From repeated to continuous quantum interactions

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Abstract

We consider the general physical situation of a quantum system $\mathcal{H}_0$ interacting with a chain of exterior systems $\bigotimes_{\mathbb{N}^*} \mathcal{H}$, one after the other, during a small interval of time h and following some Hamiltonian H on $\mathcal{H}_0 \otimes \mathcal{H}$. We discuss the passage to the limit to continuous interactions ($h \to 0$) in a setup which allows to compute the limit of this Hamiltonian evolution in a single state space: a continuous field of exterior systems $\bigotimes_{\mathbb{R}^+} \mathcal{H}$. Surprisingly, the passage to the limit shows the necessity for 3 different time scales in H. The limit evolution equation is shown to spontaneously produce quantum noises terms: we obtain a quantum Langevin equation as limit of the Hamiltonian evolution. For the very first time, these quantum Langevin equations are obtained as the effective limit from repeated to continuous interactions and not only as a model. These results justify the usual quantum Langevin equations considered in continual quantum measurement or in quantum optics. We show that the three time scales correspond to the normal regime, the weak coupling limit and the low density limit. Our approach allows to consider these two physical limits altogether for the first time. Their combination produces an effective Hamiltonian on the small system, which had never been described before. We apply these results to give an Hamiltonian description of the von Neumann measurement. We also consider the approximation of continuous time quantum master equations by discrete time ones. In particular we show how any Lindblad generator is obtained as the limit of completely positive maps.

Contents

I. Introduction

Quantum Langevin equations as a model for quantum open systems have been considered for at least 40 years (for example [FKM], [FLO], [AFL]). They have been given many different meanings in terms of several definitions of quantum noises or quantum Brownian motions (for example [G-Z], [H-P], [GSI]). One of the most developed and useful mathematical languages developed for that purpose is the quantum stochastic calculus of Hudson and Parthasarathy and their quantum stochastic differential equations ([H-P]). The quantum Langevin equations they allow to consider have been used very often to modelize typical situations of quantum open systems: continual quantum measurement ([Ba1], [B-B]), quantum optics ([F-R], [FRS] [Ba2]), electronic transport [BRSW], thermalization ([M-R], [L-M]), etc.

The justification for such quantum Langevin equation is often given in terms of some particular approximations of the true Hamiltonian interaction dynamic: rotating wave approximation, Markov approximation, large band approximation (cf [G-Z] chapter 11).

They are also often justified as natural dilations of quantum master equations on the small system. That is, for any (good) semigroup of completely positive maps on the small system (with Lindblad generator $\mathcal{L}$), one can dilate the small system with an appropriate Fock space, and obtain an explicit quantum stochastic differential equation on the whole space. The unique solution of this equation is a unitary evolution (in interaction picture) such that the trace on the small system of the induced evolution yields the original semigroup. This corresponds, at the quantum level, to the well-known way of realizing a concrete Markov process from a given semigroup (or generator) by adding a noise space to the (classical) system space and solving an adequate stochastic differential equation.

Some quantum stochastic differential equations have also been obtained in the so-called *stochastic limit* from explicit Hamiltonian dynamics ([A-L], [AGL], [ALV]). This shows some similarities with the results described here, but the limits considered in those articles are in the sense of the convergence of processes living in a different space than the one of the Hamiltonian dynamic.

In this article we consider the effective Hamiltonian dynamic describing the repeated interactions, during short time intervals of length h, of a small system $\mathcal{H}_0$ with a chain of exterior systems $\otimes_{\mathbb{N}^*}\mathcal{H}$. We embed all these chains as particular subspaces, attached to the parameter h, of a continuous field

$$\bigotimes_{\mathbb{R}^+}\mathcal{H}$$

in such a way that the subspaces associated to the chain increase and fill the field when h tends to 0.

By developing an appropriate language of the chain $\otimes_{\mathbb{N}^*}\mathcal{H}$ and of the field $\otimes_{\mathbb{R}^+}\mathcal{H}$ and by describing the discrete time Hamiltonan evolution generated by the repeated interactions, we are able to pass to the limit when $h \to 0$ and prove that the limit evolution equation is solution of a quantum stochastic differential equation. This limit is obtained in the weak topology of operators and in a single space: the continuous field $\bigotimes_{\mathbb{R}^+}\mathcal{H}$.

Of course, such a limit cannot be obtained without assumptions on the elementary interaction Hamiltonian H. This is similar to the central limit theorem: a random walk gives a trivial limit when its time step h goes to zero and it is only when suitably renormalized (by a factor $\sqrt{h}$) that it yields a Gaussian. Other normalizations give either trivial limits or no limit at all.

In our Hamiltonian context the situation is going to be the same. For a non-trivial limit of these repeated interactions to exist, we will need the Hamiltonian H to satisfy some renormalization properties. The surprise here is that the necessary renormalization factor is not global, it is different following some parts of the Hamiltonian operator. We identify 3 different time scales in H: one of order 1, one of order $\sqrt{h}$, one of order h.

We describe a class of Hamiltonian which seems to be typical for the above conditions to be satisfied. These typical Hamiltonians are clearly a combination of free evolution, weak coupling limit typical hamiltonians and low density limit typical Hamiltonians. This physically explains the three different time scales. But the originality of our approach allows to consider both limits together; to our knowledge this constitutes a novelty in the literature. As a consequence, the combination of the two limits shows an effective Hamiltonian for the small system which is very surprising: it contains a new term

$$V^*D^{-2}(\sin D - D)V$$

which comes from the presence of both the weak coupling and the low density limit in the Hamiltonian. It seems that such a term had never been described before.

This article is structured as follows.

In section II we present the exact mathematical model of repeated quantum interactions and end up with the associated evolution equation (subsection II.1). We then introduce a mathematical setup for the study of the space $\bigotimes_{\mathbb{N}^*} \mathcal{H}$ which will help much for passing to the continuous field. In particular this includes a particular choice of an orthonormal basis of the phase space and a particular choice of a basis for the operators on that phase space (subsection II.2). Finally we show how the typical evolution equations obtained in II.1 are the general model for the unitary dilation of any given discrete semigroup of completely positive maps (subsection II.3).

Section III is devoted to presenting the whole formalism of the continuous atom field. In subsection III.1 we present the space which is candidate for representing the continuous field limit of the atom chain. It is actually a particular Fock space on which we develop an unusual structure which clearly shows the required properties. In subsection III.2 we present the natural quantum noises on the continuous field and the associated quantum stochastic integrals, the quantum Ito formula and the quantum stochastic differential equations. In subsection III.3 we concretely realize the atom chain of section II as a strict subspace of the atom field. Not only do we realize it as a subspace, but also realize the action of its basic operators inside the atom field. All these atom chain subspaces are related to a partition of $\mathbb{R}^+$. When the diameter of the partition goes to 0, we show that the corresponding subspace completely fills the continuous field and the basic operators of the chain converge to the quantum noises of the field (with convenient normalizations). Finally, considering the projection of the continuous atom field onto an atom chain subspace, we state a formula for the projection of a general quantum stochastic integral.

In section IV all the pieces of the puzzle fit together. By computing the projection on the atom chain of a quantum stochastic differential equation we show that the typical evolution equation of repeated interactions converges in the field space to the solution of a quantum Langevin equation, assuming the fact that the associated Hamiltonian satisfies some particular renormalization property corresponding to three different time scales. It is to that result and to some of its extensions that subsection IV.1 is devoted. In subsection IV.2 we describe a family of Hamiltonians which seems to be typical of the conditions obtained above. We show that this family of Hamiltonians describes altogether free evolution, weak coupling limit and low density limit terms. Computing the associated quantum Langevin equation at the limit, we obtain an effective Hamiltonian on $\mathcal{H}_0$ which contains a new term. This new term appears only when weak coupling and low density limits are in presence together in the Hamiltonian. In subsection IV.3, we apply these results to describe the von Neumann measurement apparatus in the Hamiltonian framework of repeated quantum interactions. In subsection IV.4 we explicitly compute a simple example. In subsection IV.5 we show that our approximation theorem puts into evidence a natural way that completely positive maps have to converge to Lindblad generators.

II. Discrete dynamics on the atom chain

II.1 Repeated quantum interactions

We here give a precise description of our physical model: repeated quantum interactions.

We consider a small quantum system $\mathcal{H}_0$ and another quantum system $\mathcal{H}$ which represents a piece of environment, a measuring apparatus, incoming photons... We consider the space $\mathcal{H}_0 \otimes \mathcal{H}$ in order to couple the two systems, an Hamiltonian H on $\mathcal{H}_0 \otimes \mathcal{H}$ which describes the interaction and the associated unitary evolution during the interval $[0, h]$ of time:

$$\mathbb{L} = e^{-ihH}.$$

This single interaction is therefore described in the Schrödinger picture by

$$\rho \mapsto \mathbb{L}\, \rho\, \mathbb{L}^*$$

and in the Heisenberg picture by

$$X \mapsto \mathbb{L}^* X \mathbb{L}.$$

Now, after this first interaction, we repeat it but this time coupling the same $\mathcal{H}_0$ with a new copy of $\mathcal{H}$. This means that that new copy was kept isolated until then; similarly the previously considered copy of $\mathcal{H}$ will remain isolated for the rest of the experience. One can think of many physical examples where this situations arises: in repeated quantum measurement where a family of identical measurement devices are presented one after the other before the system (or a single device is refreshened after every use), in quantum optics where a sequence of independent atoms arrives one after the other to interact with a field in some cavity for a short time. More generally it can be seen as a good model if it is assumed that perturbations in $\mathcal{H}$ due to the interaction are dissipated after every time h.

The sequence of interactions can be described in the following way: the state space for the whole system is

$$\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H}$$

Index for a few lines only the copies of $\mathcal{H}$ as $\mathcal{H}_1$, $\mathcal{H}_2$, ... Define then a unitary operator $\mathbb{L}_n$ as the canonical ampliation to $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ of the operator which acts as $\mathbb{L}$ on $\mathcal{H}_0 \otimes \mathcal{H}_n$; that is, $\mathbb{L}_n$ acts as the identity on copies of $\mathcal{H}$ other than $\mathcal{H}_n$.

The effect of the n-th interaction in the Schrödinger picture writes then

$$\rho \mapsto \mathbb{L}_n\, \rho\, \mathbb{L}_n^*,$$

for every density matrix ρ, so that the effect of the n first interactions is

$$\rho \mapsto u_n\, \rho\, u_n^*$$

where $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H})$ which satisfies the equations

$$\begin{cases} u_{n+1} = \mathbb{L}_{n+1}\, u_n \\ u_0 \;\;= I. \end{cases} \tag{1}$$

It is evolution equations such as (1) that we are going to study in this article.

II.2 Structure of the atom chain

We here describe some useful mathematical structure on the space $\otimes_{\mathbb{N}^*}\mathcal{H}$ which will constitute the main ingredient of our approach.

Let us fix a particular Hilbertian basis $(X^i)_{i\in\Lambda\cup\{0\}}$ for the Hilbert space $\mathcal{H}$, where we assume (for notational purposes) that $0 \notin \Lambda$. This particular choice of notations is motivated by physical interpretations: indeed, we see the X^i, $i \in \Lambda$, as representing for example the different possible excited states of an atom. The vector X^0 represents the "ground state" or "vacuum state" of the atom and will usually be denoted Ω.

Let $T\Phi$ be the tensor product $\bigotimes_{\mathbb{N}^*} \mathcal{H}$ with respect to the stabilizing sequence Ω. In other words, this means simply that an orthonormal basis of $T\Phi$ is given by the family

$$\{X_A;\ A \in \mathcal{P}_{\mathbb{N}^*,\Lambda}\}$$

where

– the set $\mathcal{P}_{\mathbb{N},\Lambda}$ is the set of finite subsets

$$\{(n_1, i_1), \dots, (n_k, i_k)\}$$

of $\mathbb{N}^* \times \Lambda$ such that the n_i's are mutually different. Another way to describe the set $\mathcal{P}_{\mathbb{N}^*,\Lambda}$ is to identify it to the set of sequences $(A_n)_{n\in\mathbb{N}^*}$ with values in $\Lambda \cup \{0\}$ which take a value different from 0 only finitely often.

– X_A denotes the vector

$$\Omega \otimes \dots \otimes \Omega \otimes X^{i_1} \otimes \Omega \otimes \dots \otimes \Omega \otimes X^{i_2} \otimes \dots$$

where X^{i_1} appears in n_1-th copy of $\mathcal{H}$...

The physical signification of this basis is easy to understand: we have a chain of atoms, indexed by $\mathbb{N}^*$. The space $T\Phi$ is the state space of this chain, the vector X_A with $A = \{(n_1, i_1), \dots, (n_k, i_k)\}$ representing the state in which exactly k atoms are excited: atom n_1 in the state i_1, etc, all other atoms being in the ground state.

This particular choice of a basis gives $T\Phi$ a particular structure. If we denote by $T\Phi_{\mathrm{n]}}$ the space generated by the X_A such that $A \subset \{1, \dots, n\} \times \Lambda$ and by $T\Phi_{\mathrm{[m}}$ the one generated by the X_A such that $A \subset \{m, m+1, \dots\} \times \Lambda$, we get an obvious natural isomorphism between $T\Phi$ and $T\Phi_{\mathrm{n-1]}} \otimes T\Phi_{\mathrm{[n}}$ given by

$$[f \otimes g](A) = f\left(A \cap \{1, \dots, n-1\} \times \Lambda\right)\, g\left(A \cap \{n, \dots\} \times \Lambda\right).$$

Put $\{a^i_j; i, j \in \Lambda \cup \{0\}\}$ to be the natural basis of $\mathcal{B}(\mathcal{H})$, that is,

$$a^i_j(X^k) = \delta_{ik} X^j.$$

We denote by $a^i_j(n)$ the natural ampliation of the operator a^i_j to $T\Phi$ which acts on the copy number n as a^i_j and the identity elsewhere. That is, in terms of the basis X_A,

$$a^i_j(n)X_A = \mathbb{1}_{(n,i)\in A} X_{A\setminus(n,i)\cup(n,j)}$$

if neither i nor j is zero, and

$$\begin{aligned} a^i_0(n)X_A &= \mathbb{1}_{(n,i)\in A} X_{A\setminus(n,i)}, \\ a^0_j(n)X_A &= \mathbb{1}_{(n,0)\in A} X_{A\cup(n,j)}, \\ a^0_0(n)X_A &= \mathbb{1}_{(n,0)\in A} X_A, \end{aligned}$$

where $(n,0)\in A$ actually means "for any i in Λ, $(n,i)\notin\Lambda$".

II.3 Unitary dilation of completely positive maps

The evolution equations

$$u_n = \mathbb{L}_n \dots \mathbb{L}_1$$

obtained in the physical setup of repeated quantum interactions are actually of mathematical interest on their own for they provide a canonical way of dilating discrete semigroups of completely positive maps into unitary automorphisms.

The mathematical setup is the same. Let $\mathbb{L}$ be any operator on $\mathcal{H}_0\otimes\mathcal{H}$. Let $T\Phi = \otimes_{\mathbb{N}^*}\mathcal{H}$ and $(\mathbb{L}_n)_{n\in\mathbb{N}^*}$ be defined as in the above section. We then consider the associated evolution equations

$$u_n = \mathbb{L}_n \dots \mathbb{L}_1 \tag{1}$$

with $u_0 = I$.

The following result is obvious.

Proposition 1. – *The solution $(u_n)_{n\in\mathbb{N}}$ of (1) is made of unitary (resp. isometric, contractive) operators if and only if $\mathbb{L}$ is unitary (resp. isometric, contractive).* ■

Note that if $\mathbb{L}$ is unitary, then the mappings

$$j_n(H) = u_n^* H u_n$$

are automorphisms of $\mathcal{B}(\mathcal{H}_0\otimes\mathcal{H})$.

Let $\mathbb{E}_0$ be the partial trace on $\mathcal{H}_0$ defined by

$$<\phi\,,\, \mathbb{E}_0(H)\,\psi> = <\phi\otimes\Omega\,,\, H\,\psi\otimes\Omega>$$

for all $\phi,\psi\in\mathcal{H}_0$ and every operator H on $\mathcal{H}_0\otimes T\Phi$.

Unitary dilations of completely positive semigroups are obtained in the following theorem. Recall that, by Kraus' theorem, any completely positive operator ℓ on $\mathcal{B}(\mathcal{H}_0)$ is of the form

$$\ell(X) = \sum_{i\in\mathbb{N}} A_i^* X A_i$$

where the summation ranges over $(\Lambda\cup\{0\})^2$, the A_i are bounded operators and the sum is strongly convergent. Conversely, any such operator is completely positive.

Remark: Of course the Kraus form of an operator is *a priori* indifferent to the specificity of the value $i=0$. The special role played by one of the indices will appear later on.

Theorem 2. – *Let $\mathbb{L}$ be any unitary operator on $\mathcal{H}_0\otimes\mathcal{H}$. Consider the coefficients $(\mathbb{L}^i_j)_{i,j\in\Lambda\cup\{0\}}$, which are operators on $\mathcal{H}_0$, of the matrix representation of $\mathbb{L}$ in the basis Ω, X^i, $i\in\Lambda$ of $\mathcal{H}$.*

Then, for any $X\in\mathcal{B}(\mathcal{H}_0)$ we have

$$\mathbb{E}_0[j_n(X\otimes I)] = \ell^n(X)$$

where ℓ is the completely positive map on $\mathcal{B}(\mathcal{H}_0)$ given by

$$\ell(X) = \sum_{i\in\Lambda\cup\{0\}} (\mathbb{L}^0_i)^* X \mathbb{L}^0_i.$$

Conversely, consider any completely positive map

$$\ell(X) = \sum_{i\in\Lambda\cup\{0\}} A_i^* X A_i$$

on $\mathcal{B}(\mathcal{H}_0)$ such that $\ell(I)=I$. Then there exists a unitary operator $\mathbb{L}$ on $\mathcal{H}_0\otimes\mathcal{H}$ such that the associated unitary family of automorphisms

$$j_n(H) = u_n^* H u_n$$

satisfies

$$\mathbb{E}_0[j_n(X\otimes I)] = \ell^n(X),$$

for all $n\in\mathbb{N}$.

Proof

Consider $\mathbb{L} = (\mathbb{L}^i_j))_{i,j\in\Lambda\cup\{0\}}$ such as in the above statements. Consider the unitary family

$$u_n = \mathbb{L}_n \dots \mathbb{L}_1.$$

Note that

$$u_{n+1} = \mathbb{L}_{n+1} u_n.$$

Put $j_n(H) = u_n^* H u_n$ for every operator H on $\mathcal{H}_0\otimes\mathcal{H}$. Then, for any operator X on $\mathcal{H}_0$ we have

$$j_{n+1}(X\otimes I) = u_n^* \mathbb{L}_{n+1}^* (X\otimes I)\mathbb{L}_{n+1} u_n.$$

When considered as a matrix of operators on $\mathcal{H}_0$, in the basis Ω, X^i, $i\in\Lambda$ of $\mathcal{H}$, the matrix associated to $X\otimes I$ is of diagonal form. We get

$$\mathbb{L}_{n+1}^*(X\otimes I)\mathbb{L}_{n+1} =$$

$$= \begin{pmatrix} (\mathbb{L}^0_0)^* & (\mathbb{L}^0_1)^* & \dots \\ (\mathbb{L}^1_0)^* & (\mathbb{L}^1_1)^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X & 0 & \dots \\ 0 & X & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathbb{L}^0_0 & \mathbb{L}^1_0 & \dots \\ \mathbb{L}^0_1 & \mathbb{L}^1_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

which is the matrix $\mathbf{L}_{n+1}(X) = \left(B^i_j(X)\right)_{i,j\in\Lambda\cup\{0\}}$ with

$$B^i_j(X) = \sum_{k\in\Lambda\cup\{0\}} (\mathbb{L}^j_k)^* X \mathbb{L}^i_k.$$

Note that the operator $\mathbf{L}_{n+1}(X)$ acts non trivialy only on the tensor product of $\mathcal{H}_0$ with the $(n+1)$-th copy of $\mathcal{H}$. When represented as an operator on

$$\mathcal{H}_0 \otimes \mathrm{T\Phi_{n+1]}} = \left(\mathcal{H}_0 \otimes \mathrm{T\Phi_{n]}}\right) \otimes \mathcal{H}$$

as a matrix with coefficients in $\mathcal{B}(\mathcal{H}_0 \otimes \mathrm{T\Phi_{n]}})$ it writes exactly in the same way as above, just replacing $B^i_j(X)$ (which belongs to $\mathcal{B}(\mathcal{H}_0)$) by

$$B^i_j(X) \otimes I_{|\mathrm{T\Phi_{n]}}}.$$

Also note that, as can be proved by an easy induction, the operator u_n acts on $\mathcal{H}_0 \otimes \mathrm{T\Phi_{n]}}$ only. As an operator on $\mathcal{H}_0 \otimes \mathrm{T\Phi_{n+1]}}$ it is represented by a diagonal matrix. Thus $j_{n+1}(X) = u_n^* \mathbf{L}_{n+1}(X) u_n$ can be written on $\mathcal{H}_0 \otimes \mathrm{T\Phi_{(n+1)]}} = \mathcal{H}_0 \otimes \mathrm{T\Phi_{n]}} \otimes \mathcal{H}$ as a matrix of operators on $\mathcal{H}_0 \otimes \mathrm{T\Phi_{n]}}$ by

$$\left(j_{n+1}(X\otimes I)\right)^i_j = j_n(B^i_j(X)\otimes I).$$

Note that $B^0_0(X) = \sum_{i\in\Lambda\cup\{0\}} (\mathbb{L}^0_i)^* X \mathbb{L}^0_i$ which is the mapping $\ell(X)$ of the statement.

Put $T_n(X) = \mathbb{E}_0[j_n(X\otimes I)]$. We have, for all $\phi, \Psi \in \mathcal{H}_0$

$$\begin{aligned}
<\phi\,,\, T_{n+1}(X)\Psi> &= <\phi\otimes\Omega\,,\, j_{n+1}(X\otimes I)\,\Psi\otimes\Omega> \\
&= <\phi\otimes\Omega\,,\, \left(j_n(B^i_j(X)\otimes I)\right)_{i,j}\,\Psi\otimes\Omega> \\
&= <\phi\otimes\Omega_{\,\mathrm{T\Phi_{n]}}}\otimes\Omega_{\,\mathcal{H}}\,,\, \left(j_n(B^i_j(X)\otimes I)\right)_{i,j}\,\Psi\otimes\Omega_{\,\mathrm{T\Phi_{n]}}}\otimes\Omega_{\,\mathcal{H}}> \\
&= <\phi\otimes\Omega_{\,\mathrm{T\Phi_{n]}}}\,,\, j_n(B^0_0(X)\otimes I)\Psi\otimes\Omega_{\,\mathrm{T\Phi_{n]}}}> \\
&= <\phi\,,\, T_n(\ell(X))\Psi>.
\end{aligned}$$

This proves that $T_{n+1}(X) = T_n(\ell(X))$ and the first part of the theorem is proved.

Conversely, consider a decomposition of a completely positive map ℓ of the form

$$\mathcal{L}(X) = \sum_{i\in\Lambda\cup\{0\}} A_i^* X A_i$$

for a familly $(A_i)_{i\in\Lambda\cup\{0\}}$ of bounded operators on $\mathcal{H}_0$ such that

$$\sum_{i\in\Lambda\cup\{0\}} A_i^* A_i = I.$$

We claim that there exists a unitary operator $\mathbb{L}$ on $\mathcal{H}_0\otimes\mathcal{H}$ of the form

$$\mathbb{L} = \begin{pmatrix} A_0 & \cdots & \cdots \\ A_1 & \cdots & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Indeed, the condition $\sum_{i\in\Lambda\cup\{0\}} A_i^* A_i = I$ guarantees that the first columns of $\mathbb{L}$ are made of orthonormal vectors of $\mathcal{H}_0\otimes\mathcal{H}$. We can thus complete the matrix by completing it into an orthonormal basis of $\mathcal{H}_0 \otimes \mathcal{H}$. This makes out a unitary matrix $\mathbb{L}$ the coefficients of which we denote by $(A^i_j)_{i,j\in\Lambda\cup\{0\}}$. Note that $A^0_i = A_{i+1}$. We now conclude easily by the first part of the theorem.

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III From the atom chain to the atom field

III.1 Structure of the atom field

We now describe the structure of the continuous version of $T\Phi$. The structure we are going to present here is rather original and not much expanded in the literature. It is very different from the usual presentation of quantum stochastic calculus ([H-P]), but it actually constitutes a very natural language for our purpose: approximation of the atom field by atom chains. This approach is taken from [At1].

We first start with a heuristical discussion.

By a continuous version of the atom chain $T\Phi$ we mean a Hilbert space with a structure which makes it the space

$$\Phi = \bigotimes_{\mathbb{R}^+} \mathcal{H}.$$

We have to give a meaning to the above notation. This could be achieved by invoquing the framework of continous tensor products of Hilbert spaces (see [Gui]), but we prefer to give a self-contained presentation which fits better with our approximation procedure.

Let us make out an idea of what it should look like by mimicking, in a continuous time version, what we have described in $T\Phi$.

The countable orthonormal basis X_A, $A \in \mathcal{P}_{\mathbb{N}^*,\Lambda}$ is replaced by a continuous orthonormal basis $d\chi_\sigma$, $\sigma \in \mathcal{P}_{\mathbb{R},\Lambda}$, where $\mathcal{P}_{\mathbb{R},\Lambda}$ is the set of finite subsets of $\mathbb{R}^+ \times \Lambda$. With the same idea as for $T\Phi$, this means that each copy of $\mathcal{H}$ is equipped with an orthonormal basis $\Omega, d\chi^i_t$, $i \in \Lambda$ (where t is the parameter attached to the copy we are looking at). The orthonormal basis above is just the one obtained by specifying a finite number of sites $t_1, \ldots, t_n$ which are going to be excited, the other ones being supposed to be in the fundamental state Ω, and by specifying their level of excitation.

The representation of an element f of $T\Phi$:

$$f = \sum_{A\in\mathcal{P}_{\mathbb{N}^*,\Lambda}} f(A)\, X_A$$

$$||f||^2 = \sum_{A\in\mathcal{P}_{\mathbb{N}^*,\Lambda}} |f(A)|^2$$

is replaced by an integral version of it in Φ:

$$f = \int_{\mathcal{P}_{\mathbb{R},\Lambda}} f(\sigma)\, d\chi_\sigma$$

$$||f||^2 = \int_{\mathcal{P}_{\mathbb{R},\Lambda}} |f|^2 \; d\sigma.$$

This last integral has to be explained: the measure $d\sigma$ is a "Lebesgue measure" on $\mathcal{P}_{\mathbb{R},\Lambda}$, as will be explained later. From now on, the notation $\mathcal{P}$ will denote, depending on the context, spaces of the type $\mathcal{P}_{\mathbb{N}^*,\Lambda}$ or $\mathcal{P}_{\mathbb{R},\Lambda}$.

A good basis of operators acting on Φ can be obtained by mimicking the operators $a^i_j(n)$ of $\mathrm{T}\Phi$. We will here have a set of infinitesimal operators $da^i_j(t)$, $i, j \in \Lambda \cup \{0\}$, acting on the "t-th" copy of $\mathcal{H}$ by:

$$\begin{aligned}
da^0_0(t)\, d\chi_\sigma &= d\chi_\sigma\, dt\, \mathbb{1}_{t\notin\sigma}\\
da^0_i(t)\, d\chi_\sigma &= d\chi_{\sigma\cup\{(t,i)\}}\, \mathbb{1}_{t\notin\sigma}\\
da^i_0(t)\, d\chi_\sigma &= d\chi_{\sigma\setminus\{(t,i)\}}\, dt\, \mathbb{1}_{(t,i)\in\sigma}\\
da^i_j(t)\, d\chi_\sigma &= d\chi_{\sigma\setminus\{(t,i)\}\cup\{(t,j)\}}\, \mathbb{1}_{(t,i)\in\sigma}
\end{aligned}$$

for all $i, j \in \Lambda$.

We shall now describe a rigourous setup for the above heuristic discussion.

We recall the structure of the bosonic Fock space Φ and its basic structure (cf [At1] for more details and [At3] for a complete study of the theory and its connections with classical stochastic processes).

Let $\mathcal{H}$ be, as before, a Hilbert space with an orthonormal basis X^i, $i \in \Lambda\cup\{0\}$ and let $\mathcal{H}'$ be the closed subspace generated by vectors X^i, $i \in \Lambda$ (or simply said, the orthogonal of X^0).

Let $\Phi = \Gamma_s(L^2(\mathbb{R}^+, \mathcal{H}'))$ be the symmetric (bosonic) Fock space over the space $L^2(\mathbb{R}^+, \mathcal{H}')$. We shall here give a very efficient presentation of that space, the so-called *Guichardet interpretation* of the Fock space.

Let $\mathcal{P}$ $(= \mathcal{P}_{\mathbb{R},\Lambda})$ be the set of finite subsets $\{(s_1, i_1), \dots, (s_n, i_n)\}$ of $\mathbb{R}^+ \times \Lambda$ such that the s_i are two by two different. Then $\mathcal{P} = \cup_n \mathcal{P}_n$ where $\mathcal{P}_n$ is the subset of $\mathcal{P}$ made of n-elements subsets of $\mathbb{R}^+ \times \Lambda$. By ordering the $\mathbb{R}^+$-part of the elements of $\sigma \in \mathcal{P}_n$, the set $\mathcal{P}_n$ can be identified to the increasing simplex $\Sigma_n = \{0 < t_1 < \dots < t_n\} \times \Lambda$ of $\mathbb{R}^n \times \Lambda$. Thus $\mathcal{P}_n$ inherits a measured space structure from the Lebesgue measure on $\mathbb{R}^n$ times the counting measure on Λ. This also gives a measure structure on $\mathcal{P}$ if we specify that on $\mathcal{P}_0 = \{\emptyset\}$ we put the measure $\delta_\emptyset$. Elements of $\mathcal{P}$ are often denoted by σ, the measure on $\mathcal{P}$ is denoted $d\sigma$. The σ-field obtained this way on $\mathcal{P}$ is denoted $\mathcal{F}$.

We identify any element $\sigma \in \mathcal{P}$ with a family $\{\sigma_i,\, i \in \Lambda\}$ of (two by two disjoint) subsets of $\mathbb{R}^+$ where

$$\sigma_i = \{s \in \mathbb{R}^+; (s, i) \in \sigma\}.$$

The *Fock space* Φ is the space $L^2(\mathcal{P}, \mathcal{F}, d\sigma)$. An element f of Φ is thus a measurable function $f : \mathcal{P} \to \mathbb{C}$ such that

$$||f||^2 = \int_{\mathcal{P}} |f(\sigma)|^2\, d\sigma < \infty.$$

One can define, in the same way, $\mathcal{P}_{[a,b]}$ and $\Phi_{[a,b]}$ by replacing $\mathbb{R}^+$ with $[a, b] \subset \mathbb{R}^+$. As in discrete time, there is a natural isomorphism between $\Phi_{[0,t]} \otimes \Phi_{[t,+\infty[}$ given by $h \otimes g \mapsto f$ where $f(\sigma) = h(\sigma \cap [0, t])g(\sigma \cap (t, +\infty[)$.

We shall use the following notations:

$$\Phi_{t]} = \Phi_{[0,t]}, \qquad \Phi_{[t} = \Phi_{[t,+\infty[}.$$

Define Ω to be the *vacuum vector*, that is, $\Omega(\sigma) = \delta_\emptyset(\sigma)$.

We now define a particular family of curves in Φ, which is going to be of great importance here. Define $\chi_t^i \in \Phi$ by

$$\chi_t^i(\sigma) = \begin{cases} 1\!\!1_{[0,t]}(s) & \text{if } \ \sigma = \{(s,i)\} \\ 0 & \text{otherwise.} \end{cases}$$

Then notice that for all $t \in I\!\!R^+$ we have that χ_t^i belongs to $\Phi_{[0,t]}$. We actually have much more than that: we have

$$\chi_t^i - \chi_s^i \ \in \ \Phi_{[s,t]} \text{ for all } s \leq t.$$

This last property can be checked immediately from the definitions, and it is going to be of great importance in our construction. Also notice that χ_t^i and χ_s^j are orthogonal elements of Φ as soon as $i \neq j$. As we will see later on, apart from trivialities, the curves $(\chi_t^i)_{t\geq 0}$ are the only ones to share these properties.

These properties allow to define the so-called *Ito integral* on Φ. Indeed, let $g = \{(g_t^i)_{t\geq 0},\ i \in \Lambda\}$ be families of elements of Φ indexed by both $I\!\!R_+$ and Λ, such that

i) $t \mapsto \|g_t^i\|$ is measurable, for all i,

ii) $g_t^i \in \Phi_{[0,t]}$ for all t,

iii) $\sum_{i\in\Lambda} \int_0^\infty \|g_t^i\|^2 \, dt < \infty$

then one says that g is *Ito integrable* and we define its *Ito integral*

$$\sum_{i\in\Lambda} \int_0^\infty g_t^i \, d\chi_t^i$$

to be the limit in Φ of

$$\sum_{i\in\Lambda} \sum_{j\in I\!\!N} \widetilde{g}_{t_j}^i \otimes \left(\chi_{t_{j+1}}^i - \chi_{t_j}^i \right) \tag{2}$$

where $\mathcal{S} = \{t_j,\ j\in I\!\!N\}$ is a partition of $I\!\!R^+$ which is understood to be refining and to have its diameter tending to 0, and $(\widetilde{g}_\cdot^i)_i$ is an Ito integrable family in Φ, such that for each i, $t \mapsto \widetilde{g}_t^i$ is a step process, and which converges to $(g_\cdot^i)_i$ in $L^2(I\!\!R^+ \times \mathcal{P})$.

Note that by assumption we always have that $\widetilde{g}_{t_j}^i$ belongs to $\Phi_{t_j]}$ and $\chi_{t_{j+1}}^i - \chi_{t_j}^i$ belongs to $\Phi_{[t_j,t_{j+1}]}$, hence the tensor product symbol in (2).

Also note that, as an example, one can take

$$\widetilde{g}_t^i = \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} P_{t_j} g_s^i \, ds$$

where P_t is the orthogonal projection onto $\Phi_{[0,t]}$.

One then obtains the following properties ([At1], Proposition 1.4):

Theorem 3. – *The Ito integral* $I(g) = \sum_i \int_0^\infty g_t^i \, d\chi_t^i$, *of an Ito integrable family* $g = (g_\cdot^i)_{i\in\Lambda}$, *is the element of* Φ *given by*

$$I(g)(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g^i_{\vee\sigma}(\sigma \setminus (\vee\sigma, i)) & \text{if } \vee\sigma \in \sigma_i. \end{cases}$$

It satisfies the Ito isometry formula*:*

$$||I(g)||^2 = \left\| \sum_{i\in\Lambda} \int_0^\infty g_t^i \, d\chi_t^i \right\|^2 = \sum_{i\in\Lambda} \int_0^\infty \left\| g_t^i \right\|^2 \, dt \ . \tag{3}$$

■

In particular, consider a family $f = (f^i)_{i\in\Lambda}$ which belongs to $L^2(\mathcal{P}_1) = L^2(\mathbb{R}^+ \times \Lambda)$, then the family $(f^i(t)\Omega)$, $t \in \mathbb{R}^+$, $i \in \Lambda$, is clearly Ito integrable. Computing its Ito integral we find that

$$I(f) = \sum_{i\in\Lambda} \int_0^\infty f^i(t)\Omega \, d\chi_t^i$$

is the element of the first particle space of the Fock space Φ associated to the function f, that is,

$$I(f)(\sigma) = \begin{cases} f^i(s) & \text{if } \sigma = \{s\}_i \\ 0 & \text{otherwise.} \end{cases}$$

Let us define the "adjoint" mapping of the Ito integral. For all $f \in \Phi$, all i in Λ and all $t \in \mathbb{R}^+$, consider the following mapping on $\mathcal{P}$:

$$\left[D_t^i f\right](\sigma) = f(\sigma \cup \{(s,i)\}) \mathbb{1}_{\sigma\subset[0,s[}.$$

We then have the following result ([At1], Theorem 1.6).

Theorem 4. [Fock space predictable representation property] – *For all* $f \in \Phi$, *all* $i \in \Lambda$ *and for almost all* $t \in \mathbb{R}^+$, *the mapping* $D_t^i f$ *belongs to* $\Phi = L^2(\mathcal{P})$. *Furthermore, the family* $(D_\cdot^i f)_i$ *is always Ito integrable and we have the representation*

$$f = f(\emptyset)\Omega + \sum_{i\in\Lambda} \int_0^\infty D_t^i f \, d\chi_t^i \tag{4}$$

with the isometry formula

$$||f||^2 = |f(\emptyset)|^2 + \sum_{i\in\Lambda} \int_0^\infty \left|\left|D_t^i f\right|\right|^2 dt. \tag{5}$$

■

As an immediate corollary we get the following.

Corollary 5. – *The representation (4) of* f *is unique. In particular, if* $g \in \Phi$ *is of the form*

$$g = c\,\Omega + \sum_{i\in\Lambda} \int_0^\infty h_t^i \, d\chi_t^i$$

then for almost all t, all i in Λ,

$$D_t^i g = h_t^i \,.$$

■

Let $f\in L^2(\mathcal{P}_n)$, one can easily define the *iterated Ito integral* on Φ:

$$I_n(f) = \int_{\mathcal{P}_n} f(\sigma)\, d\chi_\sigma$$

by iterating the definition of the Ito integral:

$$I_n(f) = \sum_{i_1,\ldots,i_n\in\Lambda} \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f^{i_1,\ldots,i_n}(t_1,\ldots,t_n)\Omega\; d\chi_{t_1}^{i_1}\,\ldots\, d\chi_{t_n}^{i_n}.$$

We obtain this way an element of Φ which is actually the representant of f in the n-particle subspace of Φ, that is

$$[I_n(f)](\sigma) = \begin{cases} f^{i_1,\ldots,i_n}(t_1,\ldots,t_n) & \text{if } \sigma = \{t_1\}_{i_1}\cup\ldots\cup\{t_n\}_{i_n} \\ 0 & \text{otherwise.}\end{cases}$$

For any $f\in\mathcal{P}$ we put

$$\int_{\mathcal{P}} f(\sigma)\, d\chi_\sigma$$

to denote the series of iterated Ito integrals

$$f(\emptyset)\Omega + \sum_{n=1}^\infty \sum_{i_1,\ldots,i_n\in\Lambda} \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f^{i_1,\ldots,i_n}(t_1,\ldots,t_n)\Omega\, d\chi_{t_1}^{i_1}\,\ldots\, d\chi_{t_n}^{i_n}.$$

We then have the following representation ([At1], Theorem 1.7).

Theorem 6. [Fock space chaotic representation property] – *Any element f of Φ admits an* abstract chaotic representation

$$f = \int_{\mathcal{P}} f(\sigma)\; d\chi_\sigma \tag{6}$$

satisfying the isometry formula

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2\; d\sigma \qquad (7).$$

This representation is unique.

■

The above theorem is the exact expression of the heuristics we wanted in order to describe the space

$$\Phi = \bigotimes_{\mathbb{R}^+} \mathcal{H}.$$

Indeed, we have, for each $t\in\mathbb{R}^+$, a family of elementary orthonormal elements Ω, $d\chi_t^i$, $i\in\Lambda$ (a basis of $\mathcal{H}$) whose (tensor) products $d\chi_\sigma$ form a continuous basis of Φ (formula (6)) and, even more, form an orthonormal continuous basis (formula (7)).

The attentive reader will have noticed that the only property of the curves $\chi^i_\cdot$ that we really used is the fact that $\chi^i_t - \chi^i_s$ belongs to $\Phi_{[s,t]}$ for all $s \le t$. One can naturally wonder if there exists another such family, which will then allow another Ito integral and furnish another continuous basis for Φ via another chaotic expansion property.

Of course there are obvious curves that can be obtained from the $\chi^i_\cdot$: for any function f on $\mathbb{R}^+$ and any $g \in L^2(\mathbb{R}^+ \times \Lambda)$ put

$$y_t = f(t)\Omega + \sum_i \int_0^t g^i(s)\Omega \, d\chi^i_s$$

for all $t \in \mathbb{R}^+$. Then one easily checks that $(y_\cdot)$ satisfies the same property, namely, $y_t - y_s$ belongs to $\Phi_{[s,t]}$ for all $s \le t$. But clearly the Ito integration theory obtained from y is the same as the one from χ, except that scalar factors $g^i(s)$ will appear in the integration.

One can wonder if there exist more complicated examples, giving rise to a different Ito integration. The following result shows that there are no more examples. In particular, there is only one Ito integral, one chaotic expansion and one natural continuous basis ([At1], Theorem 1.8).

Theorem 7. – *Let $(y_t)_{t\ge 0}$ be a curve in Φ such that $y_t - y_s$ belongs to $\Phi_{[s,t]}$ for all $s \le t$. Then there exist a function f on $\mathbb{R}^+$ and $g \in L^2(\mathbb{R}^+ \times \Lambda)$ such that*

$$y_t = f(t)\Omega + \sum_i \int_0^t g^i(s)\Omega \, d\chi^i_s$$

for all $t \in \mathbb{R}^+$.

■

III.2 The quantum noises

The space Φ we have constructed is the natural space for defining quantum noises. These quantum noises are the natural, continuous-time, extensions of the basis operators $a^i_j(n)$ we met in the atom chain $T\Phi$.

As indicated in the heuristic discussion above, we shall deal with a family of infinitesimal operators $da^i_j(t)$ on Φ which act on the continuous basis $d\chi_\sigma$ in the same way as they discrete-time counterparts $a^i_j(n)$ act on the X_A. The integrated version of the above heuristic infinitesimal formulas easily gives an exact formula for the action of the operators $a^i_j(t)$ on Φ:

$$[a^0_i(t)f](\sigma) = \sum_{\substack{s\in\sigma_i \\ s\le t}} f(\sigma \setminus \{s\}_i),$$

$$[a^i_0(t)f](\sigma) = \int_0^t f(\sigma \cup \{s\}_i) \, ds,$$

$$[a^i_j(t)f](\sigma) = \sum_{\substack{s\in\sigma_j \\ s\le t}} f(\sigma \setminus \{s\}_j \cup \{s\}_i)$$

$$[a^0_0(t)f](\sigma) = t\, f(\sigma)$$

for $i, j \neq 0$.

All these operators, except $a_0^0(t)$, are unbounded. But note that a good common domain to all these operators is

$$\mathcal{D} = \left\{ f \in \Phi \ ; \ \int_{\mathcal{P}} |\sigma| \ |f(\sigma)|^2 \ d\sigma < \infty \right\}.$$

This family of operators is characteristic and universal in a sense which is close to the one of the curves χ_t^i. Indeed, one can easily check that in the decomposition of $\Phi \simeq \Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$, the operators $a_j^i(t) - a_j^i(s)$ are all of the form

$$I \otimes (a_j^i(t) - a_j^i(s))_{|\Phi_{[s,t]}} \otimes I.$$

This property is fundamental for the definition of the quantum stochastic integrals and, in the same way as for $(\chi_\cdot^i)$, these operator families are the only ones to share that property (cf [Coq]).

This property allows to consider Riemann sums:

$$\sum_k H_{t_k} \left(a_j^i(t_{k+1}) - a_j^i(t_k) \right) \tag{8}$$

where $\mathcal{S} = \{0 = t_0 < t_1 < \ldots < t_k < \ldots\}$ is a partition of $\mathbb{R}^+$, where $(H_t)_{t \geq 0}$ is a family of operators on Φ such that

– each H_t is an operator of the form $H_t \otimes I$ in the tensor product space $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ (we say that H_t is a *t-adapted operator* and that $(H_t)_{t \geq 0}$ is an *adapted process of operators*),

– $(H_t)_{t \geq 0}$ is a *step process*, that is, it is constant on intervals:

$$H_t = \sum_k H_{t_k} \mathbb{1}_{[t_k, t_{k+1}]}(t),$$

and where the operator product $H_{t_k} \left(a_j^i(t_{k+1}) - a_j^i(t_k) \right)$ is actually a tensor product of operators

$$H_{t_k} \otimes \left(a_j^i(t_{k+1}) - a_j^i(t_k) \right).$$

Note that, in particular, the above "product" is commutative and does not impose any new domain constraint.

The resulting operator associated to the Riemann sum (8) is denoted by

$$\int_0^\infty H_s \, da_j^i(s).$$

If we denote by T the above operator and by T_t the operator

$$\int_0^t H_s \, da_j^i(s) = \int_0^\infty H_s \mathbb{1}_{[0,t]}(s) \, da_j^i(s)$$

we can then compute the action of T on a "good" vector f of its domain and we obtain (cf [A-M] for more details)

$$Tf = \sum_{k \in \Lambda} \int_0^\infty T_t D_t^k f \, d\chi_t^k + \int_0^\infty H_t D_t^i f \, d\chi_s^j \tag{9}$$

with the notations: $D_t^0 = P_t$ and $d\chi_t^0 = dt$. For general operator processes (still adapted but not step process anymore) and general f, it is equation (9) which is kept as a definition for the domain and for the action of the operator

$$T = \int_0^\infty H_s \, da_j^i(s).$$

The maximal domain and the explicit action of the above operator can be described but is not worth developing here. The interested reader may refer to [At3], chapter 12 or to [A-L]. There are particular domains where the definition simplifies. The one we shall use here is the case of coherent vectors.

Indeed, if ϕ is any element of $L^2(\mathbb{R}^+, \mathcal{H}')$, consider the associated coherent vector $\varepsilon(\phi)$ in Φ. That is,

$$[\varepsilon(\phi)](\sigma) = \prod_i \prod_{s\in\sigma_i} \phi_i(s).$$

Put $\phi_0(s) = 1$ for all s. If ϕ is such that

$$\int_0^t |\phi_j(s)|^{(1-\delta_{0i})(2-\delta_{0j})} \, \|H_s\varepsilon(\phi)\|^{2-\delta_{0j}} \; ds < \infty$$

then $\int_0^t H_s \, da_j^i(s)$ is well-defined on $\varepsilon(\phi)$ with

$$<\varepsilon(\psi)\, , \int_0^t H_s \, da_j^i(s) \, \varepsilon(\phi)> = \int_0^t \overline{\psi}_j(s)\phi_i(s)<\varepsilon(\psi)\, , \, H_s \, \varepsilon(\phi)> ds.$$

for all $\psi \in L^2(\mathbb{R}^+, \mathcal{H}')$.

III.3 Embedding and approximation by the Toy Fock space

We now describe the way the atom chain and its basic operators can be realized as a subspace of the Fock space and a projection of the quantum noises. The subspace associated to the atom chain is attached to the choice of some partition of $\mathbb{R}^+$ in such a way that the expected properties are satisfied:

– the associated subspaces increase when the partition refines and they constitute an approximation of Φ when the diameter of the partition goes to 0,

– the associated basic operators are restrictions of the others when the partition increases and they constitute an approximation of the quantum noises when the diameter of the partition goes to 0.

Note that this approximation has deep interpretations in terms of approximations of n-dimensional classical noises by extremal random walks in $\mathbb{R}^n$ whose jumps take $n+1$ different values. This aspect is developed in [A-P].

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a partition of $\mathbb{R}^+$ and $\delta(\mathcal{S}) = \sup_i |t_{i+1} - t_i|$ be the diameter of $\mathcal{S}$. For $\mathcal{S}$ being fixed, define $\Phi_n = \Phi_{[t_{n-1}, t_n]}$, $n \in \mathbb{N}^*$. We clearly have that Φ is naturally isomorphic to the countable tensor product $\otimes_{n\in\mathbb{N}^*} \Phi_n$ (which is understood to be defined with respect to the stabilizing sequence $(\Omega)_{n\in\mathbb{N}}$).

For all $n \in \mathbb{N}^*$, define for $i, j \in \Lambda$

$$\begin{aligned}
X^i(n) &= \frac{\chi^i_{t_n} - \chi^i_{t_{n-1}}}{\sqrt{t_n - t_{n-1}}} \in \Phi_n \ , \\
a^i_0(n) &= \frac{a^i_0(t_n) - a^i_0(t_{n-1})}{\sqrt{t_n - t_{n-1}}} \circ P_{1]}, \\
a^i_j(n) &= P_{1]} \circ \big(a^i_j(t_n) - a^i_j(t_{n-1})\big) \circ P_{1]}, \\
a^0_i(n) &= P_{1]} \circ \frac{a^0_i(t_n) - a^0_i(t_{n-1})}{\sqrt{t_n - t_{n-1}}} \ , \\
a^0_0(n) &= \qquad P_{0]},
\end{aligned}$$

where for $i = 0, 1$, $P_{i]}$ is the orthogonal projection onto $L^2(\mathcal{P}_i)$ and where the above definitions are understood to be valid on Φ_n only, the corresponding operator acting as the identity operator I on the others Φ_m's.

For every $A \in \mathcal{P} = \mathcal{P}_{\mathbb{N}^*,\Lambda}$, define X_A from the $X^i(n)$'s in the same way as for $\mathrm{T}\Phi$:

$$X_A = \Omega \otimes \ldots \otimes \Omega \otimes X^{i_1}(n_1) \otimes \Omega \otimes \ldots \otimes \Omega \otimes X^{i_2}(n_2) \otimes \ldots$$

in $\otimes_{n \in \mathbb{N}^*} \mathcal{H}_n$.

Define $T\Phi(\mathcal{S})$ to be the space of $f \in \Phi$ which are of the form

$$f = \sum_{A \in \mathcal{P}} f(A) X_A$$

(note that the condition $\|f\|^2 = \sum_{A \in \mathcal{P}} |f(A)|^2 < \infty$ is automatically satisfied).

The space $T\Phi(\mathcal{S})$ is thus clearly identifiable to the spin chain $\mathrm{T}\Phi$. The space $T\Phi(\mathcal{S})$ is a closed subspace of Φ. We denote by $P_{\mathcal{S}}$ the operator of orthogonal projection from Φ onto $T\Phi(\mathcal{S})$. One can prove for example that the projection of an exponential vector is an "exponential vector" of the embedded toy Fock space: indeed, a direct computation shows that for any ϕ in $L^2(\mathbb{R}^+, \mathcal{H}')$,

$$\big(P_{\mathcal{S}}\, \varepsilon(\phi)\big)(A) = \prod_i \prod_{n \in A_i} \tilde{\phi}_i(n)$$

where the function $\tilde{\phi}$ belongs to $l^2(\mathbb{N}^*, \mathcal{H}')$ and is defined by

$$\tilde{\phi}_i(n) = \frac{1}{\sqrt{t_n - t_{n-1}}} \int_{t_{n-1}}^{t_n} \phi_i(s)\, ds.$$

We will denote by $e(\tilde{\phi})$ such a discrete time version of a coherent vector.

We shall now check that the above operators $a^i_j(n)$ act on $\mathrm{T}\Phi(\mathcal{S})$ in the same way as the the basic operators of $\mathrm{T}\Phi$.

Proposition 8. – *We have, for all $i, j \in \Lambda$*

$$\begin{cases} a_0^i(n)\, X^j(n) = \delta_{ij}\Omega \\ a_0^i\, \Omega = 0 \end{cases}$$

$$\begin{cases} a_j^i(n)\, X^k(n) = \delta_{ik} X^j(n) \\ a_j^i\, \Omega = 0 \end{cases}$$

$$\begin{cases} a_i^0(n)\, X^j(n) = 0 \\ a_i^0(n)\, \Omega = X^i(n) \end{cases}$$

$$\begin{cases} a_0^0(n)\, X^k(n) = 0 \\ a_j^i\, \Omega = \Omega. \end{cases}$$

Proof

This is a direct application of the definitions and computations using equation (9), cf [At2] for details. For example:

$$\begin{aligned} a_0^i(n) X^j(n) &= \frac{1}{t_n - t_{n-1}} \left(a_0^i(t_n) - a_0^i(t_{n-1})\right) \int_{t_{n-1}}^{t_n} \Omega\ d\chi_t^j \\ &= \frac{1}{t_n - t_{n-1}} \left[\sum_{k\in\Lambda} \int_{t_{n-1}}^{t_n} \left(a_0^i(t) - a_0^i(t_{n-1})\right) \Omega\ d\chi_t^k + \int_{t_{n-1}}^{t_n} \delta_{ij}\Omega\ dt \right] \\ &= \frac{1}{t_n - t_{n-1}} \left(0 + (t_n - t_{n-1})\delta_{ij}\Omega\right) = \delta_{ij}\Omega. \end{aligned}$$

And so on for the other cases. ■

Thus the action of the operators a_j^i on the $X^i(n)$ is exactly the same as the action of the corresponding operators on the spin chain of section II; the operators $a_j^i(n)$ act on $T\Phi(\mathcal{S})$ exactly in the same way as the corresponding operators do on TΦ. We have completely embedded the toy Fock space in the Fock space.

The action of operators $a_j^i(n)$ on discrete exponential vectors as defined above will be most useful in the sequel. The following lemma is deduced immediately from Proposition 8.

Lemma 9. – *For any ϕ, ψ in $L^2(\mathbb{R}_+, \mathcal{H})$ and for any t_n-adapted operator H_n the bracket*

$$\left\langle e(\tilde{\phi}),\, H_n a_j^i(n+1) e(\tilde{\psi}) \right\rangle$$

is equal to

$$\overline{\tilde{\phi}_j}(n+1)\tilde{\psi}_i(n+1) \left\langle e(\tilde{\phi}_{n]}),\, H_n e(\tilde{\psi}_{n]}) \right\rangle \left\langle e(\tilde{\phi}_{[n+2}),\, e(\tilde{\psi}_{[n+2}) \right\rangle .$$

This lemma is the basis for our future computations involving discrete-time quantum stochastic integrals (for more precise treatment of this subject see [Pa1] or [Pa3]).

We are now going to see that the Fock space Φ and its basic operators $a^i_j(t)$, $i, j \in \Lambda \cup \{0\}$ can be approached by the toy Fock spaces $T\Phi(\mathcal{S})$ and their basic operators $a^i_j(n)$.

We are given a sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of partitions which are getting finer and finer and whose diameter $\delta(\mathcal{S}_n)$ tends to 0 when n tends to $+\infty$. Let $T\Phi(n) = T\Phi(\mathcal{S}_n)$ and $P_n = P_{\mathcal{S}_n}$, for all $n\in\mathbb{N}$.

Theorem 10. –

i) The orthogonal projectors P_n strongly converge to the identity operator I on Φ. That is, any $f \in \Phi$ can be approached in Φ by a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in$ TΦ(n) for all $n \in \mathbb{N}$.

ii) If $\mathcal{S}_n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_k < \cdots\}$, then for all $t\in\mathbb{R}^+$, the operators $\sum_{k;t^n_k\leq t} a^i_j(k)$, $\sum_{k;t^n_k\leq t} \sqrt{t^n_k - t^n_{k-1}}\, a^i_0(k)$, $\sum_{k;t^n_k\leq t} \sqrt{t^n_k - t^n_{k-1}}\, a^0_i(k)$, and $\sum_{k;t^n_k\leq t}(t^n_k - t^n_{k-1})\, a^0_0(k)$ converge strongly on $\mathcal{D}$ to $a^i_j(t)$, $a^i_0(t)$, $a^0_i(t)$ and $a^0_0(t)$ respectively.

Proof

i) As the $\mathcal{S}_n$ are refining then the $(P_n)_n$ forms an increasing family of orthogonal projection in Φ. Let $P_\infty = \vee_n P_n$. Clearly, for all $s \leq t$, we have that $\chi^i_t - \chi^i_s$ belongs to RanP_∞. But by the construction of the Ito integral and by Theorem 6, we have that the $\chi^i_t - \chi^i_s$ generate Φ. Thus $P_\infty = I$.

ii) Let us check the case of a^0_i. A direct computation shows that, for $f\in\mathcal{D}$

$$\Big[\sum_{k;t^n_k\leq t} \sqrt{t^n_k - t^n_{k-1}}\, a^0_i(k) f\Big](\sigma) = \sum_{k;t^n_k\leq t} \mathbb{1}_{|\sigma_i\cap[t^n_{k-1},t^n_k]|=1} \sum_{s\in\sigma_i\cap[t^n_{k-1},t^n_k]} f(\sigma\setminus\{s\}_i).$$

Put $t^n = \sup\{t^n_k\in\mathcal{S}_n\ ; t^n_k \leq t\}$. We have

$$\begin{aligned}
&\Big\| \Big(\sum_{k;t^n_k\leq t} \sqrt{t^n_k - t^n_{k-1}}\, a^0_i(k) - a^0_i(t)\Big) f\Big\|^2 \\
&= \int_{\mathcal{P}} \Big| \sum_{k;t^n_k\leq t} \mathbb{1}_{|\sigma_i\cap[t^n_{k-1},t^n_k]|=1} \sum_{s\in\sigma_i\cap[t^n_{k-1},t^n_k]} f(\sigma\setminus\{s\}_i) - \sum_{s\in\sigma_i\cap[0,t]} f(\sigma\setminus\{s\}_i)\Big|^2 \, d\sigma \\
&\leq 2\int_{\mathcal{P}} \Big| \sum_{s\in\sigma\cap[t^n,t]} f(\sigma\setminus\{s\}_i)\Big|^2 \, d\sigma + 2\int_{\mathcal{P}} \Big| \sum_{k;t^n_k\leq t} \mathbb{1}_{|\sigma_i\cap[t^n_{k-1},t^n_k]|\geq 2} \\
&\qquad\qquad \times \sum_{s\in\sigma_i\cap[t^n_{k-1},t^n_k]} f(\sigma\setminus\{s\}_i)\Big|^2 \, d\sigma.
\end{aligned}$$

For any fixed σ, the terms inside each of the integrals above converge to 0 when n

tends to $+\infty$. Furthermore we have, for n large enough,

$$\begin{aligned}\int_{\mathcal{P}}\Big|\sum_{s\in\sigma\cap[t^n,t]} f(\sigma\setminus\{s\}_i)\Big|^2\,d\sigma &\leq \int_{\mathcal{P}}|\sigma|\sum_{\substack{s\in\sigma\\ s\leq t+1}}|f(\sigma\setminus\{s\}_i)|^2\,d\sigma\\ &=\int_0^{t+1}\int_{\mathcal{P}}(|\sigma|+1)|f(\sigma)|^2\,d\sigma\,ds\\ &\leq (t+1)\int_{\mathcal{P}}(|\sigma|+1)|f(\sigma)|^2\,d\sigma\end{aligned}$$

which is finite for $f\in\mathcal{D}$;

$$\begin{aligned}&\int_{\mathcal{P}}\Big|\sum_{k;t_k^n\leq t}1\!\!1_{|\sigma_i\cap[t_{k-1}^n,t_k^n]|\geq 2}\sum_{s\in\sigma_i\cap[t_{k-1}^n,t_k^n]} f(\sigma\setminus\{s\}_i)\Big|^2\,d\sigma\\ &\qquad\leq\int_{\mathcal{P}}\Big(\sum_{k;t_k^n\leq t}1\!\!1_{|\sigma_i\cap[t_{k-1}^n,t_k^n]|\geq 2}\Big|\sum_{s\in\sigma_i\cap[t_{k-1}^n,t_k^n]} f(\sigma\setminus\{s\}_i)\Big|\Big)^2\,d\sigma\\ &\qquad\leq\int_{\mathcal{P}}\Big(\sum_{k;t_l^n\leq t}\sum_{s\in\sigma_i\cap[t_{k-1}^n,t_k^n]}|f(\sigma\setminus\{s\}_i)|\Big)^2\,d\sigma\\ &\qquad=\int_{\mathcal{P}}\Big(\sum_{\substack{s\in\sigma_i\\ s\leq t^n}}|f(\sigma\setminus\{s\}_i)|\Big)^2\,d\sigma\\ &\qquad=\int_{\mathcal{P}}|\sigma|\sum_{\substack{s\in\sigma_i\\ s\leq t^n}}|f(\sigma\setminus\{s\}_i)|^2\,d\sigma\\ &\qquad\leq (t+1)\int_{\mathcal{P}}(|\sigma|+1)\big|f(\sigma)\big|^2\,d\sigma<\infty\end{aligned}$$

in the same way as above. So we can apply Legesgue's theorem. This proves the result.

The other cases are treated in the same way. See [At2] for details. ■

We have fulfilled our duties: not only the space $T\Phi(\mathcal{S})$ recreates $T\Phi$ and its basic operators as a subspace of Φ and a projection of its quantum noises, but, when $\delta(\mathcal{S})$ tends to 0, this realisation constitutes an approximation of the space Φ and of its quantum noises.

To any operator H on Φ we can associate the projected operator $P_{\mathcal{S}}HP_{\mathcal{S}}$ which acts on the atom chain only and which approximates H (if H is bounded for example).

We wish to compute the corresponding projections of the quantum stochastic integral operators. We reduce our computations to the case where integrals are of the type $H=\int_0^\infty H_t^\epsilon\,da_j^i(s)$, with $(i,j)\neq(0,0)$, and satisfy the following conditions **(HS)**:

– the operator H is bounded and

– the integrands $H^i_j(t)$ are bounded for all t and $t \mapsto \left\|H^i_j(t)\right\|$ is square integrable if one of i or j is zero, essentially bounded otherwise.

Even though they are rather restrictive, these hypotheses will suffice for our neeeds.

The following result is a consequence of the theory and the computations developed in [Pa1] and [Pa2] (adapted here to the case of higher multiplicity) and we do not reproduce the proof here. It is also stated in the following form in [Pa3], chapter 4.

Theorem 11. – *Let $(i,j) \neq (0,0)$ be fixed. Let $H = \int H^i_j(t)da^i_j(t)$ be a quantum stochastic integral on Φ that satisfies the assumptions* **(HS)**. *Then $P_{\mathcal{S}} H P_{\mathcal{S}}$ is an operator on $\mathrm{T}\Phi$ of the form*

$$\sum_{k,l} \sum_n h^k_l(n)\, a^k_l(n+1)$$

where the sum is over all couples (k,l) in $(\Lambda \cup \{0\})^2$ different from $(0,0)$ and is meaningful in the weak sense. The operators h^k_l are given by :

– if both i and j are nonzero,

$$h^k_l(n) = \delta_{ki}\delta_{lj} \frac{1}{t_{n+1}-t_n} P_{\mathcal{S}} \int_{t_n}^{t_{n+1}} P_{t_n} H^i_j(t)\, dt$$

– if $i=0$,

$$h^0_l(n) = \delta_{lj} \frac{1}{\sqrt{t_{n+1}-t_n}} P_{\mathcal{S}} \int_{t_n}^{t_{n+1}} P_{t_n} H^0_j(t)\, dt$$

and for all $k \neq 0$,

$$h^k_l(n) = \delta_{lj} \frac{1}{t_{n+1}-t_n} P_{\mathcal{S}} \int_{t_n}^{t_{n+1}} P_{t_n} H^0_j(t) \left(a^0_k(t) - a^0_k(t_n)\right)\, dt$$

– if $j=0$,

$$h^k_0(n) = \delta_{ki} \frac{1}{\sqrt{t_{n+1}-t_n}} P_{\mathcal{S}} \int_{t_n}^{t_{n+1}} P_{t_n} H^i_0(t)\, dt$$

and for all $l \neq 0$,

$$h^k_l(n) = \delta_{ki} \frac{1}{t_{n+1}-t_n} P_{\mathcal{S}} \int_{t_n}^{t_{n+1}} P_{t_n} \left(a^l_0(t) - a^l_0(t_n)\right) H^i_0(t)\, dt.$$

■

III.4 Quantum Langevin equations

In this article what we call quantum Langevin equation is actually a restricted version of what is usually understood in the literature (cf [G-Z]); by this we mean that we here study the so-called quantum stochastic differential equations as defined by Hudson and Parthasarathy and heavily studied by further authors.

This type of quantum noise perturbation of the Schrödinger equation is exactly the type of equation which we will get as the continuous limit of our Hamiltonian description of repeated quantum interactions.

The aim of quantum stochastic differential equations is to study equations of the form

$$dU_t = \sum_{i,j\in\Lambda\cup\{0\}} L^i_j U_t\, da^i_j(t), \tag{10}$$

with initial condition $U_0 = I$. The above equation has to be understood as an integral equation

$$U_t = I + \int_0^t \sum_{i,j\in\Lambda\cup\{0\}} L^i_j U_t\, da^i_j(t),$$

for operators on $\mathcal{H}_0 \otimes \Phi$, the operators L^i_j being bounded operators on $\mathcal{H}_0$ alone which are ampliated to $\mathcal{H}_0 \otimes \Phi$.

The main motivation and application of that kind of equation is that it gives an account of the interaction of the small system $\mathcal{H}_0$ with the bath Φ in terms of quantum noise perturbation of a Schrödinger-like equation. Indeed, the first term of the equation

$$dU_t = L^0_0 U_t\, dt + \dots$$

describes the induced dynamics on the small system, all the other terms are quantum noises terms.

One of the main application of equations such as (10) is that they give explicit constructions of unitary dilations of semigroups of completely positive maps of $\mathcal{B}(\mathcal{H}_0)$ (see [H-P]). Let us here only recall one of the main existence, uniqueness and boundedness theorems connected to equations of the form (10). The literature is huge about those equations; we refer to [Par] for the result we mention here.

Theorem 12. – *If $\mathcal{H}_0$ is separable and*

$$\|L\| = \Big(\sum_{i,j\in\Lambda\cup\{0\}} \|L^i_j\|^2\Big)^{1/2} < +\infty,$$

then the quantum stochastic differential equation

$$U_t = I + \sum_{i,j} \int_0^t L^i_j U_t\, da^i_j(t)$$

admits a unique solution defined on the space of coherent vectors.

The solution $(U_t)_{t\geq 0}$ is made of unitary operators if and only if there exist, on $\mathcal{H}_0$, a self-adjoint operator H, operators L_i, $i \in \Lambda$ and operators S^i_j, $i,j \in \Lambda$ such that $(S^i_j)_{i,j\in\Lambda}$ is unitary and the coefficients L^i_j are of the form

$$L^0_0 = -(iH + \frac{1}{2}\sum_{k\in\Lambda} L^*_k L_k)$$

$$L^0_j = L_j$$

$$L_0^i = -\sum_{k\in\Lambda} L_k^* S_i^k$$
$$L_j^i = S_j^i - \delta_{ij} I.$$

■

IV Convergence theorems

IV.1 Convergence to quantum Langevin equations

We are now ready to assemble together all the pieces of the puzzle and prove that the Hamiltonian dynamic associated to repeated quantum interactions spontaneously converges to a quantum Langevin equation under some normalization conditions on the Hamiltonian. Notice that we no longer assume that $\mathbb{L}(h)$ has been conveniently constructed for our needs; in particular $\mathbb{L}$ is not assumed to be unitary.

Let h be a parameter in $\mathbb{R}^+$, which is thought of as representing a small time interval. Let $\mathbb{L}(h)$ be an operator on $\mathcal{H}_0 \otimes \mathcal{H}$, with coefficients $\mathbb{L}_j^i(h)$ as a matrix of operators on $\mathcal{H}_0$. Let $u_n(h)$ be the associated solution of

$$u_{n+1}(h) = \mathbb{L}_{n+1}(h) u_n(h)$$

with the same notation as in section II.3. In the following we will drop dependency in h and write simply $\mathbb{L}$, or u_n. Besides, we denote

$$\varepsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})$$

for all i, j in $\Lambda \cup \{0\}$. That is,

$$\varepsilon_{i0} = e_{0j} = \frac{1}{2}, \quad \varepsilon_{ij} = 0, \quad \varepsilon_{00} = 1.$$

Note that from now on we take the embedding of TΦ in Φ for granted and we consider, without mentionning it, all the repeated quantum interactions to happen in TΦ(h), the subspace of Φ associated to the partition $\mathcal{S} = \{t_i = ih; i \in \mathbb{N}\}$.

Theorem 13. – *Assume that there exist bounded operators L_j^i, $i, j \in \Lambda \cup \{0\}$ on $\mathcal{H}_0$ such that*

$$\sum_{i,j\in\Lambda\cup\{0\}} \left\|L_j^i\right\|^2 < +\infty$$

and

$$\lim_{h\to 0} \sum_{i,j\in\Lambda\cup\{0\}} \left\| \frac{\mathbb{L}_j^i(h) - \delta_{ij} I}{h^{\varepsilon_{ij}}} - L_j^i \right\|^2 = 0.$$

for all $i, j = 0, \ldots, N-1$. Assume that the quantum stochastic differential equation

$$dU_t = \sum_{i,j} L_j^i U_t \, da_j^i(t)$$

with initial condition $U_0 = I$ admits a unique solution $(U_t)_{t\geq 0}$ which is a process of bounded operators with locally uniform norm bound.

Then, for almost all t, for every ϕ, ψ in $L^\infty([0,t])$, the quantity

$$< a\otimes\varepsilon(\phi)\,,\, P_{\mathcal{S}} u_{[t/h]} P_{\mathcal{S}}\, b\otimes\varepsilon(\psi) >$$

converges to

$$< a\otimes\varepsilon(\phi)\,,\, U_t\, b\otimes\varepsilon(\psi) >$$

when h goes to 0.

Moreover, the convergence is uniform for a, b in any bounded ball of $\mathcal{H}$, uniform for t in a bounded interval of $\mathbb{R}_+$.

Remarks

– This is where we particularize the index zero : the above hypotheses of convergence simply mean that, among the coefficients of $\mathbb{L}$,

$(\mathbb{L}_0^0(h) - I)/h$ converges,

$\mathbb{L}_j^i(h)/\sqrt{h}$ converges if either i or j is zero,

$\mathbb{L}_j^i(h) - \delta_{i,j}$ converges if neither i nor j is zero.

We here meet the announced three time scales appearing in the Hamiltonian. We shall discuss the physical meaning of these normalizations in next section.

– In the case where the operator $\mathbb{L}$ is unitary and satisfies the convergence assumptions of the above theorem, then one can see that the limiting operators L_j^i are of the form given in the second part of Theorem 11.

– In that case, the solution $(U_t)_{t\in\mathbb{R}_+}$ enjoys a particular algebraic property which we won't define here: it is a *cocycle* (see [H-P] or chapter 6 of [Pa3]). This property traduces the fact that the evolution of the system is, in the limit, memory-less.

Consider the quantum stochastic differential equation on $\mathcal{H}_0\otimes\Phi$:

$$dU_s = \sum_{i,j} L_j^i U_s\, da_j^i(s)$$

where the L_j^i are the bounded operators on the initial space $\mathcal{H}_0$ given by our assumptions.

We consider that h is fixed and the associated partition $\mathcal{S} = \{0 = t_0 < t_1 = h < \ldots < t_k = kh < \ldots\}$ is also fixed. Note that we have chosen a regular partition only for simplicity and that all our results hold with general partitions when the mesh size tends to 0. We fix some bounded interval $[0,T]$ of $\mathbb{R}_+$.

We will proceed by successive simplifications. Consider the operator on the atom chain defined by

$$w_k = P_{\mathcal{S}} U_{t_k} P_{\mathcal{S}}.$$

The following lemma will be used over and again.

Lemma 14. – *For any $r < s$, any vectors $a\otimes\varepsilon(\phi)$, $b\otimes\varepsilon(\psi)$ with*

$$\phi,\psi \in L^2(\mathbb{R}^+;\mathcal{H}')\cap L^\infty(\mathbb{R}^+;\mathcal{H}'),$$

we have

$$|< a\otimes\varepsilon(\phi)\,,\,(U_s - U_r)\, b\otimes\varepsilon(\psi) >| \le C\,\|a\|\,\|b\|\,(s-r)$$

where C depends only on $\|L\|$, defined in Theorem 12, and on the L^2 and L^∞ norms of ϕ and ψ.

Proof

$$
\begin{aligned}
&|< a\otimes\varepsilon(\phi)\,,\,(U_s-U_r)b\otimes\varepsilon(\psi)>|\\
&\qquad\le \sum_{i,j}\int_r^s \left|\overline{\phi}_i(u)\right|\,|\psi_j(u)|\,\left|< a\otimes\varepsilon(\phi)\,,\,L^i_jU_ub\otimes\varepsilon(\psi)>\right|\,du\\
&\qquad\le \|L\|\int_r^s \|\phi(u)\|\,\|\psi(u)\|\,\|a\otimes\varepsilon(\phi)\|\,\|U_ub\otimes\varepsilon(\psi)\|\,du
\end{aligned}
$$

from which the estimate follows, using the fact that $\|\phi(u)\|\le\|\phi\|_\infty$ and $\|\psi(u)\|\le\|\psi\|_\infty$, the L^i_j are bounded and U is locally uniformly bounded. ∎

The following lemma shows that $(w_k)_k$ converges to $(U_t)_{t\ge0}$, in the same weak sense as in the theorem, as h goes to 0.

Lemma 15. – *For any $t_k<s$, any vectors $a\otimes\varepsilon(\phi)$, $b\otimes\varepsilon(\psi)$ with*

$$\phi,\psi\in L^2(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+),$$

we have

$$
\begin{aligned}
&|< a\otimes\varepsilon(\phi)\,,\,(w_k-U_s)b\otimes\varepsilon(\psi)>|\\
&\qquad\le C\,\|a\|\,\|b\|\,\left((s-t_k)+\|(I-P_{\mathcal S})\varepsilon(\phi)\|+\|(I-P_{\mathcal S})\varepsilon(\psi)\|\right).
\end{aligned}
$$

where C depends only on $\|L\|$ and on the L^2 and L^∞ norms of ϕ and ψ.

Proof

We have

$$
\begin{aligned}
&|< a\otimes\varepsilon(\phi)\,,\,(w_k-U_s)b\otimes\varepsilon(\psi)>|\le\\
&\qquad\le |< a\otimes\varepsilon(\phi)\,,\,(U_{t_k}-U_s)b\otimes\varepsilon(\psi)>|\\
&\qquad\quad+|< a\otimes\varepsilon(\phi)\,,\,(P_{\mathcal S}U_{t_k}P_{\mathcal S}-U_{t_k}P_{\mathcal S})b\otimes\varepsilon(\psi)>|\\
&\qquad\quad+|< a\otimes\varepsilon(\phi)\,,\,(U_{t_k}P_{\mathcal S}-U_{t_k})b\otimes\varepsilon(\psi)>|\\
&\qquad\le \sum_{i,j}\int_{t_k}^s \left|\overline{\phi}_i(u)\right|\,|\psi_j(u)|\,\left|< a\otimes\varepsilon(\phi)\,,\,L^i_jU_ub\otimes\varepsilon(\psi)>\right|\,du\\
&\qquad\quad+\|(I-P_{\mathcal S})a\otimes\varepsilon(\phi)\|\,\|U_{t_k}P_{\mathcal S}b\otimes\varepsilon(\psi)\|\\
&\qquad\quad+\left\|U^*_{t_k}a\otimes\varepsilon(\phi)\right\|\,\|(I-P_{\mathcal S})b\otimes\varepsilon(\psi)\|
\end{aligned}
$$

and we conclude as in the previous lemma. ∎

We can now prove Theorem 13.

Proof of Theorem 13

Let $\omega^i_j(h)$ be such that

$$\mathbb{L}^i_j(h)-\delta_{ij}I=h^{\varepsilon_{ij}}(L^i_j+\omega^i_j(h))$$

for all i, j in $\Lambda \cup \{0\}$. In particular we have that,

$$\sum_{i,j \in \Lambda \cup \{0\}} \left\| \omega_j^i(h) \right\|^2$$

converges to 0 when h tends to 0.

Consider the solution $(u_n)_{n \in \mathbb{N}^*}$ of

$$u_{n+1} = \mathbb{L}_{n+1} u_n$$

with the notations of section II.3. Note that if A denotes the matrix $\mathbb{L} - (\delta_{ij} I)_{i,j}$ we then have

$$u_{n+1} - u_n = A_{n+1} u_n.$$

Let F be the matrix $(h^{\varepsilon_{ij}} L_j^i + \widehat{\delta}_{ij} h L_0^0)_{i,j}$ where

$$\widehat{\delta}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } (i,j) \neq (0,0), \\ 0 & \text{if } i \neq j \text{ or } (i,j) = (0,0) \end{cases}$$

and consider the solution $(v_n)_{n \in \mathbb{N}}$ of the equation

$$v_{n+1} - v_n = F_{n+1} v_n.$$

Note that

$$A_{n+1} = \sum_{i,j} A_j^i a_j^i(n+1)$$

and similarly for F_{n+1}.

Also note that $a_j^i(n+1)$ commutes with u_n (*resp.* v_n), for they do not act on the same part of the space $\mathrm{T}\Phi$. Thus we get another useful way to write the above equations in terms of the basis $a_j^i(n)$:

$$u_{n+1} - u_n = \sum_{i,j} A_j^i u_n \, a_j^i(n+1).$$

and

$$v_{n+1} - v_n = \sum_{i,j} \left(h^{\varepsilon_{ij}} L_j^i + \widehat{\delta}_{ij} h L_0^0 \right) v_n \, a_j^i(n+1).$$

From the above lemma it is enough to prove the convergence to zero of $u_n - w_n$. We actually start with $w_n - v_n$.

From the fact that

$$U_{t_{k+1}} - U_{t_k} = \sum_{i,j} \int_{t_k}^{t_{k+1}} L_j^i U_s \, da_j^i(s)$$

and thanks to the formulas for projections of Fock space integrals onto the toy Fock space in Theorem 11, one obtains the following expression for $w_{k+1} - w_k$ (be careful that the $da_0^0(t)$ integrals gives rise to $a_i^i(k)$ terms for *all* i, for $I = \sum_i a_i^i$):

$$\begin{aligned} w_{k+1} - w_k = & \sum_{i,j \neq (0,0)} h^{\varepsilon_{ij}} L_j^i \left(\frac{1}{h} P_{\mathcal{S}} \int_{t_k}^{t_{k+1}} P_{t_k} U_t \, dt \right) a_j^i(k+1) P_{\mathcal{S}} \\ & + \sum_i h L_0^0 \left(\frac{1}{h} P_{\mathcal{S}} \int_{t_k}^{t_{k+1}} U_t \, dt \right) a_i^i(k+1) P_{\mathcal{S}} \end{aligned}$$

$$+\sum_{i\in\Lambda}\sum_{j\in\Lambda} P_{\mathcal{S}}\left(\frac{1}{h}\int_{t_k}^{t_{k+1}}\Big(L_0^i P_{t_k} U_t(a_0^j(t)-a_0^j(t_k))\Big)\,dt\right.$$
$$\left.+\frac{1}{h}\int_{t_k}^{t_{k+1}}\big(L_j^0 P_{t_k} U_t(a_i^0(t)-a_i^0(t_k))\big)\,dt\right) a_j^i(k+1)P_{\mathcal{S}}.$$

As a consequence

$$\begin{aligned}w_n-v_n=&\sum_{k<n}\sum_{i,j\neq(0,0)} h^{\varepsilon_{ij}}L_j^i\,a_j^i(k+1)\left(\frac{1}{h}\,P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}P_{t_k}U_t\,dtP_{\mathcal{S}}-v_k\right)\\ &+\sum_{k<n}\sum_{i} hL_0^0\,a_i^i(k+1)\left(\frac{1}{h}P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}U_t\,dtP_{\mathcal{S}}-v_k\right)\\ &+\sum_{k<n}\sum_{i\in\Lambda}\sum_{j\in\Lambda}P_{\mathcal{S}}\left(\frac{1}{h}\int_{t_k}^{t_{k+1}}\Big(L_0^iP_{t_k}U_t(a_0^j(t)-a_0^j(t_k))\Big)\,dt\right.\\ &\left.+\frac{1}{h}\int_{t_k}^{t_{k+1}}\big(L_j^0P_{t_k}U_t(a_i^0(t)-a_i^0(t_k))\big)\,dt\right)P_{\mathcal{S}}\,a_j^i(k+1).\end{aligned}\tag{11}$$

We first wish to replace $\frac{1}{h}\,P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}P_{t_k}U_t\,dt$ or $\frac{1}{h}\,P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}U_t\,dt$ in the first two terms by w_k. Lemma 14 allows us to estimate the error term. Consider two essentially bounded functions ϕ, ψ in $L^2(\mathbb{R}_+)$ and two vectors a, b in the unit ball $\mathcal{H}_1$ of $\mathcal{H}$; we expand the first terms in (11) and estimate each term :

$$\left|<a\otimes e(\tilde{\phi})\,,\,\sum_{k<n}h^{\varepsilon_{ij}}L_j^i\left(\frac{1}{h}P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}P_{t_k}U_t\,dt-w_k\right)a_j^i(k+1)b\otimes e(\tilde{\psi})>\right|$$
$$\leq\sum_{k<n}\left|\tilde{\phi}_j(k)\right|\left|\tilde{\psi}_i(k)\right|\left|<a\otimes e(\tilde{\phi}_k)\,,\,P_{\mathcal{S}}h^{\varepsilon_{ij}}L_j^i\frac{1}{h}\int_{t_k}^{t_{k+1}}(U_s-U_{t_k})ds\,b\otimes e(\tilde{\psi}_k)>\right|,$$

where we have omitted a uniformly bounded factor

$$\left\langle a\otimes e(\tilde{\phi}_{[k+2}),b\otimes e(\tilde{\psi}_{[k+2})\right\rangle.$$

From the fact that $\sum_i\left|\tilde{\phi}_i(k)\right|\leq\sqrt{h}\,\|\phi\|_\infty$, $\sum_i\left|\tilde{\psi}_i(k)\right|\leq\sqrt{h}\,\|\psi\|_\infty$ we obtain an estimation of the error term of the form

$$C\,\|a\|\,\|b\|\,\sqrt{h}$$

for some constant C which depends only on the $\|L\|$ and on the L^2 and L^∞ norms of ϕ and ψ.

On the other hand, the error term obtained by replacing $\frac{1}{h}\,P_{\mathcal{S}}\int_{t_k}^{t_{k+1}}U_t\,dt$ by w_k in (11) is clearly dominated by $C\,h$ in norm because $\sum_i hL_0^0a_i^i(k+1)$ is just hL_0^0.

We now seek to evaluate the third sum in (11); for that consider again two functions ϕ, ψ in $L^\infty([0,t])$ and two vectors a, b in the unit ball $\mathcal{H}_1$ of $\mathcal{H}$; we have, up to an uniformly bounded factor,

$$\sum_{i,j\in\Lambda}\left|<a\otimes\varepsilon(\phi)\,,\,\frac{1}{h}L_0^iP_{\mathcal{S}}\Big(\int_{t_k}^{t_{k+1}}P_{t_k}U_s(a_0^j(s)-a_0^j(t_k))\,ds\Big)\,a_j^i(k+1)\,P_{\mathcal{S}}\,b\otimes\varepsilon(\psi)>\right|$$

$$= \sum_{i,j\in\Lambda} \left|\tilde{\phi}_j(k)\tilde{\psi}_i(k)\right| \left|< a\otimes e(\widetilde{\phi}_k)\,,\, \frac{1}{h}L_0^i \int_{t_k}^{t_{k+1}} U_s(a_0^j(s)-a_0^j(t_k))\, b\otimes e(\widetilde{\psi}_k)\, ds >\right|$$

$$\leq Ch^{3/2}\, \|L\|\, \|\phi\|_\infty\, \|\psi\|_\infty\,,$$

for $a_0^j(s)-a_0^j(t_k)$ is bounded on Φ_{t_k}, with norm $\sqrt{s-t_k}$. One obtains similarly

$$\sum_{i,j\in\Lambda} \left|< a\otimes\varepsilon(\phi)\,,\, \frac{1}{h}L_j^0\, P_{\mathcal{S}} \int_{t_k}^{t_{k+1}} P_{t_k} U_s(a_i^0(s)-a_i^0(t_k))\, ds\, a_j^i(k+1)\, P_{\mathcal{S}} b\otimes\varepsilon(\psi) >\right|$$

$$\leq h^{3/2}\, \|L\|\, \|\phi\|_\infty\, \|\psi\|_\infty\,,$$

so that the third sum in (11) is bounded by $C\sqrt{h}\,2t\,\|\phi\|_\infty\,\|\psi\|_\infty$.

We have shown that, putting $F_k = \sum_{i,j}(h^{\varepsilon_{ij}}L_j^i + \widehat{\delta}_{ij} h L_0^0)\, a_j^i(k+1)$,

$$|< a\otimes\varepsilon(\phi)\,,\, P_{\mathcal{S}}(w_n-v_n)P_{\mathcal{S}} b\otimes\varepsilon(\psi) >|$$
$$= \sum_{k<n} \left|< a\otimes e(\tilde{\phi})\,,\, F_k(w_k-v_k)\, b\otimes e(\tilde{\psi}) >\right| + o(1) \qquad (12)$$

where $o(1)$ is a term which converges to zero as h goes to zero *uniformly for a, b in $\mathcal{H}_1$ and for t in a bounded interval.* That uniform convergence property will be important in the sequel.

Expanding F_k in equation (12) gives

$$\left|< a\otimes\varepsilon(\tilde{\phi}_n)\,,\, (w_n-v_n)P_{\mathcal{S}}\, b\otimes\varepsilon(\tilde{\psi}_n) >\right|$$
$$\leq \sum_{i,j}\sum_{k<n} h^{\varepsilon_{ij}} \left|\tilde{\phi}_j(k)\right| \left|\tilde{\psi}_i(k)\right| \left|< (L_j^i)^* a\otimes e(\tilde{\phi}_k)\,,\, (w_k-v_k) b\otimes e(\tilde{\psi}_k) >\right|$$
$$+ \sum_{i\in\Lambda}\sum_{k<n} h \left|\tilde{\phi}_i(k)\right| \left|\tilde{\psi}_i(k)\right| \left|< (L_0^0)^* a\otimes e(\tilde{\phi}_k)\,,\, (w_k-v_k) b\otimes e(\tilde{\psi}_k) >\right|$$
$$+ o(1),$$

where in each term we have omitted an uniformly bounded factor and the notation $o(1)$ indicates a function of h which converges to zero as h goes to zero.

Since ϕ and ψ are essentially bounded, the quantities $\sum_{i,j} h^{\varepsilon_{ij}} \left|\tilde{\phi}_j(k)\right| \left|\tilde{\psi}_i(k)\right|$ and $\sum_{i\in\Lambda} h \left|\tilde{\phi}_i(k)\right| \left|\tilde{\psi}_i(k)\right|$ are again of order h at most, with an estimate which is independent on k. Besides, remark that normalizing all operators would only imply an additional constant factor, so that we can assume all the L_j^i to be contractions. In that case the above implies that for all n,

$$\sup_{a,b\,\in\,\mathcal{H}_1} \left|< a\otimes e(\tilde{\phi}_n)\,,\, (w_n-v_n) b\otimes e(\tilde{\psi}_n) >\right|$$
$$\leq hC \sum_{k<n} \sup_{a,b\,\in\,\mathcal{H}_1} \left|< a\otimes e(\tilde{\phi}_k)\,,\, (w_k-v_k) b\otimes e(\tilde{\psi}_k) >\right| + o(1)$$

for some constant C. Here we have used our earlier remark that all convergences are uniform in a, $b\in\mathcal{H}_1$. This implies that

$$\sup_{a,b\,\in\,\mathcal{H}_1} \left|< a\otimes e(\tilde{\phi}_n)\,,\, (w_n-v_n) b\otimes e(\tilde{\psi}_n) >\right| \leq (1+Ch)^n \times o(1)$$

and since nh converges to t, the quantity $(1+Ch)^n$ is bounded so that

$$\sup_{a,b\in\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_n)\,,\,(w_n-v_n)b\otimes e(\tilde{\psi}_n)>\right|$$

converges to zero as h goes to zero.

We have proved the desired convergence property for the process $(w_k)_{k\geq 0}$. Now we will prove that

$$\sup_{a,b\in\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_n)\,,\,(u_n-v_n)b\otimes e(\tilde{\psi}_n)>\right|$$

converges to zero as h goes to zero. We have

$$\begin{aligned}
u_n-v_n &= \sum_{k<n} F_k(u_k-v_k) + \sum_{k<n}\left(\sum_{i,j} h^{\varepsilon_{ij}}(\omega^i_j(h)-h\widehat{\delta}_{ij}L^0_0)a^i_j(k+1)\right)u_k \\
&= \sum_{k<n}\left(F_k + \sum_{i,j} h^{\varepsilon_{ij}}(\omega^i_j(h)-h\widehat{\delta}_{ij}L^0_0)a^i_j(k+1)\right)(u_k-v_k) \\
&\quad + \sum_{k<n}\left(\sum_{i,j} h^{\varepsilon_{ij}}(\omega^i_j(h)-h\widehat{\delta}_{ij}L^0_0)a^i_j(k+1)\right)v_k
\end{aligned} \tag{13}$$

Replace for simplicity $\omega^i_j(h)$ by $\omega^i_j(h)-h\widehat{\delta}_{ij}L^0_0$ (which does not change the validity of the earlier estimates). The interest of the second form lies therein, that the term without recurring u_k-v_k can now be estimated thanks to our previous result:

$$\left|< a\otimes e(\tilde{\phi}_k)\,,\,\sum_{k<n}\left(\sum_{i,j} h^{\varepsilon_{ij}}\omega^i_j(h)a^i_j(k+1)\right)v_k\,b\otimes e(\tilde{\psi}_k)>\right|$$

is bounded by

$$\sum_{k<n}\left(\sum_{i,j} h^{\varepsilon_{ij}}\left|\tilde{\phi}_j(k)\right|\left|\tilde{\psi}_i(k)\right|\left\|\omega^i_j(h)\right\|\right)\sup_{a,b\in\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_k)\,,\,v_k b\otimes e(\tilde{\psi}_k)>\right| \tag{14}$$

with

$$\begin{aligned}
&\sup_{a,b\in\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_k)\,,\,v_k b\otimes e(\tilde{\psi}_k)>\right| \\
&\quad\leq \sup_{a,b\in\mathcal{H}_1}\Big(\left|< a\otimes e(\tilde{\phi}_k)\,,\,(v_k-w_k)b\otimes e(\tilde{\psi}_k)>\right| \\
&\qquad\qquad + \left|< a\otimes e(\tilde{\phi}_k)\,,\,w_k b\otimes e(\tilde{\psi}_k)>\right|\Big)
\end{aligned}$$

and our estimate shows that the first term on the right-hand-side converges to zero uniformly in k as h goes to zero. The second term is, in turn, bounded by $\|\varepsilon(\phi)\|\,\|\varepsilon(\psi)\|$ since any w_k is $P_{\mathcal{S}}U_{t_k}P_{\mathcal{S}}$ and as such has bounded norm.

Thanks to our assumptions on the perturbative operators, the bound (14) we are interested in converges to zero as h goes to zero, uniformly for a, b in $\mathcal{H}_1$.

Besides, for the recurring term in (13) we obtain as before

$$\sup_{a,b\in\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_n)\,,\,\sum_{k<n}\Big(F_k+\sum_{i,j}h^{\varepsilon_{ij}}\omega^i_j(h)a^i_j(k+1)\Big)(w_k-v_k)b\otimes e(\tilde{\psi}_n)>\right|$$

$$\leq h\,C\sum_{k<n}\sup_{a,b\,\in\,\mathcal{H}_1}\left|< a\otimes e(\tilde{\phi}_k)\,,\,(v_k-w_k)b\otimes e(\tilde{\psi}_k)>\right|,$$

thanks to the fact that the operators ω^ϵ are assumed to have norms which converge to zero uniformly with h. We conclude as in the previous case.

This ends the proof. ■

Under some additional assumptions, which are verified in many applications, we can very much improve the convergence.

Theorem 16. – *Consider the same assumptions and the same notations as in Theorem 13. If furthermore $\|u_k\|$ is locally uniformly bounded, then $u_{[t/h]}$ converges weakly to U_t on all $\mathcal{H}_0\otimes\Phi$.*

Proof

Theorem 13 allows us to perform a $\epsilon/3$ argument with an approximation of any vectors of $\mathcal{H}_0\otimes\mathrm{T}\Phi$ by combinations of vectors $a\otimes\varepsilon(\phi)$, $b\otimes\varepsilon(\psi)$ with essentially bounded functions ϕ, ψ. ■

One of the main application of this last theorem is the case when the matrices $\mathbb{L}(h)$ give rise at the limit to a matrix L such as in Theorem 12 (the case of a unitary solution $(U_t)_{t\geq 0}$). We shall show that the associated discrete evolution $(u_n)_{n\in\mathbb{N}}$ satisfy the conditions of the above theorem, so that the convergence of $u_{[t/h]}$ towards U_t is weak.

Theorem 17. – *Consider a matrix $\mathbb{L}$ on $\mathcal{H}_0\otimes\mathcal{H}$ with coefficients*

$$\begin{aligned}
\mathbb{L}^0_0 &= I-h(iH+\frac{1}{2}\sum_k L^*_kL_k)+h\omega^0_0\\
\mathbb{L}^0_j &= \sqrt{h}L_j+h\omega^0_j\\
\mathbb{L}^i_0 &= -\sqrt{h}\sum_k L^*_kS^k_j+h\omega^i_0\\
\mathbb{L}^i_j &= I+S^i_j-\delta_{ij}I+h\omega^i_j
\end{aligned}$$

where H is a bounded self-adjoint operator, $(S^i_j)_{i,j\in\mathbb{L}}$, is unitary and L_i, $i\in\mathbb{L}$ are operators on $\mathcal{H}_0$ and where all the coefficients ω^i_j converge to 0 when h tends to 0.

Then the solution $(u_n)_{n\in\mathbb{N}}$ of

$$u_{n+1}=\mathbb{L}_{n+1}u_n$$

is made of invertible operators which are locally uniformly bounded in norm.

In particular $u_{[t/h]}$ converges weakly to the solution U_t of (X).

Proof

A straightforward computation shows the special form of $\mathbb{L}$ induces many cancellations when computing the coefficients of $\mathbb{L}^*\mathbb{L} - I$ and of $\mathbb{L}\mathbb{L}^* - I$, and that they are of order h. Thus for h small enough the operators $\mathbb{L}^*\mathbb{L}$, $\mathbb{L}\mathbb{L}^*$ and thus $\mathbb{L}$ are invertible. Thus so are the operators u_n.

Furthermore the above estimates show that $\|\mathbb{L}\| \leq \sqrt{1+Ch}$. This easily gives the locally uniform boundedness of $(u_n)_{n\in\mathbb{N}}$ and thus the desired weak convergence. ∎

IV.2 Typical Hamiltonian: weak coupling and low density

We are now coming back to the initial physical motivations of Theorems 13, 16 and 17. These theorems show up quite strong conditions on the unitary operator $\mathbb{L} = e^{-ihH}$ and a natural question now is: what kind of Hamiltonian H will produce such conditions on $\mathbb{L}$? What is the typical Hamiltonian for repeated quantum interactions which will produce quantum Langevin equations at the continuous limit?

In this section we answer partly that question. We answer it as we exhibit a large family of such Hamiltonians and we conjecture that they are the typical ones. We do not fully answer the question for we are not able to prove that they are the only ones.

We keep here the notations of section II.1-II.3.

On $\mathcal{H}_0 \otimes \mathcal{H}$ consider the following Hamiltonian

$$H = H_0 \otimes I + I \otimes H_S + \frac{1}{\sqrt{h}} \sum_{i\in\Lambda} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i\right) + \frac{1}{h} \sum_{i,j\in\Lambda} D_{ij} \otimes a_j^i \qquad (15)$$

where H_0, V_i and D_{ij} are bounded operators on $\mathcal{H}_0$ (with H_0 hermitian and $D_{ij} = D_{ji}^*$), and H_S is bounded hermitian on $\mathcal{H}$.

The condition that H_S is bounded can be felt as a weakness in our conditions. But one has to keep in mind that $\mathcal{H}$ is just a "small piece" of the bath system; it is only $\otimes_{\mathbb{R}^+}\mathcal{H}$ which represents the bath. In general $\mathcal{H}$ is finite dimensional and the resulting continuous field is a Fock space.

Put $\mathcal{H}'$ to be the closed subspace of $\mathcal{H}$ generated by the basis elements X^i, $i \in \Lambda$; that is, the orthogonal of $X^0 = \Omega$. Consider the "column operator"

$$V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \end{pmatrix}$$

as an operator from $\mathcal{H}_0$ to $\mathcal{H}_0 \otimes \mathcal{H}'$. Assume that this operator is bounded.

The adjoint of V is then the "row operator"

$$V^* = \begin{pmatrix} V_1 & V_2 & \dots \end{pmatrix}$$

from $\mathcal{H}_0 \otimes \mathcal{H}'$ to $\mathcal{H}_0$.

Define the "matrix operator" $D = \left(D_j^i\right)_{ij}$ as an operator from $\mathcal{H}_0 \otimes \mathcal{H}'$ to $\mathcal{H}_0 \otimes \mathcal{H}'$. We also assume D to be bounded.

Let us relate the above Hamiltonian with the usual literature on weak coupling and low density limits.

Recall that, in the literature about weak coupling limit, the bath is usually made of several harmonic oscillators (a Fock space) with associated creation operators $a^*(g)$ and annihilation operators $a(g)$, where g runs over a Hilbert space $\mathcal{H}$. The Hamiltonian which is considered is then of the form

$$H = H_0 \otimes I + I \otimes H_S + \lambda\,(V \otimes a^*(g) + V^* \otimes a(g))$$

The part $V \otimes a^*(g) + V^* \otimes a(g)$ corresponds to the typical dipole Hamiltonian usually considered in the weak coupling limit or van Hove limit (cf [Dav], [D-J]).

This Hamiltonian meets (15) when considering an orthonormal basis (e_i) of $\mathcal{H}$ and when one writes

$$(V \otimes a^*(g) + V^* \otimes a(g)) = \sum_i \left(<e_i\,,\,g>V \otimes a^*(e_i) + <g\,,\,e_i>V^* \otimes a(e_i)\right)$$

Our time renormalization term $1/\sqrt{h}$ corresponds to the usual time renormalisation for the weak coupling limit $(t/\lambda^2 = \tau)$.

On the other hand, interaction Hamiltonians of the form $D \otimes a^*(f)a(h) + D^* \otimes a^*(h)a(f)$ are typical of the low density limit (cf [APV], [AFL]). They meet (15) when decomposing in the same way as above:

$$\begin{aligned}(D \otimes a^*(f)a(h) + D^* \otimes a^*(h)a(f)) = \sum_{i,j} \big(&<e_i\,,\,f><h\,,\,e_j>D \otimes a^*(e_i)a(e_j) + \\ &+ <f\,,\,e_i><e_j\,,\,h>D^* \otimes a^*(e_j)a(e_i)\big)\,.\end{aligned}$$

Our time renormalization $1/h$ is also the typical one for this limit.

We are now back to our general Hamiltonian (15).

We shall take compact notations for the unitary quantum Langevin equations of Theorem 10. Consider the equation

$$dU_t = \sum_{i,j} L^i_j U_t\, da^i_j(t) \tag{16}$$

with

$$\begin{aligned}
L^0_0 &= -(iK + \frac{1}{2}\sum_{k\in\Lambda} L^*_k L_k)\\
L^0_j &= L_j\\
L^i_0 &= -\sum_{k\in\Lambda} L^*_k S^k_i\\
L^i_j &= S^i_j - \delta_{ij} I,
\end{aligned}$$

where K is a bounded self-adjoint operator and $(S^i_j)_{i,j\in\Lambda}$ is unitary.

We write W for the column operator

$$\begin{pmatrix} L_1 \\ L_2 \\ \vdots \end{pmatrix}$$

and S for the matrix operator $(S^i_j)_{i,j}$. Then, with obvious notations

$$\begin{aligned}
L^0_0 &= -(iK + \frac{1}{2}W^*W)\\
L^0_\cdot &= W\\
L^\cdot_0 &= -W^*S\\
L^\cdot_\cdot &= S - I.
\end{aligned}$$

With the Hamiltonian H given by (15), put

$$U = e^{-ihH}$$

and consider the evolution equation for repeated interactions associated to U:

$$u_{n+1} = U_{n+1}u_n.$$

Theorem 18. – *The solution $(u_n)_{n\in\mathbb{N}^*}$ of the discrete time evolution equation converges weakly in Φ to the solution $(U_t)_{t\geq 0}$ of the quantum Langevin equation*

$$dU_t = \sum_{i,j\in\Lambda\cup\{0\}} L^i_j U_t\, da^i_j(t)$$

where, with the same notations as above

$$\begin{aligned}
K &= H_0 + <\Omega\,,\, H_S\Omega> I + V^*D^{-2}(\sin D - D)V\\
W &= D^{-1}(e^{-iD} - I)V\\
S &= e^{-iD}.
\end{aligned}$$

The above expressions $D^{-2}(\sin D - D)$ and $D^{-1}(e^{iD} - I)$ have to be understood has a short notation for the associated convergent series, even if D is not invertible.

Proof

Put $k^i_j = <X^j\,,\, H_S X^i>$ for all $i, j \in \Lambda\cup\{0\}$. We consider the column operator

$$|\,k\rangle = \begin{pmatrix} k^0_1 I\\ k^0_2 I\\ \vdots \end{pmatrix},$$

the row operator

$$\langle k\,| = (\, k^1_0 I \quad k^2_0 I \quad \dots)$$

and the matrix operator

$$k = (k^i_j I)_{i,j\in\Lambda}\,.$$

They all are bounded operators.

Put $\widetilde{H} = H_0 + k^0_0 I$ and $M = (M^i_j)_{i,j\in\Lambda}$ with $M^i_j = \delta_{ij}H_0 + k^i_j I$. We then have

$$H = \begin{pmatrix} \widetilde{H} & \frac{1}{\sqrt{h}}V^* + \langle k\,| \\ \frac{1}{\sqrt{h}}V + |\,k\rangle & \frac{1}{h}D + M \end{pmatrix}$$

as an operator on $\mathcal{H}_0 \otimes \mathcal{H}$ which is decomposed as an operator on $\mathcal{H}_0 \otimes (\mathbb{C}\Omega \oplus \mathcal{H}')$. In particular

$$hH = \begin{pmatrix} h\widetilde{H} & \sqrt{h}V^* + h\langle k\,| \\ \sqrt{h}V + h|\,k\rangle & D + hM \end{pmatrix}.$$

Let α be a bound for $\left\|\widetilde{H}\right\|$, $\|V\|$, $\|D\|$, $\|M\|$, $\|\langle k\,|\|$, $\||\,k\rangle\|$.

Lemma 19. – *For all $m \in \mathbb{N}$ we have*

$$(hH)^m = \begin{pmatrix} hA_m + h^{3/2}R_m^1 & \sqrt{h}B_m + hR_m^2 \\ \sqrt{h}C_m + hR_m^3 & D_m + hR_m^4 \end{pmatrix}$$

with

$$\|X_m\| \leq \alpha^m$$

for all $X = A, B, C, D$ and

$$\left\|R_m^i\right\| \leq 7^{m-1}\alpha^m$$

for all $i = 1, 2, 3, 4$; with

$$\begin{aligned} A_{m+1} &= V^* C_m, \qquad A_0 = I, A_1 = \widetilde{H} \\ B_{m+1} &= V^* D_m, \qquad B_0 = 0, B_1 = V^* \\ C_{m+1} &= DC_m, \qquad C_0 = 0, C_1 = V \\ D_{m+1} &= DD_m, \qquad D_0 = I. \end{aligned}$$

Proof of the lemma

For $m = 0$ the statements are clearly satisfied. For $m = 1$ we find the announced A_1, B_1, C_1, D_1 and $R_1^1 = 0$, $R_1^2 = \langle k\,|$, $R_1^3 = |\,k\rangle$, $R_1^4 = M$. The norm estimates are then clearly satisfied. Now, applying the induction hypothesis and computing $(hH)^{m+1}$ from $(hH)^m$ we get

$$\begin{aligned} & A_{m+1} = V^* C_m \qquad \text{and } \|A_{m+1}\| \leq \alpha\alpha^m = \alpha^{m+1} \\ & R_{m+1}^1 = \sqrt{h}\widetilde{H}A_m + h\widetilde{H}R_m^1 + V^*R_m^3 + \langle k\,|C_m + \sqrt{h}\langle k\,|R_m^3 \\ & \left\|R_{m+1}^1\right\| \leq 5(7^{m-1}\alpha^{m+1}) \leq 7^m\alpha^{m+1} \\ & B_{m+1} = V^* D_m \qquad \text{and } \|B_{m+1}\| \leq \alpha\alpha^m = \alpha^{m+1} \\ & R_{m+1}^2 = \sqrt{h}\widetilde{H}B_m + h\widetilde{H}R_m^2 + \sqrt{h}V^*R_m^4 + \langle k\,|D_m + h\langle k\,|R_m^4 \\ & \left\|R_{m+1}^2\right\| \leq 5(7^{m-1}\alpha^{m+1}) \leq 7^m\alpha^{m+1} \\ & C_{m+1} = DC_m \qquad \text{and } \|C_{m+1}\| \leq \alpha\alpha^m = \alpha^{m+1} \\ & R_{m+1}^3 = \sqrt{h}VA_m + hVR_m^1 + DR_m^3 + \sqrt{h}MC_m + hMR_m^3 \\ & \left\|R_{m+1}^3\right\| \leq 5(7^{m-1}\alpha^{m+1}) \leq 7^m\alpha^{m+1} \\ & D_{m+1} = DD_m \qquad \text{and } \|D_{m+1}\| \leq \alpha\alpha^m = \alpha^{m+1} \\ & R_{m+1}^4 = VB_m + hVR_m^2 + \sqrt{h}|\,k\rangle B_m + h|\,k\rangle R_m^2 + DR_m^4 + MD_m + hMR_m^4 \\ & \left\|R_{m+1}^4\right\| \leq 7(7^{m-1}\alpha^{m+1}) \leq 7^m\alpha^{m+1}. \end{aligned}$$

The lemma is proved.

In particular, we get

$$\begin{aligned} D_m &= D^m \\ B_m &= V^* D^{m-1} \\ C_m &= D^{m-1} V \\ A_m &= V^* D^{m-2} V. \end{aligned}$$

By the norm estimates of Lemma 19 we have that the series

$$\sum_m \frac{(-i)^m}{m!} R_m^j$$

are norm convergent. Let us denote by R^k their repective limit.

We get

$$\mathbb{L} = \sum_m \frac{(-i)^m}{m!} (hH)^m =$$

$$= \begin{pmatrix} \begin{matrix} I - ih\widetilde{H} + h\sum_{m=2}^\infty \frac{(-i)^m}{m!} V^* D^{m-2} V \\ +h^{3/2} R^1 \end{matrix} & \begin{matrix} \sqrt{h} \sum_{m=1}^\infty \frac{(-i)^m}{m!} V^* D^{m-1} \\ +hR^2 \end{matrix} \\ & \\ \begin{matrix} \sqrt{h} \sum_{m=1}^\infty \frac{(-i)^m}{m!} D^{m-1} V \\ +hR^3 \end{matrix} & \begin{matrix} I + \sum_{m=1}^\infty \frac{(-i)^m}{m!} D^m \\ +hR_m^4 \end{matrix} \end{pmatrix}.$$

This gives

$$U = \begin{pmatrix} \begin{matrix} I - ih\widetilde{H} + hV^* D^{-2}(e^{-iD} - I + iD)V \\ +h^{3/2} R^1 \end{matrix} & \begin{matrix} \sqrt{h} V^* D^{-1}(e^{-iD} - I) \\ +hR^2 \end{matrix} \\ & \\ \begin{matrix} \sqrt{h} D^{-1}(e^{-iD} - I)V \\ +hR^3 \end{matrix} & \begin{matrix} I + (e^{-iD} - I) \\ +hR_m^4 \end{matrix} \end{pmatrix}.$$

We are exactly in the conditions of Theorem 16 and we thus get the weak convergence to the solution of

$$dU_t = \sum_{i,j \in \Lambda \cup \{0\}} L_j^i U_t \, da_j^i(t)$$

where

$$\begin{aligned} L_0^0 &= -i\widetilde{H} + V^* D^{-2}(e^{-iD} - I + iD)V \\ L_\cdot^0 &= D^{-1}(e^{-iD} - I)V \\ L_0^\cdot &= V^* D^{-1}(e^{-iD} - I) \\ L_\cdot^\cdot &= e^{-iD} - I. \end{aligned}$$

If we put $W = L_\cdot^0$ and $S = e^{-iD}$, we then get

$$-W^* S = -V^*(e^{iD} - I)D^{-1} S = V^* D^{-1}(e^{-iD} - I) = L_0^\cdot$$

and

$$\begin{aligned} -\frac{1}{2} W^* W &= -\frac{1}{2} V^*(e^{iD} - I) D^{-1} D^{-1} (e^{-iD} - I)V \\ &= -\frac{1}{2} V^* D^{-2} (\cos D - I) V. \end{aligned}$$

This shows that

$$L_0^0 = -i\widetilde{H} - iV^* D^{-2}(\sin D - D)V - \frac{1}{2}W^*W.$$

The theorem is proved. ∎

Let us interpret the above theorem in terms of weak coupling and low density limit again. If in the above Hamiltonian we consider no term D_{ij}, that is,

$$H = H_0 \otimes I + I \otimes H_S + \frac{1}{\sqrt{h}} \sum_{i\in\Lambda} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i\right)$$

then we are in the usual situation of a weak coupling limit, with its typical dipole interaction Hamiltonian. The quantum Langevin equation we obtain in Theorem 16 then simplifies to

$$dU_t = -\left(iH_0 + i{<}\,\Omega\,,\, H_S\Omega\,{>}I + \frac{1}{2}\sum_i V_i^* V_i\right) U_t\, dt + \\ + \sum_i V_i U_t\, da_i^0(t) - \sum_i V_i^* U_t\, da_0^i(t).$$

This is also typical of the "diffusion" type of quantum Langevin equation one can meet in the literature for this kind of limit. The fact that only creation and annihilation quantum noises are involved is here the quantum analogue of a classical stochastic differential equation with Brownian noise. Note that if the V_i's are such that $V_i^* = -V_i$ then the above quantum Langevin equation becomes a classical stochastic differential equation with Brownian noises:

$$dU_t = -\left(iH_0 + i{<}\,\Omega\,,\, H_S\Omega\,{>}I + \frac{1}{2}\sum_i V_i^2\right) U_t\, dt + \sum_i V_i U_t\, dW_i(t).$$

We refer to [At1] for a complete discussion on the classical stochastic interpretations of the quantum noises.

On the other hand, if in the Hamiltonian we consider no term V_i, that is,

$$H = H_0 \otimes I + I \otimes H_S + \frac{1}{h} \sum_{i,j\in\Lambda} D_{ij} \otimes a_j^i$$

then we are in the usual situation of a low density limit, with its typical scattering-type interaction Hamiltonian. The quantum Langevin equation we obtain in Theorem 18 then simplifies to

$$dU_t = -i(H_0 + {<}\,\Omega\,,\, H_S\Omega\,{>}I)U_t\, dt + \sum_{i,j\in\Lambda} (S_j^i - \delta_{ij})U_t\, da_j^i(t)$$

where $S = e^{-iD}$.

This is also typical of the "Poisson" type of quantum Langevin equation one can meet in the literature for this kind of limit. The fact that only "exchange" quantum noises are involved is here the quantum analogue of a classical stochastic differential equation with Poisson noises.

Our formalism enables us to handle these two types of limits in a single setup, and this could not have been done at the classical stochastic calculus level. This seems to be the first time in the literature, for, to our knowledge, weak coupling limits and low density limits have always been considered separately, as very different objects.

The surprise in our setup in the apparition of the term

$$V^* D^{-2} (\sin D - D) V$$

only when both the limits are in presence in the Hamiltonian.

Indeed, the term

$$L_0^0 = -(iK + \frac{1}{2} W^* W)$$

is the driving term of the dynamic associated to the quantum Langevin equation. In some sense it is the generator of the dynamic on $\mathcal{H}_0$. The part $\frac{1}{2} W^* W$ is representative of the dissipation from $\mathcal{H}_0$ to the bath. The part iK is an effective Hamiltonian on $\mathcal{H}_0$. The apparition of this new contribution $V^* D^{-2} (\sin D - D) V$ is new, and we have no physical interpretation of it.

It is just clear that it results from the combined effects of the weak coupling limit and the low density limit.

IV.3 Hamiltonian description of von Neumann measurements

Our setup and approach allows to construct an Hamiltonian description for the usual von Neumann measurement procedure (collapse of the wave packet postulate).

On some quantum system state space $\mathcal{H}_0$ consider an observable A with discrete spectrum (maybe infinite). Let $P_1, P_2, \ldots$ denote the associated spectral (orthogonal) projections, with $\sum_k P_k = I$.

We want to give a model for the action on $\mathcal{H}_0$ of an exterior measurement apparatus which measures the observable A. That is, the action of the measurement apparatus on $\mathcal{H}_0$ should be to transform any observable X of $\mathcal{H}_0$ into

$$\sum_k P_k X P_k.$$

Let $\mathcal{H}$ be a Hilbert space with one more dimension than the number of projectors P_k involved above (infinite dimensional if the P_k's are in infinite number). On $\mathcal{H}$ consider an orthonormal basis $\Omega = X_0, X_1, X_2, \ldots$ and the associated creation operators a_k^0 and annihilation operators a_0^k, as in sections II.1-II.3. Consider the following Hamiltonian on $\mathcal{H}_0 \otimes \mathcal{H}$:

$$H = \frac{1}{\sqrt{h}} \sum_k \left(iP_k \otimes a_k^0 - iP_k \otimes a_0^k \right).$$

Let $U = e^{-ihH}$ and consider the process

$$u_{n+1} = \mathbb{L}_{n+1} u_n$$

of repeated quantum interactions on $\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H}$.

Theorem 20. – *The repeated quantum interaction process $(u_n)_{n\in\mathbb{N}^*}$ converges weakly in $\mathcal{H}_0\otimes\bigotimes_{\mathbb{R}^+}\mathcal{H}$, when h tends to 0, to the solution $(U_t)_{t\geq 0}$ of the quantum Langevin equation*

$$dU_t = -\frac{1}{2}U_t\,dt + \sum_k \left(iP_kU_t\,da_k^0(t) - iP_kU_t\,da_0^k(t)\right).$$

Furthermore, for any observable X on $\mathcal{H}_0$, the partial trace along the field $\bigotimes_{\mathbb{R}^+}\mathcal{H}$, in the vacuum state, of $U_t^*(X\otimes I)U_t$ converges, when t tends to $+\infty$, to

$$\sum_k P_kXP_k.$$

Proof

In the basis $\Omega = X_0, X_1, X_2, \ldots$ for $\mathcal{H}$ the Hamiltonian H writes as

$$H = \begin{pmatrix} 0 & -i\frac{1}{\sqrt{h}}P_1 & -i\frac{1}{\sqrt{h}}P_2 & \cdots \\ i\frac{1}{\sqrt{h}}P_1 & 0 & 0 & \cdots \\ i\frac{1}{\sqrt{h}}P_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, as an easy direct computation shows, we have

$$\mathbb{L} = \begin{pmatrix} \cos\sqrt{h}\,I & -\sin\sqrt{h}\,P_1 & -\sin\sqrt{h}\,P_2 & \cdots \\ \sin\sqrt{h}\,P_1 & \cos\sqrt{h}\,P_1 & 0 & \cdots \\ \sin\sqrt{h}\,P_2 & 0 & \cos\sqrt{h}\,P_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This unitary matrix clearly satisfies the conditions of Theorem 16 (one could also have directly applied Theorem 18 to the Hamiltonian), we thus have the weak convergence to the announced quantum Langevin equation

$$dU_t = -\frac{1}{2}U_t\,dt + \sum_k \left(iP_kU_t\,da_k^0(t) - iP_kU_t\,da_0^k(t)\right),$$

with $U_0 = I$.

Now, consider the evolution under $(U_t)_{t\geq 0}$ of a system observable X:

$$U_t^*(X\otimes I)U_t$$

and the partial trace along the field $\bigotimes_{\mathbb{R}^+}\mathcal{H}$ in the vacuum state:

$$P_t(X) = <\Omega\,,\,U_t^*(X\otimes I)U_t\Omega>$$

with the notation

$$<a\,,<\Omega\,,\,A\Omega>b>_{\mathcal{H}_0} = <a\otimes\Omega\,,\,A\,(b\otimes\Omega)>_{\mathcal{H}_0\otimes\bigotimes_{\mathbb{R}^+}\mathcal{H}}$$

for any operator A on $\mathcal{H}_0\otimes\bigotimes_{\mathbb{R}^+}\mathcal{H}$, any $a, b\in\mathcal{H}_0$.

It is well known form the usual theory of quantum stochastic differential equations (cf [H-P]), that $(P_t)_{t\geq 0}$ is then a semigroup, on $\mathcal{B}(\mathcal{H}_0)$, of completely positive maps with Lindblad generator

$$\mathcal{L}(X) = -\frac{1}{2}\sum_k (P_k X + X P_k - 2P_k X P_k)\,.$$

One could also have obtained this "quantum master equation" by simply computing the discrete time quantum master equation associated to $(u_n)_{n\in\mathbb{N}^*}$, using Theorem 2 and then by passing to the limit $h \to 0$ to recover $(P_t)_{t\geq 0}$ (see section IV.5). Our approach has the advantage to also describe the exact equation for the interaction with the bath.

We thus get

$$\mathcal{L}(X) = \sum_k P_k X P_k - X$$

so that

$$\mathcal{L}^2(X) = -\mathcal{L}(X)$$

and

$$P_t(X) = e^{t\mathcal{L}}(X) = \big(I + (1-e^{-t})\mathcal{L}\big)\,(X).$$

That is,

$$P_t(X) = (1-e^{-t})\sum_k P_k X P_k + e^{-t}X$$

which converges to

$$\sum_k P_k X P_k$$

when t tends to $+\infty$. ∎

We thus have proved the a von Neumann measurement apparatus can be described in an Hamiltonian setup by, first considering an Hamiltonian description of a repeated quantum interaction, secondly passing to the limit to continuous quantum interactions ($h \to 0$) and thirdly passing to the limit to large times ($t \to +\infty$).

IV.4 One example

We shall here follow a very basic example. It is actually the simplest non-trivial physical example and it already gives very interesting consequences.

Assume $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$ that is, both are two-level systems with basis states Ω (the fundamental state) and X (the excited state).

During the small amount of time h the two systems are in contact and they evolve in the following way:

– if the states of the two systems are the same (both fundamental or both excited) then nothing happens;

– if they are different (one fundamental and the other one excited) then they can either be exchanged or stay as they are.

In the base $\{\Omega\otimes\Omega, \Omega\otimes X, X\otimes\Omega, X\otimes X\}$ the operator $\mathbb{L}$ is taken to be of the form

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The associated Hamiltonian is thus

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\alpha/h & 0 \\ 0 & i\alpha/h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $\mathbb{L} = e^{-ihH}$.

For the choice $\alpha = \sqrt{h}$, that is,

$$H = \frac{1}{\sqrt{h}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\sqrt{h} & -\sin\sqrt{h} & 0 \\ 0 & \sin\sqrt{h} & \cos\sqrt{h} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Repeating this interaction leads to an Hamiltonian equation of the form

$$u_{n+1} = \mathbb{L}_{n+1} u_n$$

with coefficients

$$\mathbb{L}_0^0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\sqrt{h} \end{pmatrix}, \ \mathbb{L}_1^0 = \begin{pmatrix} 0 & \sin\sqrt{h} \\ 0 & 0 \end{pmatrix},$$

$$\mathbb{L}_0^1 = \begin{pmatrix} 0 & 0 \\ -\sin\sqrt{h} & 0 \end{pmatrix}, \ \mathbb{L}_1^1 = \begin{pmatrix} \cos\sqrt{h} & 0 \\ 0 & 1 \end{pmatrix}$$

for $\mathbb{L}$. We then have

$$\lim_{h\to 0} \frac{\mathbb{L}_0^0 - I}{h} = \begin{pmatrix} 0 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\lim_{h\to 0} \frac{\mathbb{L}_1^0}{\sqrt{h}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\lim_{h\to 0} \frac{\mathbb{L}_0^1}{\sqrt{h}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\lim_{h\to 0} \mathbb{L}_1^1 - I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We are exactly in the condition for applying Theorem 15 since for all h the matrix $\mathbb{L}(h)$ is unitary. We get that for almost all t, $u_{[t/h]}$ converges weakly to U_t where $(U_t)_{t\in\mathbb{R}_+}$ is the unitary solution of

$$dU_t = -\frac{1}{2}V^*V\,U_t\,dt + VU_t\,da_1^0(t) - V^*U_t\,da_0^1(t)$$

with $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This equation is the well-known quantum Langevin equation associated to the spontaneous decay into the ground state in the Wigner-Weisskopf model for the two-level atom (see [M-R] for example).

IV.5 From completely positive maps to Lindbladians

Recall, from section II.3 that the solution $(u_n)_{n\in\mathbb{N}}$ of the equation

$$u_{n+1} = \mathbb{L}_{n+1}u_n$$

with $u_0 = I$ induces a completely positive evolution on the small system. Namely, in the Heisenberg picture, one has for any a, b in $\mathcal{H}_0$, any X in $\mathcal{B}(\mathcal{H}_0)$,

$$< a\otimes\Omega\,,\, u_n^*Xu_n\,b\otimes\Omega> = < a\,,\,\ell^n(X)\,b>$$

where

$$\ell(X) = \sum_{i\in\mathbb{N}}(\mathbb{L}_i^0)^*X\mathbb{L}_i^0.$$

Theorem 21. – *Let $\mathbb{L}(h) = \left(\mathbb{L}_j^i(h)\right)_{i,j=0,\dots n}$ be a family of matrices such that $\sum_i \mathbb{L}_i^0(h)^*\mathbb{L}_i^0(h) = I$ and such that*

– *$(\mathbb{L}_0^0(h) - I)/h$ converges to some L_0^0,*

– *$\mathbb{L}_i^0/\sqrt{h}$ converges to some L_i^0 for all $i = 1,\dots,n$.*

Then there exists a self-adjoint operator H in $\mathcal{B}(\mathcal{H}_0)$ such that for all $t\in\mathbb{R}^+$,

$$\ell^{[t/h]} \longrightarrow e^{t\mathcal{L}}$$

in operator norm, where $\mathcal{L}$ is the Lindblad generator

$$\mathcal{L}(X) = i[H,X] + \frac{1}{2}\sum_{i\in\mathbb{N}}\left(2L_i^{0^*}XL_i^0 - L_i^{0^*}L_i^0X - XL_i^{0^*}L_i^0\right)$$

and ℓ is defined above.

Proof

It is a straightforward computation that

$$\ell(X) = X + h(L_0^{0^*}X + XL_0^0) + h\sum_{i=1}^n L_i^{0^*}XL_i^0 + o(h\,\|X\|).$$

The equality $\ell(I) = I$ entails

$$(L_0^{0^*} + L_0^0) + \sum_{i\in\mathbb{N}} L_i^{0^*}L_i^0 = 0$$

so that
$$i\left(L_0^0+\frac{1}{2}\sum_{i\in\mathbb{N}}L_i^{0*}L_i^0\right)$$
is self-adjoint. We denote it by H. Then ℓ is of the form
$$\ell(X)=X+h\left(i[H,X]+\frac{1}{2}\sum_{i\in\mathbb{N}}\left(2L_i^{0*}XL_i^0-L_i^{0*}L_i^0X-XL_i^{0*}L_i^0\right)\right)+o(h\,\|X\|),$$
which is, with the notations of the statements
$$\ell(X)=X+h\mathcal{L}(X)+o(h\,\|X\|).$$
The above mentioned convergence is therefore clear. ■

As a consequence, it is very easy to obtain approximations of solutions of continuous-time master equations:
$$\frac{dX_t}{dt}=\mathcal{L}(X_t)$$
by solutions of discrete-time ones:
$$x_{n+1}=\ell(x_n).$$
Yet notice that the master equation gives no information whatsoever on the interaction between the small system and the environment or on the environment itself. On the other hand, the associated quantum Langevin equation contains the information of the whole system; this justifies our effort.

Notice that, in the above proposition, no hypothesis is needed on the other coefficients of the matrix $\mathbb{L}(h)$. Their properties actually depend on the choice of additional features of the matrices $\mathbb{L}(h)$, for example their unitarity. The possibility of choosing $\mathbb{L}(h)$ to be unitary and obtain in the end the desired Lindbladian $\mathcal{L}$ is described by Parthasarathy in exercises 29.12 and 29.13 of [Par].

What's more, these manipulations show that the hypotheses of convergence of Theorem 16 are not as artificial as it seems, and are not only convenient assumptions we set up in order to obtain the right convergence. Indeed, to $\mathbb{L}(h)$ is associated both a dynamic on the observables, as we have seen, and an evolution τ, defined by
$$<a\,,\,\tau_n b> \,=\, <a\otimes\Omega\,,\,u_n\,b\otimes\Omega>$$
for all a,b in $\mathcal{H}_0$, which turns our to be
$$\tau_n=(\mathbb{L}_0^0)^n.$$
If one assumes that $\tau_{[t/h]}$ converges for almost all t and that $\mathbb{L}_0^0$ is assumed to be continuous at $h=0$ then the assumption on $\mathbb{L}_0^0$ in Theorem 16 is to be fulfilled; this implies that $\sum_{i=1}^n \mathbb{L}_0^{i\,*}\mathbb{L}_0^i=-h(L_0^{0*}+L_0^0)+o(h)$,

so that the other assumptions of convergence of Theorem 16 are natural.

The other conditions described in Theorem 12 are in turn necessary if one wants the process $(U_t)_{t\geq 0}$ obtained in the limit to be unitary (see Theorem 11) or alternatively the matrices $\mathbb{L}(h)$ to be sufficiently close to unitarity.

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