Universality of Scaling of Entropy in 1D Gap-less Models

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We consider gap-less models in one dimension (in thermodynamic limit). At small temperature these models can be described by conformal field theory. According to the third law of thermodynamics at zero temperature the entropy of the whole infinite ground state is zero. But the entropy can be positive for a part of the ground state. Let us consider Bose gas with delta interaction as an example. A gas on a space interval $(0, x)$ has some entropy. We study the scaling of the entropy as the size of this subsystem x increases. Our main example is the Hubbard model. For spin chains we prove the formula, discovered by G.Vidal, J.I. Latorre, E.Rico and A. Kitaev. Our proof is based on the second law of thermodynamics.

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Conformal field theory [1] is useful for the description of low temperature behavior of gap-less models in one space and one time dimensions. It is closely related to Luttinger liquid [2] approach. We are interested in specific entropy s [entropy per unit length]. Let us start with specific heat $C = Tds/dT$. Low temperature behavior was obtained in [8, 9]:

$$C = \frac{\pi T c}{3v}, \quad \text{as} \quad T \to 0. \tag{1}$$

Here c is a central charge of corresponding Virasoro algebra and v is Fermi velocity. We are more interested in s. We can integrate the equation and fix the integration constant from the third law of thermodynamics ($s = 0$ at $T = 0$). So for specific entropy we have the same low temperature behavior:

$$s = \frac{\pi T c}{3v}, \quad \text{as} \quad T \to 0. \tag{2}$$

For quantum spin chains this formula agrees with [6] . To formulate the problem precisely let us consider Bose gas with delta interaction. The Hamiltonian of the model is:

$$H = \int dx \left[\partial\psi_x^\dagger \partial\psi_x + g\psi^\dagger\psi^\dagger\psi\psi\right]. \tag{3}$$

Here ψ is a canonical Bose field and $g > 0$ is a coupling constant. The model was solved in [10]. Physics of the model is described in detail in book [4]. First let us consider the model at zero temperature and in the infinite volume. The ground state is unique $|gs\rangle$. We consider positive density case. We are interested in the entropy $S(x)$ of the part of the gas present on the space interval $(0, x)$. Formally we can define it by means of the density matrix

$$\rho = \text{tr}_\infty \left(|gs\rangle\langle gs|\right) \tag{4}$$

Here we traced out the 'external' degrees of freedom, they describe the gas on the rest of the ground state: on the unification of the intervals $(-\infty, 0)$ and (x, ∞). The density matrix ρ describes gas on the interval $(0, x)$. Now we can calculate von Neumann entropy $S(x)$ of the part of the gas on the interval $(0, x)$:

$$S(x) = -\text{tr}_x \rho \ln \rho \tag{5}$$

Here we are taking the trace with respect to the degrees of freedom representing the part of the gas on the interval $(0, x)$. In the major test books it is shown that the laws of thermodynamics can be derived from statistical mechanics, see for example [13, 14] . Second law of thermodynamics states that the entropy is extensive parameter: the entropy of a subsystem $S(x)$ is proportional to the system size x:

$$S(x) = sx \quad \text{at} \quad T > 0 \tag{6}$$

Thermodynamics is applicable to the subsystem of macroscopical size, meaning large x. Here specific entropy s depends on the temperature. For small temperatures the dependence simplifies, see (2):

$$S(x) = \frac{\pi T c}{3v} x, \quad x > \frac{1}{T}. \tag{7}$$

Let us try to find out how $S(x)$ depends on x for zero temperature. It is some functions of the size:

$$S(x) = f(x), \quad \text{at} \quad T = 0 \tag{8}$$

Now let us apply the ideas of conformal field theory, see [1, 8, 9] and also Chapter XVIII of [4] . We can arrive to small temperatures from zero temperature by conformal mapping $\exp[2\pi T z/v]$. It maps the whole complex plane of z without the origin to a strip of the width $1/T$. This replaces zero temperature by finite temperature. The mapping replaces variable x by $[v/\pi T]\sinh[\pi T x/v]$. Now the entropy of the subsystem is given by the formula:

$$S(x) = f\left(\frac{v}{\pi T}\sinh\left[\frac{\pi T x}{v}\right]\right), \quad \text{at} \quad T > 0 \tag{9}$$

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In order to match this to formula (7) we have to consider asymptotic of large x. The formula simplifies:

$$S(x) = f\left(\exp\left[\frac{\pi T(x-x_0)}{v}\right]\right), \quad Tx \to \infty. \qquad (10)$$

Here $\pi T x_0/v = -\ln(v/2\pi T)$.

Formulae (7) and (10) should coincide. Both represent the entropy of the subsystem for small positive temperatures. This provides an equation for f:

$$f\left(\exp\left[\frac{\pi T(x-x_0)}{v}\right]\right) = \frac{\pi T c}{3v}(x-x_0) \qquad (11)$$

This formula describes asymptotic for large x, so we added $-x_0$ to the right hand side. We are considering the region $x > 1/T$ and $x_0 \sim \ln(1/T)$, so $x >> x_0$ at $T \to 0$. In order to solve the equation for f, let us introduce a new variable $y = \exp\left[\pi T(x-x_0)/v\right]$. Then the equation (11) reads:

$$f\left(y\right) = \frac{c}{3}\ln y \qquad (12)$$

So we found the function f in (8). Now we know that at zero temperature entropy of the gas containing on the interval $(0, x)$ is:

$$S\left(x\right) = \frac{c}{3}\ln x \quad \text{as} \quad x \to \infty \qquad (13)$$

Let us remind that for Bose gas $c = 1$, see [4]. Our result agrees with the third law of thermodynamics. Specific entropy is a limit of the ratio $S(x)/x$ as $x \to \infty$. The limit is zero.

The proof presented here is universal. It is also applicable to quantum spin chains. For example to XXZ spin chain. The Hamiltonian of the model is:

$$\mathbf{H}_{XXZ} = -\sum_m \left\{\sigma_x^{(m)}\sigma_x^{(m+1)} + \sigma_y^{(m)}\sigma_y^{(m+1)} + \Delta(\sigma_z^{(m)}\sigma_z^{(m+1)} - 1)\right\} . \qquad (14)$$

It describes interaction of spins 1/2. Here σ are Pauli matrices. The model can be described by a conformal field theory with $c = 1$ in a disordered regime ($-1 \leq \Delta < 1$), see [4].

For the spin chains the formula (13) was discovered by G.Vidal, J.I. Latorre, E.Rico and A. Kitaev in the paper [5]. The authors emphasized the role of the entropy of a subsystem in quantum computing (it describes entanglement).

In a special case $\Delta = 0$ the model is called XX0. For this model the formula (13) was proved in [3]. More detailed form of asymptotic was obtained in that paper:

$$S\left(x\right) = \frac{c}{3}\ln\frac{x}{x_a} \qquad (15)$$

Here x_a defines a characteristic scale. Asymptotic stars when $x > x_a$. Explicit analytical expression for x_a was obtained in [3]. In XX0 the scale x_a is small. Formula (15) provides a good approximation for the entropy of the subsystem not only for large x, but for the full range of sizes of the subsystem.

In the information theory not only von Neumann entropy is used. Sometimes different generalizations of Neumann entropy are useful. The paper [3] shows that Rényi entropy also scales logarithmically with the size [also the coefficient in front of the logarithm is different].

HUBBARD MODEL

The Hamiltonian for the Hubbard model H can be represented as

$$H = -\sum_{\substack{j=1\\ \sigma=\uparrow,\downarrow}} (c^\dagger_{j,\sigma}c_{j+1,\sigma} + c^\dagger_{j+1,\sigma}c_{j,\sigma}) + u\sum_{j=1} n_{j,\uparrow}n_{j,\downarrow} \quad (16)$$

Here $c^\dagger_{j,\sigma}$ is a canonical Fermi operator on the lattice [operator of creation of an electron] and operator $n_{j,\sigma} = c^\dagger_{j,\sigma}c_{j,\sigma}$ is an operator on number of electrons in cite number j with spin σ. Summation in the Hamiltonian goes through the whole infinite lattice. We are considering repulsive case $u > 0$ below half filling [less then one electron per lattice cite]. The model was solved in [11]. A collection of important papers on the model can be found in [12]. Charge and spin separate in the model. The model is gap-less. Both charge and spin degrees of freedom can be described by Virasoro algebra with central charge equal to 1, see [7] :

$$c_c = 1 \quad \text{and} \quad c_s = 1. \qquad (17)$$

Also Fermi velocities are different for spin v_s and charge v_c degrees of freedom. Both spin and charge degrees of freedom contribute to the specific entropy s. So for small temperature we have:

$$s = \frac{\pi T}{3v_s} + \frac{\pi T}{3v_c}. \qquad (18)$$

For small positive temperature the entropy of the subsystem is proportional to the size of the system (second law):

$$S(x) = \left(\frac{\pi T}{3v_s} + \frac{\pi T}{3v_c}\right)x \qquad (19)$$

Now we have to apply conformal arguments separately to spin and charge degrees of freedom. The results add up:

$$S\left(x\right) = \frac{2}{3}\ln x \qquad (20)$$

This describes the entropy of electrons on the interval $(0, x)$ in the infinite ground state at zero temperature. This is actually an asymptotic for large x.

SUMMARY

In the paper we described universal properties of the entropy in one dimensional gap-less models. We studied the entropy of the subsystem. We used the Hubbard model and Bose gas with delta interaction as examples to illustrate the scaling properties of the entropy. The second law of thermodynamics tells us that for positive temperature the entropy of the subsystem is proportional to its size, see (6) and (19). At zero temperature the coefficient at x vanishes $s=0$, now the entropy of the subsystem grows logarithmically with the system size. The formulae (13) and (20) can be considered as a resolution of the third law of thermodynamics. The ratio

$$\frac{S\,(x)}{x} \rightarrow s, \quad \text{as} \quad x \rightarrow \infty \tag{21}$$

approaches specific entropy s for large x. The specific entropy $s=0$ at $T=0$ according to the third law. Our result provides the detailed description of how specific entropy scales down to zero with the subsystem size.

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