Connecting lattice and relativistic models via conformal field theory.

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Abstract

We consider the quantum group invariant XXZ-model. In infrared limit it describes Conformal Field Theory (CFT) with modified energy-momentum tensor. The correlation functions are related to solutions of level -4 of qKZ equations. We describe these solutions relating them to level 0 solutions. We further consider general matrix elements (form factors) containing local operators and asymptotic states. We explain that the formulae for solutions of qKZ equations suggest a decomposition of these matrix elements with respect to states of corresponding CFT.

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1 Quantum group invariant XXZ-model.

Let us recall some well known facts concerning XXZ-model and its continuous limit. Usually XXZ-model is considered as thermodynamic limit of finite spin chain. Consider the space $(\mathbb{C}^2)^{\otimes N}$. The finite spin chain in question is described by the Hamiltonian:

$$H_{XXZ} = \sum_{k=1}^{N} (\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3) \tag{1}$$

where the periodic boundary conditions are implied: $\sigma_{N+1} = \sigma_1$. We consider the critical case $|\Delta| < 1$ and parametrize it as follows:

$$\Delta = \cos \pi\nu$$

It is well-known that in the infrared limit the model describes Conformal Field Theory (CFT) with $c = 1$ and coupling constant equal to ν. The correlation functions in the thermodynamic limit were found by Jimbo and Miwa [2].

It is equally matter of common knowledge that the model is closely related to the R-matrix:

$$R(\beta,\nu) = \begin{pmatrix} a(\beta) & 0 & 0 & 0 \\ 0 & b(\beta) & c(\beta) & 0 \\ 0 & c(\beta) & b(\beta) & 0 \\ 0 & 0 & 0 & a(\beta) \end{pmatrix} \tag{2}$$

where

$$a(\beta) = R_0(\beta), \quad b(\beta) = -R_0(\beta)\frac{\sinh \nu\beta}{\sinh \nu(\pi i - \beta)}$$

$$c(\beta) = R_0(\beta)\frac{\sinh \nu\pi i}{\sinh \nu(\pi i - \beta)}$$

$$R_0(\beta) = \exp\left\{ i \int_0^\infty \frac{\sin(\beta k)\sinh \frac{\pi k(\nu-1)}{2\nu}}{k \sinh \frac{\pi k}{2\nu} \cosh \frac{\pi k}{2}} \right\}$$

The coupling constant ν will be often omitted from $R(\beta,\nu)$. The relation between R-matrix and XXZ-model is explained later.

From the point of view of mathematics the R-matrix (2) is the R-matrix for two-dimensional evaluation representations of the quantum affine algebra $U_q(\widehat{sl}_2)$.

The latter algebra contains two sub-algebras $U_q(sl_2)$. Let us perform a gauge transformation with the R-matrix in order to make the invariance with respect to one of them transparent:

$$\mathcal{R}(\beta_1,\beta_2,\nu)=e^{\frac{\nu}{2}\beta_1\sigma^3}\otimes e^{\frac{\nu}{2}\beta_2\sigma^3}\,R(\beta_1-\beta_2,\nu)\,e^{-\frac{\nu}{2}\beta_1\sigma^3}\otimes e^{-\frac{\nu}{2}\beta_2\sigma^3}=$$
$$=\frac{R_0(\beta_1-\beta_2)}{2\sinh\nu(\pi i-\beta_1+\beta_2)}\left(e^{\nu(\beta_1-\beta_2)}R_{21}^{-1}(q)-e^{\nu(\beta_2-\beta_1)}R_{12}(q)\right) \tag{3}$$

where

$$q=e^{2i\pi(\nu+1)}$$

Adding 1 to ν is important since we will use fractional powers of q. Here $R(q)$ is usual R-matrix for $U_q(sl_2)$:

$$R_{12}(q)=\begin{pmatrix} q^{\frac12} & 0 & 0 & 0\\ 0 & 1 & q^{\frac12}-q^{-\frac12} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & q^{\frac12}\end{pmatrix}$$

We want to use this quantum group symmetry. Unfortunately, the Hamiltonian (1) is not invariant with respect to the action of the quantum group which is represented in the space $(\mathbb{C}^2)^{\otimes N}$ by

$$S^3=\sum_{k=1}^{N}\sigma_k^3$$
$$S^{\pm}=\sum_{k=1}^{N}q^{-\frac{\sigma_1^3}{4}}\cdots q^{-\frac{\sigma_{k-1}^3}{4}}\sigma_k^{\pm}q^{\frac{\sigma_{k+1}^3}{4}}\cdots q^{\frac{\sigma_N^3}{4}}$$

A solution of this problem of quantum group invariance was found by Pasquier and Saleur [3]. They proposed to consider another integrable model on the finite lattice with Hamiltonian corresponding to open boundary conditions:

$$H_{RXXZ}=\sum_{k=1}^{N-1}(\sigma_k^1\sigma_{k+1}^1+\sigma_k^2\sigma_{k+1}^2+\Delta\sigma_k^3\sigma_{k+1}^3)+i\sqrt{1-\Delta}\,(\sigma_1^3-\sigma_N^3) \tag{4}$$

This Hamiltonian is manifestly invariant under the action of quantum group on the finite lattice. After the thermodynamic limit one obtains a model with the same spectrum as original XXZ, but different scattering (this point will be described later). The infrared limit corresponds to CFT with modified energy-momentum tensor of central charge

$$c = 1 - \frac{6\nu^2}{1-\nu}$$

especially interesting when ν is rational and additional restriction takes place.

In the present paper we shall consider RXXZ-model. We shall propose formulae for correlators for this model showing their similarity with correlators for XXX-model. The latter can be expressed in terms of values of Riemann zeta-function at odd natural arguments. We shall obtain an analogue of this statement for RXXZ-model.

Let us say few words about hypothetic relation of XXZ and RXXZ models in thermodynamic limit. The argument that this limit should not depend on the boundary conditions must be dismissed in our situation since we consider a critical model with long-range correlations. Still we would expect that the following relation between two models in infinite volume exists. The quantum group $U_q(sl_2)$ acts on infinite XXZ-model and commute with the Hamiltonian. Consider a projector $\mathcal{P}$ on the invariant subspace. We had XXZ-vacuum $|\mathrm{vac}\rangle_{XXZ}$. We suppose that the RXXZ-model is obtained by projection, in particular:

$$|\mathrm{vac}\rangle_{RXXZ} = \mathcal{P}|\mathrm{vac}\rangle_{XXZ}$$

The correlators in RXXZ-model are

$$_{RXXZ}\langle\mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle_{RXXZ} = \ _{XXZ}\langle\mathrm{vac}|\mathcal{POP}|\mathrm{vac}\rangle_{XXZ}$$

which can be interpreted in two ways: either as correlator in RXXZ-model or as correlator of $U_q(sl_2)$-invariant operator $\mathcal{POP}$ in XXZ-model. This assumption explains the notation RXXZ standing for Restricted XXZ-model. So, we assume that in the lattice case a phenomenon close to the one taking place in massive models occurs [12].

Let us explain in some more details the set of operators in XXZ model for which we are able to calculate the correlators in simple form provided the above reasoning holds. Under $\mathcal{O}$ we understand some local operator of XXZ-chain, i.e. a product of several local spins σ_k^a, $a = 1, 2, 3$. Under the above action of quantum group these spins transform with respect to 3-dimensional adjoint representation. The projection $\mathcal{POP}$ extracts all the invariant operators, i.e. projects over the subspace of singlets in the tensor product of 3-dimensional representations.

Let us explain more explicitly the relation between the the R-matrix and XXZ, RXXZ Hamiltonians. Both of them can be constructed form the transfer-matrix with different boundary conditions constructed via the monodromy matrix:

$$R_{01}(\lambda)R_{02}(\lambda)\cdots R_{0,N-1}(\lambda)R_{0,N}(\lambda)$$

In some cases it is very convenient to consider inhomogeneous model for which the monodromy matrix contains a fragment:

$$R_{0k}(\lambda-\lambda_k)\cdots R_{0,k+n}(\lambda-\lambda_{k+n})$$

As we shall see many formulae become far more transparent for inhomogeneous case.

2 QKZ on level -4 and correlators.

The main result of Kyoto group [1, 2] is that the correlators in XXZ-model are related to solutions of QKZ-equations [6, 8] on level -4. We formulate the equations first and then explain the relation. The equations for the function $g(\beta_1,\cdots,\beta_{2n})\in\mathbb{C}^{\otimes 2n}$ are

$$\begin{aligned}&R(\beta_j-\beta_{j+1})g(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_{2n})=\\&=\ g(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})\end{aligned}\tag{5}$$

$$g(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i)=g(\beta_{2n},\beta_1,\cdots,\beta_{2n-1})\tag{6}$$

For application to correlators a particular solution is needed which satisfies additional requirement:

$$g(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})|_{\beta_{j+1}=\beta_j-\pi i}=s_{j,j+1}\otimes g(\beta_1,\cdots,\beta_{j-1},\beta_{j+2},\cdots,\beta_{2n})\tag{7}$$

where $s_{j,j+1}$ is the vector $(\uparrow\downarrow)+(\downarrow\uparrow)$ in the tensor product of j-th and $(j+1)$-th spaces.

The relation of these equations to correlators is conjectured by Jimbo and Miwa [2]. It cannot be proved for critical model under consideration as it was done for the XXZ-model with $|q|<1$ in [1]. However, later arguments based on Bethe Anzatz technique were proposed by Maillet and collaborators [4, 5] which can be considered as a proof of Jimbo and Miwa conjecture.

Jimbo and Miwa find the solution needed [2] in the form:

$$g(\beta_1,\cdots,\beta_{2n})=\frac{1}{\sum e^{\beta_j}}\prod_{i<j}\zeta^{-1}(\beta_i-\beta_j)\int\limits_{-\infty}^{\infty}d\alpha_1\cdots\int\limits_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i,\beta_j,\nu)$$
$$\times\prod_{i<j}\frac{A_i^2-A_j^2}{a_i-a_jq}\,D(a_1,\cdots a_{n-1}|b_1,\cdots,b_{2n})$$

where

$$\varphi(\alpha,\beta,\nu)=\exp\left\{-(1+\nu)\frac{\alpha+\beta}{2}-2\int\limits_0^{\infty}\frac{\sin^2(\frac{\alpha-\beta}{2}k)\sinh\frac{\pi k(\nu+1)}{2\nu}}{k\sinh\frac{\pi k}{2\nu}\sinh\pi k}\right\}$$

$\zeta(\beta)$ is some complicated function, we shall not need it. We use the notations:

$$a_j=e^{2\nu\alpha_j},\quad b_j=e^{2\nu\beta_j},\quad A_j=e^{\alpha_j}\quad B_j=e^{\beta_j}$$

$D(a_1,\cdots a_{n-1}|b_1,\cdots,b_{2n})$ is a Laurent polynomial of all its variables taking values in $\mathbb{C}^{2n}$. We shall not use explicit formula for this polynomial in the present paper.

For application to correlators in homogeneous XXZ-model one has to specify:

$$\beta_1=\beta_2=\cdots=\beta_n=-\frac{\pi i}{2}$$

$$\beta_{n+1}=\beta_{n+2}=\cdots=\beta_{2n}=\frac{\pi i}{2}$$

Then

$$g\left(-\frac{\pi i}{2},\cdots,-\frac{\pi i}{2},\frac{\pi i}{2},\cdots,\frac{\pi i}{2}\right)=$$
$$=\int\limits_{-\infty}^{\infty}d\alpha_1\cdots\int\limits_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i<j}\frac{A_i^2-A_j^2}{a_i-a_jq}\prod_i\frac{1}{A_i+A_i^{-1}}\ \widetilde{D}(a_1,\cdots,a_{n-1})\qquad(8)$$

with some Laurent polynomial $\widetilde{D}(a_1,\cdots,a_{n-1})$. The trouble with this integral is that it is essentially multi-fold one. In our previous papers we have shown that the integrals can be simplified and essentially reduced to products of one-fold ones in XXX case. For the moment we cannot state the same for XXZ-model, but we shall explain that the simplification can be done in RXXZ case. Let us consider this in some more details.

According to the understanding of relation between XXZ and RXXZ models explained in the Introduction we expect that the correlators for RXXZ model are related to certain invariant under the quantum group solution of the same equations (6,7). In order to make the quantum group symmetry transparent we make the transformation:

$$\widehat{g}(\beta_1,\cdots,\beta_{2n}) = \exp\left(\frac{\nu}{2}\sum\beta_j\sigma_j^3\right) g(\beta_1,\cdots,\beta_{2n})$$

With this notation the equations (6,7) take the form:

$$\begin{aligned}&\mathcal{R}(\beta_j,\beta_{j+1})\widehat{g}(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_{2n}) = \\ &= \ \widehat{g}(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})\end{aligned} \tag{9}$$

$$\widehat{g}(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i) = -q^{-\frac{1}{2}\sigma_{2n}^3}\widehat{g}(\beta_{2n},\beta_1,\cdots,\beta_{2n-1}) \tag{10}$$

and

$$\widehat{g}(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})|_{\beta_{j+1}=\beta_j-\pi i} = i\ \widehat{s}_{j,j+1}\otimes\widehat{g}(\beta_1,\cdots,\beta_{j-1},\beta_{j+2},\cdots,\beta_{2n}) \tag{11}$$

where $\widehat{s}_{j,j+1}$ is the quantum group singlet in the tensor product of corresponding spaces:

$$q^{\frac{1}{4}}(\uparrow\downarrow) - q^{-\frac{1}{4}}(\downarrow\uparrow)$$

These equations respect the invariance under the quantum group. This fact is obvious for the first and the third equations. To see this in the second equation one has to keep in mind that $q^{\frac{1}{2}\sigma^3}$ gives in two-dimensional representation the element which realizes the square of antipode as inner authomorphism.

From Jimbo-Miwa solution (8) one can obtain a solution to (10,11) by projection on $U_q(sl_2)$- invariant subspace which will suffer of the same problems related to denominators. The main goal of this paper is to show that at least in this case corresponding to RXXZ-model another form of solution is possible.

3 QKZ on level 0.

Consider the qKZ equations on level 0 which are the same as two out of three basic equations (axioms) for the form factors. We write these equations in $U_q(sl_2)$-invariant form which corresponds to form factors of RSG-model [12]. Consider a co-vector $\widehat{f}(\beta_1,\cdots,\beta_{2n})\in(\mathbb{C}^{\otimes 2n})^*$. The equations are

$$\widehat{f}(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_{2n})=$$
$$=\ \ \widehat{f}(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})\mathcal{R}(\beta_j-\beta_{j+1})$$

$$\widehat{f}(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i)=-q^{-\frac{1}{2}\sigma^3_{2n}}\widehat{f}(\beta_{2n},\beta_1,\cdots,\beta_{2n-1})$$

We need solution belonging to the singlet with respect to the action of $U_q(sl_2)$ subspace as has been explained in level -4 case. The application to form factors imposes additional requirement which connects sectors with different number of particles:

$$\begin{aligned}&2\pi i\,\mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i}\widehat{f}(\beta_1\cdots,\beta_{2n-2},\beta_{2n-1},\beta_{2n})=\\&=\widehat{s}^{\,*}_{2n-1,2n}\otimes\widehat{f}(\beta_1\cdots,\beta_{2n-2})\left(1-\mathcal{R}(\beta_{2n-1}-\beta_1)\cdots\mathcal{R}(\beta_{2n-1}-\beta_{2n-2})\right)\end{aligned}\tag{12}$$

The difference with level -4 case seems to be minor, but the formulae for solutions are much nicer. Many solutions can be written which are counted sets of integers: $\{k_1,\cdots,k_{n-1}\}$ such that $0\le k_1<,\cdots<k_{n-1}\le 2n-2$:

$$\begin{aligned}&f^{\{k_1,\cdots,k_{n-1}\}}(\beta_1,\cdots,\beta_{2n})=\prod_{i<j}\zeta(\beta_i-\beta_j)\int\limits_{-\infty}^{\infty}d\alpha_1\cdots\int\limits_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i,\beta_j)\\&\times\det\|A_i^{k_j}\|_{1\le i,j\le n-1}\ \ h(a_1,\cdots a_{n-1}|b_1,\cdots,b_{2n})\prod_j a_jA_j\end{aligned}$$

where h is skew-symmetric w.r. to α's polynomial. Notice that there are no denominators mixing the integration variables in the integrant, so, effectively the integral is reduced to one-fold integrals of the form:

$$\langle P\mid p\rangle=\int\limits_{-\infty}^{\infty}\prod_j\varphi(\alpha,\beta_j)\ P(A)\ p(a)aAd\alpha\tag{13}$$

where $p(\alpha)$ and $P(A)$ are polynomials. This is what we would like to have for the correlators!

Again we do not describe explicitly the functions h which take values in $(\mathbb{C}^{\otimes 2n})^*$. As has been said they are skew-symmetric polynomials of $a's$ they are also rational functions of $b's$ with simple poles at $b_i = qb_j$ only. But there is one important property of h which we need to mention.

First, the integral (13) is such that the degree of any polynomial $s(a)$ can be reduced to $2n-2$ or less. For the polynomials of degree $\leq 2n-2$ there is a basis (choice is not unique)

$$s_j(\alpha),\ j = -(n-1), \cdots, (n-1),\ \deg(s_j) = j + (n-1)$$

with special properties described later. We shall not write down explicit formulae. Then

$$h(a_1, \cdots, a_{n-1}) = \sum_{j_1 \neq 0, \cdots j_{n-1} \neq 0} h_{j_1, \cdots, j_{n-1}} \det \| s_{j_p}(a_q) \|_{1 \leq p,q \leq n-1}$$

and the skew-symmetric tensor h belongs to subspace of maximal irreducible representation of symplectic group $Sp(2n-2)$ of dimension

$$\dim(\mathcal{H}_{\text{irreducible}}) = \binom{2n-2}{n-1} - \binom{2n-2}{n-3}$$

Let

$$J = 1, \cdots, \binom{2n-2}{n-1} - \binom{2n-2}{n-3}$$

Consider the basis e^J in $\mathcal{H}_{\text{irreducible}}$ with components $e^J_{j_1, \cdots j_{n-1}}$. Then we define h_J by

$$h_{j_1, \cdots, j_{n-1}} = \sum_J h_J \, e^J_{j_1, \cdots j_{n-1}}$$

Recall that $h(a_1, \cdots, a_{n-1})$ takes values in singlet subspace, so, it has components

$$h^I(a_1, \cdots, a_{n-1})$$

where I counts basis of this subspace:

$$I = 1, \cdots, \binom{2n}{n} - \binom{2n}{n-1}$$

Notice that

$$\binom{2n}{n} - \binom{2n}{n-1} = \binom{2n-2}{n-1} - \binom{2n-2}{n-3}$$

which means that there is a square matrix h^I_J defined by

$$h^I(a_1, \cdots, a_{n-1}) = \sum_J h^I_J s^J(a_1, \cdots a_{n-1})$$

where $s_J(a_1, \cdots a_{n-1})$ are the following anti-symmetric polynomials:

$$s^J(a_1, \cdots a_{n-1}) = \sum_{j_1, \cdots, j_{n-1}} e^J_{j_1, \cdots j_{n-1}} \text{det} \| s_{j_p}(a_q) \|_{1 \leq p, q \leq n-1}$$

If we do not consider $s_0(a)$ the degrees of polynomials $P(A)$ can be reduced to $2n-3$ or less. We consider a special basis

$$S_j(A), \quad |j| = 1, \cdots, (n-1),$$

$$\text{deg}(S_{-k}) = 2k-1, \ k = 1, \cdots n-1,$$
$$\text{deg}(S_k) = 2k-2, \ k = 1, \cdots n-1$$

which we do not describe explicitly, again.

The most important property of the integrals $\langle S_i \mid s_j \rangle$ is deformed Riemann bilinear relation:

$$\sum_{k=1}^{n-1} \left(\langle S_k \mid s_i \rangle \langle S_{-k} \mid s_j \rangle - \langle S_k \mid s_j \rangle \langle S_{-k} \mid s_i \rangle \right) = \delta_{i,-j}$$
$$\sum_{k=1}^{n-1} \left(\langle S_i \mid s_k \rangle \langle S_j \mid s_{-k} \rangle - \langle S_j \mid s_k \rangle \langle S_i \mid s_{-k} \rangle \right) = \delta_{i,-j}$$

These relations and properties of $h(\alpha_1, \cdots, \alpha_{n-1})$ imply that among $f^{\{k_1, \cdots, k_{n-1}\}}$ only $\text{dim}(\mathcal{H}_{\text{irreducible}})$ are linearly independent which are span by action of $Sp(2n-2)$ on $\{1, 3, \cdots, 2n-3\}$. The basis in this space is denoted by

$$S_J(A_1, \cdots, A_{n-1}) = \sum_{j_1, \cdots, j_{n-1}} e_J^{j_1, \cdots, j_{n-1}} \text{det} \| S_{j_p}(A_q) \|_{p,q=1,\cdots,n-1}$$

The result is that the solutions are combined into square matrix (there is the same number of solutions as the dimension of space):

$$F^J_I = P^K_I H^J_K$$

where H_{KJ} is polynomial function of β_j, the transcendental dependence on β_j is hidden in the period matrix P_I^J which is defined as

$$P_I^J = \langle S^J \mid s_I \rangle$$

where the notations has obvious meaning:

$$\langle P_1 \wedge \cdots \wedge P_{n-1} \mid p_1 \wedge \cdots \wedge p_{n-1} \rangle = \mathrm{det} \| \langle P_i \mid p_j \rangle \|_{1 \leq i,j \leq n-1}$$

4 New formula for level -4 from level 0.

Recall that solutions to QKZ on level 0 are co-vectors while solutions on level -4 are vectors. Consider the scalar product for two solutions:

$$f(\beta_1, \cdots, \beta_{2n}) g(\beta_1, \cdots, \beta_{2n})$$

it is a quasi-constant (symmetric function of e^{β_j}).
So, we can construct singlet solutions of QKZ on level -4 from those on level 0. Indeed we have square matrix F:

$$G = F^{-1} = H^{-1} P^{-1}$$

The matrix H^{-1} is complicated but rational function of β_j.
Due to deformed Riemann relation it is easy to invert P! Indeed

$$\left(P^{-1}\right)_J^I = \langle S^I \mid s_J^{\dagger} \rangle$$

where $s_J^{\dagger}$ is obtained from s_J replacing all

$$s_j \longrightarrow \mathrm{sgn}(j) s_{-j}$$

So, the transcendental part almost does not change, and we prove that the new formula for solutions on level -4 is possible:

$$g^{\{k_1, \cdots, k_{n-1}\}}(\beta_1, \cdots, \beta_{2n}) = \prod_{i<j} \zeta(\beta_i - \beta_j) \int_{-\infty}^{\infty} d\alpha_1 \cdots \int_{-\infty}^{\infty} d\alpha_{n-1} \prod_{i,j} \varphi(\alpha_i, \beta_j)$$
$$\times \mathrm{det} \| A_i^{k_j} \|_{1 \leq i,j \leq n-1} \ \tilde{h}(a_1, \cdots a_{n-1} | b_1, \cdots, b_{2n}) \prod_j a_j A_j$$

where $\tilde{h}$ are skew-symmetric w.r. to a_i polynomials. Actually, they are polynomials in b_j as well. The proof is based on the following calculation:

$$H^{-1} = H^* (HH^*)^{-1}$$

The operator HH^* is nicer than H itself because it acts from $\mathcal{H}_{\text{irreducible}}$ to itself. We were able to calculate its determinant:

$$\det(HH^*) = Const \left(\prod_{i,j} (b_i - qb_j) \right)^{-\left(\binom{2n-4}{n-2} - \binom{2n-4}{n-4}\right)}$$

Also the rank of the residue of H at $b_j = q\beta_i$ equals the dimension of singlet subspace in $\mathbb{C}^{\otimes(2n-2)}$.

5 Cohomological meaning of new formula.

"Classical" limit: $\nu \to 0$ and β_j are rescaled in such a way that b_j are finite. In this limit

$$\langle P \mid p \rangle = \int_{-\infty}^{\infty} \prod_j \varphi(\alpha, \beta_j) \; P(A) \; p(a) d\alpha \to$$
$$\to \int_\gamma \frac{p(a)}{c} da$$

where the hyper-elliptic surface X is defined by

$$c^2 = \prod (a - b_j),$$

The genus equals $n - 1$. The contour γ is defined by P. In particular,

$$S_{-k} \leftrightarrow \mathrm{b}_k, \quad S_k \leftrightarrow \mathrm{a}_k$$

Consider

$$\mathrm{Symm}(X^{n-1})$$

the points on this variety are divisors:

$$\{P_1, \cdots, P_{n-1}\} \qquad P_j = \{a_j, c_j\} \in X$$

Consider the non-compact variety

$$\text{Symm}(X^{n-1}) - D$$

where

$$D = \{\{P_1, \cdots, P_{n-1}\} | P_j = \infty^{\pm}, P_i = \sigma(P_j)\}$$

This is an affine variety isomorphic to affine Jacobian.

The integrant of the classical limit of invariant part of Jimbo-Miwa solution (8) gives a $(n-1)$-differential form (maximal dimension) on

$$\text{Symm}(X^{n-1}) - D$$

of the kind:

$$\Omega = \frac{F(a_1, c_1, \cdots, a_{n-1}, c_{n-1})}{\prod_{i<j}(a_i - a_j)} \frac{da_1}{c_1} \wedge \cdots \wedge \frac{da_{n-1}}{c_{n-1}}$$

where the polynomial $F(a_1, c_1, \cdots, a_{n-1}, c_{n-1})$ vanishes when $a_i = a_j$ and $c_i = c_j$. The question arises concerning cohomologies.

Theorem (A. Nakayashiki) The elements of $H^{(n-1)}$ can be realized as

$$\Omega_{k_1, \cdots, k_{n-1}} = \det \| a_p^{k_q} \|_{p,q=1,\cdots,n-1} \quad \frac{da_1}{c_1} \wedge \cdots \wedge \frac{da_{n-1}}{c_{n-1}}$$

where $k_q = 0, \cdots, 2n-2$.

Remark. Actually some of these forms are linearly dependent (mod exact forms), we do not describe all details.

6 Back to correlattors.

Comparing with Jimbo-Miwa solution one makes sure that the solution needed for correlators is

$$g^{\{0,2,4,\cdots,2n-4\}}$$

so, it corresponds to "a-cycles". We need to put

$$\beta_k = \lambda_k - \frac{\pi i}{2} + i\delta_k, \quad \beta_{2n-k+1} = \lambda_k + \frac{\pi i}{2} - i\delta_k$$

and to take he limit $\delta_k \to 0$. The calculation of integrals is similar to XXX case, the result can be expressed in terms of the function:

$$\chi(\alpha) = \frac{d}{d\alpha}\left(\log\frac{\varphi(\alpha - \frac{\pi i}{2})}{\varphi(\alpha + \frac{\pi i}{2})}\right) = i\int\limits_0^\infty \frac{\cos(\alpha k)\sinh\frac{\pi k(\nu-1)}{2\nu}}{\sinh\frac{\pi k}{2\nu}\cosh\frac{\pi k}{2}}dk =$$

$$= i\sum_{m=0}^\infty \alpha^{2m}\,\frac{(-1)^m}{(2m)!}\int\limits_0^\infty \frac{k^{2m}\sinh\frac{\pi k(\nu-1)}{2\nu}}{\sinh\frac{\pi k}{2\nu}\cosh\frac{\pi k}{2}}dk$$

Finally for the correlator in inhomogeneous case:

$$g(\lambda_1 - \frac{\pi i}{2}\cdots\lambda_n - \frac{\pi i}{2}, \lambda_n + \frac{\pi i}{2}\cdots\lambda_1 + \frac{\pi i}{2})^{\epsilon_1\cdots\epsilon_n\epsilon_{n+1}\cdots\epsilon_{2n}} =$$

$$= \sum_{m=0}^{\left[\frac{n}{2}\right]}\sum_{k_1,\cdots,k_{2m}} Q^{\epsilon_1\cdots\epsilon_n\epsilon_{n+1}\cdots\epsilon_{2n}}_{k_1k_2\cdots k_{2m-1}k_{2m}}(\lambda_1,\cdots,\lambda_n)$$

$$\times\,\chi(\lambda_{k_1} - \lambda_{k_2})\cdots\chi(\lambda_{k_{2m-1}} - \lambda_{k_{2m}})$$

7 General matrix elements.

When we pass to description of XXZ-model in terms of particles a common phenomenon known nowadays as "modular double" [18] occurs. The essence of this phenomenon is that another quantum group with dual q enters the game. In a sense RXXZ model is invariant with respect to "modular double" which is quite non-trivial, and not completely understood, combination of two quantum groups. The particle description of the model is as follows.

For coupling constants not very far from 0 the spectrum of the model contains one particle (magnon). This particle is parametrized by rapidity θ carrying momentum and energy:

$$p(\theta) = \log\tanh\frac{1}{2}\left(\theta - \frac{\pi i}{2}\right), \quad e(\theta) = \frac{dp(\theta)}{d\theta}$$

The particle has internal degrees living in isotopic space $\mathbb{C}^2$. The S-matrix is given by

$$S(\theta_1 - \theta_2) = R(\theta_1 - \theta_2,\ \tfrac{\nu}{1-\nu})$$

This is where the second quantum group appears. The RXXZ model is invariant under the action of of two quantum groups:

$$U_q(sl_2)\ ,\ U_{\widetilde{q}}(sl_2), \quad \text{with} \quad q = e^{2\pi i(\nu+1)}\ ,\ \widetilde{q} = e^{\frac{2\pi i}{1-\nu}}$$

For the asymptotic states it means that they must be taken as invariant under the action of the second quantum group. All that is familiar from consideration of massive models and its restrictions [12].

Consider the matrix elements

$$_{RXXZ}\langle \text{vac} \mid \mathcal{O} \mid \theta_1, \cdots, \theta_n \rangle_{RXXZ}$$

where $\mathcal{O}$ is some operator of the type

$$E_{\epsilon_1}^{\epsilon_1'} \cdots E_{\epsilon_n}^{\epsilon_n'}$$

It can be obtained from "Kyoto generalization" which is the function

$$\widehat{f}(\beta_1, \cdots, \beta_{2n}, \theta_1, \cdots, \theta_{2m}) \in \mathbb{C}^{\otimes 2n} \otimes (\mathbb{C}^*)^{\otimes 2m}$$

which satisfies level -4 qKZ with R-matrix $\mathcal{R}(\cdot, \nu)$ (denoted by $\mathcal{R}(\cdot)$) with respect to β's and level 0 qKZ with gauge transformed S-matrix $\mathcal{R}(\cdot, \frac{\nu}{1-\nu})$ (denoted by $\mathcal{S}(\cdot)$) with respect to θ's. Actually, both equations are slightly modified. In addition it must satisfy the following normalization conditions. All together we have:

$$\begin{aligned} &\mathcal{R}(\beta_{j+1} - \beta_j)\widehat{f}(\beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_{2n}, \theta_1, \cdots, \theta_{2m}) = \\ &= \ \widehat{f}(\beta_1, \cdots, \beta_j, \beta_{j+1}, \cdots, \beta_{2n}, \theta_1, \cdots, \theta_{2m}) \end{aligned} \tag{14}$$

$$\begin{aligned} &\widehat{f}(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i, \theta_1, \cdots, \theta_{2m}) = \\ &= -\prod_{j=1}^{2m} \tanh \frac{1}{2}\left(\beta_{2n} - \theta_j + \frac{\pi i}{2}\right) q^{\frac{1}{2}\sigma_{2n}^3} \widehat{f}(\beta_{2n}, \beta_1, \cdots, \beta_{2n-1}, \theta_1, \cdots, \theta_{2m}) \end{aligned} \tag{15}$$

$$\begin{aligned} &\widehat{f}(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}, \theta_1, \cdots, \theta_{2m})|_{\beta_{2n}=\beta_{2n-1}+\pi i} = \\ &= \widehat{s}_{2n-1,2n} \otimes \widehat{f}(\beta_1, \cdots, \beta_{2n-2}, \theta_1, \cdots, \theta_{2m}) \end{aligned} \tag{16}$$

$$\widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_{j+1},\theta_j,\cdots,\theta_{2m}) = \\ = \widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_j,\theta_{j+1},\cdots,\theta_{2m})\mathcal{S}(\theta_j-\theta_{j+1}) \tag{17}$$

$$\widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_{2m-1},\theta_{2m}+2\pi i) = \\ = -\prod_{j=1}^{2n}\tanh\frac{1}{2}\left(\theta_{2m}-\beta_j+\frac{\pi i}{2}\right)\widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_{2m},\theta_1,\cdots,\theta_{2m-1})q^{-\frac{1}{2}\sigma^3_{2m}} \tag{18}$$

$$2\pi i\mathrm{res}_{\theta_{2m}=\theta_{2m-1}+\pi i}\widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_1\cdots,\theta_{2m-2},\theta_{2m-1},\theta_{2m}) = \\ = \widehat{s}^{\,*}_{2m-1,2m}\otimes\widehat{f}(\beta_1,\cdots,\beta_{2n},\theta_1\cdots,\theta_{2m-2}) \\ \times\left(1-\prod_{j=1}^{2n}\tanh\frac{1}{2}\left(\theta_{2m-1}-\beta_j+\frac{\pi i}{2}\right)\mathcal{S}(\theta_{2m-1}-\theta_1)\cdots\mathcal{S}(\theta_{2m-1}-\theta_{2m-2})\right) \tag{19}$$

The equations (14,15,17,18) are slightly different from respectively level -4 and level 0 qKZ equations because of multipliers containing $\tanh$'s. This difference, however, is easily taken care of by multiplier

$$\prod_{i=1}^{2n}\prod_{j=1}^{2m}\psi(\beta_i,\theta_j)$$

where the function

$$\psi(\beta,\theta)=2^{-\frac{3}{4}}\exp\left(-\frac{\beta+\theta}{4}-\int_0^\infty\frac{\sin^2\frac{1}{2}(\beta-\theta+\pi i)k+\sinh^2\frac{\pi k}{2}}{k\sinh\pi k\cosh\frac{\pi k}{2}}dk\right)$$

satisfies the equations:

$$\psi(\beta,\theta+2\pi i)=\tanh\frac{1}{2}(\theta-\beta+\frac{\pi i}{2})\psi(\beta,\theta)$$
$$\psi(\beta,\theta)\psi(\beta,\theta+\pi i)=\frac{1}{e^\beta-i\epsilon^\theta}$$

For the function $f(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_{2m})$ in XXZ-model Jimbo-Miwa give a formula of the following kind:

$$f(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_{2m})=\prod_{i<j}\zeta(\theta_i-\theta_j,\tfrac{\nu}{1-\nu})\prod_{i<j}\zeta^{-1}(\beta_i-\beta_j,\nu)\prod_{i,j}\psi(\beta_i,\tau_j)$$

$$\times\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_n\int_{-\infty}^{\infty}d\sigma_1\cdots\int_{-\infty}^{\infty}d\sigma_m\prod\varphi(\alpha_i-\beta_j,\nu)\prod\varphi(\sigma_i-\theta_j,\tfrac{\nu}{1-\nu})$$

$$\times\prod_{i<j}\frac{A_i^2-A_j^2}{a_i-qa_j}\prod_{i<j}\frac{S_i^2-S_j^2}{\mathrm{s}_i-\widetilde{q}\mathrm{s}_j}\prod\frac{1}{A_i^2-S_j^2}$$

$$\times D(a_1,\cdots,a_n|b_1,\cdots,b_{2n})F(\mathrm{s}_1,\cdots,\mathrm{s}_m|\mathrm{t}_1,\cdots\mathrm{t}_{2m})$$

where we use the notations:

$$a_j=e^{2\nu\alpha_j},\quad b_j=e^{2\nu\beta_j},\quad A_j=e^{\alpha_j},\quad B_j=e^{\beta_j}$$
$$\mathrm{s}_j=e^{\frac{2\nu}{1-\nu}\sigma_j},\quad \mathrm{t}_j=e^{\frac{2\nu}{1-\nu}\theta_j},\quad S_j=e^{\sigma_j},\quad ,T_j=e^{\theta_j}$$

The functions D, F are polynomials of their variables. For us the main problem with this formula is in denominators. Here we are concerned not only about the denominators a_i-qa_j and $\mathrm{s}_i-\widetilde{q}\mathrm{s}_j$ which are unpleasant for technical reasons as explained above. Our main trouble is in the denominators $A_i^2-S_j^2$ because due to certain physical intuition we would expect another kind of formula. Let us explain the point.

At this point it would be more clear to talk about lattice SOS-model instead of RXXZ-model. These two models are equivalent due to usual Onzager relation between 2D classical statistical physics and 1D quantum mechanics. The advantage of the lattice model is due to the fact that it allows intuitively clear relation to Euclidian Quantum Field Theory. Our physical intuition about general matrix element is based on the following picture:

Let us give some explanations. Suppose we consider instead of critical model SOS-model out of criticality corresponding to elliptic R-matrix. Suppose further that we are very close to the critical temperature. Then microscopically we have already critical lattice SOS-model. On the scales much bigger than the lattice size but much less than the correlation length we have massless relativistic field theory which is nothing but CFT with the central charge

$$c = 1 - \frac{6\nu^2}{\nu - 1}$$

Finally on the scales of the order of correlation length we have massive relativistic field theory which is RSG-model with the coupling constant $\frac{1-\nu}{\nu}$. The role of CFT is clear: it describes infrared limit of the lattice model and ultraviolet limit of massive model. The local operators of the massive model are counted by the states of CFT. These local operators are described by form factors in asymptotic states description. On the other hand one should be able to consider the lattice critical model with boundary conditions corresponding to different states of CFT. That is why we expect the following kind of formula for general matrix element:

$$_{RXXZ}\langle \text{vac}|\mathcal{O}|\theta_1, \cdots, \theta_{2m}\rangle = \sum_{\Psi} {}_{RXXZ}\langle \text{vac}|\mathcal{O}|\Psi\rangle\langle \Psi\, |\theta_1, \cdots, \theta_{2m}\rangle \qquad (20)$$

where Ψ are states of CFT, $\langle \Psi\, |\theta_1, \cdots, \theta_{2m}\rangle$ are form factors of local operator corresponding to Ψ in RSG-model, $_{RXXZ}\langle \text{vac}|\mathcal{O}|\Psi\rangle$ are correlators of local operator $\mathcal{O}$ in the lattice model (of usual kind $E_{\epsilon_1}^{\epsilon_1'} \cdots E_{\epsilon_n}^{\epsilon_n'}$). The latter object requires more careful definition, we hope to return to it in feature.

Notice that the formula (20) is in nice correspondence with the system of equations (14, 15, 16, 17, 18, 19) because passing from the Kyoto generalize correlator to the usual one we put $\beta_j = \beta_{2n-j+1} + \pi i$, so, the th in equations with respect to th_j cancel, and we get usual Form Factor Axioms. In our case of RSG-model a complete set of solutions to these axioms is known [15, 20], so, a formula of the kind (20) must hold.

So, there must be a formula of the type:

$$f(\beta_1,\cdots,\beta_{2n},\theta_1,\cdots,\theta_{2m})=\prod_{i<j}\zeta(\theta_i-\theta_j,\tfrac{\nu}{1-\nu})\prod_{i<j}\zeta^{-1}(\beta_i-\beta_j,\nu)\prod_{i,j}\psi(\beta_i,\tau_j)$$

$$\times\int\limits_{-\infty}^{\infty}d\alpha_1\cdots\int\limits_{-\infty}^{\infty}d\alpha_n\int\limits_{-\infty}^{\infty}d\sigma_1\cdots\int\limits_{-\infty}^{\infty}d\sigma_m\prod\varphi(\alpha_i-\beta_j,\nu)\prod\varphi(\sigma_i-\theta_j,\tfrac{\nu}{1-\nu})$$

$$\times M(A_1,\cdots,A_{n-1}|T_1,\cdots,T_{m-1})$$
$$\times\widetilde{h}(a_1,\cdots,a_n|b_1,\cdots,b_{2n})h(\mathrm{s}_1,\cdots,\mathrm{s}_m|\mathrm{t}_1,\cdots\mathrm{t}_{2m}) \tag{21}$$

where $M(A_1,\cdots,A_{n-1}|S_1,\cdots,S_{m-1})$ is skew-symmetric with respect to $A_1,\cdots,A_{n-1}$ and $T_1,\cdots,T_{m-1}$ polynomial which depends on B_j, S_j as on parameters. This polynomial must satisfy certain equations in order that the relations (16, 19) hold. We do not write down explicitly these bulky equations, but fortunately they coincide with equations for similar polynomials for quite different problem which is the calculation of form factors for massless flows [19]. The solution to these equations is not unique, but there is a "minimal" one which has minimal possible degree with respect to variables A_j and S_j. Our conjecture is that this is the solution we need. It satisfies all simple checks that we were able to carry on. Denote the sets $S=\{1,\cdots,2n\}$, $S'=\{1,\cdots,2m\}$. The polynomial is:

$$\begin{aligned}&M(A_1,\cdots,A_{n-1}|S_1,\cdots,S_{m-1})=\\&=\prod_{i<j}(A_i-A_j)\prod_{i<j}(S_i-S_j)\prod_{j=1}^{2n}B_j\prod_{j=1}^{n-1}A_j\\&\times\sum_{\substack{T\subset S\\ \#T=n-1}}\ \sum_{\substack{T'\subset S'\\ \#T'=m-1}}\prod_{j\in T}B_j\prod_{i=1}^{n-1}\prod_{j\in T}(A_i+iB_j)\prod_{i=1}^{m-1}\prod_{j\in T'}(S_i+iT_j)\\&\times\prod_{\substack{i,j\in S\setminus T\\ i<j}}(B_i+B_j)\prod_{\substack{i,j\in S'\setminus T'\\ i<j}}(T_i+T_j)\prod_{\substack{i\in T\\ j\in S\setminus T}}\frac{1}{B_i-B_j}\prod_{\substack{i\in T'\\ j\in S'\setminus T'}}\frac{1}{T_i-T_j}\\&\times\prod_{\substack{i\in T\\ j\in S'\setminus T'}}(B_i+iT_j)\prod_{\substack{i\in T'\\ j\in S\setminus T}}(T_i+iB_j)X_{T,T'}(B_1,\cdots,B_{2n}|T_1,\cdots,B_{2m})\end{aligned}$$

where

$$X_{T,T'}(B_1,\cdots,B_{2n}|T_1,\cdots,B_{2m})=$$
$$=\sum_{i_1,i_2\in S\setminus T}\prod_{p=1}^{2}\left(\frac{\prod_{j\in T}(B_{i_p}+B_j)\prod_{j\in T'}(B_{i_p}+iT_j)}{\prod_{j\in S'\setminus T'\setminus\{i_1,i_2\}}(B_{i_p}-B_j)\prod_{j\in S'\setminus T'}(B_{i_p}-iT_j)}\right)$$

Obviously, the formula (21) is in agreement with the intuitive formula (20). After specialization $\beta_k=\lambda_k+\frac{\pi i}{2}$, $\beta_{2n-k+1}=\lambda_k-\frac{\pi i}{2}$ (21) will turn into a sum of form factors of RSG-model with coefficients constructed via the functions $\chi(\lambda_i-\lambda_j)$ which correspond to correlators of RXXZ-model with boundary conditions. The identification of RSG-form factors with operators counted by CFT is known at least to some extent [20, 21, 22]. So, it should be possible to make the correspondence between (21) and (20) more explicit, but this problem goes beyond the scope of the present paper.

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