 6\,C\,e \sup_{i_0 \in N} \sum_{k \ge 1} \frac{(6\,e)^k}{k!} \\
&\qquad \times \sum_{\substack{i_1, \dots, i_k \in N \setminus \{i_0\} \\ i_h \ne i_{h'},\, h \ne h'}} \Big(\inf_{h = 0, \dots, k} |\Lambda_{i_h}| \Big) \exp\Big\{ -a\,\delta\, \frac{1}{2} \inf_{\pi \in \Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}\big(\Lambda_{i_{\pi(l-1)}}, \Lambda_{i_{\pi(l)}}\big) \Big\} \\
&\le \; 6\,C\,e \sup_{i_0 \in N} \sum_{k \ge 1} \frac{(6\,e)^k}{k!} \sum_{\substack{i_1, \dots, i_k \in N \setminus \{i_0\} \\ i_h \ne i_{h'},\, h \ne h'}} \sum_{\pi \in \Pi_0(k)} \Big(\inf_{h = 0, \dots, k} |\Lambda_{i_h}| \Big) \prod_{l=1}^{k} e^{-a\,\delta\, \frac{1}{2} \mathrm{d}(\Lambda_{i_{\pi(l-1)}}, \Lambda_{i_{\pi(l)}})} \\
&\le \; 6\,C\,e \sum_{k=1}^{\infty} (6\,e)^k \Big(\sup_{i \in I} \sum_{j \in I \setminus \{i\}} |\Lambda_i| \wedge |\Lambda_j|\, e^{-a\,\delta\, \frac{1}{2} \mathrm{d}(\Lambda_j, \Lambda_i)} \Big)^k \\
&\le \; 6\,C\,e\, \frac{6\,e\,\lambda}{1 - 6\,e\,\lambda} \; \le \; \frac{1}{3}
\end{aligned}$$

where, in the last line, we used (2.13). □

Proof of Proposition 3.3. We note that from the bound (3.22) it follows $e^{\theta_I} \le e^{\bar{\theta}_I} \le e$. By letting $\varepsilon := 1/3$ and $\widetilde{\theta}_I := (3e)^{|I|} \bar{\theta}_I$ and using Lemma 3.5, we can apply the estimate in [6, Appendix B] and get

$$\sum_{k \ge 1} \sum_{\underline{I} \in \mathcal{I}_k} \prod_{I \in \underline{I}} \bar{\theta}_I\, e^{\theta_I} \le \sum_{k \ge 1} \sum_{\underline{I} \in \mathcal{I}_k} \prod_{I \in \underline{I}} \varepsilon^{|I|}\, \widetilde{\theta}_I \le \varepsilon \widetilde{K} \left[1 + \frac{e^{\widetilde{K}} - 1}{1 + \varepsilon^2 e^{\widetilde{K}} - 2\varepsilon e^{\widetilde{K}}} \right] \; \le \; 1$$

since $\varepsilon \le 1/3$ and $\widetilde{K} \le 1/3$. □

It is now straightforward to conclude the proof of Theorem 2.1.

Proof of the bound (2.14). For $\underline{I} \in \mathcal{I}_k$ we have, recalling (3.16)

$$\prod_{I \in \underline{I}} \theta_I \, e^{\theta_I} = \prod_{I \in \underline{I}} e^{-a\,(1-\delta)\,\mathcal{T}(I)} \, \bar{\theta}_I \, e^{\theta_I} \leq e^{-a\,(1-\delta)\,\mathcal{T}(f_1;\ldots;f_n)} \prod_{I \in \underline{I}} \bar{\theta}_I \, e^{\theta_I}$$

and the bound (2.14) follows from Lemmata 3.1, 3.2 and Proposition 3.3. □

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