On Effective Hamiltonians for Adiabatic Perturbations of Magnetic Schrödinger Operators

Mouez Dimassi, Jean-Claude Guillot and James Ralston

Abstract. *We construct almost invariant subspaces and the corresponding effective Hamiltonian for magnetic Bloch bands. We also discuss the question of the dynamics related to the effective Hamiltonian. We assume that the magnetic and electric potentials are slowly varying perturbations of the potential of a constant magnetic field and a periodic lattice potential, respectively.*

1. Introduction

In [5] we constructed wave packets for adiabatic perturbations of Schrödinger operators in periodic media. The recent work of Panati-Spohn-Teufel, [13], led us to consider the relation of those constructions to effective Hamiltonians. In §3 of this article we give a simple derivation of effective Hamiltonians for these problems.

The main simplification in our method is the omission of the Floquet-Bloch transformation. This transformation has many nice properties. In particular, it is unitary, and this makes in useful in studying spectral properties of operators. In the work of Helffer-Sjöstrand [9] and Gérard-Martinez-Sjöstrand [7] this transformation was used quite effectively in the computation of spectra, both of perturbed and effective Hamiltonians. However, if one is simply interested in effective Hamiltonians, the Floquet-Bloch transformation requires that one transform the Hamiltonian by a Fourier integral unitary operator only to transform it back at the end of the calculation. In this article we only need to assume that eigenspaces of the unperturbed Hamiltonian depend smoothly on quasi-momentum. The behavior of these eigenspaces on fundamental domains for the dual lattice does not play a role, even though they are sometimes nontrivial bundles over those (toroidal) domains.

There is also the interesting question of how one interprets the lower order terms in the effective Hamiltonian. If one considers the propagation of observables in the Heisenberg picture, it is natural to think of these terms as lower order corrections to the dynamics. This point of view is adopted in [13], and it is implicit in [3] and [4]. However, the highest order contributions of these terms to the wave packets are in a precession of the phase. Thus in [5] we did not include them in the dynamics, and did not see how to reconcile our results with those of [3] and [4]. It now appears that the two points of view complement each other instead of conflicting.

2. Preliminaries

The Hamiltonian for an electron in a crystal lattice Γ in $\mathbb{R}^3$ in the presence of a constant magnetic field $\omega = (\omega_1, \omega_2, \omega_3)$ is given by

$$H_0 = \frac{1}{2m}\left(-ih\frac{\partial}{\partial x} + e\frac{\omega \times x}{2}\right)^2 + V(x), \tag{1}$$

where V is a smooth, real-valued potential, periodic with respect to Γ. Here m and e are the mass and charge of the electron. To simplify notation we will use units in which $h = 2m = e = 1$.

We will assume that Γ is generated by the basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$,

$$\Gamma = e_1\mathbb{Z} + e_2\mathbb{Z} + e_3\mathbb{Z}, \tag{2}$$

and let E be the fundamental domain $\{\sum_{j=1}^{3} t_j e_j, t_j \in [0,1)\}$. We will use the dual lattice $\Gamma^* = e_1^*\mathbb{Z} + e_2^*\mathbb{Z} + e_3^*\mathbb{Z}$, where $e_j^* \cdot e_k = 2\pi\delta_{jk}$, with the fundamental domain $E^* = \{\sum_{j=1}^{3} t_j e_j^*, t_j \in [0,1)\}$.

To realize H_0 as a self-adjoint operator in $L^2(\mathbb{R}^3)$ we define it first on the Schwartz functions $\mathcal{S}(\mathbb{R}^3)$, and then take the Friedrichs extension. The resulting operator commutes with the magnetic translations introduced by Zak [18],

$$T_\gamma f(x) = e^{i\langle \omega \times x, \gamma\rangle/2} f(x-\gamma) \tag{3}$$

for $\gamma \in \Gamma$. We assume that

$$\langle \omega, \Gamma \times \Gamma \rangle \subset 4\pi\mathbb{Z}.$$

With this assumption $G = \{T_\gamma, \gamma \in \Gamma\}$ is an abelian group, and we can reduce H_0 by the eigenspaces of G, i.e. setting

$$\mathcal{D}_k = \{u \in H^2_{loc}(\mathbb{R}^3), T_\gamma u = e^{-ik\cdot\gamma}u, \gamma \in \Gamma\}, \tag{4}$$

considered as a subspace of $L^2(E)$, H_0 restricted to $\mathcal{D}_k$ is self-adjoint with compact resolvant. We denote its spectrum by

$$E_1(k) \le E_2(k) \le \dots$$

Then by standard results the spectrum of H_0 as an operator in $L^2(\mathbb{R}^3)$ is equal to

$$\cup_{k\in E^*} \cup_{m=1}^{\infty} E_m(k).$$

Note that, since $\mathcal{D}_{k+\gamma^*} = \mathcal{D}_k$ for $\gamma^* \in \Gamma^*$, $E_m(k+\gamma^*) = E_m(k)$.

Standard perturbation theory shows that the function $E_m(k)$ is continuous for $k \in \mathbb{R}^3$ and real analytic in a neighborhood of any k such that

$$E_{m-1}(k) < E_m(k) < E_{m+1}(k) \tag{5}$$

The closed interval $\Lambda_m = \cup_{k\in E^*} E_m(k)$ is known as the "m-th magnetic Bloch band" in the spectrum of H_0.

In what follows it will be convenient to replace H_0 acting on $\mathcal{D}_k$ by

$$H_0(k) = e^{-ik\cdot x} H_0 e^{ik\cdot x} = \left(-i\frac{\partial}{\partial x} + \frac{\omega \times x}{2} + k\right)^2 + V(x)$$

with the domain

$$\mathcal{D} = \{u \in H^2_{loc}(\mathbb{R}^3), T_\gamma u = u, \gamma \in \Gamma\}.$$

for all k. As with $\mathcal{D}_k$, we consider $\mathcal{D}$ as a subspace of $L^2(E)$.

Assumption A. *For a given m we will assume that E_m satisfies (5) for all k.*

Under this assumption we can choose the eigenfunction $\Psi(x,k)$ associated to $E_m(k)$ to be a real-analytic function of k with values in $\mathcal{D}$, such that

$$H_0(k)\Psi(k) = E_m(k)\Psi(k) \text{ for all k } \int_E |\Psi(x,k)|^2 dx = 1$$

and

$$\Psi(x, k + \gamma^*) = e^{i(\gamma^* \cdot x + \theta(k,\gamma^*))}\Psi(x, k), \gamma^* \in \Gamma^*.$$

The function $\theta(k, \gamma^*)$ is real-valued, and it determines the structure of the family of eigenspaces as a complex line bundle over the torus, $\mathbb{R}^3/\Gamma^*$.

Remark 1. *The general method of constructing effective Hamiltonians which we give here will apply under the weaker hypothesis: for a given m there exist p and q such that*

$$E_{m-p-1}(k) < E_{m-p}(k) \text{ and } E_{m+q}(k) < E_{m+q+1}(k) \text{ for all } k.$$

However, in this case the effective Hamiltonian will be a matrix operator acting on functions with values in $\mathbb{C}^{p+q+1}$, as in [6],[7], [9] and [13].

3. Main Result

The adiabatically perturbed Hamiltonian is

$$H_\epsilon = \left(-i\frac{\partial}{\partial x} + \frac{\omega \times x}{2} + A(\epsilon x)\right)^2 + V(x) + W(\epsilon x),$$

where W and $A = (A_1, A_2, A_3)$ are smooth, and bounded together with all of their derivatives. As before, we define H_ϵ first on $\mathcal{S}(\mathbb{R}^3)$, and then take the Friedrichs extension to get a self-adjoint operator in $L^2(\mathbb{R}^3)$.

The essential step in applying multi-scale techniques is simply to consider $y = \epsilon x$ as a new independent variable in H_ϵ. Let

$$\tilde{H}_\epsilon = \left(-i\frac{\partial}{\partial x} - i\epsilon\frac{\partial}{\partial y} + \frac{\omega \times x}{2} + A(y)\right)^2 + V(x) + W(y).$$

Then, for $u(x, y)$ we can define $w(x) = u(x, \epsilon x)$ and conclude that

$$[\tilde{H}_\epsilon u](x, \epsilon x) = [H_\epsilon w](x). \tag{6}$$

The identity (6) enables us to solve the Schrödinger equation for H_ϵ uniformly in ϵ by solving the Schrödinger equation for $\tilde{H}_\epsilon$ uniformly in (y, ϵ). The latter might sound more difficult, but it turns out not to be.

Let $\mathcal{B}$ denote the subspace of $C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$ consisting of functions of the form

$$P(x, y, k, \epsilon) = P_0(x, y, k) + \epsilon P_1(x, y, k) + \cdots + \epsilon^N P_N(x, y, k)$$

such that $P(x + \gamma, y, k, \epsilon) = e^{i\langle\omega \times x, \gamma\rangle/2} P(x, y, k, \epsilon)$ and

$$\sup_{y,k} \|\partial_y^\alpha \partial_k^\beta P_j(\cdot, y, k)\|_{L^2(E)} < \infty, \text{ for all } \alpha, \beta \in \mathbb{N}^3.$$

To $P \in \mathcal{B}$ we associate the ϵ-pseudo-differential operator

$$P(x, y, \epsilon D_y, \epsilon) f(x, y, \epsilon) = (2\pi\epsilon)^{-3} \int e^{ik\cdot(y-z)/\epsilon} P(x, y, k, \epsilon) f(z) dz dk, f \in \mathcal{S}(\mathbb{R}^3).$$

Note that here we are using the standard quantization – as opposed to the Weyl quantization. Our main result is the following:

Theorem. *For every $N \in \mathbb{N}$ there exist $P_N = F_0 + \epsilon F_1 + \cdots + \epsilon^N F_N \in \mathcal{B}$ and $H_{eff}^N = h_0 + \epsilon h_1 + \cdots + \epsilon^N h_N \in \mathcal{B}$ (independent on x) such that*

$$\tilde{H}_\epsilon(P_N(x,y,\epsilon D_y,\epsilon)u) - P_N(x,y,\epsilon D_y,\epsilon)H_{eff}^N(y,\epsilon D_y)u = \mathcal{O}(\epsilon^{N+1}) \tag{7}$$

for $u \in \mathcal{S}(\mathbb{R}^3)$. Moreover, considered as an operator from $L^2(\mathbb{R}^3)$ into $L^2(E \times \mathbb{R}^3)$, P_N is approximately isometric, i.e. $P_N^* P_N = I + \mathcal{O}(\epsilon^{N+1})$.

We interpret H_{eff}^N as the effective Hamiltonian up to order ϵ^N. The leading term in its symbol is $h_0(y,k) = E_m(k + A(y)) + W(y)$. This is the well-known "Peierls substitution", [14]. The symbol of h_1 is also quite interesting, and we discuss it in §4.

Proof. As in §2 it will be convenient to work with

$$\tilde{H}_\epsilon(k) = e^{-ik\cdot x}\tilde{H}_\epsilon e^{ik\cdot x} = \left(-i\frac{\partial}{\partial x} - i\epsilon\frac{\partial}{\partial y} + \frac{\omega \times x}{2} + A(y) + k\right)^2 + V(x) + W(y),$$

acting on functions in $\mathcal{B}$, in place of $\tilde{H}_\epsilon$. Note that

$$\tilde{H}_\epsilon(P_N(x,y,\epsilon D_y,\epsilon)u) = (2\pi\epsilon)^{-3}\int e^{ik\cdot(y-z)/\epsilon}\tilde{H}_\epsilon(k)P_N(x,y,k,\epsilon)u(z)dzdk.$$

The Hamiltonian $\tilde{H}_\epsilon(k)$ can be written as $\tilde{H}_\epsilon(k) = \tilde{H}_0(k) + \epsilon\tilde{H}_1(k) + \epsilon^2\tilde{H}_2(k)$, where

$$\tilde{H}_0(k) = (-i\partial_x + \frac{\omega \times x}{2} + k + A(y))^2 + V(x) + W(y) = H_0(k + A(y)) + W(y)$$

$$\tilde{H}_1(k) = -2i(-i\partial_x + \frac{\omega \times x}{2} + k + A(y))\cdot\partial_y - i\partial_y \cdot A(y), \tag{8}$$

and $\tilde{H}_2(k) = -\Delta_y$.

We will simply construct the symbols of the pairs (h_0, F_0), $(h_1, F_1), \ldots$, successively so that (7) holds to order $\mathcal{O}(\epsilon^{N+1})$ in the ϵ-pseudo-idfferential calculus. To cancel the order zero terms in (7) we set $h_0(y,k) = E_m(k + A(y)) + W(y)$ and $F_0(x,y,k) = \Psi(x, k + A(y))$. Then, since the symbol of $F_0(x,y,\epsilon D_y)h_0(y,\epsilon D_y)$ is

$$F_0(x,y,k)h_0(y,k) + \frac{\epsilon}{i}\frac{\partial F_0}{\partial k}(x,y,k)\cdot\frac{\partial h_0}{\partial y}(y,k) + \mathcal{O}(\epsilon^2),$$

we must have

$$(\tilde{H}_0(k) - h_0)F_1 - \frac{1}{i}\frac{\partial F_0}{\partial k}\cdot\frac{\partial h_0}{\partial y} + (\tilde{H}_1(k) - h_1)F_0 = 0. \tag{9}$$

By the Fredholm Alternative in $L^2(E)$, we can solve (9) for F_1 if and only if

$$\langle F_0(\cdot,y,k), -\frac{1}{i}\frac{\partial F_0}{\partial k}(\cdot,y,k)\cdot\frac{\partial h_0}{\partial y} + (\tilde{H}_1(k) - h_1(y,k))F_0(\cdot,y,k)\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(E)$. Hence, we choose

$$h_1(y,k) = \langle F_0(\cdot,y,k), \frac{-1}{i}\frac{\partial F_0}{\partial k}(\cdot,y,k)\cdot\frac{\partial h_0}{\partial y}(y,k) + \tilde{H}_1(k)F_0(\cdot,y,k)\rangle \text{ and} \tag{10}$$

$$F_1(x,y,k) = (\tilde{H}_0(k) - h_0(y,k))^{-1}(\frac{1}{i}\frac{\partial F_0}{\partial k}\cdot\frac{\partial h_0}{\partial y} + h_1F_0 - \tilde{H}_1(k)F_0) + a_1(y,k)F_0, \tag{11}$$

where $(\tilde{H}_0(k) - h_0(y,k))^{-1}$ denotes the inverse which maps the orthogonal complement of $F_0(\cdot,y,k)$ in $L^2(E)$ onto itself. We recall that

$$\tilde{H}_0(k) - h_0(y,k) = H_0(k + A(y)) - E_m(k + A(y)).$$

We determine $a_1(y,k)$ by the requirement that $P_N^* P_N = I + \mathcal{O}(\epsilon^{N+1})$. Given $f, g \in \mathcal{S}(\mathbb{R}^3)$, this implies

$$\int_E dx \int_{\mathbb{R}^3} [\overline{(F_0(x,y,\epsilon D_y) + \epsilon F_1(x,y,\epsilon D_y))f}][(F_0(x,y,\epsilon D_y) + \epsilon F_1(x,y,\epsilon D_y))g]dy =$$

$$\int_{\mathbb{R}^3} \overline{f} g dy + \mathcal{O}(\epsilon^2). \tag{12}$$

Since $\int_E |F_0(x,y,k)|^2 dx = 1$ for all (k,y), one can calculate $\mathrm{Re}\{a_1(y,k)\}$ from (12), and see that it is smooth and bounded, by the pseudo-differential calculus. We choose $\mathrm{Im}\{a_1(y,k)\} = 0$.

The calculation of (F_j, h_j) for $j > 1$ proceeds in the same manner: we calculate the terms of order ϵ^j in the symbol of the left hand side of (7) which come from (F_l, h_l) for $l < j$ and choose (F_j, h_j) so that the left hand side of (7) has the desired form up to terms of order ϵ^{j+1}. At each stage we use the Fredholm Alternative in $L^2(E)$, and F_j is only determined modulo a term of the form $a_j(y,k)F_0(x,y,k)$. Then we choose the real part of $a_j(y,k)$ so that $P_j^* P_j = I + \mathcal{O}(\epsilon^{j+1})$, and take the imaginary part of a_j to be zero. Continuing in this way we complete the proof of the Theorem.

Remark 2 *If we set* $\Pi_N = P_N P_N^*$, *then* Π_N *is approximately a projection:* $\Pi_N^2 = \Pi_N + \mathcal{O}(\epsilon^{N+1})$ *and* $\Pi_N = \Pi_N^*$. *The equation (7) implies that*

$$P_N^* \tilde{H}_\epsilon = H_{eff} P_N^* + \mathcal{O}(\epsilon^{N+1}).$$

Hence

$$P_N P_N^* \tilde{H}_\epsilon = P_N H_{eff} P_N^* + \mathcal{O}(\epsilon^{N+1}).$$

If we replace $P_N H_{eff}$ *in the equality above by the left hand side of (7),we obtain:*

$$P_N P_N^* \tilde{H}_\epsilon = \tilde{H}_\epsilon P_N P_N^* + \mathcal{O}(\epsilon^{N+1}).$$

Thus Π_N *is a projection which commutes with* $\tilde{H}_\epsilon$ *to order* $\mathcal{O}(\epsilon^{N+1})$ *as in [13].*

4. Relations with Previous Work

To relate the results here to what has already been done we need to complete the calculation of the effective Hamiltonian up to terms of order ϵ^2, i.e. to compute the symbol $h_1(y,k)$ from (10). The explicit computation of

$$\langle F_0(\cdot,y,k), \tilde{H}_1(k)F_0(\cdot,y,k)\rangle =_{def} E_1(y,k)$$

is contained in the computations in [5] (it is also in [8] with a small error – see Remark 1 in [5]). When one replaces $k(y,s)$ and $\Psi(x,y,s)$ in [5, pp. 7601-3] by $k+A(y)$ and $\Psi(x,k+A(y))$ respectively, $E_1(y,k)$ is "ih" in the notation of [5, (25)] and we have

$$E_1 = \frac{1}{2i}\frac{\partial}{\partial y}\cdot\frac{\partial E_m(\tilde{k})}{\partial k} - L\cdot B - i\langle\Psi(\cdot,\tilde{k}), \frac{\partial \Psi(\cdot,\tilde{k})}{\partial y}\rangle\cdot\frac{\partial E_m(\tilde{k})}{\partial k},$$

where $\tilde{k} = k + A(y)$. Here $B(y) = \nabla\times A(y)$ and $L =$

$$L = \mathrm{Im}\left(\langle M(y,k)\frac{\partial\Psi}{\partial k_2}, \frac{\partial\Psi}{\partial k_3}\rangle, \langle M(y,k)\frac{\partial\Psi}{\partial k_3}, \frac{\partial\Psi}{\partial k_1}\rangle, \langle M(y,k)\frac{\partial\Psi}{\partial k_1}, \frac{\partial\Psi}{\partial k_2}\rangle\right).$$

with $M(y,k) = \tilde{H}_0(k) - h_0(y,k)$. The vector L is an angular momentum and $L\cdot B$ contributes the "Rammal-Wilkinson" term to the energy, cf. [1]. Adding the additional term from (10) to E_1 to obtain h_1, we obtain

$$h_1(y,k) = \frac{1}{2i}\frac{\partial}{\partial y}\cdot\frac{\partial E_m(k+A(y))}{\partial k} - L\cdot B - i\langle\Psi(\cdot,k+A(y), \dot{\Psi}(\cdot,k+A(y))\rangle, \quad (13)$$

where

$$\dot{\Psi}(x,k+A(y)) = \frac{\partial\Psi(x,k+A(y))}{\partial y}\cdot\dot{y} + \frac{\partial\Psi(x,k+A(y))}{\partial k}\cdot\dot{k}$$

and $\dot{y}$ and $\dot{k}$ are defined by the Hamiltonian system

$$\dot{y} = \frac{\partial(E_m(k+A(y))+W(y))}{\partial k} \qquad \dot{k} = -\frac{\partial(E_m(k+A(y))+W(y))}{\partial y}.$$

Thus one recognizes $i\langle\Psi(\cdot,k+A(y)),\dot{\Psi}(\cdot,k+A(y))\rangle$ as the term generating the Berry phase precession, cf. [12], [16]. Comparing (13) with [13, (22)] (in the case $l=1$), one sees that they agree completely when one takes into account the difference in the choice of sign in the magnetic potential, $A(y)$, and the use of Weyl quantization in [13]. The sign of the Berry phase term in (13) may appear inconsistent with [5, (29)], but it is not. In [5] $(\dot{y},\dot{k})$ was the vector field from the Hamiltonian $-E_m(k+A(y))$.

In [5] and [8] instead of introducing effective Hamiltonians we constructed wave packets. These packets are nonetheless related to effective Hamiltonians in that one can compute what the effective Hamiltonian must be – assuming that there is one – from the packets. To see this one can proceed as follows. The packets have the form (here $s=\epsilon t$ and $W(y)=0$)

$$u(x,y,s,\epsilon) = e^{i\phi(y,s)/\epsilon}[f(y,s)\Psi(x,\frac{\partial\phi}{\partial y}+A(y))+\mathcal{O}(\epsilon)],$$

where ϕ and f are solutions of

$$\frac{\partial\phi}{\partial s} = E_m(\frac{\partial\phi}{\partial y}+A(y)) \text{ and } \frac{\partial f}{\partial s} = \frac{\partial E_m}{\partial k}(\frac{\partial\phi}{\partial y}+A(y))\cdot\frac{\partial f}{\partial y}+(D-iL\cdot B+\langle\Psi,\dot{\Psi}\rangle)f. \quad (14)$$

Here all functions of (k,y) are evaluated at $k=\tilde{k}(y,s)=\partial_y\phi$, and $D=(1/2)\partial_y\cdot(\partial_k E_m(\partial_y\phi+A(y)))$. Assuming that the evolution of f is governed by an effective

Hamiltonian $H_{eff} = h_0(y, \epsilon D_y) + \epsilon h_1(y, \epsilon D_y) + \mathcal{O}(\epsilon^2)$, we must have (on bounded intervals in s)

$$[e^{isH_{eff}/\epsilon} e^{i\phi(\cdot,0)/\epsilon} f(\cdot, 0)](y, s) = e^{i\phi(y,s)/\epsilon} f(y, s) + \mathcal{O}(\epsilon^2). \tag{15}$$

Differentiating (15) with respect to s, one concludes

$$\frac{i}{\epsilon} H_{eff}(e^{i\phi(y,s)/\epsilon} f(y, s)) = (\frac{i}{\epsilon}\frac{\partial \phi}{\partial s} f + \frac{\partial f}{\partial s}) e^{i\phi(y,s)/\epsilon} + \mathcal{O}(\epsilon). \tag{16}$$

Using the symbol expansion from the pseudo-differential calculus

$$e^{-i\phi/\epsilon} H_{eff}(e^{i\phi/\epsilon} f) = h_0(y, \tilde{k}) +$$

$$\epsilon[\frac{1}{i}\frac{\partial h_0}{\partial k}(y, \tilde{k}) \cdot \frac{\partial f}{\partial y}(y) + \frac{1}{2i}\sum_{j,l} \frac{\partial^2 h_0}{\partial k_j \partial k_l}(y, \tilde{k}) \frac{\partial^2 \phi}{\partial y_j \partial y_l}(y) + h_1(y, \tilde{k})] f(y) + \mathcal{O}(\epsilon^2). \tag{17}$$

Substituting (17) into (16) and comparing the result with (14), one recovers the formulas given earlier for $h_0(y, k)$ and $h_1(y, k)$.

In [8] we were unable to reconcile our results with those of Chang and Niu, see [3], [4] and also [17]). We thought that this might have resulted from different choices of scales. This is partially true, since Chang and Niu do not distinguish the scale $y = \epsilon x$, but, as Panati, Spohn and Teufel point out in [13], the differences largely disappear when one considers the Heisenberg formulation of quantum dynamics. Letting $a(y, \epsilon D_y)$ be an observable, the propagation of a in the Heisenberg picture is given by

$$a(s) = e^{-isH_{eff}/\epsilon} a e^{isH_{eff}/\epsilon} \text{ or } \epsilon\frac{da}{ds} = i[a, H_{eff}].$$

If one considers this propagation at the symbol level, then the symbol of a is propagating along the trajectories of the Hamiltonian system

$$\dot{y} = \frac{\partial H_{eff}}{\partial k} \qquad \dot{k} = -\frac{\partial H_{eff}}{\partial y}.$$

Hence one can consider the contribution of h_1 as an order ϵ correction to the classical Peierls dynamics arising from h_0. This is the point of view taken in [13]. However, it is worth noting that the wave packets are propagating along the trajectories from h_0 with a precession in their phases arising from (the imaginary part of) h_1.

Acknowledgements We wish to thank Professors Panati, Spohn and Teufel for sending us the preprint version of [13] and for several helpful discussions of this work.

References

[1] J. Bellissard and R. Rammal, *An algebric semi-classical approach to Bloch electrons in a magnetic field* J. Pysique France **51**(1990), 1803.

[2] V. S. Buslaev, *Semi-classical approximation for equations with periodic coefficients.* Russian. Math. Surveys, **42** (1987), 97–125.

[3] M. C. Chang and Q. Niu, *Berry phase, hyperorbits, and the Hofstadter spectrum.* Phys. Rev. lett. **75**(1996), 1348-1351.

[4] M. C. Chang and Q. Niu, *Berry phase, hyperorbits, and the Hofstadter spectrum: semiclassical in magnetic Bloch bands* Phys. Rev. B **53**(1996) 7010-7022.

[5] M. Dimassi, J.-C. Guillot and J. Ralston, *Semi-Classical Asymptotics in Magnetic Bloch Bands.* J. Phys. A: Math. G., **35** (2002), 7597–7605 .

[6] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit.* London Math. Soc. Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999.

[7] C. Gérard, A. Martinez and J. Sjöstrand, *A Mathematical Approach to the effective Hamiltonian in perturbed periodic Problems.* Commun. Math. Phys., **142** (1991), 217–244.

[8] J. C. Guillot, J. Ralston and E. Trubowitz, *Semi-classical methods in solid state physics.* Commun. Math. Phys., **116** (1988), 401–415.

[9] B. Helffer and J. Sjöstrand, *On diamagnetism and the de Haas - van Alphen effect.* Annales I.H.P. (Physique théorique)**52**(1990), 303-375.

[10] F. Hôvermann, H. Spohn and S. Teufel, *Semi-classical limit for the Schrödinger equation with a short scale periodic potential.* Comm. Math. Phys. **215** (2001), no. 3, 609–629.

[11] W. Horn, *Semi-classical construction in solid state physics.* Commun. P.D.E. **16**(1993) 255-290.

[12] M. Kohomoto, *Berry's phase of Bloch electrons in electromagnetic fields.* J. Phys. Soc. Japan **62**(1993), 659-663.

[13] G. Panati, H. Spohn, S. Teufel, *Effective dynamics for Bloch electrons: Peierls substitution and beyond.* mp-arc 02-516.

[14] R. Peierls, *Zur Theorie des diamagnetimus von leitungselektronen* Z. Phys. **80**(1933), 763-791.

[15] J. Ralston, *Magnetic breakdown* Astérisque **210**(1992), 263-2282.

[16] B. Simon, *Holonomy, the quantum adiabatic theorem, and Berry's phase*, Phys. Rev. Lett. **51**(1983), 2167-2170.

[17] G. Sundaram and Q. Niu, *Wave packet dynamics in slowly perturbed crystals: Gradient corrections and Berry phase effects,* Phys. Rev. B **59**(1999), 14915-14925.

[18] J. Zak, *Dynamics of electrons in solids in external fields.* Phy. Rev. **168**(1968), 686-695.

M. Dimassi and J.-C. Guillot
Département de Mathématiques, Université Paris 13, Villetaneuse, France
email: dimassi@math.univ-paris13.fr
email: guillot@math.univ-paris13.fr

J. Ralston
University of California, Los Angeles, CA 90095, USA
email: ralston@math.ucla.edu