Mass renormalization in nonrelativistic QED

Fumio Hiroshima *and Herbert Spohn †

October 17, 2003

Abstract

In nonrelativistic QED the charge of an electron equals its bare value, whereas the self-energy and the mass have to be renormalized. In our contribution we study perturbative mass renormalization, including second order in the fine structure constant α, in the case of a single, spinless electron. As well known, if m denotes the bare mass and m_{eff} the mass computed from the theory, to order α one has

$$\frac{m_{\mathrm{eff}}}{m} = 1 + \frac{8\alpha}{3\pi}\log(1+\frac{1}{2}(\Lambda/m)) + O(\alpha^2)$$

which suggests that $m_{\mathrm{eff}}/m = (\Lambda/m)^{8\alpha/3\pi}$ for small α. If correct, in order α^2 the leading term should be $\frac{1}{2}((8\alpha/3\pi)\log(\Lambda/m))^2$. To check this point we expand m_{eff}/m to order α^2. The result is $\sqrt{\Lambda/m}$ as leading term, suggesting a more complicated dependence of m_{eff} on m.

1 Introduction

Nonperturbative renormalization in relativistic QED remains as a mathematical challenge. Thus it is of interest to study simplified candidates, an obvious one being nonrelativistic QED. In this theory, with comparable little effort, one can start from a self-adjoint Hamiltonian operator and thus has a well-defined mathematical framework. As an additional simplification, there is no charge renormalization because of the absence of positrons. Nevertheless, even in nonrelativistic QED, energy and mass renormalization remain poorly understood. Our, admittedly modest, contribution is to study mass renormalization including order α^2.

*Department of Mathematics and Physics, Setsunan University, 572-8508, Osaka, Japan. email: hiroshima@mpg.setsunan.ac.jp

†Zentrum Mathematik and Physik Department, TU München, D-80290, München, Germany. email: spohn@ma.tum.de

Let us first explain the basic Hamiltonian. We consider a single, spinless free electron coupled to the quantized radiation field. We will use relativistic units and employ immediately the total momentum representation. Then the Hilbert space of states is the symmetric Fock space, $\mathcal{F}$, over the one-particle space $L^2(\mathbb{R}^3 \times \{1, 2\})$, i.e.

$$\mathcal{F} = \oplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}).$$

The inner product in $\mathcal{F}$ is denoted by $(\cdot, \cdot)$ and the Fock vacuum by Ω. On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k, j)^* a(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}). \tag{1.1}$$

Operators $a(f)$ and $a(f)^* = a^*(f)$ are densely defined and satisfy the CCR

$$\begin{aligned} &[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3 \times \{1,2\})}, \\ &[a(f), a(g)] = 0, \\ &[a^*(f), a^*(g)] = 0. \end{aligned}$$

The kinetic energy of the photon is given by

$$H_{\mathrm{f}} = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \tag{1.2}$$

which is the second quantization of $\omega(k) = |k|$ considered as a multiplication operator on $L^2(\mathbb{R}^3)$. Similarly the momentum of the photon field is

$$P_{\mathrm{f}} = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk. \tag{1.3}$$

The coupling of the electron to the Maxwell field is mediated through the transverse vector potential $A_{\hat{\varphi}}$ defined by

$$A_{\hat{\varphi}} = \frac{1}{\sqrt{2}} (a(f) + a^*(f)), \tag{1.4}$$

where

$$f(k, j) = \frac{1}{\sqrt{\omega}} \hat{\varphi}(k) e(k, j) \tag{1.5}$$

with $k/|k|, e(k, 1), e(k, 2)$ forming a right-handed dreibein. $\hat{\varphi}$ is the form factor which, as a minimal assumption, satisfies

$$\int_{\mathbb{R}^3} |\hat{\varphi}(k)|^2 (\omega(k)^{-2} + \omega(k)) dk < \infty. \tag{1.6}$$

Later on, we will make more specific choice of $\hat{\varphi}$.

With these definitions the Hamiltonian under study is

$$H_{\hat{\varphi}}(p) = \frac{1}{2} :(p - P_{\mathrm{f}} - eA_{\hat{\varphi}})^2: + H_{\mathrm{f}}, \quad p \in \mathbb{R}^3, \tag{1.7}$$

where p is the total momentum, e the charge, to be definite $e \geq 0$, and $:X:$ denotes the Wick order of X. According to [5], $H_{\hat{\varphi}}(p)$ with domain $D(H_{\mathrm{f}} + \frac{1}{2}P_{\mathrm{f}}{}^2)$ is self-adjoint for all e and p, provided (1.6) holds. The energy-momentum relation is defined as the bottom of the spectrum of $H_{\hat{\varphi}}(p)$,

$$E_{\hat{\varphi}}(p) = \inf \sigma(H_{\hat{\varphi}}(p)). \tag{1.8}$$

In (1.7) the bare mass m of the electron is still missing. It appears in two places. Firstly the form factor depends on m. Let us assume a sharp ultraviolet cutoff Λ. Then

$$\begin{aligned} &\hat{\varphi}(k) = \hat{\varphi}_0(mck/\Lambda), \quad \Lambda > 0, \\ &\hat{\varphi}_0(k) = \begin{cases} (2\pi)^{-3/2} & \text{for } |k| \leq 1, \\ 0 & \text{for } |k| > 1, \end{cases} \end{aligned} \tag{1.9}$$

with $1/mc$ the Compton wave length. Secondly energy is to be measured in units of mc^2 and momentum in units of mc. We henceforth set $c = 1$ (and also $\hbar = 1$). Thus the true energy-momentum relation of the Pauli-Fierz Hamiltonian is

$$E_{m,\Lambda}(p) = mE_{\hat{\varphi}}(p/m), \quad \hat{\varphi} \text{ of (1.9)}. \tag{1.10}$$

Note that equivalently $E_{m,\Lambda}(p)$ is given through

$$E_{m,\Lambda}(p) = \inf \sigma(\frac{1}{2m} :(p - P_{\mathrm{f}} - eA_{\hat{\varphi}_0(\cdot/\Lambda)})^2: + H_{\mathrm{f}}).$$

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \to \infty, \quad m \to 0 \tag{1.11}$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. A convenient criterion for nondegeneracy is the curvature of $E_{m,\Lambda}(p)$ at $p = 0$, in other words the inverse effective mass. Let us assume for a moment an infrared cutoff

$$\hat{\varphi}(k) = 0 \quad \text{for } |k| < \kappa/m$$

with some $0 < \kappa$. Then it is known [7] that, for $|e| < e_*$, $|p| < p_*$ with suitable $e_* > 0$ and $p_* > 0$, $H_{\hat{\varphi}}(p)$ has a nondegenerate ground state $\psi_{\rm g}(p)$ separated by a gap from the continuum, i.e.

$$H_{\hat{\varphi}}(p)\psi_{\rm g}(p) = E_{\hat{\varphi}}(p)\psi_{\rm g}(p), \quad \psi_{\rm g}(p) \in \mathcal{F},$$

has a unique solution. Let us set

$$E_{m,\Lambda}(p) - E_{m,\Lambda}(0) = \frac{1}{2m_{\rm eff}}p^2 + \mathcal{O}(|p|^3) \tag{1.12}$$

for small p. Then, using second order perturbation theory in (1.10), one obtains

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}\sum_{\mu=1,2,3} \frac{(\psi_{\rm g}(0), (P_{\rm f} + eA_{\hat{\varphi}})_\mu (H_{\hat{\varphi}}(0) - E_{\hat{\varphi}}(0))^{-1}(P_{\rm f} + eA_{\hat{\varphi}})_\mu \psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}. \tag{1.13}$$

We assume that this formula remains valid also when $\kappa = 0$.

On the basis of (1.13), mass renormalization can be discussed more precisely. From (1.13) it trivially follows that $m/m_{\rm eff}$ depends only on the ratio Λ/m. It is convenient to express this dependence in the form

$$\frac{m_{\rm eff}}{m} = h(\Lambda/m). \tag{1.14}$$

Clearly $h \geq 1$ and $h(0) = 1$. Let us set

$$\lambda = \frac{\Lambda}{m} \tag{1.15}$$

One expects that h is increasing in λ, because with increasing Λ more photons are bound to the electron which makes $m_{\rm eff}$ larger.

Let us distinguish several cases. If h has a finite limit as $\lambda \to \infty$, then the mass renormalization is finite,

$$m_{\rm eff} = h(\infty)m.$$

Such kind of behavior occurs in the Nelson model [10]. Secondly let us consider the case that $h(\lambda)$ increases linearly for large λ. We set

$$h(\lambda) = 1 + b_0\lambda$$

with $b_0 > 0$. Then

$$m_{\rm eff} = m + b_0\Lambda. \tag{1.16}$$

Hence mass renormalization is additive. This behavior is found in the dipole approximation to the Pauli-Fierz Hamiltonian, e.g. [9], and in the classical Abraham model [13]. If $m_{\mathrm{eff}} > 0$ is imposed, then as $\Lambda \to \infty$ necessarily $m \to -\infty$. As soon as $m < 0$, $mH_{\hat{\varphi}}(p/m)$ is not bounded from below. Therefore we regard the theory as not renormalizable. Thus the case of interest is when for large λ

$$h(\lambda) \simeq b_0 \lambda^\gamma, \quad b_0 > 0, \quad 0 < \gamma < 1, \tag{1.17}$$

which defines the scaling exponent γ and the amplitude b_0. γ depends on e, as does b_0. Inserting (1.17) in (1.14), one obtains for sufficiently large λ,

$$\frac{m_{\mathrm{eff}}}{m} \simeq b_0 \left(\frac{\Lambda}{m}\right)^\gamma. \tag{1.18}$$

Thus the choice

$$m = \Lambda^{-\gamma/(1-\gamma)} b_1^{1/(1-\gamma)} \tag{1.19}$$

yields

$$\lim_{\Lambda\to\infty} m_{\mathrm{eff}}(\Lambda) = m^* = b_0 b_1. \tag{1.20}$$

Here b_0 is fixed by $h(\lambda)$ and b_1 is a parameter which can be adjusted to yield the experimentally determined mass of the electron.

Of course, the difficulty with our discussion is that, while the scaling function is well defined, at present we have no technique to find out its behavior for large λ. For that reason we turn to perturbative renormalization which, through the interchange of the limits $\Lambda \to \infty$ and $e \to 0$, tries to guess the proper value of γ. The details will be given in the following sections, but let us summarize briefly our findings. The fine structure constant is defined through

$$\alpha = \frac{e^2}{4\pi}. \tag{1.21}$$

To first order one finds

$$h(\lambda) = 1 + \frac{8\alpha}{3\pi} \log(1 + \frac{1}{2}\lambda) + \mathcal{O}(\alpha^2), \tag{1.22}$$

which suggests

$$h(\lambda) \simeq \lambda^{8\alpha/3\pi} \tag{1.23}$$

for sufficiently large λ and therefore

$$\gamma = \frac{8\alpha}{3\pi}, \qquad \alpha \ll 1. \tag{1.24}$$

To have a control check, one assumes that to second order

$$\begin{aligned} h(\lambda) &\simeq \lambda^{(8/3\pi)\alpha+b\alpha^2} \\ &\simeq 1+\frac{8\alpha}{3\pi}\log\lambda+\frac{1}{2}(\frac{8\alpha}{3\pi}\log\lambda)^2+b\alpha^2\log\lambda+\mathcal{O}(\alpha^3) \end{aligned} \tag{1.25}$$

for small α. Therefore by expanding m_{eff}/m to order α^2, one should find a term $(\log\lambda)^2$ with an already determined prefactor and a term propotional to $\log\lambda$, together with lower order terms. As to explained, this guess is not confirmed. *Instead* we prove that

$$h(\lambda)=1+\frac{8\alpha}{3\pi}\log(1+\frac{1}{2}\lambda)+c_0\alpha^2\sqrt{\lambda}+\mathcal{O}(\alpha^3), \quad c_0>0, \tag{1.26}$$

which could suggest $\gamma=\frac{1}{2}$ independent of e.

Some aspects of the effective mass and its renormalization have been studied before. Spohn [14] investigates the effective mass of the Nelson model [10] from a functional integral point of view. Lieb and Loss [9] study mass renormalization and binding energies for various models of matter coupled to the radiation field including the Pauli-Fierz model. Hainzl [3] and Hainzl and Seiringer [4] compute the leading order of the effective mass of the Pauli-Fierz Hamiltonian with spin 1/2.

Our paper is organized in the following way. In Section 2 we review under which conditions $E_{\hat\varphi}(p)=E_{\hat\varphi}(p,e)$ is jointly analytic in p and e. In Section 3 we set up the perturbation theory for the effective mass and work out explicitely the terms including α^2. Their asymptotics with $\Lambda\to\infty$ is studied in Section 4.

2 Ground state and its analytic properties

Throughout this paper we assume that

$$\hat\varphi(k)=\begin{cases} 0 & \text{for } |k|<\kappa/m, \\ (2\pi)^{-3/2} & \text{for } \kappa/m\le|k|\le\Lambda/m, \\ 0 & \text{for } |k|>\Lambda/m. \end{cases}$$

For notational convenience, we shall use notations $H(p)$, A and $E(p)$ instead of $H_{\hat\varphi}(p)$, $A_{\hat\varphi}$ and $E_{\hat\varphi}(p)$, respectively.

It is proven in [5] that $H(p)$ is self-adjoint on $D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$ and essentially self-adjoint on any core of $\frac{1}{2}P_{\mathrm{f}}{}^2+H_{\mathrm{f}}$. The bottom of the continuous spectrum of $H(p)$ is denoted by $E_c(p)$. The following lemma can be proven in the similar manner as in [2].

Lemma 2.1 [2] *There exists a constant $p_* > 0$ independent of e such that for $p \in \mathbb{R}^3$ with $|p| < p_*$,*

$$E_c(p) - E(p) > 0.$$

In particular there exists a ground state $\psi_{\rm g}(p)$ of $H(p)$ for $p \in \mathbb{R}^3$ provided $|p| < p_*$.

To have uniqueness, one proves that for any ground state $\psi_{\rm g}(p)$, one has

$$(\psi_{\rm g}(p), \Omega) > \frac{1}{2}$$

provided $|p| < p_*$ and $|e| < e_*$.

Lemma 2.2 [7] *There exists a constant $e_* > 0$ such that for $(p, e) \in \mathbb{R}^3 \times \mathbb{R}$ with $|p| < p_*$ and $|e| < e_*$, the ground state of $H(p)$ is unique up to multiple constants.*

Remark 2.3 *In the case $\kappa = 0$ and for sufficiently small e, Chen [1] proves the absence of a ground state of $H(p)$ in $\mathcal{F}$ for $p \neq 0$ and the existence of a ground state of $H(0)$.*

We also need the analytic properties of $\psi_{\rm g}(p) = \psi_{\rm g}(p, e)$ and $E(p) = E(p, e)$ with respect to $(p, e) \in \mathbb{R}^3 \times \mathbb{R}$ near $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}$.

Lemma 2.4 *There exists an open neighborhood $\mathcal{O}$ containing $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}$ such that $\psi_{\rm g}(p, e)$ is strongly analytic and $E(p, e)$ analytic on $\mathcal{O}$.*

Proof: We split $H(p)$ as

$$H(p) = H_0(p) + H_I(p), \tag{2.1}$$

where

$$H_0(p) = \frac{1}{2}(p - P_{\rm f})^2 + H_{\rm f},$$
$$H_I(p) = -e(p - P_{\rm f}) \cdot A + e^2 \frac{1}{2} :A^2: .$$

Then we obtain that

$$\|H_I(p)\Psi\| \leq c_4 \|H_0(p)\Psi\| + c_5 \|\Psi\| \tag{2.2}$$

for $\Psi \in D(H_0(p)) = D(H_{\rm f}) \cap D({P_{\rm f}}^2)$. Then $H(p)$ is an analytic family of type (A) for e near $e = 0$ (see [12, p.16]). Thus by [12, Theorem XII.9], $H(p)$ is an analytic family in the sense of Kato, which implies that by [12, Theorem XII.8], together with Lemmas 2.1 and 2.2, $\psi_{\rm g}(p,e)$ is strongly analytic and $E(p,e)$ analytic for e near $e=0$. Alternatively we split $H(p)$ as

$$H(p) = H_0' + p\cdot H_I' + \frac{1}{2}p^2,$$

where

$$\begin{aligned} H_0' &= :(P_{\rm f}+eA_{\hat\varphi})^2: + H_{\rm f}, \\ H_I' &= -(P_{\rm f}+eA_{\hat\varphi}). \end{aligned}$$

Then we have

$$\|H_I'\Psi\| \le c_6\|H_0'\Psi\| + c_7\|\Psi\| \tag{2.3}$$

with some constants c_6 and c_7 for $\Psi\in D(H_{\rm f})\cap D(P_{\rm f}{}^2)$. Thus $H(p)$ is an analytic family of type (A) for $p\in\mathbb{R}^3$ near $p=0$. We can see that $\psi_{\rm g}(p,e)$ is strongly analytic and $E(p,e)$ analytic for p near $p=0$ in the similar manner as for e. $\square$

3 Effective mass

3.1 Formulae

In what follows we assume that $(p,e)\in\mathcal{O}$. The ground state energy has the symmety $E(p,e)=E(p,-e)=E(-p,e)$. Therefore

$$\frac{m}{m_{\rm eff}} = \frac{1}{3}\Delta_p E(p,e)\Big\lceil_{p=0}, \tag{3.1}$$

since

$$\frac{\partial}{\partial p_\mu}E(p,e)\Big\lceil_{p_\mu=0} = 0, \quad \mu=1,2,3. \tag{3.2}$$

Moreover, $E(p,e)$ is a function of e^2, the odd derivatives with respect to e vanish, i.e.

$$\frac{d^{2m-1}}{de^{2m-1}}E(p,e)\Big\lceil_{e=0} = 0. \tag{3.3}$$

Lemma 3.1 *We have*

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}\sum_{\mu=1,2,3}\frac{(\psi_{\rm g}(0),(P_{\rm f}+eA)_\mu(H(0)-E(0))^{-1}(P_{\rm f}+eA)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0),\psi_{\rm g}(0))}.$$

Proof: In this proof, $f'(p)_\mu$ means strong derivative of $f(p)$ with respect to p_μ. Since

$$H(p)\psi_{\rm g}(p) = E(p)\psi_{\rm g}(p),$$

we have

$$H'(p)_\mu\psi_{\rm g}(p) + H(p)\psi'_{\rm g}(p)_\mu = E'(p)_\mu\psi_{\rm g}(p) + E(p)\psi'_{\rm g}(p)_\mu \tag{3.4}$$

and

$$\begin{aligned} & H''(p)_\mu\psi_{\rm g}(p) + 2H'(p)_\mu\psi'_{\rm g}(p)_\mu + H(p)\psi''_{\rm g}(p)_\mu \\ & = E''(p)_\mu\psi_{\rm g}(p) + 2E'(p)_\mu\psi'_{\rm g}(p)_\mu + E(p)\psi''_{\rm g}(p)_\mu. \end{aligned} \tag{3.5}$$

By (3.2) it follows that $E'(0)_\mu = 0$, and by (3.4) with $p = 0$,

$$\begin{aligned} & (P_{\rm f} + eA)_\mu\psi_{\rm g}(0) \in D((H(0) - E(0))^{-1}), \\ & \psi'_{\rm g}(0)_\mu = (H(0) - E(0))^{-1}(P_{\rm f} + eA)_\mu\psi_{\rm g}(0). \end{aligned}$$

Therefore, using (3.1) and (3.5), we have

$$\begin{aligned} \frac{m}{m_{\rm eff}} &= \frac{1}{3}\sum_{\mu=1,2,3}\frac{(\psi_{\rm g}(0), E''(0)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))} \\ &= \frac{1}{3}\sum_{\mu=1,2,3}\left\{1 + \frac{(\psi_{\rm g}(0), 2H'(0)_\mu\psi'_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}\right\} \\ &= 1 - \frac{2}{3}\sum_{\mu=1,2,3}\frac{((P_{\rm f} + eA)_\mu\psi_{\rm g}(0), (H(0) - E(0))^{-1}(P_{\rm f} + eA)_\mu\psi_{\rm g}(0))}{(\psi_{\rm g}(0), \psi_{\rm g}(0))}. \end{aligned}$$

Thus the lemma follows. □

From this lemma we obtain the following corollary.

Corollary 3.2 *Let $|e| < e_*$. Then $m_{\rm eff} \geq m$.*

3.2 Perturbative expansions

Let

$$\psi_{\rm g}(0) = \sum_{n=0}^{\infty}\frac{e^n}{n!}\varphi_n, \quad E(0) = \sum_{n=0}^{\infty}\frac{e^{2n}}{(2n)!}E_{2n}.$$

Let us define

$$\mathcal{F}^{\rm even} = \bigoplus_{m=0}^{\infty}\mathcal{F}^{(2m)}, \quad \mathcal{F}^{\rm odd} = \bigoplus_{m=0}^{\infty}\mathcal{F}^{(2m+1)}.$$

Lemma 3.3 *We have*

$$\varphi_{2m} \in \mathcal{F}^{\rm even}, \quad \varphi_{2m+1} \in \mathcal{F}^{\rm odd}.$$

Proof: Let T be the unitary operator of $\mathcal{F}$ onto itself defined by for $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}$,

$$(T\Psi)^{(n)} = \begin{cases} \Psi^{(n)}, & n = 2m, \\ -\Psi^{(n)}, & n = 2m+1. \end{cases}$$

We see that if $T\Psi = -\Psi$ (resp. $T\Psi = +\Psi$) then $\Psi \in \mathcal{F}^{\rm odd}$ (resp. $\Psi \in \mathcal{F}^{\rm even}$). To emphasize the dependence of e, let us denote $\psi_{\rm g}(0)$ and $H(0)$ as $\psi_{\rm g}(0,e)$ and $H(0,e)$, respectively. We have

$$T^{-1}H(0,e)T = H(0,-e).$$

Since each ground state of $H(0,e)$ and $H(0,-e)$ is unique, $T\psi_{\rm g}(0,e)$ is the ground state of $H(0,-e)$. Then

$$T\psi_{\rm g}(0,e) = \theta\psi_{\rm g}(0,-e) = \theta\sum_{n=0}^\infty \frac{(-e)^n}{n!}\varphi_n \tag{3.6}$$

with some phase constant $\theta \in \{e^{i\psi} | 0 \le \psi < 2\pi\}$. Moreover

$$T\psi_{\rm g}(0,e) = \sum_{n=0}^\infty \frac{e^n}{n!} T\varphi_n. \tag{3.7}$$

In particular $\Omega = T\varphi_0 = \theta\Omega$ follows from comparing (3.6) and (3.7). Then $\theta = 1$. Hence (3.6) and (3.7) yield that $T\varphi_n = (-1)^n\varphi_n$. Thus the lemma follows. $\square$

We want to get the explicit form of φ_n. Let

$$\mathcal{F}_{\rm fin} = \{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} | \Psi^{(m)} = 0 \text{ for } m \ge \ell \text{ with some } \ell\},$$
$$\mathcal{F}_0 = \Big\{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_{\rm fin} \Big| \text{(i) } \Psi^{(0)} = 0,$$
$$\text{(ii) } {\rm supp}_{(k_1,\dots,k_n)\in\mathbb{R}^{3n}} \Psi^{(n)}(k_1,\dots,k_n,j_1,\dots,j_n) \not\ni \{(0,\dots,0)\}\Big\}.$$

Lemma 3.4 *We see that* $\mathcal{F}_0 \subset D(H_0^{-1})$.

Proof: Let $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_0$. Since

$$\begin{aligned}&(H_0\Psi)^{(n)}(k_1,\dots,k_n,j_1,\dots,j_n)\\ &= \left[\frac{1}{2}(k_1+\cdots+k_n)^2 + \sum_{j=1}^n \omega(k_j)\right]\Psi^{(n)}(k_1,\dots,k_n,j_1,\dots,j_n),\end{aligned}$$

we see that

$$
\begin{aligned}
&(H_0^{-1}\Psi)^{(n)}(k_1,...,k_n,j_1,...,j_n)\\
&=\left[\frac{1}{2}(k_1+\cdots+k_n)^2+\sum_{j=1}^{n}\omega(k_j)\right]^{-1}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n).
\end{aligned}
$$

Since $\operatorname{supp}_{(k_1,\ldots,k_n)\in\mathbb{R}^{3n}}\Psi^{(n)}(k_1,...,k_n,j_1,...,j_n)\not\ni\{(0,...,0)\}$, we obtain that

$$
\|H_0^{-1}\Psi\|_{\mathcal{F}}^2=\sum_{n=1}^{\text{finite}}\|(H_0^{-1}\Psi)^{(n)}\|_{\mathcal{F}^{(n)}}^2<\infty
$$

and the lemma follows. $\square$

We define A^- and A^+ by

$$
A^-=\frac{1}{\sqrt{2}}a(f),\quad A^+=\frac{1}{\sqrt{2}}a^*(f).
$$

Then $A=A^++A^-$. Moreover A_μ^- and A_μ^+ are defined by A^- and A^+ with $e(k,j)$ replaced by $e^\mu(k,j)$. We split $H(0)$ as

$$
H(0)=H_0+eH_1+\frac{e^2}{2}H_2,
$$

where

$$
\begin{aligned}
H_0&=\frac{1}{2}{P_{\mathrm{f}}}^2+H_{\mathrm{f}},\\
H_1&=\frac{1}{2}(P_{\mathrm{f}}\cdot A+A\cdot P_{\mathrm{f}})=P_{\mathrm{f}}\cdot A=A\cdot P_{\mathrm{f}},\\
H_2&=:A^2:=A^+\cdot A^++A^-\cdot A^-+2A^+\cdot A^-.
\end{aligned}
$$

Lemma 3.5 *We have* $E_0=E_1=E_2=E_3=0$ *and*

$$
\varphi_0=\Omega,\quad \varphi_1=0,\quad \varphi_2=-H_0^{-1}H_2\Omega,\quad \varphi_3=3H_0^{-1}H_1H_0^{-1}H_2\Omega.
$$

In particular $\varphi_2\in\mathcal{F}^{(2)}$ *and* $\varphi_3\in\mathcal{F}^{(1)}\cap\mathcal{F}^{(3)}$.

Proof: Let us set $H(0)$, $E(0)$ and $\psi_{\mathrm{g}}(0)$ as H, E and ψ_{g}, respectively. It is obvious that $E_0=0$ and $\varphi_0=a\Omega$ with arbitrary $a\in\mathbb{C}$, and by (3.3), $E_1=E_3=0$. Set $a=1$. We denote the strong derivative of $f=f(e)$ with respect to e by f'. We have

$$
H'\psi_{\mathrm{g}}+H\psi_{\mathrm{g}}'=E'\psi_{\mathrm{g}}+E\psi_{\mathrm{g}}' \tag{3.8}
$$

and

$$H''\psi_{\rm g} + 2H'\psi'_{\rm g} + H\psi''_{\rm g} = E''\psi_{\rm g} + 2E'\psi'_{\rm g} + E\psi''_{\rm g}. \tag{3.9}$$

From (3.9) it follows that

$$(\psi_{\rm g}, H''\psi_{\rm g}) + (\psi_{\rm g}, 2H'\psi'_{\rm g}) + (\psi_{\rm g}, H\psi''_{\rm g}) = E''(\psi_{\rm g}, \psi_{\rm g}) + (\psi_{\rm g}, 2E'\psi'_{\rm g}) + (\psi_{\rm g}, E\psi''_{\rm g}). \tag{3.10}$$

Put $e = 0$ in (3.10). Then

$$(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega). \tag{3.11}$$

Since the left-hand side of (3.11) vanishes, we have $E_2 = 0$. From (3.8) with $e = 0$ and the fact $E_0 = E_1 = 0$, it follows that

$$H_1\Omega + H_0\varphi_1 = 0,$$

from which it holds that $H_0\varphi_1 = 0$. Since H_0 has the unique eigenvector Ω (the ground state) with eigenvalue zero, it follows that $\varphi_1 = b\Omega$ with some constant b. By Lemma 3.3, $\varphi_1 \in \mathcal{F}^{\rm odd}$, which implies $b = 0$. Hence $\varphi_1 = 0$ follows. By (3.9) with $e = 0$, we have

$$H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.$$

Since $H_2\Omega \in \mathcal{F}_0$, we see that by Lemma 3.4, $H_2\Omega \in D(H_0^{-1})$. Thus we have $\varphi_2 = -H_0^{-1}H_2\Omega$. From the identity

$$H'''\psi_{\rm g} + 3H''\psi'_{\rm g} + 3H'\psi''_{\rm g} + H\psi'''_{\rm g} = E'''\psi_{\rm g} + 3E''\psi'_{\rm g} + 3E'\psi''_{\rm g} + E\psi'''_{\rm g} \tag{3.12}$$

it follows that at $e = 0$,

$$3H_1\varphi_2 + H_0\varphi_3 = 0.$$

Since $H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0$, Lemma 3.4 ensures that $H_1\varphi_2 \in D(H_0^{-1})$. Hence $\varphi_3 = -3H_0^{-1}H_1\varphi_2 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega$. Then the lemma is proven. □

In the similar manner as Lemma 3.5, we can prove the following proposition.

Proposition 3.6 *(In the case of $n = 2m$)*

$$E_{2m} = \binom{2m}{2}(\Omega, H_2\varphi_{2m-2}),$$

$$\varphi_{2m} = H_0^{-1}\left\{-\sum_{j=1,2}\binom{2m}{j}H_1\varphi_{2m-j} + \sum_{j=2}^{m}\binom{2m}{2j}E_{2j}\varphi_{2m-2j}\right\}.$$

Moreover $\varphi_{2m} \in \mathcal{F}^{(2)} \cap \mathcal{F}^{(4)} \cap \cdots \cap \mathcal{F}^{(2m)}$.

(In the case of $n = 2m+1$*)*

$$E_{2m+1} = 0,$$
$$\varphi_{2m+1} = H_0^{-1} \left\{ - \sum_{j=1,2} \binom{2m+1}{j} H_j \varphi_{2m+1-j} + \sum_{j=2}^{m-1} \binom{2m+1}{2j} E_{2j} \varphi_{2m-2j+1} \right\}.$$

Moreover $\varphi_{2m+1} \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)} \cap \cdots \cap \mathcal{F}^{(2m+1)}$.

3.3 Effective mass up to order e^4

In this subsection we expand m/m_{eff} up to order e^4.

Lemma 3.7 *We have*

$$\begin{aligned}\frac{m}{m_{\text{eff}}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1,2,3} \left(\Omega, A_\mu H_0^{-1} A_\mu \Omega \right) \\ &\quad - e^4 \frac{2}{3} \sum_{\mu=1,2,3} \Big\{ 2 \left(\Psi_3^\mu, H_0^{-1} \Psi_1^\mu \right) + \left(\Psi_2^\mu, H_0^{-1} \Psi_2^\mu \right) - 2 \left(\Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \\ &\qquad - \frac{1}{2} \left(\Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu \right) + \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \Big\} + \mathcal{O}(e^6),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\Psi_1^\mu &= A_\mu \Omega, \\ \Psi_2^\mu &= -\frac{1}{2} P_{\text{f}\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, \\ \Psi_3^\mu &= \frac{1}{2} \left\{ -A_\mu H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{\text{f}\mu} H_0^{-1} (P_\text{f} \cdot A + A \cdot P_\text{f}) H_0^{-1} (A^+ \cdot A^+) \Omega \right\}.\end{aligned}$$

Proof: By Lemma 3.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_\text{f} + eA)_\mu \psi_\text{g}(0), (H(0) - E(0))^{-1} (P_\text{f} + eA)_\mu \psi_\text{g}(0))}{(\psi_\text{g}(0), \psi_\text{g}(0))}. \tag{3.14}$$

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$\begin{aligned}(H(0) - E(0))^{-1} &= \sum_{n=0}^\infty \left(-H_0^{-1} \sum_{l=1}^\infty \frac{e^l}{l!} H_l \right)^n H_0^{-1} \\ &= \sum_{n=0}^\infty (-1)^n \sum_{k=n}^\infty e^k \sum_{\substack{l_1,\ldots,l_n=1 \\ l_1+\cdots+l_n=k}}^k \frac{1}{l_1! \cdots l_n!} H_0^{-1} H_{l_1} H_0^{-1} H_{l_2} \cdots H_0^{-1} H_{l_n} H_0^{-1}.\end{aligned}$$

Here we set

$$\mathcal{H}_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$$

In particular

$$\begin{aligned}(H(0) - E(0))^{-1} &= H_0^{-1} - eH_0^{-1}\mathcal{H}_1H_0^{-1} \\ &+e^2\left(-\frac{1}{2}H_0^{-1}\mathcal{H}_2H_0^{-1} + H_0^{-1}\mathcal{H}_1H_0^{-1}\mathcal{H}_1H_0^{-1}\right) + \mathcal{O}(e^3). \quad (3.15)\end{aligned}$$

Note that by Proposition 3.6,

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$

In particular

$$\frac{1}{(\psi_\mathrm{g}, \psi_\mathrm{g})} = 1-e^4(\frac{1}{2}\varphi_2, \frac{1}{2}\varphi_2)-e^4(\Omega, \frac{1}{24}\varphi_4)+\mathcal{O}(e^6) = 1-e^4\frac{1}{4}(\varphi_2, \varphi_2)+\mathcal{O}(e^6). \quad (3.16)$$

Moreover we have

$$\begin{aligned}(P_\mathrm{f} + eA)_\mu\psi_\mathrm{g}(0) &= eA_\mu\Omega + e^2(\frac{1}{2}P_{\mathrm{f}\mu}\varphi_2) + e^3(\frac{1}{2}A_\mu\varphi_2 + \frac{1}{6}P_{\mathrm{f}\mu}\varphi_3) + \mathcal{O}(e^4) \\ &= e\Psi_1^\mu + e^2\Psi_2^\mu + e^3\Psi_3^\mu + \mathcal{O}(e^4). \quad (3.17)\end{aligned}$$

Substitute (3.15), (3.16) and (3.17) into (3.14). Then the lemma follows. □

3.4 Explicit expressions

For each $k \in \mathbb{R}^3$ let us define the projection $Q(k)$ on $\mathbb{R}^3$ by

$$Q(k) = \sum_{j=1,2} |e_j(k)\rangle\langle e_j(k)|.$$

We also set

$$m = 1,$$

since it can easily be reintroduced at the end of the computation. We set

$$\hat{\varphi}_j = \hat{\varphi}(k_j), \quad \omega_j = \omega(k_j), \quad Q(k_j) = Q_j, \quad j = 1, 2.$$

Let

$$\begin{aligned}
\frac{1}{E_j} &= \frac{1}{|k_j|^2/2+\omega_j}, \quad j=1,2,\\
\frac{1}{E_{12}} &= \frac{1}{|k_1+k_2|^2/2+\omega_1+\omega_2}, \quad k_1,k_2\in\mathbb{R}^3,\\
\frac{1}{F_j} &= \frac{1}{r_j^2/2+r_j}, \quad j=1,2,\\
\frac{1}{F_{12}} &= \frac{1}{(r_1^2+2r_1r_2X+r_2^2)/2+r_1+r_2}, \quad r_1,r_2\geq 0, \quad -1\leq X\leq 1.
\end{aligned}$$

Lemma 3.8 *We have*

$$\frac{m}{m_{\text{eff}}} = 1-\alpha a_1(\Lambda,\kappa)-\alpha^2 a_2(\Lambda,\kappa)+\mathcal{O}(\alpha^3),$$

where

$$a_1(\Lambda,\kappa)=\frac{8}{3\pi}\log\left(\frac{\Lambda+2}{\kappa+2}\right) \tag{3.18}$$

and

$$\begin{aligned}
a_2(\Lambda,\kappa) &= (4\pi)^2\frac{2}{3}\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\times\\
&\times\Bigg\{-\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\frac{1}{E_{12}}(1+s^2)+\left(\frac{1}{E_{12}}\right)^3\frac{|k_1+k_2|^2}{2}(1+s^2)\\
&+\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\left(\frac{1}{E_{12}}\right)^2(k_1\cdot k_2)(-1+s^2)-\frac{1}{E_1}\frac{1}{E_2}(1+s^2)\\
&+\left(\frac{|k_1|^2}{E_1^2}+\frac{|k_2|^2}{E_2^2}\right)\frac{1}{E_{12}}(1-s^2)+\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_{12}}(k_1\cdot k_2)(-1+s^2)\Bigg\},
\end{aligned} \tag{3.19}$$

where $s=(\hat k_1,\hat k_2)$. Changing variables to polar coordinates we also have

$$\begin{aligned}
a_2(\Lambda,\kappa) &= \frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\int_{-1}^{1}\mathrm{d}X\int_\kappa^\Lambda\mathrm{d}r_1\int_\kappa^\Lambda\mathrm{d}r_2\pi r_1r_2\times\\
&\times\Bigg\{-\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\frac{1}{F_{12}}(1+X^2)+\left(\frac{1}{F_{12}}\right)^3\frac{r_1^2+2r_1r_2X+r_2^2}{2}(1+X^2)\\
&+\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2 r_1r_2X(-1+X^2)-\frac{1}{F_1}\frac{1}{F_2}(1+X^2)\\
&+\left(\frac{r_1^2}{F_1^2}+\frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}}(1-X^2)+\frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}}r_1r_2X(-1+X^2)\Bigg\}.
\end{aligned} \tag{3.20}$$

Proof: Note that

$$\begin{aligned} a_1(\Lambda,\kappa) &= \frac{2}{3}(\sqrt{4\pi})^2(A_\mu^+\Omega, H_0^{-1}A_\mu^+\Omega) \\ &= \frac{2}{3}(\sqrt{4\pi})^2 2\int \frac{\hat{\varphi}(k)^2}{2\omega(k)}\frac{1}{|k|^2/2+|k|}\mathrm{d}^3k \\ &= \frac{2}{3}(\sqrt{4\pi})^2\frac{1}{(2\pi)^3}4\pi\int_\kappa^\Lambda \frac{1}{r/2+1}\mathrm{d}r \\ &= \frac{8}{3\pi}\log\left(\frac{\Lambda+2}{\kappa+2}\right). \end{aligned}$$

Thus (3.18) follows. To check $a_2(\Lambda,\kappa)$ we exactly compute the five terms on the right-hand side of (3.13) separately.

(1) We have

$$\begin{aligned} 2\left(\Psi_3^\mu, H_0^{-1}\Psi_1^\mu\right) &= \left(\Omega, -(A^-\cdot A^-)H_0^{-1}A_\mu H_0^{-1}A_\mu^+\Omega\right) \\ &\quad +\frac{1}{2}\left(\Omega, (A^-\cdot A^-)H_0^{-1}(P_\mathrm{f}\cdot A + A\cdot P_\mathrm{f})H_0^{-1}P_{\mathrm{f}\mu}H_0^{-1}A_\mu^+\Omega\right). \end{aligned} \tag{3.21}$$

Since $P_{\mathrm{f}\mu}H_0^{-1}A_\mu\Omega = H_0^{-1}A_\mu P_{\mathrm{f}\mu}\Omega = 0$, the second term of the righ-hand side of (3.21) vanishes, we have

$$\begin{aligned} 2\left(\Psi_3^\mu, H_0^{-1}\Psi_1^\mu\right) &= -\left(\Omega, (A^-\cdot A^-)H_0^{-1}A_\mu^+H_0^{-1}A_\mu^+\Omega\right) \\ &= -\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat{\varphi}_1|^2}{2\omega_1}\frac{|\hat{\varphi}_2|^2}{2\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}2\mathrm{tr}(Q_1Q_2) \\ &= -\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat{\varphi}_1|^2}{2\omega_1}\frac{|\hat{\varphi}_2|^2}{2\omega_2}\frac{1}{E_{12}}(\frac{1}{E_1}+\frac{1}{E_2})\mathrm{tr}(Q_1Q_2). \end{aligned} \tag{3.22}$$

(2) We have

$$\begin{aligned} \left(\Psi_2^\mu, H_0^{-1}\Psi_2^\mu\right) &= \left(\frac{1}{2}\right)^2\left(P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega\right) \\ &= \left(\frac{1}{2}\right)^2\left(\Omega, (A^-\cdot A^-)\left(H_0^{-1}\right)^3(P_\mathrm{f}\cdot P_\mathrm{f})(A^+\cdot A^+)\Omega\right) \\ &= \left(\frac{1}{2}\right)^2\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat{\varphi}_1|^2}{2\omega_1}\frac{|\hat{\varphi}_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^3|k_1+k_2|^2 2\mathrm{tr}(Q_1Q_2). \end{aligned} \tag{3.23}$$

(3) We have

$$
\begin{aligned}
&-2\left(\Psi_2^\mu, H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right)\\
&=\frac{1}{2}\left(P_{\mathrm{f}\mu}H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}(P_\mathrm{f}\cdot A+A\cdot P_\mathrm{f})H_0^{-1}A_\mu^+\Omega\right)\\
&=\sum_{\nu=1,2,3}\left(\Omega,(A^-\cdot A^-)H_0^{-1}P_{\mathrm{f}\mu}H_0^{-1}P_{\mathrm{f}\nu}A_\nu^+H_0^{-1}A_\mu^+\Omega\right)\\
&=\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^2\frac{1}{E_2}2(k_2,Q_1Q_2k_1)\\
&=\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\left(\frac{1}{E_{12}}\right)^2(\frac{1}{E_1}+\frac{1}{E_2})(k_2,Q_1Q_2k_1).
\end{aligned}\tag{3.24}
$$

(4) We have

$$
\begin{aligned}
&-\frac{1}{2}\left(\Psi_1^\mu, H_0^{-1}H_2H_0^{-1}\Psi_1^\mu\right)\\
&=-\frac{1}{2}\left(A_\mu^+\Omega, H_0^{-1}((A^+\cdot A^+)+2(A^+\cdot A^-)+(A^-\cdot A^-))H_0^{-1}A_\mu^+\Omega\right)\\
&=-\left(\Omega, A_\mu^-H_0^{-1}(A^+\cdot A^-)H_0^{-1}A_\mu^+\Omega\right)\\
&=-\iint \mathrm{d}^3k_1\mathrm{d}^3k_2\frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_1}\frac{1}{E_2}\mathrm{tr}(Q_1Q_2).
\end{aligned}\tag{3.25}
$$

(5) We have

$$
\begin{aligned}
&\left(\Psi_1^\mu, H_0^{-1}H_1H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right)\\
&=\left(\frac{1}{2}\right)^2\left(A_\mu^+\Omega, H_0^{-1}(P_\mathrm{f}\cdot A+A\cdot P_\mathrm{f})H_0^{-1}(P_\mathrm{f}\cdot A+A\cdot P_\mathrm{f})H_0^{-1}A_\mu^+\Omega\right)\\
&=\left(A_\mu^+\Omega, H_0^{-1}(P_\mathrm{f}\cdot A)H_0^{-1}(P_\mathrm{f}\cdot A)H_0^{-1}A_\mu^+\Omega\right)\\
&=\sum_{\nu,\kappa=1,2,3}\left(A_\mu^+\Omega, H_0^{-1}P_{\mathrm{f}\nu}A_\nu^+H_0^{-1}P_{\mathrm{f}\kappa}A_\kappa^-H_0^{-1}A_\mu^+\Omega\right)\\
&\qquad+\sum_{\nu,\kappa=1,2,3}\left(A_\mu^+\Omega, H_0^{-1}P_{\mathrm{f}\nu}A_\nu^-H_0^{-1}P_{\mathrm{f}\kappa}A_\kappa^+H_0^{-1}A_\mu^+\Omega\right).
\end{aligned}\tag{3.26}
$$

Since

$$P_{\mathrm{f}\kappa}A_\kappa^-H_0^{-1}A_\mu^+\Omega=0,$$

the first term on the last line in (3.26) vanishes. Then we have

$$
\begin{aligned}
&\left(\Psi_1^\mu, H_0^{-1}H_1H_0^{-1}H_1H_0^{-1}\Psi_1^\mu\right)\\
&=\sum_{\nu,\kappa=1,2,3}\left(\Omega, A_\mu^-H_0^{-1}P_{\mathrm{f}\nu}A_\nu^-H_0^{-1}P_{\mathrm{f}\kappa}A_\kappa^+H_0^{-1}A_\mu^+\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \iint \mathrm{d}^3k_1\mathrm{d}^3k_2 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\left\{\left(\frac{1}{E_2}\right)^2 2(k_2,Q_1k_2) + \frac{1}{E_1}\frac{1}{E_2}(k_2,Q_1Q_2k_1)\right\} \\
&= \iint \mathrm{d}^3k_1\mathrm{d}^3k_2 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\left\{\left(\frac{1}{E_1}\right)^2 (k_1,Q_2k_1) + \left(\frac{1}{E_2}\right)^2 (k_2,Q_1k_2)\right\} \\
&\qquad + \iint \mathrm{d}^3k_1\mathrm{d}^3k_2 \frac{|\hat\varphi_1|^2}{2\omega_1}\frac{|\hat\varphi_2|^2}{2\omega_2}\frac{1}{E_{12}}\frac{1}{E_1}\frac{1}{E_2}(k_2,Q_1Q_2k_1). \qquad (3.27)
\end{aligned}
$$

(3.19) follows from Lemma 3.7, (3.22), (3.23), (3.24), (3.25), (3.27) and the facts

$$
\begin{aligned}
\mathrm{tr}[Q(k_1)Q(k_2)] &= \sum_{j,j'=1,2}(e_j(k_1)e_{j'}(k_2))^2 = 1 + (\hat k_1,\hat k_2)^2, \\
(k_1, Q(k_2)Q(k_1)k_2) &= (k_1,k_2)((\hat k_1,\hat k_2)^2 - 1), \\
(k_1, Q(k_2)k_1) &= |k_1|^2(1-(\hat k_1,\hat k_2)^2).
\end{aligned}
$$

Thus the proof is complete. □

4 Main theorem

By (3.20) we can see that

$$
a_2(\Lambda,\kappa) = \frac{(4\pi)^2}{(2\pi)^6}\frac{2}{3}\sum_{j=1}^{6} b_j(\Lambda), \qquad (4.1)
$$

where

$$
\begin{aligned}
b_1(\Lambda) &= -\int_{-1}^{1}\mathrm{d}X(1+X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1 r_2\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\frac{1}{F_{12}}, \\
b_2(\Lambda) &= \int_{-1}^{1}\mathrm{d}X(1+X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1 r_2\left(\frac{1}{F_{12}}\right)^3\frac{r_1^2+2r_1r_2X+r_2^2}{2}, \\
b_3(\Lambda) &= \int_{-1}^{1}\mathrm{d}X X(-1+X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1^2 r_2^2\left(\frac{1}{F_1}+\frac{1}{F_2}\right)\left(\frac{1}{F_{12}}\right)^2, \\
b_4(\Lambda) &= -\int_{-1}^{1}\mathrm{d}X(1+X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1 r_2\frac{1}{F_1}\frac{1}{F_2}, \\
b_5(\Lambda) &= \int_{-1}^{1}\mathrm{d}X(1-X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1 r_2\left(\frac{r_1^2}{F_1^2}+\frac{r_2^2}{F_2^2}\right)\frac{1}{F_{12}}, \\
b_6(\Lambda) &= \int_{-1}^{1}\mathrm{d}X X(-1+X^2)\int_\kappa^\Lambda \mathrm{d}r_1\int_\kappa^\Lambda \mathrm{d}r_2 \pi r_1^2 r_2^2\frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_{12}}.
\end{aligned}
$$

Our main theorem is stated as follows.

Theorem 4.1 *There exist strictly positive constants c_1 and c_2 such that*

$$c_1 \leq \lim_{\Lambda\to\infty} \frac{a_2(\Lambda,\kappa)}{\sqrt{\Lambda}} \leq c_2.$$

To prove Theorem 4.1 we estimate the lower and upper bounds of $a_2(\Lambda,\kappa)/\sqrt{\Lambda}$ as $\Lambda \to \infty$ in what follows.

Let $\rho_\Lambda(\cdot,\cdot) : [0,\infty) \times [-1,1] \to \mathbb{R}$ be defined by

$$\rho_\Lambda = \rho_\Lambda(r,X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,$$

where

$$\Delta = \Lambda^2(1-X^2) + 2\Lambda(1-X) - 1. \tag{4.2}$$

Lemma 4.2 *There exist constants C_1, C_2, C_3 and C_4 such that for sufficiently large $\Lambda > 0$,*

$$\begin{aligned}
&(1)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)} \leq C_1\frac{1}{\Lambda},\\
&(2)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 \leq C_2\frac{1}{\Lambda^{5/2}},\\
&(3)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)}\frac{1}{r+2} \leq C_3\frac{\log\Lambda}{\Lambda^2},\\
&(4)\ \int_{-1}^{1}\mathrm{d}X\int_0^\Lambda \mathrm{d}r \left(\frac{1}{\rho_\Lambda(r,X)}\right)^2(1-X^2) \leq C_4\frac{1}{\Lambda^3}.
\end{aligned}$$

Proof: See Appendix A. □

4.1 Upper bounds

Lemma 4.3 *There exists a constant $C_{\max}$ such that*

$$\lim_{\Lambda\to\infty}\left|\frac{a(\Lambda,\kappa)}{\sqrt{\Lambda}}\right| < C_{\max}.$$

Proof: Note that for a continuous function f,

$$\frac{d}{d\Lambda}\int_\kappa^\Lambda \mathrm{d}r_1 \int_\kappa^\Lambda \mathrm{d}r_2 f(r_1,r_2) = \int_\kappa^\Lambda f(\Lambda,r)\mathrm{d}r + \int_\kappa^\Lambda f(r,\Lambda)\mathrm{d}r. \tag{4.3}$$

In this proof, C denotes some sufficiently large constant and is not necessarily the same number.

(1) We have

$$\frac{d}{d\Lambda}b_1(\Lambda) = 8\pi\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}\left(\frac{\Lambda}{r+2}+\frac{r}{\Lambda+2}\right)(1+X^2).$$

Since by Lemma 4.2 (3) and (1),

$$\begin{aligned}\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}\frac{\Lambda}{r+2} &\leq C\frac{\log\Lambda}{\Lambda},\\ \int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}\frac{r}{\Lambda+2} &\leq C\frac{1}{\Lambda},\end{aligned}$$

we have

$$\left|\frac{d}{d\Lambda}b_1(\Lambda)\right| \leq C\frac{\log\Lambda}{\Lambda}. \tag{4.4}$$

(2) We see that by Lemma 4.2 (2),

$$\begin{aligned}\frac{d}{d\Lambda}b_2(\Lambda) &= 8\pi\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\left(\frac{1}{\rho_\Lambda(r,X)}\right)^3 r\Lambda(\Lambda^2+r^2+2\Lambda rX)(1+X^2)\\ &\leq 8\pi\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\left(\frac{1}{\rho_\Lambda(r,X)}\right)^2 r\Lambda(1+X^2)\\ &\leq 8\pi\Lambda^2\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\left(\frac{1}{\rho_\Lambda(r,X)}\right)^2(1+X^2)\\ &\leq C\frac{1}{\sqrt{\Lambda}}.\end{aligned}$$

Hence

$$\left|\frac{d}{d\Lambda}b_2(\Lambda)\right| \leq C\frac{1}{\sqrt{\Lambda}}. \tag{4.5}$$

(3) We have

$$\left|\frac{d}{d\Lambda}b_3(\Lambda)\right| = 16\pi\left|\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}rX(X^2-1)\left(\frac{1}{\rho_\Lambda(r,X)}\right)^2\left(\frac{\Lambda^2 r}{r+2}+\frac{r^2\Lambda}{\Lambda+2}\right)\right|.$$

Since by Lemma 4.2 (3) and (4),

$$\left|\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}rX(X^2-1)\left(\frac{1}{\rho_\Lambda(r,X)}\right)^2\frac{\Lambda^2 r}{r+2}\right| \leq C\frac{1}{\Lambda},$$

$$\left|\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}rX(X^2-1)\left(\frac{1}{\rho_\Lambda(r,X)}\right)^2\frac{r^2\Lambda}{\Lambda+2}\right| \leq C\frac{1}{\Lambda},$$

we have

$$\left|\frac{d}{d\Lambda}b_3(\Lambda)\right| \leq C\frac{1}{\Lambda}. \tag{4.6}$$

(4) It is trivial that

$$|b_4(\Lambda)| \leq C[\log\Lambda]^2. \tag{4.7}$$

(5) We have

$$\frac{d}{d\Lambda}b_5(\Lambda) = 16\pi\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}(1-X^2)r\Lambda\left\{\left(\frac{1}{\Lambda+2}\right)^2+\left(\frac{1}{r+2}\right)^2\right\}.$$

Since by Lemma 4.2 (1),

$$\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}(1-X^2)r\Lambda\left(\frac{1}{\Lambda+2}\right)^2 \leq C\frac{1}{\Lambda},$$
$$\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r\frac{1}{\rho_\Lambda(r,X)}(1-X^2)r\Lambda\left(\frac{1}{r+2}\right)^2 \leq C\frac{1}{\Lambda},$$

we see that

$$\left|\frac{d}{d\Lambda}b_5(\Lambda)\right| \leq C\frac{1}{\Lambda}. \tag{4.8}$$

(6) We have by Lemma 4.2 (1)

$$\left|\frac{d}{d\Lambda}b_6(\Lambda)\right| = 16\pi\left|\int_{-1}^{1}\mathrm{d}X\int_{\kappa}^{\Lambda}\mathrm{d}r(X^2-1)X\frac{1}{\rho_\Lambda(r,X)}\frac{r}{r+2}\frac{\Lambda}{\Lambda+2}\right| \leq C\frac{1}{\Lambda}.$$

Then we have

$$\left|\frac{d}{d\Lambda}b_6(\Lambda)\right| \leq C\frac{1}{\Lambda}. \tag{4.9}$$

From (4.4)-(4.9) it follows that

$$|b_j(\Lambda)| \leq C[\log\Lambda]^2, \quad j=1,4,$$

$$|b_2(\Lambda)| \leq C\Lambda^{1/2}, \quad |b_j(\Lambda)| \leq C\log\Lambda, \quad j=3,5,6. \tag{4.10}$$

Then the lemma follows. □

4.2 Lower bounds

Lemma 4.4 *There exists a positive constant $C_{\min} > 0$ such that*

$$C_{\min} \leq \lim_{\Lambda\to\infty}\frac{b_2(\Lambda)}{\sqrt{\Lambda}}.$$

From this lemma and (4.10), i.e.,

$$\lim_{\Lambda\to\infty} \frac{b_j(\Lambda)}{\sqrt{\Lambda}} = 0, \quad j = 1, 3, 4, 5, 6,$$

the following corollary follows.

Corollary 4.5 *It follows that*

$$C_{\min} \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda,\kappa)}{\sqrt{\Lambda}}.$$

Proof of Lemma 4.4: We have

$$\frac{d}{d\Lambda} b_2(\Lambda) = 8\pi \int_{-1}^{1} \mathrm{d}X \int_{\kappa}^{\Lambda} \mathrm{d}r (1+X^2) \left(\frac{1}{\rho_\Lambda(r,X)}\right)^3 (r^2 + 2r\Lambda X + \Lambda^2) r\Lambda, \qquad (4.11)$$

where, recall that

$$\begin{aligned} \rho_\Lambda(r,X) &= (r + \Lambda X + 1)^2 + \Delta, \\ \Delta &= \Lambda^2(1-X^2) + \Lambda(1-X) - 1. \end{aligned}$$

Note that $\Delta > 0$ for $X \le 0$ and a sufficiently large Λ. Since the integrand of (4.11)

$$T_R(r) = \left(\frac{1}{\rho_\Lambda(r,X)}\right)^3 (r^2 + 2r\Lambda X + \Lambda^2) r\Lambda$$

is positive, it is enough to prove that

$$\lim_{\Lambda\to\infty} \sqrt{\Lambda} \frac{d}{d\Lambda} b_2(\Lambda) = \lim_{\Lambda\to\infty} \sqrt{\Lambda} \int_{-1+(1/\Lambda)}^{0} \mathrm{d}X \int_{\kappa}^{\Lambda} \mathrm{d}r T_R(r)(1+X^2) > 0. \qquad (4.12)$$

We simply set $\rho = \rho_\Lambda(r,X)$. Since

$$(r^2 + 2r\Lambda X + \Lambda^2) r\Lambda = \Lambda \left\{ (r-2)\rho + (4 + 4\Lambda X - 2\Lambda) r + 2(\Lambda^2 + 2\Lambda) \right\},$$

we have

$$\begin{aligned} \int_{\kappa}^{\Lambda} \mathrm{d}r T_R(r) &= \Lambda \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{r-2}{\rho^2} + \Lambda(4 + 4\Lambda X - 2\Lambda) \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{r}{\rho^3} + 2\Lambda(\Lambda^2 + 2\Lambda) \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{1}{\rho^3} \\ &= \Lambda \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{r}{\rho^2} + \Lambda^2(4X - 2) \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{r}{\rho^3} + 2\Lambda^3 \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{1}{\rho^3} + t_1(\Lambda), \end{aligned}$$

where

$$t_1(\Lambda) = -2\Lambda \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{1}{\rho^2} + 4\Lambda \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{r}{\rho^3} + 2\Lambda^2 \int_{\kappa}^{\Lambda} \mathrm{d}r \frac{1}{\rho^3}.$$

Moreover since

$$\int_\kappa^\Lambda \mathrm{d}r\frac{r}{\rho^2} = -\frac{1}{2}\left[\frac{1}{\rho}\right]_\kappa^\Lambda - \int_\kappa^\Lambda \mathrm{d}r\frac{\Lambda X+1}{\rho^2},$$
$$\int_\kappa^\Lambda \mathrm{d}r\frac{r}{\rho^3} = -\frac{1}{4}\left[\frac{1}{\rho^2}\right]_\kappa^\Lambda - \int_\kappa^\Lambda \mathrm{d}r\frac{\Lambda X+1}{\rho^3},$$

we have

$$\int_\kappa^\Lambda \mathrm{d}r T_R(r) = -\Lambda^2 X\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2} + \Lambda^3(2-X(4X-2))\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^3} + t_1(\Lambda) + t_2(\Lambda),$$

where

$$\begin{aligned} t_2(\Lambda) &= \Lambda(-\frac{1}{2})\left[\frac{1}{\rho}\right]_\kappa^\Lambda - \Lambda\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2} \\ &\quad +\Lambda^2(4X-2)(-\frac{1}{4})\left[\frac{1}{\rho^2}\right]_\kappa^\Lambda - \Lambda^2(4X-2)\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^3}. \end{aligned}$$

Note that

$$\int\frac{1}{(x^2+a^2)^{n+1}}\mathrm{d}x = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\frac{1}{a^2}\int\frac{1}{(x^2+a^2)^n}\mathrm{d}x, \quad n\geq 1.$$

Then

$$\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2} = \left[\frac{r+\Lambda X+1}{2\Delta}\frac{1}{\rho}\right]_\kappa^\Lambda + \frac{1}{2\Delta^{3/2}}\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda,$$
$$\begin{aligned}\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^3} &= \left[\frac{r+\Lambda X+1}{4\Delta}\frac{1}{\rho^2}\right]_\kappa^\Lambda + \frac{3}{8}\left[\frac{r+\Lambda X+1}{\Delta^2}\frac{1}{\rho}\right]_\kappa^\Lambda \\ &\quad +\frac{3}{8}\frac{1}{\Delta^{5/2}}\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_\kappa^\Lambda \mathrm{d}r T_R(r) &= -\Lambda^2 X\frac{1}{2\Delta^{3/2}}\left\{\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda + \left[\sqrt{\Delta}\frac{r+\Lambda X+1}{\rho}\right]_\kappa^\Lambda\right\} \\ &+\frac{3}{8}\Lambda^3 2(2X+1)(1-X)\frac{1}{\Delta^{5/2}}\left\{\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda + \left[\sqrt{\Delta}\frac{r+\Lambda X+1}{\rho}\right]_\kappa^\Lambda\right\} \\ &\quad +t_1(\Lambda)+t_2(\Lambda)+t_3(\Lambda),\end{aligned}$$

where

$$t_3(\Lambda) = \Lambda^3 2(2X+1)(1-X)\left[\frac{r+\Lambda X+1}{4\Delta}\frac{1}{\rho^2}\right]_\kappa^\Lambda.$$

It is proven in Lemma B.1 of Appendix B that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0}(1+X^2)(t_1(\Lambda)+t_2(\Lambda)+t_3(\Lambda))dX=0. \tag{4.13}$$

From this it is enough to show that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{-1+1/\Lambda}^{0}dX(1+X^2)\frac{1}{\Delta^{3/2}}\left[\sqrt{\Delta}\frac{r+\Lambda X+1}{\rho}\right]_{\kappa}^{\Lambda}\times$$
$$\times\frac{1}{4}\left\{-2X+3(1-X)(2X+1)\frac{\Lambda}{\Delta}\right\}\geq 0 \tag{4.14}$$

and that there exists a positive constant $\xi>0$ such that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0}dX(1+X^2)\left\{-\Lambda^2X\frac{1}{2\Delta^{3/2}}\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_{\kappa}^{\Lambda}\right.$$
$$\left.+\frac{3}{8}\Lambda^3 2(2X+1)(1-X)\frac{1}{\Delta^{5/2}}\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_{\kappa}^{\Lambda}\right\}$$
$$=\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{-1+1/\Lambda}^{0}dX(1+X^2)\frac{1}{\Delta^{3/2}}\left[\arctan\frac{r+\Lambda X+1}{\sqrt{\Delta}}\right]_{\kappa}^{\Lambda}\times$$
$$\times\frac{1}{4}\left\{-2X+3(1-X)(2X+1)\frac{\Lambda}{\Delta}\right\}>\xi \tag{4.15}$$

Changing variable X to $-y$, we shall prove (4.15), i.e.,

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{0}^{1-(1/\Lambda)}dy(1+y^2)\frac{1}{\Delta^{3/2}}\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_{\kappa}^{\Lambda}\frac{1}{4}a_\Lambda(y)dy>\xi,$$

where

$$a_\Lambda(y)=2y+\frac{6}{\Lambda}+b_\Lambda(y),$$
$$b_\Lambda(y)=\frac{3}{\Lambda}\left(1+\frac{2}{\Lambda}\right)\frac{y+\frac{2\Lambda+3}{2\Lambda+4}}{\left(y-\frac{1}{2\Lambda}\right)^2-\frac{(2\Lambda+3)(2\Lambda-1)}{4\Lambda^2}}.$$

The function $b_\Lambda(\cdot)$ satisfies the following properties:

(1) $b''_\Lambda(y)<0$ for $0\leq y\leq 1-1/\Lambda$, i.e., $b_\Lambda(y)$ is concave for $0\leq y\leq 1-1/\Lambda$,

(2) $\lim_{\Lambda\to\infty}b_\Lambda(1-1/\Lambda)=-3/2$,

(3) $\lim_{\Lambda\to\infty}b_\Lambda(y)=0$ for $y\neq 1$ and $\lim_{\Lambda\to\infty}b_\Lambda(1)=-3$.

By (1)–(3) we have

$$\inf_{0\le y\le 1-1/\Lambda} b_\Lambda(y) = \min\{b_\Lambda(0), b_\Lambda(1-1/\Lambda)\}$$

and then for sufficiently large Λ,

$$\inf_{0\le y\le 1-1/\Lambda} b_\Lambda(y) = b_\Lambda(1-1/\Lambda) > -\frac{7}{4}.$$

Hence

$$\inf_{15/16\le y\le 1-1/\Lambda} a_\Lambda(y) \ge \frac{15}{8} - \frac{7}{4} = \frac{1}{8} > 0.$$

Moreover

$$\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda = \arctan\frac{(1-y)\Lambda+1}{\sqrt{\Delta}} - \arctan\frac{\kappa-\Lambda y+1}{\sqrt{\Delta}} > 0,$$

$$\lim_{\Lambda\to\infty}\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda = \arctan\frac{1-y}{\sqrt{1-y^2}} + \arctan\frac{y}{\sqrt{1-y^2}} > 0$$

for $0\le y\le 1$. Then

$$\delta = \inf_{\Lambda>1}\inf_{0\le y\le 1}\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda > 0$$

Then we have

$$\begin{aligned}
&\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_0^{1-(1/\Lambda)} dy(1+y^2)\frac{1}{\Delta^{3/2}}\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda \frac{1}{4}a_\Lambda(y)\\
&\ge \lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{15/16}^{1-(1/\Lambda)} \mathrm{d}y(1+y^2)\frac{1}{\Delta^{3/2}}\left[\arctan\frac{r-\Lambda y+1}{\sqrt{\Delta}}\right]_\kappa^\Lambda \frac{1}{4}a_\Lambda(y)\\
&\ge \frac{1}{8}\times\frac{1}{4}\times\delta\times\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{15/16}^{1-(1/\Lambda)}\mathrm{d}y\frac{1}{\Delta^{3/2}}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_{15/16}^{1-(1/\Lambda)}\mathrm{d}y\frac{1}{\Delta^{3/2}}\\
&= \lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}\int_{15/16}^{1-(1/\Lambda)}\mathrm{d}y\frac{1}{\left\{(1-y^2)+\frac{1}{\Lambda}(1+y)-\frac{1}{\Lambda^2}\right\}^{3/2}}\\
&\ge \lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}\int_{15/16}^{1-(1/\Lambda)}\mathrm{d}y\frac{1}{\left\{(1-y)+\frac{1}{\Lambda}\right\}^{3/2}}\frac{1}{(1+y)^{3/2}}\\
&\ge \lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}\int_{15/16}^{1-(1/\Lambda)}\mathrm{d}y\frac{1}{\left\{(1-y)+\frac{1}{\Lambda}\right\}^{3/2}}\frac{1}{2^{3/2}}\\
&= \lim_{\Lambda\to\infty}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\Lambda}}\left(\frac{1}{\sqrt{2/\Lambda}} - \frac{1}{\sqrt{1/16+1/\Lambda}}\right) = \frac{1}{2}.
\end{aligned}$$

Then we proved that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\frac{d}{d\Lambda}b_2(\Lambda) > 4\pi\times\frac{1}{8}\times\frac{1}{4}\times\frac{1}{2}\times\delta = \frac{\pi\delta}{16} > 0.$$

Then (4.15) follows. We shall show (4.14). Since the left-hand side of (4.14) is

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\Lambda^2\int_0^{1-(1/\Lambda)}dy(1+y^2)\frac{1}{\Delta^{3/2}}\left[\sqrt{\Delta}\frac{r-\Lambda y+1}{\rho}\right]_\kappa^\Lambda\frac{1}{4}a_\Lambda(y)dy, \tag{4.16}$$

it is enough to show that $[\cdots]_\kappa^\Lambda$ in (4.16) is nonnegative. We can directly see that

$$\left[\sqrt{\Delta}\frac{r-\Lambda y+1}{\rho}\right]_\kappa^\Lambda = \frac{\sqrt{\Delta}K}{\{(\Lambda+\Lambda X+1)^2+\Delta\}\{(\kappa+\Lambda X+1)^2+\Delta\}},$$

where, for $0\le y\le 1$,

$$K = (-2y^2+y+1)\Lambda^3+(1+4y)\Lambda^2-2\Lambda$$

$$+\kappa((y^2-2)\Lambda^2+(-2y-2)\Lambda+1)+\kappa^2((-y+1)\Lambda+1).$$

Since $K>0$ for a sufficiently large Λ, (4.14) follows. □

Proof of Theorem 4.1: The theorem follows from Lemma 4.3 and Corollary 4.5. □

Remark 4.6 *(1) $a_2(\Lambda,\kappa)/\sqrt{\Lambda}$ converges to a nonnegative constant as $\Lambda\to\infty$.*
(2) By (4.1), we can define $a_2(\Lambda,0)$ since $b_j(\Lambda)$ with $\kappa=0$ are finite. Moreover $a_2(\Lambda,0)$ also satisfies Theorem 4.1.

Appendix

A Proof of Lemma 4.2

Proof of Lemma 4.2
By the definition of Δ it follows that for sufficiently large Λ,

$$\frac{1}{\Delta} \leq \frac{1}{\Lambda}, \quad \text{for } X \leq 0.$$

Let

$$\delta = \delta(k) = \frac{1}{\Lambda^k}, \quad 0 < k \leq 2.$$

Then for sufficiently large Λ,

$$\Delta \geq \Lambda^2(1 - X^2) - 1 > 0, \quad \text{for } -1 + \delta(k) < X \leq 0, \quad 0 < k \leq 2.$$

In particular we obtain

$$\frac{1}{\Delta} \leq \frac{a}{\Lambda^2} \frac{1}{1 - X^2}, \quad \text{for } -1 + \delta(k) \leq X \leq 0, \quad 0 < k \leq 2,$$

with some constant a independent of Λ. In this proof C denotes some sufficiently large constant and it is not necessarily the same number. We divide $\int_{-1}^{1} \cdots \mathrm{d}X$ as

$$\int_{-1}^{1} \cdots \mathrm{d}X = \int_{0}^{1} + \int_{-1+\delta}^{0} + \int_{-1}^{-1+\delta}.$$

(1) It is trivial that $\left| \int_{0}^{1} \cdots \mathrm{d}X \right| \leq \frac{C}{\Lambda}$. Note that

$$\begin{aligned} \left| \int_{0}^{\Lambda} \mathrm{d}r \frac{1}{\rho_\Lambda(r, X)} \right| &= \frac{1}{\sqrt{\Delta}} \left| \arctan \frac{\Lambda + \Lambda X + 1}{\sqrt{\Delta}} - \arctan \frac{\Lambda X + 1}{\sqrt{\Delta}} \right| \\ &\leq \pi \frac{1}{\sqrt{\Delta}}. \end{aligned}$$

Let $\delta = \delta(1/2) = 1/\sqrt{\Lambda}$. Hence we have

$$\begin{aligned} \left| \int_{-1+\delta}^{0} \cdots \mathrm{d}X \right| &\leq \frac{C}{\Lambda} \arcsin(1 - \delta), \\ \left| \int_{-1}^{-1+\delta} \cdots \mathrm{d}X \right| &\leq \frac{C}{\sqrt{\Lambda}} \delta. \end{aligned}$$

Thus (1) follows.

(2) It is trivial that $\left|\int_0^1 \cdots \mathrm{d}X\right| \leq \dfrac{C}{\Lambda^3}$. Note that

$$
\begin{aligned}
&\left|\int_0^\Lambda \mathrm{d}r \frac{1}{\rho_\Lambda(r,X)^2}\right| \\
&= \frac{1}{2\Delta}\left|\int_0^\Lambda \frac{1}{\rho_\Lambda(r,X)}\mathrm{d}r + \left(\frac{\Lambda+\Lambda X+1}{(\Lambda+\Lambda X+1)^2+\Delta} - \frac{\Lambda X+1}{(\Lambda X+1)^2+\Delta}\right)\right| \\
&\leq \begin{cases} \frac{C}{\Lambda^2}\left(\frac{1}{\Lambda(1-X^2)^{3/2}}+\frac{1}{\Lambda}\right), & -1+\delta \leq X \leq 0, \\ \frac{C}{\Lambda}\left(\frac{1}{\sqrt{\Lambda}}+\frac{1}{\Lambda}\right), & -1 \leq X \leq -1+\delta. \end{cases}
\end{aligned}
$$

Let $\delta = \delta(1) = 1/\Lambda$. Hence we have

$$
\begin{aligned}
\left|\int_{-1+\delta}^0 \cdots \mathrm{d}X\right| &\leq \frac{C}{\Lambda^3}\int_{-1+\delta}^0 \mathrm{d}X \left(\frac{1}{(1-X^2)^{3/2}}+1\right) \leq \frac{C}{\Lambda^3}\left(\frac{1}{\sqrt{\delta}}+1\right) \\
\left|\int_{-1}^{-1+\delta} \cdots \mathrm{d}X\right| &\leq \frac{C}{\Lambda}\left(\frac{1}{\sqrt{\Lambda}}+\frac{1}{\Lambda}\right)\delta.
\end{aligned}
$$

Then (2) follows.

(3) We see that

$$
\frac{1}{\rho_\Lambda(r,X)}\frac{1}{r+2} = \frac{l_1}{r+2} + \frac{l_2}{\rho_\Lambda(r,X)},
$$

where

$$
l_1 = \frac{1}{\Lambda^2}\frac{1}{(4X-1)/\Lambda-1}, \quad l_2 = \frac{1}{\Lambda^2}\frac{r+2\Lambda X}{(4X-1)/\Lambda-1}.
$$

We have

$$
\begin{aligned}
&\left|\int_{-1}^1 \mathrm{d}X \int_0^\Lambda \mathrm{d}r \frac{l_1}{r+2}\right| \leq \frac{\log(\Lambda+2)}{\Lambda^2}\int_{-1}^1 \mathrm{d}X \frac{1}{(4X-1)/\Lambda-1} \leq C\frac{\log\Lambda}{\Lambda^2}, \\
&\left|\int_{-1}^1 \mathrm{d}X \int_0^\Lambda \mathrm{d}r \frac{l_2}{\rho_\Lambda(r,X)}\right| \leq \frac{\Lambda}{\Lambda^2}\int_{-1}^1 \mathrm{d}X \int_0^\Lambda \frac{1}{\rho_\Lambda(r,X)}\frac{1+2X}{(4X-1)/\Lambda-1} \leq \frac{C}{\Lambda^2}.
\end{aligned}
$$

Hence (3) follows.

(4) It is trivial that $\left|\int_0^1 \cdots \mathrm{d}X\right| \leq \dfrac{C}{\Lambda^3}$. Let $\delta = \delta(3/2) = 1/\Lambda^{3/2}$. From the proof of (2) it follows that

$$
\begin{aligned}
\left|\int_{-1+\delta}^0 \cdots \mathrm{d}X\right| &\leq \frac{C}{\Lambda^3}\int_{-1+\delta}^0 d\mathrm{X}\left\{\frac{(1-X^2)}{(1-X^2)^{3/2}}+(1-X^2)\right\} \\
&\leq \frac{C}{\Lambda^3}\left(\arcsin(1-\delta)+1\right), \\
\left|\int_{-1}^{-1+\delta} \cdots \mathrm{d}X\right| &\leq \frac{C}{\Lambda}\left(\frac{1}{\sqrt{\Lambda}}+\frac{1}{\Lambda}\right)\delta.
\end{aligned}
$$

Hence (4) follows. $\square$

B Proof of (4.13)

Lemma B.1 *We have*

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0}(1+X^2)(t_1(\Lambda)+t_2(\Lambda)+t_3(\Lambda))dX=0, \tag{2.1}$$

where

$$\begin{aligned}
t_1(\Lambda)&=-2\Lambda\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2}+4\Lambda\int_\kappa^\Lambda \mathrm{d}r\frac{r}{\rho^3}+2\Lambda^2\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^3},\\
t_2(\Lambda)&=\Lambda(-\frac{1}{2})\left[\frac{1}{\rho}\right]_\kappa^\Lambda-\Lambda\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2},\\
t_3(\Lambda)&=\Lambda^3 2(2X+1)(1-X)\left[\frac{r+\Lambda X+1}{4\Delta}\frac{1}{\rho^2}\right]_\kappa^\Lambda.
\end{aligned}$$

Proof: In this proof C also denotes some sufficiently large constant, which is not necessarily the same number. We have

$$\begin{aligned}
\sqrt{\Lambda}\Lambda\left|\int_{-1+(1/\Lambda)}^{0}\mathrm{d}X\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^2}\right|&\le C\sqrt{\Lambda}\Lambda\frac{1}{\Lambda^{5/2}}=C\frac{1}{\Lambda},\\
\sqrt{\Lambda}\Lambda\left|\int_{-1+(1/\Lambda)}^{0}\mathrm{d}X\int_\kappa^\Lambda \mathrm{d}r\frac{r}{\rho^3}\right|&\le C\sqrt{\Lambda}\Lambda^2\frac{1}{\Lambda^{7/2}}=C\frac{1}{\Lambda},\\
\sqrt{\Lambda}\Lambda^2\left|\int_{-1+(1/\Lambda)}^{0}\mathrm{d}X\int_\kappa^\Lambda \mathrm{d}r\frac{1}{\rho^3}\right|&\le C\sqrt{\Lambda}\Lambda^2\frac{1}{\Lambda^{7/2}}=C\frac{1}{\Lambda}.
\end{aligned}$$

Then

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0}\mathrm{d}X(1+X^2)t_1(\Lambda)=0$$

follows. Next we shall show that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0}\mathrm{d}X(1+X^2)t_2(\Lambda)=0. \tag{2.2}$$

Note that

$$\left|\left[\frac{1}{\rho}\right]_\kappa^\Lambda\right|=\left|\frac{1}{(\Lambda+\Lambda X+1)^2+\Delta}-\frac{1}{(\kappa+\Lambda X+1)^2+\Delta}\right|\le\frac{2}{\Delta}\le\frac{2}{\Lambda^2}\frac{1}{1-X^2}.$$

Then

$$\left|\int_{-1+1/\Lambda}^{0} \mathrm{d}X \left[\frac{1}{\rho}\right]_{\kappa}^{\Lambda}\right| \le C\frac{\log\Lambda}{\Lambda^2}.$$

Similarly we can see that

$$\left|\int_{-1+1/\Lambda}^{0} \mathrm{d}X \left[\frac{1}{\rho^2}\right]_{\kappa}^{\Lambda}\right| \le C\frac{1}{\Lambda^3},$$

which implies that

$$\sqrt{\Lambda}\Lambda\left|\int_{-1+1/\Lambda}^{0} \mathrm{d}X \left[\frac{1}{\rho}\right]_{\kappa}^{\Lambda}\right| \le C\frac{\log\Lambda}{\sqrt{\Lambda}},$$

$$\sqrt{\Lambda}\Lambda^2\left|\int_{-1+1/\Lambda}^{0} \mathrm{d}X \left[\frac{1}{\rho^2}\right]_{\kappa}^{\Lambda}\right| \le C\frac{1}{\sqrt{\Lambda}}.$$

Hence (2.2) follows. Finally we shall show that

$$\lim_{\Lambda\to\infty}\sqrt{\Lambda}\int_{-1+1/\Lambda}^{0} \mathrm{d}X(1+X^2)t_3(\Lambda) = 0. \tag{2.3}$$

We divide $\int_{-1+1/\Lambda}^{0} \mathrm{d}X$ as

$$\int_{-1+1/\Lambda}^{0} \mathrm{d}X = \int_{-1+1/\Lambda}^{-1/2} \mathrm{d}X + \int_{-1/2}^{0} \mathrm{d}X.$$

Since

$$\frac{1}{\Delta} \le \frac{1}{\Lambda^2}\frac{1}{1-X^2} \le \frac{1}{\Lambda^2}\frac{4}{3}, \quad \text{for } -\frac{1}{2} \le X \le 0,$$

we see that

$$\sqrt{\Lambda}\Lambda^3\left|\int_{-1/2}^{0} \mathrm{d}X \left[\frac{r+\Lambda X+1}{\Delta}\frac{1}{\rho^2}\right]_{\kappa}^{\Lambda}\right| \le C\sqrt{\Lambda}\Lambda^3\frac{\Lambda}{\Lambda^2}\frac{1}{\Lambda^3} = C\frac{1}{\sqrt{\Lambda}}. \tag{2.4}$$

On the other hand

$$\begin{aligned}&\left[\frac{r+\Lambda X+1}{\Delta}\frac{1}{\rho^2}\right]_{\kappa}^{\Lambda} \\ &= \frac{\Lambda+\Lambda X+1}{\Delta}\frac{1}{\{(\Lambda+\Lambda X+1)^2+\Delta\}^2} - \frac{\kappa+\Lambda X+1}{\Delta}\frac{1}{\{(\kappa+\Lambda X+1)^2+\Delta\}^2}.\end{aligned}$$

Since

$$\left|\frac{\Lambda+\Lambda X+1}{\Delta}\right| \le \frac{C}{\Lambda}, \quad \left|\frac{\kappa+\Lambda X+1}{(\kappa+\Lambda X+1)^2+\Delta}\right| \le \frac{C}{\Lambda},$$

we have

$$\left|\int_{-1+1/\Lambda}^{-1/2} \mathrm{d}X \left[\frac{r+\Lambda X+1}{\Delta}\frac{1}{\rho^2}\right]_\kappa^\Lambda\right| \leq C\frac{1}{\Lambda}\int_{-1+1/\Lambda}^{-1/2}\frac{1}{\Delta^2} \leq C\frac{1}{\Lambda^4}.$$

Then we obtain that

$$\sqrt{\Lambda}\Lambda^3\left|\int_{-1+1/\Lambda}^{-1/2} \mathrm{d}X \left[\frac{r+\Lambda X+1}{\Delta}\frac{1}{\rho^2}\right]_\kappa^\Lambda\right| \leq C\frac{1}{\sqrt{\Lambda}}. \tag{2.5}$$

Thus (2.3) follows from (2.4) and (2.5). □

Acknowledgment F. H. thanks a kind hospitality of TU München in 2003 and Grant-in-Aid 13740106 for Encouragement of Young Scientists from the Ministry of Education, Science, Sports and Culture for financial support.

References

[1] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp-arc 01-301, preprint, 2001.

[2] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159–198.

[3] C. Hainzl, Increase of the binding energy of an electron by coupling to a photon field, math-ph/0204052, preprint, 2002.

[4] C. Hainzl and R. Seiringer, Mass Renormalization and Energy Level Shift in Non-Relativistic QED, math-ph/0205044, preprint, 2002.

[5] F. Hiroshima, Functional integral representation of nonrelativistic QED with a fixed total momentum, in preparation.

[6] F. Hiroshima and H. Spohn, Enhanced binding through coupling to a quantum field, *Ann. Henri Poincaré* **2** (2001), 1159–1187.

[7] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[8] E. Lieb and M. Loss, Self-energy of electrons in non-perturbative QED, preprint, 1999.

[9] E. Lieb and M. Loss, A bound on binding energies and mass renormalization in models of quantum electrodynamics, *J. Stat. Phys.* **108**, 1057–1069 (2002).

[10] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1975.

[12] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.

[13] H. Spohn, Dynamics of charged particles and their radiation field, preprint, mp-arc 99-316 1999.

[14] H. Spohn, Effective mass of the polaron: A functional integral approach, *Ann. Phys.* **175** (1987), 278–318.