ANALOGS OF THE M-FUNCTION IN THE THEORY OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

BARRY SIMON*

To Norrie Everitt, on his 80th birthday,
a bouquet to the master of the m-function

ABSTRACT. We show that the multitude of applications of the Weyl-Titchmarsh m-function leads to a multitude of different functions in the theory of orthogonal polynomials on the unit circle that serve as analogs of the m-function.

1. INTRODUCTION

Use of the Weyl-Titchmarsh m-function has been a constant theme in Norrie Everitt's opus, so I decided a discussion of the analogs of these ideas in the theory of orthogonal polynomials on the unit circle (OPUC) was appropriate. Interestingly enough, the uses of the m-functions are so numerous that OPUC has multiple analogs of the m-function!

m-functions are associated to solutions of

$$-u'' + qu = zu \tag{1.1}$$

with q a real function on $[0,\infty)$ and z a parameter in $\mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$. The most fundamental aspect of the m-function is its relation to the spectral measure, ρ, for (1.1) by

$$m(z) = c + \int d\rho(x) \left[\frac{1}{x-z} - \frac{x}{1+x^2} \right] \tag{1.2}$$

where c is determined by (see Atkinson [3], Gesztesy-Simon [13]):

$$m(z) = \sqrt{-z} + o(1) \qquad \text{as } z \to i\infty \tag{1.3}$$

(1.2) plus (1.3) allow you to compute m given $d\rho$, and $d\rho$ is determined by m via

$$\lim_{e\downarrow 0} \frac{1}{\pi} \int_a^b m(x+i\varepsilon)\, dx = \tfrac{1}{2}[\rho((a,b)) + \rho([a,b])] \tag{1.4}$$

Of course, I haven't told you what m or ρ is. This is done by defining m, in which case ρ is defined by (1.4). Under weak conditions on q at ∞, for

---

Date: October 1, 2003.

* Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592.

For special issue of J. Comp. Appl. Math.

$z \in \mathbb{C}_+$, (1.1) has a solution $u(x,z)$ which is L^2 at infinity, and it is unique up to a constant multiple. Then, m is defined by

$$m(z) = \frac{u'(0,z)}{u(0,z)} \tag{1.5}$$

With this definition, $d\rho$ is a spectral measure for $u \mapsto -u'' + qu = Hu$ in the sense that H is unitarily equivalent to multiplication by λ on $L^2(\mathbb{R}, d\rho)$. (1.5) is often written in the equivalent form,

$$\psi(x,z) + m(z)\varphi(x,z) \in L^2$$

where φ, ψ solve (1.1) with initial conditions $\varphi(0) = 0$, $\varphi'(0) = 1$, $\psi(0) = 1$, $\psi'(0) = 0$.

Note that if one defines

$$m(x;z) = \frac{u'(x,z)}{u(x,z)} \tag{1.6}$$

the m-function for $q_x(\cdot) = q(\,\cdot\, + x)$, then m obeys the Riccati equation

$$m' = q - z - m^2 \tag{1.7}$$

It could be said that this is backwards: the definition (1.5) should come first, before (1.2). I put it in this order because it is (1.2) that makes m such an important object both in classical results [2, 5, 7, 8, 9, 16, 23, 33] and very recent work [27, 10, 21, 31, 25, 4].

To describe the third role of the m-function, it will pay to switch to the case of Jacobi matrices. We now have, instead of q, two sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ with $a_n > 0$, $b_n \in \mathbb{R}$ which we will suppose uniformly bounded. Define an infinite matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.8}$$

which is a bounded selfadjoint operator. One defines

$$m(z) = \langle \delta_1, (J-z)^{-1}\delta_1 \rangle \tag{1.9}$$

In terms of the spectral measure, μ, for δ_1 for J,

$$m(z) = \int \frac{d\mu(x)}{x-z} \tag{1.10}$$

If u_n is the ℓ^2 solution of $a_{n-1}u_{n-1} + (b_n - z)u_n + a_n u_{n+1} = 0$ with $\operatorname{Im} z > 0$, one has the analog of (1.5)

$$m(z) = \frac{u_1(z)}{u_0(z)} \tag{1.11}$$

This process of going from a and b to m and then to μ can be reversed. One way is by iterating (1.15) below, which lets one go from μ to m (by (1.10)) and then gets the a's and b's as coefficients in a continued fraction expansion of m. From our point of view, an even more important way of going backwards uses orthogonal polynomials on the real line (OPRL). Given μ (of bounded support), one forms the monic orthogonal polynomials $P_n(x)$ for $d\mu$ and shows they obey a recursion relation

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n^2 P_{n-1}(x) \tag{1.12}$$

which yields the Jacobi parameters a and b. The orthonormal polynomials, $p_n(x)$, are related to P_n by

$$p_n(x) = (a_1 \dots a_n)^{-1} P_n(x) \tag{1.13}$$

and obey

$$a_{n+1}p_{n+1}(x) = (x - b_{n+1})p_n(x) - a_n p_{n-1}(x) \tag{1.14}$$

(1.7) has the analog

$$m(z; J) = (b_1 - z - a_1^2 m(z; J^{(1)}))^{-1} \tag{1.15}$$

where $J^{(1)}$ is the Jacobi matrix with parameters $\tilde{a}_m = a_{m+1}\tilde{b}_m = b_{m+1}$ (i.e., the top row and left column are removed).

If $m(x + i\varepsilon; J)$ has a limit as $\varepsilon \downarrow 0$, (1.15) says that $m(x + i\varepsilon; J^{(1)})$ has a limit, and by (1.15),

$$\frac{\operatorname{Im} m(x; J)}{\operatorname{Im} m(x; J^{(1)})} = |a_1 m(x; J)|^2 \tag{1.16}$$

$\operatorname{Im} m$ is important because if μ is given by (1.10) then

$$d\mu_{\rm ac} = \frac{1}{\pi} \operatorname{Im} m(x + i0)\, dx \tag{1.17}$$

This property of m, that its energy is the ratio of Im's, is a critical element of recent work on sum rules for spectral theory [29, 19, 30, 28, 6].

The interesting point is that, for OPUC, the analogs of the functions obeying (1.2), (1.5), and (1.16) are different! In Section 2, we will give a quick summary of OPUC. In Section 3, we discuss (1.2); in Section 4, we discuss (1.16); and finally, in Section 5, the analog of (1.5).

Happy 80th, Norrie. I hope you enjoy this bouquet.

2. Overview of OPUC

We want to discuss here the basics of OPUC, although we will only scratch the surface of a rich and beautiful subject [29]. The theory reverses the usual passage from differential/difference equations to measures, and instead follows the discussion of OPRL in Section 1. μ is now a probability measure on $\partial\mathbb{D} = \{z \mid |z| = 1\}$. We suppose μ is nontrivial, that is, not supported on a finite set. One can then form, by the Gram-Schmidt procedure, the monic orthogonal polynomials $\Phi_n(z)$ and the orthonormal polynomials, $\varphi_n(z) = \Phi_n(z)/\|\Phi_n\|$ where $\|\cdot\|$ is the $L^2(\partial\mathbb{D}, d\mu)$ norm.

Given fixed $n \in \{0,1,2,\dots\}$, we define an anti-unitary operator on $L^2(\partial\mathbb{D}, d\mu)$ by

$$f^*(z) = z^n\,\overline{f(z)} \tag{2.1}$$

The use of a symbol without "n" is terrible notation, but it is standard! If Q_n is a polynomial of degree n, Q_n^* is also a polynomial of degree n. Indeed,

$$Q_n^*(z) = z^n\,\overline{Q_n(1/\bar{z})}$$

so if $Q_n(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_0$, then $Q_n^*(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \dots + \bar{a}_n$.

Since Φ_n is monic, $\Phi_n^*(0) = 1$, and thus, $N(z) \equiv (\Phi_{n+1}^*(z) - \Phi_n^*(z))/z$ is a polynomial of degree n. Since * is anti-unitary,

$$\begin{aligned}\langle z^m, N(z)\rangle &= \langle z^{m+1}, \Phi_{n+1}^* - \Phi_n^*\rangle \\ &= \langle \Phi_{n+1}, z^{n+1-(m+1)}\rangle - \langle \Phi_n, z^{n-m-1}\rangle \\ &= 0\end{aligned}$$

for $m = 0,1,\dots,n-1$. Thus $N(z)$ must be a multiple of $\Phi_n(z)$, that is, for some $\alpha_n \in \mathbb{C}$,

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z) \tag{2.2}$$

and its *,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z) \tag{2.3}$$

(2.2)/(2.3) are the *Szegő recursion formulae* ([32]); the α_n's are the Verblunsky coefficients (after [34]). The derivation I've just given is that of Atkinson [2].

Since $\Phi_n^* \perp \Phi_{n+1}$, (2.3) implies

$$\|\Phi_{n+1}\|^2 + |\alpha_n|^2\|\Phi_n^*\|^2 = \|z\Phi_n\|^2$$

Since $\|\Phi_n^*\| = \|z\Phi_n\| = \|\Phi_n\|$, we have

$$\|\Phi_{n+1}\| = (1-|\alpha_n|^2)^{1/2}\|\Phi_n\| \tag{2.4}$$

This implies first of all that

$$|\alpha_n| < 1 \tag{2.5}$$

and if

$$\rho_n \equiv (1-|\alpha_n|^2)^{1/2} \tag{2.6}$$

then

$$\|\Phi\|_n = \rho_0\rho_1\dots\rho_{n-1} \tag{2.7}$$

so

$$\varphi_n = (\rho_0\dots\rho_{n-1})^{-1}\Phi_n \tag{2.8}$$

and (2.2), (2.3) becomes

$$z\varphi_n = \rho_n\varphi_{n+1} + \bar{\alpha}_n\varphi_n^* \tag{2.9}$$

$$\varphi_n^* = \rho_n\varphi_{n+1}^* + \alpha_n z\varphi_n \tag{2.10}$$

The α_n's not only lie in $\mathbb{D}$, but it is a theorem of Verblunsky [34] that as μ runs through all nontrivial measures, the set of α's runs through all of $\times_{n=0}^\infty \mathbb{D}$. The α's are the analogs of the a's and b's in the Jacobi case or of V in the Schrödinger case.

We will later have reason to consider Szegő's theorem in Verblunsky's form [35]:

Theorem 2.1. *Let*

$$d\mu = w\,\frac{d\theta}{2\pi} + d\mu_{\rm s} \tag{2.11}$$

Then

$$\prod_{j=0}^{\infty}(1-|\alpha_j|^2) = \exp\biggl(\int \log(w(\theta))\frac{d\theta}{2\pi}\biggr) \tag{2.12}$$

Remark. The log integral can diverge to $-\infty$. The theorem says the integral is $-\infty$ if and only if the product on the left is 0, that is, if and only if $\sum|\alpha_j|^2 = \infty$.

If

$$\sum_{j=0}^{\infty}|\alpha_j|^2 < \infty \tag{2.13}$$

we say the Szegő condition holds. This happens if and only if

$$\int |\log(w(\theta))|\,\frac{d\theta}{2\pi} < \infty \tag{2.14}$$

In that case, we define the Szegő function on $\mathbb{D}$ by

$$D(z) = \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log(w(\theta))\,\frac{d\theta}{4\pi}\biggr) \tag{2.15}$$

3. The Carathéodory and Schur Functions

Given (1.10) (and (1.2)), the natural "m-function" for OPUC is the Carathéodory function, $F(z)$,

$$F(z) = \int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta) \tag{3.1}$$

The Cauchy kernel $(e^{i\theta}+z)/(e^{i\theta}-z)$ has the Poisson kernel

$$\operatorname{Re}\biggl(\frac{e^{i\theta}+z}{e^{i\theta}-z}\biggr)\bigg|_{z=re^{i\varphi}} = \frac{1-r^2}{1+r^2-2\cos(\theta-\varphi)} \tag{3.2}$$

as its real part, and this is positive, so

$$\operatorname{Re} F(z) > 0 \text{ for } z\in\mathbb{D} \qquad F(0)=1 \tag{3.3}$$

This replaces $\operatorname{Im} m > 0$ if $\operatorname{Im} z > 0$.

One might think the "correct" analog of m is

$$R(z) = \int \frac{1}{e^{i\theta}-z}\,d\mu(\theta) \tag{3.4}$$

R and F are related by

$$R(z) = (2z)^{-1}(F(z)-1) \tag{3.5}$$

If one rotates $d\mu$ and z (i.e., $d\mu(\theta) \to d\mu(\theta - \varphi)$, $z \to e^{i\varphi} z$), F is unchanged but R is multiplied by $e^{-i\varphi}$, so the set of values R can take are essentially arbitrary — which shows F, which obeys $\operatorname{Re} F(z) > 0$, is a nicer object to take. That said, we will see R again in Section 5.

F has some important analogs of m:

(1) $\lim_{r\uparrow 1} F(re^{i\theta})$ exists for a.e. θ, and if (2.11) defines w, then

$$w(\theta) = \operatorname{Re} F(e^{i\theta}) \tag{3.6}$$

(2) θ_0 is a pure point of μ if and only if $\lim_{r\uparrow 1}(1-r)\operatorname{Re} F(re^{i\theta_0}) \neq 0$ and, in general,

$$\lim_{r\uparrow 1}(1-r)\operatorname{Re} F(re^{i\theta_0}) = \mu(\{\theta_0\})$$

(3) $d\mu_{\mathrm{s}}$ is supported on $\{\theta \mid \lim_{r\uparrow 1} F(re^{i\theta}) = \infty\}$.

In fact, the proof of the analogs of these facts for m proceeds by mapping $\mathbb{C}_+$ to $\mathbb{D}$ and using these facts for F!

These properties provide a strong analogy, but one can note a loss of "symmetry" relative to the ODE case. The m-function maps $\mathbb{C}_+$ to $\mathbb{C}_+$. F though maps $\mathbb{D}$ to $-i\mathbb{C}_+$. One might prefer a map of $\mathbb{D}$ to $\mathbb{D}$. In fact, one defines the Schur function, f, of μ via

$$F(z) = \frac{1 + zf(z)}{1 - zf(z)} \tag{3.7}$$

then f maps $\mathbb{D}$ to $\mathbb{D}$ and (3.7) sets up a one-one correspondence between F's with $\operatorname{Re} F > 0$ on $\mathbb{D}$ and $F(0) = 1$ and f mapping $\mathbb{D}$ to $\mathbb{D}$ (this fact relies on the Schwarz lemma that f maps $\mathbb{D}$ to $\mathbb{D}$ with $f(0) = 0$ if and only if $f = zg$ where g maps $\mathbb{D}$ to $\mathbb{D}$).

For at least some purposes, f is a "better" analog of m than F, for example, in regard to its analog of the recursion (1.15). If f is the Schur function associated to Verblunsky coefficients $\{\alpha_0, \alpha_1, \dots\}$ and f_n is the Schur function associated to $\{\alpha_n, \alpha_{n+1}, \dots\}$, then

$$f = \frac{\alpha_0 + zf_1}{1 + \bar{\alpha}_0 z f_1} \tag{3.8}$$

a result of Geronimus (see [29] for lots of proofs of this fact).

Interestingly enough, Schur, not knowing of the connection to OPUC, discussed (3.8) for $\alpha_0 = f(0)$ as a map of $f \to (\alpha_0, f_1)$ and, by iteration, to a parametrization of functions of $\mathbb{D}$ to $\mathbb{D}$ by parameters $\alpha_0, \dots, \alpha_n, \dots$. There is, of course, a formula relating F to F_1 that can be obtained from (3.7) and (3.8) or directly [22], but it is more complicated than (3.8).

Finally, in discussing f, we note that there is a natural family $\{d\mu_\lambda\}_{\lambda\in\partial\mathbb{D}}$ of measures related to $d\mu$ (with $d\mu_{\lambda=1} = d\mu$) that corresponds to "varying boundary conditions." We will discuss those more fully in Section 5, but we note

$$f(z; d\mu_\lambda) = \lambda f(z; d\mu) \tag{3.9}$$

while the formula for $F(d\mu_\lambda)$ is more involved.

The Schur function and Schur iterates, f_n, have been used by Khrushchev [17, 18, 14] as a powerful tool in the analysis of OPUC.

4. The Relative Szegő Function

As explained in the introduction, a critical property of m is (1.16), which is the basis of step-by-step sum rules (see [28]). The left side of (1.16) enters as the ratio of a.c. weights of $d\mu_J$ and $d\mu_{J^{(1)}}$. Thus, we are interested in $\operatorname{Im} F(e^{i\theta}; \{\alpha_j\}_{j=0}^\infty)$ divided by $\operatorname{Im} F(e^{i\theta}; \{\alpha_{j+1}\}_{j=0}^\infty)$, that is, $\operatorname{Im} F/\operatorname{Im} F_1$ in the language of the last section. Neither $|F|$ nor $|f|$ is directly related to this ratio, so we need a different object to get an analog of (1.16). The following was introduced by Simon in [29]:

$$(\delta_0 D)(z) = \frac{1-\bar\alpha_0 f}{\rho_0}\,\frac{1-zf_1}{1-zf} \tag{4.1}$$

It is called the "relative Szegő function" for reasons that will become clear in a moment.

In (4.1), f_1 is the Schur function for Verblunsky coefficients

$$\alpha_j^{(1)} = \alpha_{j+1} \tag{4.2}$$

Here is the key fact:

Theorem 4.1. *Let $d\mu$ and $d\mu^{(1)}$ be measures on $\partial\mathbb{D}$ with Verblunsky coefficients related by (4.2). Suppose $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\rm s}$ and $d\mu^{(1)} = w^{(1)}\frac{d\theta}{2\pi} + d\mu_{\rm s}$. Then*

(1) *For a.e. θ, $\lim_{r\uparrow 1}(\delta_0 D)(re^{i\theta}) \equiv \delta_0 D(e^{i\theta})$ exists.*

(2) *If $w(\theta) \neq 0$, then (for a.e. θ w.r.t. $\frac{d\theta}{2\pi}$), $w_1(\theta)\neq 0$ and*

$$\frac{w(\theta)}{w_1(\theta)} = |(\delta_0 D)(e^{i\theta})|^2 \tag{4.3}$$

Sketch of Proof. Each of the functions $1-\bar\alpha_0 f$, $1-zf_1$, and $1-zf$ takes values in $\{w \mid |w-1|<1\}$ on $\mathbb{D}$, so their arguments lie in $[-\frac{\pi}{2},\frac{\pi}{2}]$, so their logs are in all H^p, $1<p<\infty$. That is, they are outer functions, and so $\delta_0 D$ is an outer function, which means that assertion (1) holds (see Rudin [24] for a pedagogic discussion of outer functions).

To get (4.3), we note that (3.7) implies

$$\operatorname{Re} F(z) = \frac{1-|f|^2|z|^2}{|1-zf|^2}$$

so

$$\frac{\operatorname{Re} F(z)}{\operatorname{Re} F_1(z)} = \left|\frac{1-zf_1}{1-zf}\right|^2 \frac{1-|f|^2|z|^2}{1-|f_1|^2|z|^2} \tag{4.4}$$

On the other hand, (3.8) implies

$$zf_1 = \frac{f-\alpha_0}{1-\bar\alpha_0 f} \tag{4.5}$$

which implies

$$1 - |zf_1|^2 = \frac{\rho_0^2(1-|f|^2)}{|1-\bar\alpha_0 f|^2} \tag{4.6}$$

so, putting these formulae together,

$$\frac{\operatorname{Re} F(z)}{\operatorname{Re} F_1(z)} = |(\delta_0 D)(z)|^2 \left(\frac{1-|z|^2|f|^2}{1-|f|^2}\right) \tag{4.7}$$

which, as $|z| \to 1$, yields (4.3). □

In particular, one has the nonlocal step-by-step sum rule that if $w(\theta) \neq 0$ for a.e. θ, then

$$(\delta_0 D)(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{4\pi}\right) \tag{4.8}$$

and, in particular, setting $z = 0$,

$$\rho_0^2 = \exp\left(\int_0^{2\pi} \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{2\pi}\right) \tag{4.9}$$

which is not only consistent with Szegő's theorem (2.12) but, using semicontinuity of the entropy, can be used to prove it (see [19, 29]) as follows:

(1) Iterating (4.9) yields

$$(\rho_0 \dots \rho_{n-1})^2 = \exp\left(\int_0^{2\pi} \log\left(\frac{w(\theta)}{w_n(\theta)}\right) \frac{d\theta}{2\pi}\right) \tag{4.10}$$

(2) Since $\exp(\int_0^{2\pi} \log(w_n(\theta)\frac{d\theta}{2\pi}) \leq \int_0^{2\pi} w_n(\theta)\frac{d\theta}{2\pi} \leq 1$, (4.10) implies

$$(\rho_0 \dots \rho_{n-1})^2 \geq \exp\left(\int_0^{2\pi} \log(w(\theta)) \frac{d\theta}{2\pi}\right) \tag{4.11}$$

(3) If $w^{(n)}$ is the weight associated to the measure with

$$\alpha_j^{(n)} = \begin{cases} \alpha_j & j \leq n-1 \\ 0 & j \geq n \end{cases}$$

(4.10) proves

$$(\rho_0 \dots \rho_{n-1})^2 = \exp \int_0^{2\pi} \log(w^{(n)}(\theta)) \frac{d\theta}{2\pi} \tag{4.12}$$

(4) $d\mu \to \int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi}$ is an entropy, hence, weakly upper semicontinuous. Since $w^{(n)}\frac{d\theta}{2\pi} \to d\mu$ weakly as $n \to \infty$, this semicontinuity shows

$$\lim_{n\to\infty} (\rho_n \dots \rho_{n-1})^2 \leq \exp\left(\int_0^{2\pi} \log(w(\theta)) \frac{d\theta}{2\pi}\right) \tag{4.13}$$

(4.11) and (4.13) is Szegő's theorem.

Two other properties of $\delta_0 D$ that we should mention are:

(A) If $\sum_{n=0}^{\infty}|\alpha_n|^2 < \infty$, then

$$(\delta_0 D)(z) = \frac{D(z;\alpha_0,\alpha_1,\alpha_2,\dots)}{D(z;\alpha_1,\alpha_2,\alpha_3,\dots)} \tag{4.14}$$

(B) In general, one has

$$\delta_0 D(z) = \lim_{n\to\infty} \frac{\varphi_{n-1}^*(z;\alpha_1,\alpha_2,\dots)}{\varphi_n^*(z;\alpha_0,\alpha_1,\dots)} \tag{4.15}$$

5. Eigenfunction Ratios

Finally, we look at the analogs of m as a function ratio, its initial definition by Weyl and Titchmarsh. The key papers on this point of view are by Geronimo-Teplyaev [11] and Golinskii-Nevai [15]. We will see from one point of view [15] that $F(z)$ plays this role, but from other points of view [11] that other functions are more natural.

The recursion relations (2.9)/(2.10) can be rewritten as

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+1}^* \end{pmatrix} = A(\alpha_n, z) \begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} \tag{5.1}$$

where

$$A(\alpha, z) = \rho^{-1} \begin{pmatrix} z & -\bar\alpha_n \\ -\alpha_n z & 1 \end{pmatrix} \tag{5.2}$$

(with $\rho = (1-|\alpha|^2)^{1/2}$). From this point of view, the analog of the fundamental differential/difference equation in the real case is

$$\Xi_n = T_n(z)\Xi_0 \tag{5.3}$$

with

$$T_n(z) = A(\alpha_{n-1}, z)\dots A(\alpha_0, z) \tag{5.4}$$

The correct boundary conditions for the usual OPUC are $\Xi_0 = \binom{1}{1}$.

One can ask for what other initial conditions the polynomials associated with the top component of $T_n(z)\Xi_0$ are OPUC for some measure. Note that

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = U(\lambda) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.5}$$

with

$$U(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{5.6}$$

and that

$$U(\lambda)^{-1} A(\alpha, z) U(\lambda) = \rho^{-1} \begin{pmatrix} z & -\bar\alpha_n \lambda \\ -\alpha_n \lambda^{-1} z & 1 \end{pmatrix} \tag{5.7}$$

We see from this that $\bar\lambda = \lambda^{-1}$, that is, $|\lambda| = 1$ will yield $U(\lambda)^{-1} A(\alpha_1, z) U(\lambda) = A(\bar\lambda\alpha, z)$. Changing λ to $\bar\lambda$, we see that

Proposition 5.1. *Let $|\lambda| = 1$. If $\varphi_n^{(\lambda)}(z)$ are the OPUC for Verblunsky coefficients $\alpha_n^{(\lambda)} = \lambda\alpha_n$, then*

$$\begin{pmatrix} \varphi_n^{(\lambda)}(z) \\ \bar\lambda\varphi_n^{(\lambda)*}(z) \end{pmatrix} = T_n(z; \{\alpha_j\}_{j=1}^\infty) \begin{pmatrix} 1 \\ \bar\lambda \end{pmatrix} \tag{5.8}$$

This suggests that one look at the family $d\mu_\lambda$ or measures with

$$\alpha_j(d\mu_\lambda) = \lambda\alpha_j(d\mu) \tag{5.9}$$

called the family of Aleksandrov measures associated to $\{\alpha_j\}_{j=0}^\infty$ after [1]. The special case $\lambda = -1$ goes back to Verblunsky [35] and Geronimus [12], and are called the second kind polynomials, denoted $\psi_n(z)$. The following goes back to Verblunsky [35]:

Theorem 5.2. *For $z \in \mathbb{D}$, uniformly on compact subsets of $\mathbb{D}$,*

$$\lim_{n\to\infty} \frac{\psi_n^*(z)}{\varphi_n^*(z)} = F(z) \tag{5.10}$$

Clearly related to this is the following result of Golinskii-Nevai [15]:

Theorem 5.3. *Let $z \in \mathbb{D}$. Then*

$$\sum_{n=0}^\infty \left| \begin{pmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{pmatrix} + \beta \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \right|^2 < \infty \tag{5.11}$$

if and only if

$$\beta = F(z) \tag{5.12}$$

From this point of view, F is again the "correct" analog of m! Indeed, the Golinskii-Nevai [15] proof uses Weyl limiting circles to prove the theorem (one is always in limit point case!).

But this is not the end of the story. Define

$$u_k = \psi_k + F(z)\varphi_k \qquad u_k^* = -\psi_k^* + F(z)\varphi_k^* \tag{5.13}$$

so $\binom{u_k}{u_k^*}$ is the unique solution of $\Xi_n = T_n(z)\Xi_0$ which is in ℓ^2. In the OPRL case, the basic vector solution is of the form $\binom{u_n}{u_{n+1}}$, so we have the analog of (1.11),

$$\tilde m(z) = \frac{u_0^*}{u_0} = \frac{-1+F}{1+F} = zf \tag{5.14}$$

So one analog of the m-function is zf.

In particular, (5.14) implies

$$|u_k^*| < |u_k| \tag{5.15}$$

for $z \in \mathbb{D}$, and thus the rate of exponential decay of $|\binom{u_k}{u_k^*}|$ is that of u_k. If there is such exponential decay in the sense that

$$\gamma_2 = \lim_{n\to\infty} \left[\left\| \begin{pmatrix} u_n \\ u_n^* \end{pmatrix} \right\|^{1/n} \right] \tag{5.16}$$

exists, then, by (5.15),

$$\gamma_2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|m_n^+| \tag{5.17}$$

where

$$m_n^+ = \frac{u_{n+1}}{u_n} \tag{5.18}$$

For $n = 0$, $u_1 = \psi_1 + F\varphi_1$, $u_0 = 1 + F$, $\psi_1 = \rho_0^{-1}(z + \bar\alpha_0)$, $\varphi_1 = \rho_0^{-1}(z - \bar\alpha_0)$, so by a direct calculation,

$$m_0^+(z) = \rho_0^{-1} z(1 - \bar\alpha_0 f) \tag{5.19}$$

yet another reasonable choice for an m-function.

Indeed, if $\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|T_n(z)\|$ exists, the fact that $\det(T_n) = z^n$ implies that $\gamma = \log|z| - \gamma_2$, and one finds in the case of stochastic Verblunsky coefficients that [11, 29]

$$\mathbb{E}(\log|m_\omega^+(z)|) = \log|z| - \gamma(z) \tag{5.20}$$

an analog of a fundamental formula of Kotani [20, 26] that in his case uses m!

Finally, we turn to the connection of m to whole-line Green's functions. Given V on $(-\infty, \infty)$ and $z \in \mathbb{C}_+$, it is natural to look at the two solutions of (1.1), $u_\pm(x, z)$, which are ℓ^2 on $\pm(0, \infty)$ and the m-functions,

$$m_\pm(z) = \pm \frac{u'_\pm(0, z)}{u_\pm(0, z)} \tag{5.21}$$

$m_\pm$ are the m-functions for $V(\pm x) \restriction [0, \infty)$. Standard Green's function formulae show that the integral kernel, $G(x, y; z)$ of $(-\frac{d^2}{dx^2} + V - z)^{-1}$ is

$$G(x, y; z) = \frac{u_-(x_<)u_+(x_>)}{(u_+(0)u'_-(0) - u'_+(0)u_-(0))}$$

where $x_< = \min(x, y)$ and $x_> = \max(x, y)$. In particular,

$$G(0, 0; z) = -(m_+(z) + m_-(z))^{-1} \tag{5.22}$$

A complete description of the OPUC analog would require too much space, so we sketch the ideas, leaving the details to [29]. Just as the difference equation is associated to a triagonal selfadjoint matrix whose spectral measure is the one generating the OPRL, any set of α's is associated to a five-diagonal unitary matrix, called the CMV matrix, whose spectral measure is the $d\mu$ with $\alpha_j(d\mu) = \alpha_j$.

The CMV matrix is one-sided, but given $\{\alpha_j\}_{j=-\infty}^\infty$, one can define a two-sided CMV matrix, $\mathcal{E}$, in a natural way. If $G(z)$ is the 00 matrix element of $(\mathcal{E} - z)^{-1}$, then (see [11, 17, 29])

$$G(z) = \frac{f_+(z)f_-(z)}{1 - zf_+(z)f_-(z)} \tag{5.23}$$

where f_+ is the Schur function for $(\alpha_0, \alpha_1, \alpha_2, \dots)$ and f_- the Schur function for $(-\bar{\alpha}_{-1}, -\bar{\alpha}_{-2}, \dots)$. On the basis of the analogy between (5.23) and (5.22), Geronimo-Teplyaev [11] called f_+ and zf_- the m_+ and m_- functions.

6. Summary

We have thus seen that there are many analogs of the m-function in the theory of OPUC:

(1) The Carathéodory function, $F(z)$, given by (3.1), an analog of (1.2) and also related to the classic Weyl definition (5.11)/(5.12).
(2) The Schur function, $f(z)$, given by (3.7) with a recursion, (3.8), closer to the recursion (1.15) for the m-function of OPRL. f also enters via (5.23).
(3) $zf(z)$, the $\tilde{m}$-function of (5.14).
(4) The relative Szegő function, (4.1), which, via (4.3) and (1.16), is an analog of $a_1 m(z)$.
(5) The m^+-function, (5.19), which plays the role that m does in Kotani theory.

References

[1] A.B. Aleksandrov, *Multiplicity of boundary values of inner functions*, Izv. Akad. Nauk Arm. SSR **22** (1987), 490–503.
[2] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.
[3] F.V. Atkinson, *On the location of the Weyl circles*, Proc. Roy. Soc. Edinburgh Sect. A **88** (1981), 345–356.
[4] G. Borg, *Uniqueness theorems in the spectral theory of $y" + (\lambda - q(x))y = 0$*, Proc. 11th Scandinavian Congress of Mathematicians (Trondheim, 1949), pp. 276–287, Johan Grundt Tanums Forlag, Oslo, 1952.
[5] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
[6] D. Damanik and B. Simon, *Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics*, in preparation.
[7] M.S.P. Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh, 1973.
[8] M.S.P Eastham, *Antibound states and exponentially decaying Sturm-Liouville potentials*, J. London Math. Soc. (2) **65** (2002), 624–638.
[9] W.N. Everitt, M.K. Kwong, and A. Zettl, *Oscillation of eigenfunctions of weighted regular Sturm-Liouville problems*, J. London Math. Soc. (2) **27** (1983), 106–120.
[10] I.M. Gel'fand, private communication.
[11] J.S. Geronimo and A. Teplyaev, *A difference equation arising from the trigonometric moment problem having random reflection coefficients—an operator-theoretic approach*, J. Funct. Anal. **123** (1994), 12–45.
[12] J. Geronimus, *On the trigonometric moment problem*, Ann. of Math. (2) **47** (1946), 742–761.
[13] F. Gesztesy and B. Simon, *A new approach to inverse spectral theory, II. General real potentials and the connection to the spectral measure* Ann. of Math. **152** (2000), 593-643.

[14] L. Golinskii and S. Khrushchev, *Cesàro asymptotics for orthogonal polynomials on the unit circle and classes of measures*, J. Approx. Theory **115** (2002), 187–237.

[15] L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Comm. Math. Phys. **223** (2001), 223–259.

[16] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.

[17] S. Khrushchev, *Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$*, J. Approx. Theory **108** (2001), 161–248.

[18] S. Khrushchev, *Classification theorems for general orthogonal polynomials on the unit circle*, J. Approx. Theory **116** (2002), 268–342.

[19] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), 253–321.

[20] S. Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrdinger operators*, Stochastic Analysis (Katata/Kyoto, 1982), pp. 225–247, North-Holland Math. Library, 32, North-Holland, Amsterdam, 1984.

[21] B.M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.

[22] F. Peherstorfer, *A special class of polynomials orthogonal on the unit circle including the associated polynomials*, Constr. Approx. **12** (1996), 161–185.

[23] W.T. Reid, *Ordinary Differential Equations*, Wiley, New York, 1971.

[24] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1987.

[25] B. Simon, *Functional Integration and Quantum Physics*, Pure and Applied Mathematics, 86. Academic Press, New York, 1979.

[26] B. Simon, *Kotani theory for one dimensional stochastic Jacobi matrices*, Comm. Math. Phys. **89** (1983), 227–234.

[27] B. Simon, *A new approach to inverse spectral theory, I. Fundamental formalism*, Ann. of Math. (2) **150** (1999), 1029–1057.

[28] B. Simon, *A canonical factorization for meromorphic Herglotz functions on the unit disk and sum rules for Jacobi matrices*, preprint.

[29] B. Simon, *Orthogonal Polynomials on the Unit Circle*, AMS Colloquium Publications Series, expected 2004.

[30] B. Simon and A. Zlatoš, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*, to appear in Comm. Math. Phys.

[31] T. Stieltjes, *Recherches sur les fractions continues*, Anns. Fac. Sci. Univ. Toulouse **8** (1894–1895), J1–J122; **9**, A5–A47.

[32] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, R.I., 1939; 3rd edition, 1967.

[33] E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations, part 1*, second edition, Clarendon Press, Oxford, 1962.

[34] S. Verblunsky, *On positive harmonic functions: A contribution to the algebra of Fourier series*, Proc. London Math. Soc. **38** (1935), 125–157.

[35] S. Verblunsky, *On positive harmonic functions (second part)*, Proc. London Math. Soc. **40** (1936), 290–320.