Asymptotic Completeness for Compton Scattering

J. Fröhlich[1*], M. Griesemer[2†]and B. Schlein[3‡]

1. Theoretical Physics, ETH–Hönggerberg,
CH–8093 Zürich, Switzerland

2. Department of Mathematics, University of Alabama at Birmingham,
Birmingham, AL 35294, USA

3. Courant Institute, New York University,
New York, NY 10012, USA

17 November, 2003

Dedicated to Freeman Dyson on the occasion of his 80th birthday.

Abstract

Scattering in a model of a massive quantum-mechanical particle, an "electron", interacting with massless, relativistic bosons, "photons", is studied. The interaction term in the Hamiltonian of our model describes emission and absorption of "photons" by the "electron"; but "electron-positron" pair production is suppressed. An ultraviolet cutoff and an (arbitrarily small, but fixed) infrared cutoff are imposed on the interaction term. In a range of energies where the propagation speed of the dressed "electron" is strictly smaller than the speed of light, unitarity of the scattering matrix is proven, provided the coupling constant is small enough; (asymptotic completeness of Compton scattering). The proof combines a construction of dressed one–electron states with propagation estimates for the "electron" and the "photons".

Contents

[2]Work partially supported by U.S. National Science Foundation grant DMS 01-00160.
[*]juerg@itp.phys.ethz.ch
[†]marcel@math.uab.edu
[‡]schlein@cims.nyu.edu

1 Introduction

The study of collisions between photons, the field quanta of the electromagnetic field, and freely moving charged particles, in particular electrons, at energies below the threshold for electron-positron pair creation – commonly called *Compton scattering* – has played a significant rôle in establishing the reality of Einstein's photons, in the early days of quantum theory. With the development of quantum electrodynamics (QED) it became possible to calculate the cross section for Compton scattering perturbatively, using the Feynman rules of relativistic QED. The agreement between theoretical predictions and experiments is astounding.

Yet, a careful theoretical analysis of Compton scattering uncovers substantial difficulties mainly related to the so-called *infrared problem* in QED, [BN37, PF38]: When, in the course of a collision process, a charged particle, such as an electron, undergoes an accelerated motion it emits *infinitely* many photons of *finite* total energy. Unless treated carefully, a perturbative calculation of scattering amplitudes is therefore plagued by the infamous infrared divergencies.

Infrared divergencies can be eliminated by giving the photon a small mass, or, alternatively, by introducing an infrared cutoff in the interaction term. Of course, after having calculated suitable cross sections, one attempts to let the photon mass or the infrared cutoff, respectively, tend to 0. This procedure, carefully implemented, is known to work very well; see [YFS61].

If the total energy of the incoming particles, photons and an electron, is well below the threshold for electron-positron pair creation it is a fairly good approximation to neglect all terms in the Hamiltonian of relativistic QED describing pair creation- and annihilation processes in a calculation of some cross section for Compton scattering. The resulting model is a caricature of QED without positrons in which the number of electrons is conserved. It is this simplified model of QED, regularized in the infrared region by an infrared cutoff, which has inspired the analysis of Compton scattering presented in this paper.

To further simplify matters, we consider a toy model involving "scalar photons" or "phonons", and we also impose an ultraviolet cutoff in the interaction Hamiltonian. But the methods developed in this paper can be applied to the caricature of QED described above if one works in the Coulomb gauge and introduces an ultraviolet and an infrared cutoff in the interaction Hamiltonian.

The main results of this paper can be described as follows: For the toy model described above, we establish *asymptotic completeness (AC) for Compton scattering* below some threshold energy Σ, which depends on the kinematics of the electron and on the coupling constant. The latter will have to be chosen sufficiently small. This means that, on the subspace of physical state vectors containing one electron and arbitrarily many "scalar photons" (massless bosons) of total energy $\leq \Sigma$, the scattering operator of our toy model is *unitary*.

In a previous paper [FGS01], we have studied the scattering of massless bosons at an electron bound to a static nucleus, below the ionization threshold, in a similar toy model with an infrared- and an ultraviolet cutoff. In other words, we have proven AC for *Rayleigh scattering* of "photons" at an atom, below the ionization threshold, in the presence of an infrared- and ultraviolet cutoff. By combining the methods in [FGS01] with those developed in this paper, we expect to be able to establish AC in our toy model of an electron interacting with massless bosons and with a static nucleus at energies below some threshold energy Σ (depending on the kinematics of the electron), provided the coupling constant is small enough. Such a result would apply to scattering processes encountered in the analysis of the photoelectric effect (see [BKZ01]) and of Bremsstrahlung. Further possible extensions of our results are described in Sect. 10.

As quite frequently the case in mathematical physics, our methods of proof are considerably more interesting than the results we establish. We think that they illustrate *some* of the many subtleties of scattering theory in quantum field theory in a fairly illuminating way. Before we are able to describe these methods and give an outline of the strategy of our proof, we must define the model studied in this paper more precisely.

To describe the dynamics of a conserved, unbound particle, here called *electron*, coupled to a quantized field of spin-0 massless bosons, we consider the Hamiltonian

$$H_g = \Omega(p) + H_f + g\phi(G_x)$$

acting on the Hilbert space of state vectors $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$, where $\mathcal{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3, dk)$, $k \in \mathbb{R}^3$ is the momentum of a boson, $x \in \mathbb{R}^3$ the position of the electron, $p = -i\nabla_x$ the momentum of the electron, and $\Omega(p)$ is the energy of a non-interacting, free electron of momentum p. The operator $H_f = \int dk|k|a^*(k)a(k)$ is the Hamiltonian of the free bosons; $a(k)$ and $a^*(k)$ being the usual annihilation and creation operators obeying the canonical commutation relations (CCR). The operator $\phi(G_x)$ describes the interaction between an electron at position x and bosons. It is given by

$$\phi(G_x) = \int dk\,(\overline{G_x(k)}a(k) + G_x(k)a^*(k)), \quad \text{with} \tag{1}$$

$$G_x(k) = e^{-ik\cdot x}\kappa_\sigma(k). \tag{2}$$

We impose an infrared cutoff by requiring that

$$\kappa_\sigma(k) = 0 \qquad \text{if} \qquad |k| < \sigma$$

where $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ is a form factor. The constant σ must be positive but can be arbitrarily small. The smoothness and the decay assumptions on κ_σ at $|k| = \infty$ are technically convenient, but can be relaxed; see e.g. [Nel64, Amm00]. The parameter $g \in \mathbb{R}$ is a coupling constant.

The Hamiltonian H_g is invariant under translations in physical space and thus admits a decomposition over the spectrum of the total momentum $P = p + P_f$, $P_f = \int dk\, ka^*(k)a(k)$, as a direct integral

$$H_g \simeq \int_{\mathbb{R}^3}^{\oplus} H_g(P)\,dP \qquad \text{on} \qquad \mathcal{H} \simeq \int_{\mathbb{R}^3}^{\oplus} \mathcal{F}\,dP, \quad \text{with}$$
$$H_g(P) = \Omega(P - P_f) + H_f + g\phi(\kappa_\sigma),$$

where $\simeq$ indicates unitary equivalence.

On the dispersion law, $\Omega(p)$, of the free electron we only impose minimal assumptions that are sufficient for our purpose and are satisfied in examples of physical interest. We assume that $\Omega \geq 0$, that Ω is twice continuously differentiable, and that $\partial_i\partial_j\Omega$ and $|\nabla\Omega|(\Omega+1)^{-1/2}$ are bounded functions. Most importantly, we assume that, given an arbitrary $\beta > 0$, there exists a constant $O_\beta > \inf\Omega$, such that

$$|\nabla\Omega(p)| \leq \beta, \quad \text{for all } p \text{ with } \Omega(p) \leq O_\beta. \tag{3}$$

Note that these assumptions are satisfied in the examples where

$$\Omega(p) = \frac{p^2}{2M} \quad \text{(non-relativistic kinematics)}$$

and

$$\Omega(p) = \sqrt{p^2 + M^2} \quad \text{(relativistic kinematics)},$$

for some positive mass M. [We could also study an electron in a crystal interacting with phonons.]

Our assumptions on $\Omega(p)$ and the presence of an infrared cutoff $\sigma > 0$ guarantee that the Hamiltonian $H_g(P)$ has a unique one-particle eigenstate $\psi_P \in \mathcal{F}$ corresponding to the eigenvalue (energy) $E_g(P) = \inf \sigma(H_g(P))$, for each P with $\Omega(P) \leq O_\beta$, $\beta < 1$, and for $|g|$ sufficiently small, depending on β. In fact, under these assumptions, condition (3) allows us to show that

$$\inf_{|k| \geq \sigma} (E_g(P-k) + |k| - E_g(P)) > 0 \tag{4}$$

for all $\sigma > 0$. This is the key ingredient for proving that $H_g(P)$ has a one-particle eigenstate of energy $E_g(P)$ (c.f. [Frö74]). Uniqueness follows by standard Perron-Frobenius type arguments or by a suitable positive commutator estimate.

Wave packets ψ_f, $f \in L^2(\mathbb{R}^3)$, of dressed one-particle states ψ_P are defined by

$$\psi_f(P) = f(P)\psi_P \tag{5}$$

where $\operatorname{supp} f \subset \{P : \Omega(P) \leq O_\beta\}$. They minimize the energy for a given distribution $|f|^2$ of the total momentum, and they propagate according to

$$e^{-iH_g t}\psi_f = \psi_{f_t}, \qquad f_t(P) = e^{-iE_g(P)t} f(P).$$

In nature, no excited one-electron states are observed, and, correspondingly, one expects that every state $e^{-iH_g t}\psi$ eventually radiates off its excess energy and decays into a dressed one-electron wave packet ψ_f. More precisely, for any given ψ, $e^{-iH_g t}\psi$ should be well approximated, in the distant future, by a linear combination of states of the form

$$a^*(h_{1,t}) \cdot \ldots \cdot a^*(h_{n,t}) e^{-iH_g t}\psi_f \tag{6}$$

where $h_{i,t} = e^{-i|k|t}h_i$, and ψ_f is given by (5). This is called *asymptotic completeness (AC) for Compton scattering*. Mathematically more convenient characterizations of AC may be given in terms of the asymptotic field operators $a_+(h)$ and $a_+^*(h)$. Let $h \in L^2(\mathbb{R}^3, (1+|k|^{-1})dk)$ and let $\varphi \in E_\Sigma(H_g)\mathcal{H}$ for some $\Sigma < O_{\beta=1}$. Then the limit

$$a_+^{\#}(h)\varphi = \lim_{t\to\infty} e^{iH_g t} a^{\#}(h_t) e^{-iH_g t}\varphi$$

exists, and, moreover,

$$a_+^*(h_1) \cdot \ldots \cdot a_+^*(h_n)\varphi = \lim_{t\to\infty} e^{iH_g t} a^*(h_{1,t}) \cdot \ldots \cdot a^*(h_{n,t}) e^{-iH_g t}\varphi \tag{7}$$

if $h_i \in L^2(\mathbb{R}^3, (1+|k|^{-1})dk)$, $\varphi \in E_\lambda(H_g)\mathcal{H}$, and $\lambda + \sum_i M_i \leq \Sigma$, where $M_i := \sup\{|k| : h_i(k) \neq 0\}$. Let $\mathcal{H}_+$ denote the closure of the space spanned by vectors of the form $a_+^*(h_1) \cdot \ldots \cdot a_+^*(h_n)\psi_f$.

From (6) and (7) it is clear that AC means that $\mathcal{H}_+ = \mathcal{H}$. AC in this form asserts, on the one hand, that the asymptotic dynamics of bosons which are not bound to the electron corresponds to free motion, and, on the other hand, that $H_g(P)$ has no eigenvalues above $E_g(P) = \inf \sigma(H_g(P))$.

The main purpose of this paper is to show that

$$\mathcal{H}_+ \supset E_\Sigma(H_g)\mathcal{H},$$

for every $\Sigma < O_{\beta=1/3}$ provided that $|g|$ is sufficiently small depending on Σ (Theorem 17). Here $E_\Sigma(H_g)$ is the spectral projection of H_g onto vectors of energy $\leq \Sigma$. We thus prove that all vectors in $\operatorname{Ran} E_\Sigma(H_g)$ decay into states of the form (6). While the assumption $\Sigma < O_{\beta=1/3}$ may appear very restrictive, it still allows for electrons with speeds as high as one third of the speed of light ($\simeq 10^8$ m/s)!

Dressed one-electron states for the model discussed here with relativistic and non-relativistic electrons were first constructed in [Frö73, Frö74]. For similar results on the related polaron model, see [Spo88] and references therein.

First steps towards a scattering theory (construction of the Møller operators) were previously made in [Frö73], [Frö74] and, for $\sigma = 0$, very recently in [Piz00].

The scattering theory of a free electron in the framework of non-relativistic QED in the dipole approximation has been studied in [Ara83]. This model is explicitly soluble and is not translation–invariant. Arai proves asymptotic completeness after removing the infrared cutoff.

Asymptotic completeness in non-trivial models of quantum field theory was previously established in [Spe74, SZ76], [DG99, DG00] and [FGS01]. The papers [DG99] and [FGS01] are devoted to an analysis of scattering in a system of a fixed number of spatially confined particles interacting with massive relativistic bosons. Confinement is enforced by a confining (increasing) potential in [DG99] and by an energy cutoff in [FGS01]. In [DG00] asymptotic completeness is proven for spatially cutoff $P(\phi)_2$-Hamiltonians. For interesting results in the scattering theory of systems of *massless* bosons and confined electrons *without infrared cutoff* see the papers [Spo97, Gér02]. In none of these papers a translation invariant model is studied. But, such models have been analyzed in [Frö73], [Piz00].

We now present an outline of our paper and explain the key ideas underlying our proof of asymptotic completeness.

In *Sect.* 2, we introduce notation and recall some well known facts about the formalism of second quantization which will be used throughout our paper.

In *Sect.* 3, we first give a mathematically precise definition of our model and list all our hypotheses for easy reference. We then summarize our key results on the *existence and uniqueness* of *dressed one-electron states*, ψ_P. All proofs concerning these matters are deferred to Appendix D.1.

We also prove a fundamental *positive–commutator estimate* and a *Virial Theorem*, which, by standard arguments of Mourre theory, show that there are *no excited* dressed one-electron states; i.e., there is *no binding* between a dressed electron and bosons. See Theorems 5, 6 and 7.

In the last part of Sect. 3 we exhibit some simple properties of the interaction Hamiltonian $g\phi(G_x)$. In particular, we show that the strength of interaction between an electron at position

x and a boson localized (in the sense of Newton and Wigner) near a point $y \in \mathbb{R}^3$ tends rapidly to 0, as $|x-y| \to \infty$; (see Lemma 9). This property is important in our proof of AC.

In *Sect.* 4, we construct *Møller wave operators* as a first step towards understanding scattering in our model. Our construction is based on [FGS00]. It involves the following ideas.

(i) We prove a *propagation estimate* saying that an electron with dispersion law $\Omega(p)$ propagates with a group velocity not exceeding β, for states with a finite total energy Σ if $\| |\nabla\Omega| E_\Sigma(H_g)\| \leq \beta$, see Proposition 12 and [FGS00]. A sufficient condition for the latter assumption is that $\Sigma < O_\beta$ and that $|g|$ is sufficiently small.

(ii) This propagation estimate for the electron with $\beta < 1$ combined with a stationary phase argument for the photon guarantees that the interaction between a dressed electron and a configuration of freely moving bosons tends to 0 at large times. This can be used to establish *existence of asymptotic creation- and annihilation operators*, $a_\pm^*$, $a_\pm$:

$$a_\pm^\#(h_1)\dots a_\pm^\#(h_n)\varphi = \lim_{t\to\pm\infty} e^{iH_g t} a^\#(h_{1,t})\dots a^\#(h_{n,t}) e^{-iH_g t}\varphi,$$

for an arbitrary $\varphi \in E_\lambda(H_g)\mathcal{H}$, $h_j \in L^2_\omega(\mathbb{R}^3) = L^2(\mathbb{R}^3, (1+|k|^{-1})dk)$, $j = 1,\dots,n$, $n \in \mathbb{N}$, and $\lambda + \sum M_i \leq \Sigma$ where $\| |\nabla\Omega| E_\Sigma(H_g)\| < 1$. Here $h_t(k) := e^{-i|k|t}h(k)$, and $a^\# = a$ or a^*. See Theorem 13.

We then show that all dressed one-electron wave packets ψ_f, with $\psi_f \in E_\Sigma(H_g)\mathcal{H}$, are "*vacua*" for the asymptotic creation– and annihilation operators, in the sense that

$$a_\pm(h)\psi_f = 0,$$

for arbitrary $h \in L^2_\omega(\mathbb{R}^3)$; see Lemma 14.

(iii) We define the *scattering identification map* I by

$$\begin{aligned} I : \tilde{\mathcal{H}} \equiv \mathcal{H}\otimes\mathcal{F} &\longrightarrow \mathcal{H} \\ \varphi\otimes a^*(h_1)\dots a^*(h_n)\Omega &\longmapsto a^*(h_1)\dots a^*(h_n)\varphi \end{aligned}$$

and the *extended Hamiltonian* $\tilde{H}_g$ by

$$\tilde{H}_g = H_g \otimes 1 + 1 \otimes \mathrm{d}\Gamma(|k|).$$

To say that asymptotic creation operators exist - under the aforementioned assumptions - is equivalent to saying that the operators

$$\tilde{\Omega}_\pm\varphi = \lim_{t\to\pm\infty} e^{iH_g t} I e^{-i\tilde{H}_g t}\varphi$$

exist for φ in some dense subspace of $E_\Sigma(\tilde{H})\tilde{\mathcal{H}}$. The *Møller wave operators* are then defined by

$$\Omega_\pm = \tilde{\Omega}_\pm(P_{\mathrm{des}}\otimes 1),$$

where P_{des} is the orthogonal projection onto the subspace, $\mathcal{H}_{\mathrm{des}}$, of $\mathcal{H}$ of dressed one-electron wave packets. Since vectors in $\mathcal{H}_{\mathrm{des}}$ are vacua for $a_\pm^\#(h)$, the operators $\Omega_\pm$ are *isometric* on $\mathcal{H}_{\mathrm{des}}\otimes\mathcal{F}$; see Theorem 15.

Asymptotic completeness of scattering on states of energy $\leq \Sigma$ can be formulated as the statement that

$$\mathrm{Ran}\Omega_\pm \supset E_\Sigma(H_g)\mathcal{H}. \tag{8}$$

In *Sect.* 5 we introduce a modified Hamiltonian, H_{mod}, which agrees with H_g, except that the dispersion law, $|k|$, for soft bosons of momentum k with $|k| < \sigma$ is replaced by a new dispersion law $\omega(k)$, where $\omega \in C^\infty(\mathbb{R}^3)$, $\omega(k) \geq |k|$, $\omega(k) = |k|$, for $|k| \geq \sigma$, and $\omega(k) \geq \sigma/2$, for all k. Since bosons of momentum k with $|k| \leq \sigma$ do *not* interact with the electron, the Hamiltonians H_{mod} and H_g have the same Møller operators. But since the boson number operator is bounded by H_{mod}, it is more convenient to work with the Hamiltonian H_{mod}, instead of H_g. (Of course, this trick does not survive the limit $\sigma \to 0\,!$) *In the sections following Section 5 we work with H_{mod} exclusively and $H \equiv H_{mod}$!*

In *Sect.* 6 we establish the main propagation estimate for the bosons. Denoting by x the position of the electron and by y the Newton–Wigner position at time t of an asymptotically free boson present in a state of finite total energy, we show that

$$\frac{1}{t}|J(y/t) \cdot (\nabla\omega(k) - y/t) + h.c|\, F(|x|/t) \to 0, \qquad \text{as} \quad t \to \infty, \tag{9}$$

at an integrable rate, if $J \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $F \in C_0^\infty(\mathbb{R})$ and $\text{supp}(J) \subset \{|y| \geq \lambda\}$ while $\text{supp}(F) \subset (-\infty, \beta]$ where $\beta < \lambda$. The gradient, $\nabla\omega$, of ω is the group velocity of the bosons. By Eq. (9) the asymptotic velocity of bosons that escape the electron, is given by their group velocity $\nabla\omega$.

In *Sect.* 7, we construct the asymptotic observable W, which plays a crucial role in our proof of asymptotic completeness. Given Σ with $\sup_p |\nabla\Omega(p)|\chi(\Omega(p) \leq \Sigma) < \beta$ and g so small that $\|\nabla\Omega E_\Sigma(H_g)\| \leq \beta < 1/3$, we choose $\gamma \in (\beta, 1/3)$ and define χ_γ as depicted in Figure 1. For every energy cutoff f with $\text{supp}(f) \subset (-\infty, \Sigma]$ we define

$$W = s - \lim_{t\to\infty} e^{iHt} f(H) \mathrm{d}\Gamma(\chi_{\gamma,t}) f(H) e^{-iHt}, \tag{10}$$

where $\chi_{\gamma,t}$ denotes the operator of multiplication with $\chi_\gamma(|y|/t)$. W measures the number of bosons that propagate into the region $\{|y| \geq \gamma t\}$ as $t \to \infty$. They are asymptotically free since $\beta < \gamma$ is an upper bound on the electron propagation speed by the electron propagation estimate in Sect. 4. In fact, thanks to this propagation estimate, we may add a suitable space cutoff $F(|x|/t)$ (see Figure 1) in Eq. (10) next to $\mathrm{d}\Gamma(\chi_{\gamma,t})$ without changing the limit, if it exists. For this reason the propagation estimate (9) is sufficient to prove existence of W.

The *key result* of Sect. 7 is Theorem 27, which says that W is positive on the space of states orthogonal to $\mathcal{H}_{\text{des}}$, without soft bosons, and with energies inside the support of the energy cutoff f. This positivity is derived from an estimate of the form

$$\langle e^{-iHt_n}\psi, \mathrm{d}\Gamma(|x-y|/t_n) e^{-iHt_n}\psi\rangle \geq (1 - \beta - \varepsilon)\|f(H)\psi\|^2 + O(g), \tag{11}$$

valid for all vectors ψ with the properties specified above and for smooth energy cutoffs f with $\text{supp}(f) \subset (-\infty, \Sigma]$ and with $\||\nabla\Omega| E_\Sigma(H_g)\| \leq \beta$. Here $\{t_n\}$ is a sequence of times, depending on ε and ψ, with $t_n \to \infty$ as $n \to \infty$. Some further explanations are necessary at this point: (i) Soft bosons must be avoided because, in $H = H_{\text{mod}}$, their dispersion relation has been modified. "Without soft bosons" means "in the range of the projector $\Gamma(\chi_i)$", where $\chi_i = \chi_{|k|\geq\sigma}$. (ii) Inequality (11) would fail for some $\psi \in P_{\text{des}}^\perp \mathcal{H}$ were there *excited* one-electron states. But this has been excluded in Sect. 3. (iii) The estimate (11) does not easily translate into positivity of W because in W the photon position is measured relative to the origin, rather than relative

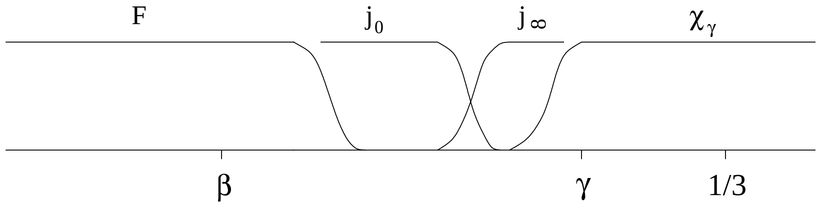

Figure 1: Typical choice of the function χ_γ, of the electron space cutoff F and of the partition in the photon space j_0, j_∞.

to x. It is due to the assumptions $\beta < 1/3$, and a suitable choice of $\gamma \in (\beta, 1/3)$ that (11) implies positivity of W. See the introduction to Section 7.1 for a detailed explanation.

The subject of *Sect.* 8 is to show that, on states of energy $\leq \Sigma$ and for sufficiently small coupling constant g (depending on the choice of Σ), the extended wave operator $\tilde{\Omega}_+$ can be *inverted*. Our proof is based on the construction of a *Deift-Simon wave operator* W_+ with the properties

$$W = \tilde{\Omega}_+ W_+, \quad \text{and} \quad (1 \otimes P_\Omega) W_+ = 0.$$

In order to construct the operator W_+, we have to split an arbitrary configuration of bosons into one staying close to the electron and a configuration of bosons escaping ballistically from the "localization cone" of the electron. This is accomplished by decomposing the space, $\mathfrak{h} = L^2(\mathbb{R}^3, dk)$, of one-boson wave functions into a direct sum of two subspaces,

$$j_t : \mathfrak{h} \ni h \longmapsto (j_{0,t} h, j_{\infty,t} h) \in \mathfrak{h} \oplus \mathfrak{h},$$

where j_0 and j_∞ are C^∞-functions on $\mathbb{R}_+$ with $j_0 + j_\infty \equiv 1$ and graphs as depicted in Figure 1, and $j_{0,t}$, $j_{\infty,t}$ are defined by

$$j_{\sharp,t}(y) := j_\sharp(y/t).$$

The operator $\breve{\Gamma}(j_t)$ is the second quantization of the operator j_t. It maps the physical Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx) \otimes \mathcal{F}$ into the extended Hilbert space $\tilde{\mathcal{H}} = L^2(\mathbb{R}^3, dx) \otimes \mathcal{F} \otimes \mathcal{F}$, and

$$I\breve{\Gamma}(j_t) = 1.$$

The Deift-Simon wave operator W_+ is a linear operator from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ defined by

$$W_+ = s - \lim_{t \to \infty} e^{i\tilde{H}t} f(\tilde{H}) \breve{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t}) f(H) e^{-iHt},$$

where $\tilde{H} = H \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega)$ is the extended modified Hamiltonian. The results of Section 8 are summarized in Theorem 28.

In *Sect.* 9, our proof of *asymptotic completeness for Compton scattering* is completed. In order to prove Eq. (8) we use an inductive argument, the induction being in the number of bosons present in a scattering state. Let $m := \sigma/2 > 0$, where σ is the infrared cutoff, and let n be an arbitrary positive integer. Our induction hypothesis is that

$$\mathrm{Ran}\Omega_+ \supset E_{(-\infty, \Sigma - nm)}(H)\mathcal{H},$$

and our claim is that $\mathrm{Ran}\Omega_+ \supset E_{(-\infty,\Sigma-(n-1)m)}(H)\mathcal{H}$. From the definition of Ω_+ it is clear that the dressed one-electron wave packets are contained in $\mathrm{Ran}\Omega_+$, and, thanks to the infrared cutoff, so are all states which differ from a given vector in $\mathrm{Ran}\,\Omega_+$ only by soft bosons. Since, moreover, $\mathrm{Ran}\Omega_+$ is closed, it is enough to show that

$$\mathrm{Ran}\Omega_+ \supset P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)E_\Delta(H_g)\mathcal{H},$$

where Δ is an arbitrary compact subinterval of $(-\infty, \Sigma-(n-1)m)$, $P_{\mathrm{des}}^{\perp} = 1-P_{\mathrm{des}}$, and P_{des} is the orthogonal projection onto the subspace, $\mathcal{H}_{\mathrm{des}}$, of $\mathcal{H}$ of dressed one-electron wave packets. Let $\psi \in P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)E_\Delta(H)\mathcal{H}$. Since our asymptotic observable W is strictly positive on this space, there exists a vector $\varphi = P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)E_\Delta(H)\varphi$ with $\psi = \Gamma(\chi_i)P_{\mathrm{des}}^{\perp}W\varphi$. By Theorem 28,

$$W\varphi = \tilde{\Omega}_+W_+\varphi = \tilde{\Omega}_+(1\otimes P_\Omega^{\perp})W_+\varphi.$$

Next, by the intertwining property of W_+ and since $\varphi \in E_{\Sigma-(n-1)m}(H)$,

$$W_+\varphi \in E_{\Sigma-(n-1)m}(\tilde{H})\tilde{\mathcal{H}}.$$

Hence,

$$(1\otimes P_\Omega^{\perp})W_+\varphi = (E_{\Sigma-nm}(H)\otimes P_\Omega^{\perp})W_+\varphi,$$

because $\tilde{H} = H\otimes 1 + 1\otimes \mathrm{d}\Gamma(\omega)$, and $\mathrm{d}\Gamma(\omega)\!\restriction\!\mathrm{Ran}P_\Omega^{\perp} \geq m$. By our induction hypothesis, $(E_{(-\infty,\Sigma-nm)}(H_g)\otimes P_\Omega^{\perp})W_+\varphi$ can be approximated, with a norm error of less than ε, by vectors of the form

$$\sum_i (\Omega_+\chi^{(i)})\otimes\varphi^{(i)},$$

where $\varepsilon > 0$ is arbitrarily small, $\varphi^{(i)} \in \mathcal{F}$ is orthogonal to Ω, and $\chi^{(i)} \in \tilde{\mathcal{H}}$. Our results in Sect. 4 (see Lemma 16, and proof thereof) then show that

$$\lim_{t\to\infty} e^{iH_gt}Ie^{-i\tilde{H}_gt}\left(\sum_i \Omega_+\chi^{(i)}\otimes\varphi^{(i)}\right)$$

exists and belongs to the range of Ω_+. Some technical details may be found in the proofs of Lemma 29 and Theorem 30 of Sect. 9.

At present, we do not see how to remove the infrared cutoff σ in the proofs of our results of Sect. 6, 7 and 8. However, it is possible to construct scattering states and wave operators in the limit $\sigma \to 0$. Elaborating on a proposal in [Frö73], this has recently been shown by Pizzo in a remarkable paper [Piz00].

A more unpleasant assumptions in our work is the energy bound $\Sigma < O_{\beta=1/3}$, forcing the electron speed to be less than one third of the speed of light. One would expect asymptotic completeness to hold true under the assumption $\Sigma < O_{\beta=1}$, which is suffices for the existence of the wave operator. The need for $\Sigma < O_{\beta=1/3}$ is due to a lack of Lorentz invariance; the speed of light is *not independent* of the frame of reference (see Section 7.1). This problem can be avoided by defining all observables relative to the electron position x, rather than relative to the origin, but then one runs into serious technical problems with non-H-bounded commutators.

In Sect. 10, we conclude with an outlook.

Some technical details are discussed in several appendices.

Acknowledgements. We thank V. Bach for his hospitality at the University of Mainz, where part of this work was done, and we are indebted to Gian Michele Graf for pointing out a serious gap in an earlier version of this paper.

2 Fock Space and Second Quantization

Let $\mathfrak{h}$ be a complex Hilbert space, and let $\otimes_s^n \mathfrak{h}$ denote the n-fold symmetric tensor product of $\mathfrak{h}$. Then the bosonic Fock space over $\mathfrak{h}$

$$\mathcal{F} = \mathcal{F}(\mathfrak{h}) = \oplus_{n\geq 0} \otimes_s^n \mathfrak{h}$$

is the space of sequences $\varphi = (\varphi_n)_{n\geq 0}$, with $\varphi_0 \in \mathbb{C}$, $\varphi_n \in \otimes_s^n \mathfrak{h}$, and with the scalar product given by

$$\langle \varphi, \psi \rangle := \sum_{n\geq 0} (\varphi_n, \psi_n),$$

where (φ_n, ψ_n) denotes the inner product in $\otimes_s^n \mathfrak{h}$. The vector $\Omega = (1, 0, \ldots) \in \mathcal{F}$ is called the vacuum. By $\mathcal{F}_0 \subset \mathcal{F}$ we denote the dense subspace of vectors φ for which $\varphi_n = 0$, for all but finitely many n. The number operator N is defined by $(N\varphi)_n = n\varphi_n$.

2.1 Creation- and Annihilation Operators

The creation operator $a^*(h)$, $h \in \mathfrak{h}$, on $\mathcal{F}$ is defined by

$$a^*(h)\varphi = \sqrt{n}\, S(h \otimes \varphi), \qquad \text{for } \varphi \in \otimes_s^{n-1} \mathfrak{h},$$

and extended by linearity to $\mathcal{F}_0$. Here $S \in \mathbf{B}(\otimes^n \mathfrak{h})$ denotes the orthogonal projection onto the symmetric subspace $\otimes_s^n \mathfrak{h} \subset \otimes^n \mathfrak{h}$. The annihilation operator $a(h)$ is the adjoint of $a^*(h)$ restricted to $\mathcal{F}_0$. Creation- and annihilation operators satisfy the canonical commutation relations (CCR)

$$[a(g), a^*(h)] = (g, h), \qquad [a^\#(g), a^\#(h)] = 0.$$

In particular, $[a(h), a^*(h)] = \|h\|^2$, which implies that the graph norms associated with the closable operators $a(h)$ and $a^*(h)$ are equivalent. It follows that the closures of $a(h)$ and $a^*(h)$ have the same domain. On this common domain we define the self-adjoint operator

$$\phi(h) = \frac{1}{\sqrt{2}}(a(h) + a^*(h)). \tag{12}$$

The creation- and annihilation operators, and thus $\phi(h)$, are bounded relative to the square root of the number operator:

$$\|a^\#(h)(N + 1)^{-1/2}\| \leq \|h\| \tag{13}$$

More generally, for any $p \in \mathbb{R}$ and any integer n,

$$\|(N + 1)^p a^\#(h_1) \ldots a^\#(h_n)(N + 1)^{-p-n/2}\| \leq C_{n,p}\, \|h_1\| \cdot \ldots \cdot \|h_n\|.$$

2.2 The Functor Γ

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Hilbert spaces and let $b \in \mathbf{B}(\mathfrak{h}_1, \mathfrak{h}_2)$. We define $\Gamma(b) : \mathcal{F}(\mathfrak{h}_1) \to \mathcal{F}(\mathfrak{h}_2)$ by

$$\Gamma(b)\restriction \otimes_s^n \mathfrak{h}_1 = b \otimes \ldots \otimes b.$$

In general $\Gamma(b)$ is unbounded; but if $\|b\| \leq 1$ then $\|\Gamma(b)\| \leq 1$. From the definition of $a^*(h)$ it easily follows that

$$\Gamma(b)a^*(h) = a^*(bh)\Gamma(b), \qquad h \in \mathfrak{h}_1 \tag{14}$$

$$\Gamma(b)a(b^*h) = a(h)\Gamma(b), \qquad h \in \mathfrak{h}_2. \tag{15}$$

If $b^*b = 1$ on $\mathfrak{h}_1$ then these equations imply that

$$\Gamma(b)a(h) = a(bh)\Gamma(b) \qquad h \in \mathfrak{h}_1 \tag{16}$$

$$\Gamma(b)\phi(h) = \phi(bh)\Gamma(b) \qquad h \in \mathfrak{h}_1. \tag{17}$$

2.3 The Operator $\mathrm{d}\Gamma(b)$

Let b be an operator on $\mathfrak{h}$. Then $\mathrm{d}\Gamma(b) : \mathcal{F}(\mathfrak{h}) \to \mathcal{F}(\mathfrak{h})$ is defined by

$$\mathrm{d}\Gamma(b)\restriction \otimes_s^n \mathfrak{h} = \sum_{i=1}^{n} (1 \otimes \ldots b \otimes \ldots 1).$$

For example $N = \mathrm{d}\Gamma(1)$. From the definition of $a^*(h)$ we get

$$[\mathrm{d}\Gamma(b), a^*(h)] = a^*(bh) \quad [\mathrm{d}\Gamma(b), a(h)] = -a(b^*h),$$

and, if $b = b^*$,

$$i[\mathrm{d}\Gamma(b), \phi(h)] = \phi(ibh). \tag{18}$$

Note that $\|\mathrm{d}\Gamma(b)(N+1)^{-1}\| \leq \|b\|$.

2.4 The Operator $\mathrm{d}\Gamma(a, b)$

Suppose $a, b \in \mathbf{B}(\mathfrak{h}_1, \mathfrak{h}_2)$. Then we define $\mathrm{d}\Gamma(a, b) : \mathcal{F}(\mathfrak{h}_1) \to \mathcal{F}(\mathfrak{h}_2)$ by

$$\mathrm{d}\Gamma(a, b)\restriction \otimes_s^n \mathfrak{h} = \sum_{j=1}^{n} (\underbrace{a \otimes \ldots a}_{j-1} \otimes b \otimes \underbrace{a \otimes \ldots a}_{n-j}).$$

For $a, b \in \mathbf{B}(\mathfrak{h})$ this definition is motivated by

$$\Gamma(a)\mathrm{d}\Gamma(b) = \mathrm{d}\Gamma(a, ab), \qquad \text{and} \qquad [\Gamma(a), \mathrm{d}\Gamma(b)] = \mathrm{d}\Gamma(a, [a, b]).$$

If $\|a\| \leq 1$ then $\|\mathrm{d}\Gamma(a, b)(N+1)^{-1}\| \leq \|b\|$ and

$$\|N^{-1/2}\mathrm{d}\Gamma(a, b)\psi\| \leq \|\mathrm{d}\Gamma(b^*b)^{1/2}\psi\|. \tag{19}$$

Lemma 1. *Suppose $r_1 : \mathfrak{h}_1 \to \mathfrak{h}_2$, $r_2^* : \mathfrak{h}_2 \to \mathfrak{h}_3$ and $q : \mathfrak{h}_1 \to \mathfrak{h}_3$ are linear operators and $\|q\| \leq 1$. Then*

$$|\langle u, \mathrm{d}\Gamma(q, r_2^* r_1)v\rangle| \leq \langle u, \mathrm{d}\Gamma(r_2^* r_2)u\rangle^{1/2} \langle v, \mathrm{d}\Gamma(r_1^* r_1)v\rangle^{1/2}$$

for all $u \in \mathcal{F}(\mathfrak{h}_3)$ and all $v \in \mathcal{F}(\mathfrak{h}_1)$.

Proof. By definition of the inner product, of $\mathrm{d}\Gamma(q, r_2^* r_1)$, and by assumption on q,

$$\begin{aligned} |\langle u, \mathrm{d}\Gamma(q, r_2^* r_1)v\rangle| &= \left| \sum_{n\geq 0} \sum_{j=1}^{n} \langle u_n, (q \otimes \dots \underbrace{r_2^* r_1}_{jth} \otimes \dots q)v_n\rangle \right| \\ &\leq \sum_{n\geq 0} \sum_{j=1}^{n} \|(r_2)_j u_n\| \|(r_1)_j v_n\| \end{aligned}$$

where $(r_\sharp)_j = 1 \otimes \dots r_\sharp \otimes \dots 1$, $r_\sharp$ in the jth factor. The assertion now follows by the Schwarz inequality. $\square$

2.5 The Tensor Product of two Fock Spaces

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Hilbert spaces. We define a linear operator $U : \mathcal{F}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \mathcal{F}(\mathfrak{h}_1) \otimes \mathcal{F}(\mathfrak{h}_2)$ by

$$\begin{aligned} U\Omega &= \Omega \otimes \Omega \\ Ua^*(h) &= [a^*(h_{(0)}) \otimes 1 + 1 \otimes a^*(h_{(\infty)})]U \qquad \text{for } h = (h_{(0)}, h_{(\infty)}) \in \mathfrak{h}_1 \oplus \mathfrak{h}_2. \end{aligned} \tag{20}$$

This defines U on finite linear combinations of vectors of the form $a^*(h_1) \dots a^*(h_n)\Omega$. From the CCRs it follows that U is isometric. Its closure is isometric and onto, hence unitary. It follows that

$$Ua(h) = [a(h_{(0)}) \otimes 1 + 1 \otimes a(h_{(\infty)})]U. \tag{21}$$

Furthermore we note that

$$U\mathrm{d}\Gamma(b) = [\mathrm{d}\Gamma(b_0) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(b_\infty)]U \quad \text{if} \quad b = \begin{pmatrix} b_0 & 0 \\ 0 & b_\infty \end{pmatrix}. \tag{22}$$

For example $UN = (N_0 + N_\infty)U$ where $N_0 = N \otimes 1$ and $N_\infty = 1 \otimes N$.

Let $\mathcal{F}_n = \otimes_s^n \mathfrak{h}$ and let P_n be the projection from $\mathcal{F} = \oplus_{n\geq 0}\mathcal{F}_n$ onto $\mathcal{F}_n$. Then the tensor product $\mathcal{F} \otimes \mathcal{F}$ is norm-isomorphic to $\oplus_{n\geq 0} \oplus_{k=0}^n \mathcal{F}_{n-k} \otimes \mathcal{F}_k$, the corresponding isomorphism being given by $\varphi \mapsto (\varphi_{n,k})_{n\geq 0,\, k=0..n}$ where $\varphi_{n,k} = (P_{n-k} \otimes P_k)\varphi$. In this representation of $\mathcal{F} \otimes \mathcal{F}$ and with $p_i(h_{(0)}, h_{(\infty)}) = h_{(i)}$, U becomes

$$U\!\restriction \otimes_s^n (\mathfrak{h} \oplus \mathfrak{h}) = \sum_{k=0}^{n} \binom{n}{k}^{1/2} \underbrace{p_0 \otimes \dots \otimes p_0}_{n-k \text{ factors}} \otimes \underbrace{p_\infty \otimes \dots \otimes p_\infty}_{k \text{ factors}}. \tag{23}$$

2.6 Factorizing Fock Space in a Tensor Product

Suppose j_0 and j_∞ are linear operators on $\mathfrak{h}$ and $j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ is defined by $jh = (j_0 h, j_\infty h)$, $h \in \mathfrak{h}$. Then $j^*(h_1, h_2) = j_0^* h_1 + j_\infty^* h_2$ and consequently $j^* j = j_0^* j_0 + j_\infty^* j_\infty$. On the level of Fock spaces, $\Gamma(j) : \mathcal{F}(\mathfrak{h}) \to \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$, and we define

$$\check{\Gamma}(j) = U\Gamma(j) : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}.$$

It follows that $\check{\Gamma}(j)^* \check{\Gamma}(j) = \Gamma(j^* j)$ which is the identity if $j^* j = 1$. Henceforth $j^* j = 1$ is tacitly assumed in this subsection. From (14) through (17), (20) and (21) it follows that

$$\check{\Gamma}(j) a^{\#}(h) = [a^{\#}(j_0 h) \otimes 1 + 1 \otimes a^{\#}(j_\infty h)] \check{\Gamma}(j) \tag{24}$$

$$\check{\Gamma}(j) \phi(h) = [\phi(j_0 h) \otimes 1 + 1 \otimes \phi(j_\infty h)] \check{\Gamma}(j). \tag{25}$$

Furthermore, if $\underline{\omega} = \omega \oplus \omega$ on $\mathfrak{h} \oplus \mathfrak{h}$, then by (22)

$$\begin{aligned} \check{\Gamma}(j) \mathrm{d}\Gamma(\omega) &= U\Gamma(j) \mathrm{d}\Gamma(\omega) = U \mathrm{d}\Gamma(\underline{\omega}) \Gamma(j) - U \mathrm{d}\Gamma(j, \underline{\omega}\, j - j\omega) \\ &= [\mathrm{d}\Gamma(\omega) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega)] \check{\Gamma}(j) - \mathrm{d}\check{\Gamma}(j, \underline{\omega}\, j - j\omega) \end{aligned} \tag{26}$$

where the notation $\mathrm{d}\check{\Gamma}(a, b) = U \mathrm{d}\Gamma(a, b)$ is introduced. In particular $\check{\Gamma}(j) N = (N_0 + N_\infty) \check{\Gamma}(j)$. We remark that, by (23),

$$\check{\Gamma}(j) \restriction \otimes_s^n \mathfrak{h} = \sum_{k=0}^{n} \binom{n}{k}^{1/2} \underbrace{j_0 \otimes \ldots \otimes j_0}_{n-k \text{ factors}} \otimes \underbrace{j_\infty \otimes \ldots \otimes j_\infty}_{k \text{ factors}}. \tag{27}$$

Lemma 2. *If $j, k : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$, $j^* j \le 1$, and k_0, k_∞ are self-adjoint, then*

$$\begin{aligned} |\langle u, \mathrm{d}\check{\Gamma}(j, k) v \rangle| \;\le\; & \langle u, (\mathrm{d}\Gamma(|k_0|) \otimes 1)\, u \rangle^{1/2} \langle v, \mathrm{d}\Gamma(|k_0|) v \rangle^{1/2} \\ & + \langle u, (1 \otimes \mathrm{d}\Gamma(|k_\infty|))\, u \rangle^{1/2} \langle v, \mathrm{d}\Gamma(|k_\infty|) v \rangle^{1/2} \end{aligned}$$

for all $u \in \mathcal{F} \otimes \mathcal{F}$ and all $v \in \mathcal{F}$.

Proof. Write $\langle u, \mathrm{d}\check{\Gamma}(j, k) v \rangle = \langle U^* u, \mathrm{d}\Gamma(j, k^{(0)}) v \rangle + \langle U^* u, \mathrm{d}\Gamma(j, k^{(\infty)}) v \rangle$ where $k^{(0)} = (k_0, 0)$ and $k^{(\infty)} = (0, k_\infty)$. Then apply Lemma 1 to both terms. In the first term we choose $r_2 = (|k_0|^{1/2}, 0)$ and $r_1 = |k_0|^{1/2} \mathrm{sgn}(k_0)$. □

2.7 The "Scattering Identification"

An important role will be played by the scattering identification $I : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ defined by

$$\begin{aligned} I(\varphi \otimes \Omega) &= \varphi \\ I\varphi \otimes a^*(h_1) \cdots a^*(h_n) \Omega &= a^*(h_1) \cdots a^*(h_n) \varphi, \qquad \varphi \in \mathcal{F}_0, \end{aligned}$$

and extended by linearity to $\mathcal{F}_0 \otimes \mathcal{F}_0$. (Note that this definition is symmetric with respect to the two factors in the tensor product.) There is a second characterization of I which will often be used. Let $\iota : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ be defined by $\iota(h_{(0)}, h_{(\infty)}) = h_{(0)} + h_{(\infty)}$. Then $I = \Gamma(\iota) U^*$, with U as above. Since $\|\iota\| = \sqrt{2}$, the operator I is unbounded.

Lemma 3. *For each positive integer k, the operator $I(N+1)^{-k} \otimes \chi(N \leq k)$ is bounded.*

Let $j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ be defined by $jh = (j_0 h, j_\infty h)$ where $j_0, j_\infty \in \mathbf{B}(\mathfrak{h})$. If $j_0 + j_\infty = 1$, then $\check{\Gamma}(j)$ is a right inverse of I, that is,

$$I\check{\Gamma}(j) = 1. \tag{28}$$

Indeed $I\check{\Gamma}(j) = \Gamma(\iota)U^* U\Gamma(j) = \Gamma(\iota j) = \Gamma(1) = 1$.

3 The Model, Dressed One-Electron States, and Bounds on the Interaction

In this section we describe our model in precise mathematical terms and discuss its main properties. The main new result of this section is Theorem 7.

3.1 The Model

The Hamilton operator of the system described in the introduction is defined by

$$H_g = \Omega(p) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(|k|) + g\phi(G_x) \tag{29}$$

acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx) \otimes \mathcal{F}$, where $\mathcal{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3, dk)$. Here and henceforth $x \in \mathbb{R}^3$ denotes the position of the electron, k is the momentum of a boson and $p = -i\nabla_x$. In this paper we are interested in both, relativistic electrons with $\Omega(p) = \sqrt{p^2 + M^2}$ and non-relativistic ones, $\Omega(p) = p^2/2M$. Rather than treating these two cases separately, we formulate a set of assumptions that are satisfied in both cases.

> **Hypothesis 0.** $\Omega \in C^2(\mathbb{R}^3)$, $\Omega \geq 0$, and the functions $|\nabla\Omega|(\Omega+1)^{-1/2}$ and $\partial^2\Omega$ are bounded.

The boundedness of $|\nabla\Omega|(\Omega+1)^{-1/2}$ ensures that $|\nabla\Omega|^2$ is H_g-bounded.

The coupling function $G_x(k)$ has the form

$$G_x(k) = e^{-ik\cdot x}\kappa_\sigma(k)$$

with an infrared (IR) cutoff imposed on the form factor κ_σ. Specifically, we assume that

> **Hypothesis 1.** $\kappa_\sigma(k) = \kappa(k)\chi(|k|/\sigma)$, for some $\sigma > 0$. Here $\kappa \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $\kappa \geq 0$, and $\chi \in C^\infty(\mathbb{R}, [0,1])$ with $\chi(s) = 0$ if $s \leq 1$ and $\chi(s) = 1$ if $s \geq 2$.

The fact that $\int |\kappa_\sigma(k)|^2/|k|\, dk \leq \int |\kappa(k)|^2/|k|\, dk < \infty$ for all σ guarantees that the smallness assumptions on $|g|$ in Theorems 7 and 17 are independent of σ. Incidentally, we put $\kappa_{\sigma=0}(k) = \kappa(k)$ (this is used in Sect. 4 where most of the results also hold without infrared cutoff). The assumption $\kappa \geq 0$ in Hypothesis 1 is included for convenience. It allows us to give a simple proof of Lemma 38 in Appendix D.1, but it is otherwise not needed; (see the remark after Theorem 7).

By Lemma 8 below, the operator $\phi(G_x)$ is bounded relative to $(\mathrm{d}\Gamma(|k|)+1)^{1/2}$ and thus also relative to $(H_{g=0}+1)^{1/2}$. It follows that $\phi(G_x)$ is infinitesimal w.r.t. to $H_0 = H_{g=0}$, and thus the operator H_g is self-adjoint on $D(H_0)$ and bounded from below. Our main results hold on spectral subspaces $E_\Sigma(H_g)\mathcal{H}$, where $\||\nabla\Omega|E_\Sigma(H_g)\| \leq \beta$ for some $\beta < 1/3$. This bound can be derived from the following further assumption on Ω; (see Lemma 10).

Hypothesis 2. For each $\beta > 0$, there exists a constant $O_\beta > \inf_p \Omega(p)$ such that

$$|\nabla\Omega(p)| \le \beta \qquad \text{for all } p \text{ with } \Omega(p) \le O_\beta.$$

By lowering the values of O_β we may achieve that $\beta \mapsto O_\beta$ is non-decreasing and continuous from the left. Under these assumptions, for each $\Sigma < O_\beta$, there exists a $\beta' < \beta$ such that $\Sigma < O_{\beta'} < O_\beta$. A function O_β with these properties can also be defined by $O_\beta := \sup\{\lambda : f(\lambda) < \beta\}$ where $f(\lambda) := \sup\{|\nabla\Omega(P)| : \Omega(P) < \lambda\}$. Given Hypothesis 0, Hypothesis 2 is then equivalent to $f(\lambda) \to 0$ as $\lambda \to \inf\Omega(p)$.

An important consequence of Hypothesis 2 is that

$$\Omega(p-k) \ge \Omega(p) - \beta|k|, \qquad \text{if } \Omega(p) \le O_\beta, \tag{30}$$

which is obvious from a sketch of the graph of a generic function Ω satisfying Hypothesis 2.

As mentioned in the introduction, the number operator N is not bounded relative to H_g. However, by Hypothesis 1, an interacting boson has a minimal energy $\sigma > 0$ and thus the number of interacting bosons is bounded w.r. to the total energy, while the number of soft bosons with energy below σ is conserved under the time evolution. To split the soft bosons from the interacting ones, we use that $L^2(\mathbb{R}^3) = L^2(\{k : |k| > \sigma\}) \oplus L^2(\{k : |k| \le \sigma\})$ and thus that $\mathcal{F}$ is isomorphic to $\mathcal{F}_i \otimes \mathcal{F}_s$, where $\mathcal{F}_i$ and $\mathcal{F}_s$ are the Fock spaces over $L^2(|k| > \sigma)$ and $L^2(|k| \le \sigma)$, respectively. Let χ_i denote the characteristic function of the set $\{k : |k| > \sigma\}$. Then the isomorphism $U : \mathcal{F} \to \mathcal{F}_i \otimes \mathcal{F}_s$ is given by

$$\begin{aligned} U\,\Omega &= \Omega_i \otimes \Omega_s \\ U\,a^*(h) &= (a^*(\chi_i\,h) \otimes 1 + 1 \otimes a^*((1-\chi_i)h))\,U \end{aligned} \tag{31}$$

We also use the symbol U to denote the operator $1_{L^2(\mathbb{R}^3,dx)} \otimes U : \mathcal{H} \to \mathcal{H}_i \otimes \mathcal{F}_s$, where $\mathcal{H}_i = L^2(\mathbb{R}^3, dx) \otimes \mathcal{F}_i$. On the Hilbert space $\mathcal{H}_i \otimes \mathcal{F}_s$ the Hamiltonian is represented by

$$\begin{aligned} UH_gU^* &= H_i \otimes 1 + 1 \otimes \mathrm{d}\Gamma(|k|) \quad \text{with} \\ H_i &= \Omega(p) + \mathrm{d}\Gamma(|k|) + g\phi(G_x), \end{aligned}$$

and the projector $\Gamma(\chi_i)$ onto the subspace of interacting bosons becomes

$$U\Gamma(\chi_i)U^* = 1 \otimes P_{\Omega_s},$$

where P_{Ω_s} is the orthogonal projection onto the vacuum vector $\Omega_s \in \mathcal{F}_s$.

The Hamiltonian H_g commutes with translations generated by the total momentum $P = p + \mathrm{d}\Gamma(k)$. It is therefore convenient to describe H_g in a representation of $\mathcal{H}$ in which the operator P is diagonal. To this end, we define the unitary map $\Pi : \mathcal{H} \to L^2(\mathbb{R}^3_P; \mathcal{F})$, where $L^2(\mathbb{R}^3_P; \mathcal{F}) \equiv \int^\oplus dP\,\mathcal{F}$ is the space of L^2-functions with values in $\mathcal{F}$. For $\varphi = \{\varphi_n(x, k_1, \ldots, k_n)\}_{n\ge 0} \in \mathcal{H}$ we define $\Pi\varphi \in L^2(\mathbb{R}^3_P; \mathcal{F})$ by

$$(\Pi\varphi)_n(P, k_1, \ldots, k_n) = \hat{\varphi}_n(P - \sum_{i=1}^n k_i, k_1, \ldots, k_n)$$

where

$$\hat{\varphi}_n(p, k_1, \ldots, k_n) = (2\pi)^{-3/2} \int e^{-ip\cdot x}\varphi_n(x, k_1, \ldots, k_n) d^3x.$$

On $L^2(\mathbb{R}^3_P;\mathcal{F})$ the Hamiltonian H_g is given by

$$(\Pi H_g \Pi^* \psi)(P) = H_g(P)\psi(P), \quad \text{where}$$
$$H_g(P) = \Omega(P - \mathrm{d}\Gamma(k)) + \mathrm{d}\Gamma(|k|) + g\phi(\kappa_\sigma).$$

3.2 Dressed One-Electron States

Next we describe sufficient conditions for $E_g(P) = \inf\sigma(H_g(P))$ to be an eigenvalue of $H_g(P)$.

If $g = 0$ then clearly the vacuum vector is an eigenvector of $H_{g=0}(P)$ and $\Omega(P)$ is its energy. Furthermore, if $\Omega(P) \leq O_{\beta=1}$ then $\Omega(P-k) + |k| \geq \Omega(P)$ and hence

$$\Omega(P) = \inf\sigma(H_{g=0}(P)) = E_0(P).$$

At least for small g and $\Omega(P) < O_{\beta=1}$, we expect that $\inf\sigma(H_g(P))$ remains an eigenvalue, and this is what we prove below.

If $|\nabla\Omega(P)| > 1$, however, then $\Omega(P) > E_0(P)$, and the eigenvalue $\Omega(P)$ of $H_{g=0}(P)$ is expected to disappear when the interaction is turned on.

Theorem 4. *Assume Hypotheses 0–2 are satisfied. Let $H_g(P)$ be defined as above and let $E_g(P) := \inf\sigma(H_g(P))$. For every $\Sigma < O_{\beta=1}$ there exists a constant $g_\Sigma > 0$ such that, for $|g| < g_\Sigma$ and $E_g(P) \leq \Sigma$,*

(i) $E_g(P)$ is a simple eigenvalue of $H_g(P)$.

(ii) The (unique) ground state of $H_g(P)$ belongs to $\mathrm{Ran}\Gamma(\chi_i)$.

Proof. It suffices to combine results proven in Appendix D.1 to conclude Theorem 4. (i) By Hypothesis 2 and the remarks thereafter, there exists a $\beta < 1$ such that $\Sigma < O_\beta < O_1$. By Theorem 37 (i), $E_g(P)$ is an eigenvalue of $H_g(P)$ if $\Omega(P) \leq O_\beta$ and $|g| < g_\beta$. By Lemma 39, the former assumption is satisfied if $E_g(P) \leq \Sigma$ and $|g| \leq (O_\beta - \Sigma)/(O_\beta + C)$. Hence (i) holds for $g_\Sigma := \min(g_\beta, (O_\beta - \Sigma)/(O_\beta + C))$.

The uniqueness follows from Lemma 38, and part (ii) of Theorem 4 from Theorem 37, part (ii). □

Remark. If $\Omega(p) = \sqrt{p^2 + M^2}$ then $H_g(P)$ has a unique ground state for all values of $g \in \mathbb{R}$, $\sigma > 0$ and all $P \in \mathbb{R}^3$. An analogous result for $\Omega(p) = p^2/(2M)$ holds at least for all $P \in \mathbb{R}^3$ with $|P| \leq (\sqrt{3} - 1)/M$, [Frö74].

In the following we denote by $\psi_P \in \mathcal{F}$ the (up to a phase) unique ground state vector of $H_g(P)$ provided by Theorem 4. The space of dressed one-electron wave packets $\mathcal{H}_{\mathrm{des}} \subset \mathcal{H}$ is defined by

$$\Pi\mathcal{H}_{\mathrm{des}} = \{\psi \in L^2(\{P : E_g(P) \leq \Sigma\};\mathcal{F}) | \psi(P) \in \langle\psi_P\rangle\}$$

where $\langle\psi_P\rangle$ is the one-dimensional space spanned by the vector ψ_P; $\mathcal{H}_{\mathrm{des}}$ is a closed linear subspace which reduces H_g in the sense that H_g commutes with the projection P_{des} onto $\mathcal{H}_{\mathrm{des}}$. The latter is obvious from $(\Pi P_{\mathrm{des}}\Pi^*\varphi)(P) = P_{\psi_P}\varphi(P)$.

3.3 Positive Commutator and Absence of Excited States

The purpose of this section is to prove the absence of excited eigenvalues of $H_g(P)$ below a given threshold Σ if g is small enough, depending on Σ. As usual this is done by combining a positive commutator estimate with a virial theorem. A priori we only have a virial theorem on $\mathrm{Ran}\Gamma(\chi_i)$, and therefore we only get absence of excited eigenvalues for $H_g(P)$ *restricted to* $\mathrm{Ran}\Gamma(\chi_i)$ in a first step. - Recall that $\chi_i(k)$ is the characteristic function of the set $\{k \in \mathbb{R}^3 : |k| > \sigma\}$, where $\sigma > 0$ is the infrared cutoff defined in Hypothesis 1, and hence that $\Gamma(\chi_i)$ is the orthogonal projection onto the subspace of interacting bosons. - Thanks to the IR cutoff, however, this fact then allows us to show that $H_g(P)\restriction\Gamma(\chi_i)^\perp$ has no eigenvalues, at all, below Σ, and the desired result follows.

The conjugate operator we use is $A = \mathrm{d}\Gamma(a)$ where

$$a = \frac{1}{2}\left(\frac{k}{|k|}\cdot y + y \cdot \frac{k}{|k|}\right).$$

On a suitable dense subspace of $\mathcal{F}$

$$[iH_g(P), A] = N - \nabla\Omega(P - \mathrm{d}\Gamma(k)) \cdot \mathrm{d}\Gamma(k/|k|) - g\,\phi(ia\kappa_\sigma). \tag{32}$$

We use this identity to *define* the quadratic form $\langle\varphi, [iH_g(P), A]\varphi\rangle$ on $D(H_g(P)) \cap D(N)$.

Theorem 5 (Virial Theorem). *Let Hypotheses 0 and 1 (Sect. 3.1) be satisfied. If $\varphi \in \mathcal{F}$ is an eigenvector of $H_g(P)$ with $\Gamma(\chi_i)\varphi = \varphi$, then*

$$\langle\varphi, [iH_g(P), A]\varphi\rangle = 0.$$

Proof. The theorem follows directly from Lemma 40 in Appendix D.2, where we prove the Virial Theorem for a modified Hamiltonian $H_{\mathrm{mod}}(P)$, which is identical to $H_g(P)$ on states without soft bosons. □

Theorem 6. *Assume Hypotheses 0 – 2 (Sect. 3.1) are satisfied. For each $\Sigma < O_{\beta=1}$, there exist constants $\delta_\Sigma > 0$, $g_\Sigma > 0$ and C_Σ, independent of σ, such that*

$$\langle\varphi, [iH_g(P), A]\varphi\rangle \geq \delta_\Sigma\langle\varphi, N\varphi\rangle - C_\Sigma|g|\|\varphi\|^2,$$

for all $P \in \mathbb{R}^3$, $|g| < g_\Sigma$, and $\varphi \in D(N) \cap \mathrm{Ran}E_\Sigma(H_g(P))$.

Proof. Choose $f \in C_0^\infty(\mathbb{R}; [0,1])$ with $f \equiv 1$ on $[\inf_P E_0(P) - 1, \Sigma]$ and $f(s) = 0$ for $s \geq \Sigma + \varepsilon$ where $\Sigma + \varepsilon < O_{\beta=1}$. Let $f = f(H_g(P))$ and E_Σ as above. Since $fE_\Sigma = E_\Sigma$ and since $[f, N^{1/2}]$ and $[f, N^{1/2}](H_g(P) + i)^{1/2}$ are of order g, uniformly in σ, by Lemma 10,

$$\begin{aligned}
E_\Sigma\nabla\Omega(P - \mathrm{d}\Gamma(k)) \cdot \mathrm{d}\Gamma(\hat{k})E_\Sigma &\leq E_\Sigma|\nabla\Omega(P - \mathrm{d}\Gamma(k)|NE_\Sigma \\
&= E_\Sigma N^{1/2} f|\nabla\Omega(P - \mathrm{d}\Gamma(k))|fN^{1/2}E_\Sigma + O(g) \\
&\leq \||\nabla\Omega(P - \mathrm{d}\Gamma(k))|E_{\Sigma+\varepsilon}(H_g(P))\|\, E_\Sigma NE_\Sigma + O(g) \\
&\leq \||\nabla\Omega|E_{\Sigma+\varepsilon}(H_g)\|\, E_\Sigma NE_\Sigma + O(g) \\
&\leq (1 - \delta_\Sigma)E_\Sigma NE_\Sigma + O(g)
\end{aligned}$$

for some $\delta_\Sigma > 0$ and $|g|$ small enough. Here $O(g)$ is independent of σ. By Eq. (32) defining $[iH_g(P), A]$, this estimate and the boundedness of $\phi(ia\kappa_\sigma)E_\Sigma$ prove the theorem. □

In the next theorem, Theorems 5 and 6 are combined to prove absence of excited eigenvalues below Σ. This is first done for $H_g(P)\restriction \mathrm{Ran}\Gamma(\chi_i)$ (see Eq. (33)) and then for $H_g(P)$.

Theorem 7. *Assume Hypotheses 0 – 2 are satisfied and that $\Sigma < O_{\beta=1}$, with O_β given by Hypotheses 2. Then there exists a constant $g_\Sigma > 0$ such that*

$$\sigma_{pp}(H_g(P)) \cap (-\infty, \Sigma] = \{E_g(P)\},$$

for all $P \in \mathbb{R}^3$ with $E_g(P) \le \Sigma$, and all g with $|g| < g_\Sigma$.

Remark: For those P with $E_g(P) \le \Sigma$ and for $|g|$ small enough depending on Σ, the proof of this theorem shows again that $E_g(P)$ is a *non-degenerate* eigenvalue (cf. Theorem 4). Here no assumption on the sign of κ is needed.

The proof also shows that $\|\psi_P - \Omega\| = O(|g|^{1/2})$, $g \to 0$, uniformly in P for $E_g(P) \le \Sigma$.

In the case of *relativistic* electrons the theorem shows that $\sigma_{\mathrm{pp}}(H_g(P)) \cap (-\infty, \Sigma] = \{E_g(P)\}$ for *all* $\Sigma \in \mathbb{R}$ and for $|g|$ small enough, depending on Σ.

Proof. Let $\psi_g = \Gamma(\chi_i)\psi_g$ be a normalized eigenvector of $H_g(P)$ with energy $\le \Sigma$, and choose the phase of ψ_g so that $\langle \psi_g, \Omega\rangle \ge 0$. By the Virial Theorem and by Theorem 6

$$0 \ge \delta_\Sigma \langle \psi_g, (1 - P_\Omega)\psi_g\rangle - C_\Sigma |g|$$

where

$$\langle \psi_g, (1 - P_\Omega)\psi_g\rangle = 1 - |\langle \Omega, \psi_g\rangle|^2 \ge 1 - |\langle \Omega, \psi_g\rangle| = \frac{1}{2}\|\psi_g - \Omega\|^2.$$

In the last equation the choice of the phase of ψ_g was used. We conclude that $\|\psi_g - \Omega\| \le (2|g|C_\Sigma/\delta_\Sigma)^{1/2}$. Since it is impossible to have two orthonormal vectors $\psi_g^{(1)}$ and $\psi_g^{(2)}$ with $\|\psi_g^{(i)} - \Omega\| < 1/\sqrt{2}$, for $|g| < \delta_\Sigma/4C_\Sigma$ there exists only one eigenvalue of $H_g(P)\restriction \mathrm{Ran}\Gamma(\chi_i)$ below or equal to Σ, and it is *simple*. By Theorem 4, this eigenvalue is $E_g(P)$. Hence, for these values of g,

$$\sigma_{\mathrm{pp}}\left(H_g(P)\restriction \mathrm{Ran}\Gamma(\chi_i)\right) \cap (-\infty, \Sigma] = \{E_g(P)\}, \tag{33}$$

for all P with $E_g(P) \le \Sigma$. The theorem now follows if we show that

$$\sigma_{\mathrm{pp}}\left(H_g(P)\restriction \mathrm{Ran}\Gamma(\chi_i)^\perp\right) \cap (-\infty, \Sigma] = \emptyset. \tag{34}$$

To prove (34), we use that $\mathcal{F} \cong \mathcal{F}_i \otimes \mathcal{F}_s$, where $\mathcal{F}_i$ and $\mathcal{F}_s$ are the bosonic Fock spaces over $L^2(\{k : |k| > \sigma\})$ and over $L^2(\{k : |k| \le \sigma\})$, respectively, where $\sigma > 0$ is the infrared cutoff defined in Hypothesis 1; ($\mathcal{F}_i$ and $\mathcal{F}_s$ are the spaces of interacting and of soft, non-interacting bosons, respectively). Consider the restriction of $H_g(P)$ to the subspace of $\mathcal{F}_i \otimes \mathcal{F}_s$ of all vectors with exactly n soft bosons. This subspace is isomorphic to $\mathcal{F}_{s,n} = L^2_s(\mathbb{R}^{3n}, dk_1 \dots dk_n; \mathcal{F}_i)$, the space of all square integrable functions on $\mathbb{R}^{3n}$, with values in $\mathcal{F}_i$ which are symmetric with respect to permutations of the n variables. The action of $H_g(P)$ on a vector $\psi \in \mathcal{F}_{s,n}$ is given by

$$\begin{aligned}(H_g(P)\psi)(k_1, \dots k_n) &= H_P(k_1, \dots, k_n)\psi(k_1, \dots k_n) \quad \text{with} \\ H_P(k_1, \dots, k_n) &= H_g(P - k_1 - \dots - k_n) + |k_1| + \dots + |k_n|.\end{aligned}$$

The operator $H_P(k_1,\dots,k_n)$ acts on $\mathcal{F}_i$ and, by (33), its only eigenvalue in the interval $(-\infty,\Sigma]$ is given by $E_g(P-k_1-\dots-k_n)+|k_1|+\dots+|k_n|$, as long as this number is smaller than Σ, and if $|g|<\delta_\Sigma/(4C_\Sigma)$. This implies that, for $|g|<\delta_\Sigma/(4C_\Sigma)$, a number $\lambda\in(-\infty,\Sigma]$ is an eigenvalue of the restriction $H_g(P)\lceil\mathcal{F}_{s,n}$ if and only if there exists a set $M_\lambda\subset\mathbb{R}^{3n}$ of positive measure such that

$$E_g(P-k_1-\dots k_n)+|k_1|+\dots|k_n|=\lambda$$

for all $(k_1,\dots,k_n)\in M_\lambda$. Using that $|\nabla E_g(P)| = |\langle\psi_P,\nabla\Omega(P-\mathrm{d}\Gamma(k))\psi_P\rangle| \leq \sup_{P:E(P)\leq\Sigma}\|\nabla\Omega(P-P_f)\psi_P\| \leq \||\nabla\Omega|E_\Sigma(H_g)\|<1$, for $|g|$ small enough (Lemma 10), it can easily be shown that such a set M_λ does not exist. This completes the proof of the theorem. □

3.4 Bounds on the Interaction

Lemma 8. *Let* $L^2_\omega(\mathbb{R}^3)\equiv L^2(\mathbb{R}^3,(1+1/|k|)dk)=\left\{h\in L^2(\mathbb{R}^3):\int dk(1+1/|k|)|h(k)|^2<\infty\right\}$ *and let* $h\in L^2_\omega(\mathbb{R}^3)$. *Then*

$$\begin{aligned}
\|a(h)\varphi\| &\leq \left(\int dk|h(k)|^2/|k|\right)^{1/2}\|\mathrm{d}\Gamma(|k|)^{1/2}\varphi\|\\
\|a^*(h)\varphi\| &\leq \|h\|_\omega\,\|(\mathrm{d}\Gamma(|k|)+1)^{1/2}\varphi\|\\
\|\phi(h)\varphi\| &\leq \sqrt{2}\,\|h\|_\omega\,\|(\mathrm{d}\Gamma(|k|)+1)^{1/2}\varphi\|\\
\pm\phi(h) &\leq \alpha\mathrm{d}\Gamma(|k|)+\frac{1}{\alpha}\int dk\frac{|h(k)|^2}{|k|}, \qquad \alpha>0,
\end{aligned}$$

where $\|h\|^2_\omega=\int dk\,(1+1/|k|)|h(k)|^2$.

For the easy proofs, see [BFS98], where similar bounds are established.

In the analysis of electron-photon scattering it is important that the interaction between bosons and electron decays sufficiently fast with increasing distance. This decay is the subject of the next lemma.

Lemma 9. *Assume Hypothesis 1 (Sect. 3.1).*

i) For arbitrary $n,\mu\in\mathbb{N}$ *there is a constant* $C_{\mu,n}>0$ *such that*

$$\sup_{x\in\mathbb{R}^3}\|\chi(|x-y|\geq R)|x-y|^n G_x\|\leq C_{\mu,n}R^{-\mu}$$

for all $R>0$. *In particular* $\|\phi\left(|x-y|^n G_x\right)(N+1)^{-1/2}\|<\infty$, *for all* $n\in\mathbb{N}$.

ii) For every $\mu\in\mathbb{N}$ *there is a constant* $C_\mu>0$ *such that*

$$\sup_{|x|\leq R}\|\chi(|y|\geq R')G_x\|\leq C_\mu(R'-R)^{-\mu}$$

for all $R'\geq R$.

Proof. i) For all $x \in \mathbb{R}^3$

$$\begin{aligned}\|\chi(|x-y| \geq R)|x-y|^n G_x\|^2 &= \int_{|x-y|>R} dy|x-y|^{2n}|\hat{\kappa}_\sigma(x-y)|^2 = \int_{|y|>R} dy|y|^{2n}|\hat{\kappa}_\sigma(y)|^2 \\ &\leq R^{-2\mu} \int dy\,|y|^{2(n+\mu)}|\hat{\kappa}_\sigma(y)|^2 = R^{-2\mu} C_{\mu,n}\end{aligned}$$

where, by Hypothesis 1, $C_{\mu,n}$ is finite for all $\sigma \geq 0$ and all $n, \mu \in \mathbb{N}$.

Statement ii) follows from i), because if $|x| \leq R$ and $|y| \geq R'$, then $|x-y| \geq R' - R$. $\square$

The following lemma is used to apply Hypothesis 2, when we need to control the velocity of the electron $|\nabla\Omega(p)|$ by bounds on the total energy H_g.

Lemma 10. *Assume Hypotheses 0 – 2. For each $\beta > 0$ and each $\Sigma < O_\beta$, there exists a constant $g_{\beta,\Sigma} > 0$ independent of σ such that*

$$\sup_{|g| \leq g_{\beta,\Sigma}} \||\nabla\Omega| E_\Sigma(H_g)\| \leq \beta$$

for all $\sigma > 0$.

Remark. This lemma holds equally for the modified Hamiltonian H_{mod}, introduced in Section 5.

Proof. Pick $\Sigma < O_\beta$ and pick $\varepsilon > 0$ such that $\Sigma + \varepsilon < O_\beta$. Choose $f \in C_0^\infty(\mathbb{R}, [0,1])$ with $f \equiv 1$ on $[\inf \sigma(H_{g=0}) - 1, \Sigma]$ and $f(s) = 0$ for $s \geq \Sigma + \varepsilon$. Then

$$\begin{aligned}\||\nabla\Omega| E_\Sigma(H_g)\| &\leq \||\nabla\Omega| f(H_g)\| \\ &\leq \||\nabla\Omega| f(H_{g=0})\| + O(g) \\ &\leq \||\nabla\Omega| f(\Omega)\| + O(g) \leq \beta\end{aligned}$$

for g small enough, because $\||\nabla\Omega| f(\Omega)\| \leq \sup\{|\nabla\Omega(p)| : \Omega(p) \leq \Sigma + \varepsilon\} < \beta$ by Hypothesis 2 and the remarks thereafter. $\square$

For non-relativistic and relativistic electron kinematics the constants O_β and $g_{\Sigma,\beta}$ can be determined explicitly:

Lemma 11. *Let $\Sigma \in \mathbb{R}$ and $C := \int |\kappa(k)|^2/|k|\, dk$ (which is independent of the IR cutoff σ!)*

(a) If $\Omega(p) = p^2/2M$ then

$$\||\nabla\Omega| E_\Sigma(H_g)\| \leq \left(\frac{2}{M}(\Sigma + g^2 C)\right)^{1/2}. \tag{35}$$

(b) If $\Omega(p) = \sqrt{p^2 + M^2}$ then

$$\||\nabla\Omega| E_\Sigma(H_g)\| \leq \left(1 - \frac{M^2}{(\Sigma + g^2 C)^2}\right)^{1/2}. \tag{36}$$

Proof. From Lemma 8 with $\alpha = 1/g$ and from $|\kappa_\sigma| \le |\kappa|$ it follows that

$$\Omega \le H_g + g^2 \int \frac{|\kappa(k)|^2}{|k|}\, dk \tag{37}$$

in both cases.

Statement (a) follows from $|\nabla\Omega|^2 = 2\Omega/M$ and (37). In case (b) we have $|\nabla\Omega|^2 = 1 - M^2/\Omega^2$ and we need an estimate on Ω^{-2} from below. By (37), $\Omega^{-1} \ge (H + g^2 \int |\kappa(k)|^2/|k|\, dk)^{-1}$ and hence

$$E_\Sigma(H_g)\Omega^{-2}E_\Sigma(H_g) \ge (E_\Sigma(H_g)\Omega^{-1}E_\Sigma(H_g))^2 \ge (\Sigma + g^2 C)^{-2} E_\Sigma(H_g).$$

This proves (b). □

4 Propagation Estimate for the Electron and Existence of the Wave Operator

Wave operators map scattering states onto interacting states. In our model the scattering states consist of dressed one-electron (DES) wave packets and some asymptotically free outgoing bosons described by asymptotic field operators, which act on the DES. The DES were constructed in the previous section, and the existence of asymptotic field operators in models such as the present one was established in [FGS00]. We recall that the key idea in [FGS00] was to utilize Huyghens' principle in conjunction with the fact that massive relativistic particles propagate with a speed strictly less than the speed of light. In the present setting, where the electron dispersion law $\Omega(p)$ is more general, we can limit the electron speed from above by imposing a bound on the total energy. In fact, by the following propagation estimate, the electron in a state from $\mathrm{Ran} E_\Sigma(H_g)$ with $\| |\nabla\Omega| E_\Sigma(H_g)\| \le \beta$ will stay out of the region $|x| > \beta t$ in the limit $t \to \infty$. (see Proposition 6.3 in [DG97] for a similar result in N-body quantum scattering.)

No infrared cutoff is necessary in this section. From Hypothesis 1 we only need that $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ where σ may be equal to zero. Asymptotic completeness of the wave operator, stated at the end of this section, of course does require that σ is positive.

Proposition 12 (Propagation estimate for electron). *Let Hypotheses 0 and 2 (Sect. 3.1) be satisfied, and assume that $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ ($\sigma = 0$ is allowed). Suppose β, g and $\Sigma > \inf\sigma(H_g)$ are real numbers for which $\| |\nabla\Omega| E_\Sigma(H_g)\| \le \beta$. Let $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset (-\infty, \Sigma)$.*

i) If $\beta < \lambda < \lambda' < \infty$ then there exists a constant $C_{\lambda,\lambda'}$ such that

$$\int_1^\infty \frac{dt}{t} \|\chi_{[\lambda,\lambda']}(|x|/t) f(H_g) e^{-iH_g t}\varphi\|^2 \le C_{\lambda,\lambda'}\|\varphi\|^2.$$

ii) Suppose $F \in C^\infty(\mathbb{R})$ with $F' \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp}(F) \subset (\beta, \infty]$. Then

$$s - \lim_{t\to\infty} F(|x|/t) f(H_g) e^{-iH_g t} = 0$$

Remark. This proposition equally holds on the extended Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{F}$ if H_g is replaced by the extended Hamiltonian $\tilde{H}_g = H_g \otimes 1 + 1 \otimes d\Gamma(|k|)$ (see Eq. (48) below).

Furthermore, the validity of the proposition does not depend on the dispersion law of the bosons. Therefore we may replace H_g (or $\tilde{H}_g$) by the modified Hamiltonian H_{mod} (or $\tilde{H}_{\mathrm{mod}}$) to be introduced in Section 5, and the proposition continues to hold.

Proof. i) Let $\varepsilon > 0$ be so small that $\lambda - \varepsilon > \beta$. Pick $h \in C_0^\infty(\mathbb{R})$ with $h = 1$ on $[\lambda, \lambda']$ and $\mathrm{supp}(h) \subset [\lambda - \varepsilon, \lambda' + 1]$. Define $\tilde{h}(s) = \int_0^s d\tau h^2(\tau)$, and set $h = h(|x|/t)$ and $\tilde{h} = \tilde{h}(|x|/t)$. We work with the *propagation observable*

$$\phi(t) = -f(H_g)\tilde{h}f(H_g).$$

Since $\phi(t)$ is a bounded operator, uniformly in t, it is enough to prove the lower bound

$$D\phi(t) \equiv \frac{\partial\phi(t)}{\partial t} + [iH_g, \phi(t)] \geq \frac{C}{t} f h^2 f + O(t^{-2}), \tag{38}$$

for a positive constant C. To prove (38), we first note that

$$\frac{\partial\phi(t)}{\partial t} = f(H_g)h^2\frac{|x|}{t^2}f(H_g) \geq \frac{(\lambda-\varepsilon)}{t} f(H_g)h^2 f(H_g). \tag{39}$$

Furthermore, by Lemma 32,

$$\begin{aligned}
[iH_g, \phi(t)] = & -f(H_g)[i\Omega(p), \tilde{h}]f(H_g) \\
= & -\frac{1}{2t}f(H_g)\left(\nabla\Omega \cdot \frac{x}{|x|}h^2 + h^2\frac{x}{|x|}\cdot\nabla\Omega\right)f(H_g) + O(t^{-2}) \\
= & -\frac{1}{2t}f(H_g)h\left(\nabla\Omega \cdot \frac{x}{|x|} + \frac{x}{|x|}\cdot\nabla\Omega\right)hf(H_g) + O(t^{-2}).
\end{aligned}$$

and thus

$$|\langle\varphi_t, [iH_g, \phi(t)]\varphi_t\rangle| \leq \frac{1}{t}\||\nabla\Omega|hf(H_g)\varphi_t\|\|hf(H_g)\varphi_t\| + O(t^{-2}). \tag{40}$$

In order to estimate the factor $\||\nabla\Omega|hf(H_g)\varphi_t\|$, we choose $g \in C_0^\infty(\mathbb{R})$ with $gf = f$ and with $\mathrm{supp}\, g \subset (-\infty, \Sigma)$, and we note that, since $[h, g(H_g)] = O(t^{-1})$,

$$f(H_g)h|\nabla\Omega|^2hf(H_g) = f(H_g)hg(H_g)|\nabla\Omega|^2g(H_g)hf(H_g) + O(t^{-1}). \tag{41}$$

By assumption on $|\nabla\Omega|$, (41) combined with (40) shows that

$$|\langle\varphi_t, [iH_g, \phi(t)]\varphi_t\rangle| \leq \frac{\beta}{t}\|hf(H_g)\varphi_t\|^2 + O(t^{-2})$$

where we commuted $g(H_g)$ with h once again. This, together with (39) and $\lambda - \varepsilon > \beta$, implies (38) and proves the first part of the proposition.

ii) Clearly it is enough to prove that

$$\lim_{t\to\infty}\phi(t) = 0 \quad \text{where} \quad \phi(t) = \langle\varphi_t, f(H_g)F(|x|/t)f(H_g)\varphi_t\rangle, \tag{42}$$

for $\varphi \in \mathcal{H}$ and for an arbitrary F satisfying the assumptions of the proposition and such that $F(s) \geq 0$ for all s. To this end we first note that the limit $\lim_{t\to\infty}\phi(t)$ exists because

$\int_1^\infty dt|\phi'(t)| < \infty$ by part i) of this proposition. Moreover, if F has compact support, then, by i), $\int_1^\infty dt\, \phi(t)/t < \infty$ and hence $\lim_{t\to\infty} \phi(t) = 0$.

It remains to prove (42) if the support of F is not compact. Clearly it is enough to consider the case where $F(s) = 1$ for all s sufficiently large and $F' \geq 0$. For such functions F we define

$$\phi_\lambda(t) = \langle \varphi_t, fF(|x|/\lambda t) f\varphi_t\rangle,$$

for an arbitrary $\lambda \geq 1$. Computing the derivative of ϕ_λ we find

$$\begin{aligned}\frac{d}{dt}\phi_\lambda(t) &= \langle \varphi_t, f\left(-\tfrac{1}{t}F'\tfrac{|x|}{\lambda t} + \tfrac{1}{2\lambda t}(\nabla\Omega\cdot\tfrac{x}{|x|}F' + F'\tfrac{x}{|x|}\cdot\nabla\Omega) + O(\lambda^{-2}t^{-2})\right) f\varphi_t\rangle \\ &\leq O(\lambda^{-2}t^{-2})\end{aligned}$$

for λ large enough (because the sum of the terms proportional to t^{-1} is negative, if λ is large enough). Thus, for an arbitrary fixed t_0 (and for λ large enough), we have that

$$\phi_\lambda(t) = \phi_\lambda(t_0) + \int_{t_0}^t d\tau \phi_\lambda'(\tau) \leq \phi_\lambda(t_0) + \frac{C}{\lambda^2 t_0},$$

for all $t > t_0$, and, in particular, for $t \to \infty$. Since $\phi_\lambda(t_0) \to 0$ for $\lambda \to \infty$ it follows that

$$\lim_{\lambda\to\infty} \limsup_{t\to\infty} \phi_\lambda(t) = 0. \tag{43}$$

Obviously

$$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} (\phi(t) - \phi_\lambda(t)) + \lim_{t\to\infty} \phi_\lambda(t).$$

By (43) the second term can be made smaller than any positive constant, by choosing λ sufficiently large. After having fixed λ, the first term on the r.h.s. of the last equation is seen to vanish, because

$$\phi(t) - \phi_\lambda(t) = \langle \varphi_t, f\,(F(|x|/t) - F(|x|/\lambda t))\, f\varphi_t\rangle$$

and because the function $F(s) - F(s/\lambda)$ has compact support. Thus the l.h.s. of the last equation is smaller than any positive constant. Since $\phi(t) \geq 0$, for all t, Eq. (42) follows. □

Using Proposition 12 we can prove the existence of asymptotic field operators, enabling us to construct states with asymptotically free bosons. In order to prove the existence of the asymptotic field operators we have to assume that $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$; this will ensure that the photons propagating along the light cone are far away from the electron and hence move freely, as $t \to \infty$.

Theorem 13 (Existence of asymptotic field operators). *Let Hypotheses 0 and 2 be satisfied and suppose $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ ($\sigma = 0$ is allowed). Let g and Σ be real numbers for which $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$ (see Hypothesis 2 and Lemma 10). Then the following statements hold true.*

i) Let $h \in L^2_\omega(\mathbb{R}^3)$. Then the limit

$$a_+^\sharp(h)\varphi = \lim_{t\to\infty} e^{iH_g t} a^\sharp(h_t) e^{-iH_g t}\varphi$$

exists for all $\varphi \in \mathrm{Ran}E_\Sigma(H_g)$. Here $h_t(k) = e^{-i|k|t}h(k)$.

ii) Let $h, g \in L^2_\omega(\mathbb{R}^3)$. Then

$$[a_+(g), a_+^*(h)] = (g, h) \quad \textit{and} \quad [a_+^\sharp(g), a_+^\sharp(h)] = 0,$$

in the sense of quadratic forms on $\mathrm{Ran}E_\Sigma(H_g)$.

iii) Let $h \in L^2_\omega(\mathbb{R}^3)$, and let $M := \sup\{|k| : h(k) \neq 0\}$ and $m := \inf\{|k| : h(k) \neq 0\}$. Then

$$\begin{aligned} a_+^*(h)\mathrm{Ran}\chi(H_g \leq E) &\subset \mathrm{Ran}\chi(H_g \leq E + M) \\ a_+(h)\mathrm{Ran}\chi(H_g \leq E) &\subset \mathrm{Ran}\chi(H_g \leq E - m), \end{aligned}$$

if $E \leq \Sigma$.

iv) Let $h_i \in L^2_\omega(\mathbb{R}^3)$ for $i = 1, \ldots n$. Put $M_i = \sup\{|k| : h_i(k) \neq 0\}$ and assume $\varphi \in \mathrm{Ran}E_\lambda(H_g)$. Then, if $\lambda + \sum_{i=1}^n M_i \leq \Sigma$ we have $\varphi \in D(a_+^\sharp(h_1) \ldots a_+^\sharp(h_n))$ and

$$a_+^\sharp(h_1) \ldots a_+^\sharp(h_n)\varphi = \lim_{t\to\infty} e^{iH_g t} a^\sharp(h_{1,t}) \ldots a^\sharp(h_{n,t}) e^{-iH_g t}\varphi$$

and

$$\|a_+^\sharp(h_1) \ldots a_+^\sharp(h_n)(H_g + i)^{-n/2}\| \leq C\|h_1\|_\omega \ldots \|h_n\|_\omega.$$

Remark. For $\Omega(P) = \sqrt{P^2 + m^2}$ the condition $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$ is satisfied for all $\Sigma \in \mathbb{R}$ and hence $a_+^\sharp(h)$ exists on $\cup_\Sigma E_\Sigma(H_g)\mathcal{H}$, and thus on $D(|H_g + i|^{1/2})$ by (iv). In this case part (iv) holds true for $h_i \in L^2_\omega(\mathbb{R}^3)$, $i = 1, \ldots n$ and $\varphi \in D(|H_g + i|^{n/2})$, without any assumption on the support of the functions h_i.

Proof. Similar results are proven in [FGS00] for more involved models. It is easy to make the necessary adaptations of the arguments in [FGS00] to the model at hand. □

Next we show, using Proposition 12, that the DES wave packets $\varphi \in \mathcal{H}_{\mathrm{des}}$ are vacua of these asymptotic fields. It is known that $E_g(P) = \inf\sigma(H_g(P))$ is an eigenvalue of $H_g(P)$ if κ_σ is sufficiently regular at the origin (also if $\sigma = 0$). Thus $\mathcal{H}_{\mathrm{des}}$ is non–empty. However, we will not make any use of this, and no assertion about $\mathcal{H}_{\mathrm{des}}$ is made in the following Lemma.

Lemma 14. *Suppose that Hypotheses 0 and 2 are satisfied and $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ ($\sigma = 0$ is allowed). Let g and $\Sigma > \inf\sigma(H_g)$ be real numbers for which $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$ (see Hypothesis 2 and Lemma 10). Then, for all $\varphi \in E_\Sigma(H_g)\mathcal{H}_{des}$ and $h \in L^2_\omega(\mathbb{R}^3)$,*

$$a_+(h)\varphi = 0.$$

Remark. For $\Omega(P) = \sqrt{P^2 + m^2}$ one has the stronger result that $a_+(h)\varphi = 0$ for all $\varphi \in \mathcal{H}_{\mathrm{des}} \cap D(|H_g + i|^{1/2})$. This follows from the remark after Theorem 13.

Proof. The intuition behind our proof is as follows: Because of the assumption $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$ the speed of the electron is strictly less than one. Since, moreover, $\varphi \in \mathcal{H}_{\mathrm{des}}$, all bosons in φ_t are located near the electron, and thus the overlap of the bosons in φ_t with a freely propagating boson h_t will vanish in the limit $t \to \infty$, which implies that $a_+(h)\varphi = 0$.

This heuristic argument can be turned into a proof quite easily. Since $\|a(h_t)(H_g+i)^{-1/2}\| \leq C\|h\|_\omega$ uniformly in t, we may assume that $h \in C_0^\infty(\mathbb{R}^3/\{0\})$. Choose $\varepsilon > 0$ so small that

$\| |\nabla\Omega| E_\Sigma(H_g)\| \leq 1-4\varepsilon$ and pick $F \in C_0^\infty(\mathbb{R})$, with $F(s)=1$ for $s \leq 1-3\varepsilon$ and $F(s)=0$ for $s \geq 1-2\varepsilon$. Then

$$\varphi_t = F(|x|/t)\varphi_t + o(1), \qquad \text{as } t \to \infty \tag{44}$$

by Proposition 12, part ii) , with $\beta = 1-3\varepsilon$. Given $\delta > 0$, we next show that

$$\varphi_t = \Gamma(\chi_{[0,\delta]}(|x-y|/t))\varphi_t + o(1), \qquad \text{as } t \to \infty. \tag{45}$$

The operator on the right side, henceforth denoted by Q_t, is translation invariant and hence leaves the fiber spaces $\mathcal{H}_P$ invariant. On the other hand, the time evolution of the component of $\varphi \in \mathcal{H}_{\rm des}$ in $\mathcal{H}_P$ is just a phase factor. Therefore $\|Q_t\varphi_t\| = \|Q_t\varphi\|$, which converges to $\|\varphi\|$, as $t \to \infty$. Since Q_t is a projector this proves (45). Combining (44) with (45) for $\delta = \varepsilon$ we get

$$\varphi_t = \Gamma(\chi_\Delta(|y|/t))\varphi_t + o(1), \qquad \text{as } t \to \infty \tag{46}$$

with $\Delta = [0, 1-\varepsilon]$, because $|x|/t \leq 1-2\varepsilon$ and $|x-y|/t \leq \varepsilon$ imply that $|y|/t \leq 1-\varepsilon$. Let $\psi \in D(H_g)$. By (46), and because $\|a^*(h_t)\psi_t\|$ is bounded uniformly in t,

$$\begin{aligned} \langle \psi, a_+(h)\varphi \rangle &= \lim_{t\to\infty} \langle \psi_t, a(h_t)\Gamma(\chi_\Delta)\varphi_t \rangle \\ &= \lim_{t\to\infty} \langle \psi_t, \Gamma(\chi_\Delta) a(\chi_\Delta h_t)\varphi_t \rangle. \end{aligned}$$

Using the Schwarz inequality and the bound $\|a(\chi_\Delta h_t)(H_g+i)^{-1/2}\| \leq \text{const}\|\chi_\Delta h_t\|_\omega$ we get

$$\|a_+(h)\varphi\| \leq C \limsup_{t\to\infty} \|\chi_\Delta h_t\|_\omega, \tag{47}$$

where

$$\begin{aligned} \|\chi_\Delta h_t\|_\omega^2 &= \|\chi_\Delta h_t\|^2 + \langle \chi_\Delta h_t, |k|^{-1}\chi_\Delta h_t \rangle \\ &\leq 2(1+t)\|\chi_\Delta h_t\|^2 \end{aligned}$$

because $\chi_\Delta |k|^{-1}\chi_\Delta \leq (\pi/2)\,\chi_\Delta |y| \chi_\Delta \leq (\pi/2)t\chi_\Delta$, by Kato's inequality (see [Ka66], Section V.5). Since

$$\sup_{|y|/t \leq 1-\varepsilon} |\hat{h}_t(y)| \leq C_N(1+t)^{-N}$$

for any integer N, and since the support of $y \mapsto \chi_\Delta(|y|/t)$ has volume proportional to t^3, we conclude that $\|\chi_\Delta h_t\|_\omega \leq C_N(1+t)^{2-N}$. For $N=3$, this bound in conjunction with (47) completes the proof. $\square$

Next, we define the Møller wave operator Ω_+. We introduce the extended Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{F}$ and the extended Hamilton operator

$$\tilde{H}_g = H_g \otimes 1 + 1 \otimes \mathrm{d}\Gamma(|k|). \tag{48}$$

The wave operator Ω_+ will be defined on a subspace of $\tilde{\mathcal{H}}$.

Theorem 15 (Existence of the wave operator). *Let Hypotheses 0 and 2 be satisfied and assume $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ ($\sigma = 0$ is allowed). For every pair of real numbers g and Σ with $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$, the limit*

$$\Omega_+\varphi := \lim_{t\to\infty} e^{iH_g t} I e^{-i\tilde{H}_g t}(P_{des}\otimes 1)\varphi \tag{49}$$

exists, for φ in the dense subspace of $\mathrm{Ran}E_\Sigma(\tilde{H}_g)$ *spanned by finite linear combinations of vectors of the form $\gamma\otimes a^*(h_1)\dots a^*(h_n)\Omega$ with $\gamma = E_\lambda(H_g)\gamma$, $h_i \in L^2_\omega(\mathbb{R}^3)$, and $\lambda + \sum_i \sup\{|k| : h_i(k) \neq 0\} \leq \Sigma$. If $\varphi = \gamma\otimes a^*(h_1)\dots a^*(h_n)\Omega$ belongs to this space then*

$$\Omega_+\varphi = a_+^*(h_1)\dots a_+^*(h_n)P_{des}\gamma. \tag{50}$$

Furthermore $\|\Omega_+\| = 1$ and thus Ω_+ has a unique extension, also denoted by Ω_+, to $E_\Sigma(\tilde{H}_g)\tilde{\mathcal{H}}$. On $(P_{des}\otimes 1)E_\Sigma(\tilde{H}_g)\tilde{\mathcal{H}}$, the operator Ω_+ is isometric, and therefore $\mathrm{Ran}\Omega_+$ *is closed. For all $t\in\mathbb{R}$,*

$$e^{-iH_g t}\Omega_+ = \Omega_+ e^{-i\tilde{H}_g t}.$$

Remark. i) In Section 5 we will enlarge the domain of the wave operator Ω_+ to include arbitrarily many soft, non-interacting bosons, regardless of their total energy. ii) For $\Omega(p) = \sqrt{p^2+m^2}$, the wave operator can be defined as a partial isometry on the entire extended Hilbert space $\tilde{\mathcal{H}}$. This follows from the remarks after Theorem 13 and Lemma 14.

Proof. If $\varphi = \gamma\otimes a^*(h_1)\dots a^*(h_n)\Omega$, then

$$e^{iH_g t} I e^{-\tilde{H}_g t} P_{\mathrm{des}}\gamma\otimes a^*(h_1)\dots a^*(h_n) = e^{iH_g t} a^*(h_{1,t})\dots a^*(h_{n,t}) e^{-iH_g t} P_{\mathrm{des}}\gamma$$

and hence the existence of the limit (49) and equation (50) follow from Theorem 13, part iv). By Lemma 34, the space $\mathcal{D}$ spanned by vectors of the form specified in the theorem is dense in $\mathrm{Ran}E_\Sigma(\tilde{H}_g)$. From Eq. (50), in conjunction with Theorem 13, part ii) and with Lemma 14, it follows that Ω_+ is a partial isometry on $\mathcal{D}$ and therefore $\|\Omega_+\| = 1$. Hence Ω_+ has a unique extension to a partial isometry on $E_\Sigma(\tilde{H}_g)\tilde{\mathcal{H}}$. The remaining parts of the proof are straightforward. □

The next result is a generalization of equation (50) that will be needed for the proof of asymptotic completeness.

Lemma 16. *Suppose Ω_+ is defined as in the preceding theorem. Assume $\psi \in E_\lambda(\tilde{H}_g)\tilde{\mathcal{H}}$ and $h_1,\dots h_n \in L^2_\omega(\mathbb{R}^3)$, with $\lambda + \sum_{i=1}^n \sup\{|k| : h_i(k)\neq 0\} \leq \Sigma$. Then*

$$\Omega_+(1\otimes a^*(h_1)\dots a^*(h_n))\psi = a_+^*(h_1)\dots a_+^*(h_n)\Omega_+\psi. \tag{51}$$

Proof. If the vector ψ is of the form

$$\psi = \gamma\otimes a^*(f_1)\dots a^*(f_m)\Omega, \tag{52}$$

where $\gamma \in E_\eta(H_g)\mathcal{H}$, $f_1,\dots f_m \in C_0^\infty(\mathbb{R}^3)$ with $\eta + \sum_i \sup\{|k| : f_i(k)\neq 0\} \leq \lambda$, then

$$\begin{aligned}\Omega_+(1\otimes a^*(h_1)\dots a^*(h_n))\psi &= a_+^*(h_1)\dots a_+^*(h_n)a_+^*(f_1)\dots a_+^*(f_m)P_{\mathrm{des}}\gamma\\ &= a_+^*(h_1)\dots a_+^*(h_n)\Omega_+\psi\end{aligned}$$

by Eq. (50). This proves (51) for all ψ which are finite linear combinations of vectors of the form (52). These vectors span a dense subspace of $E_\lambda(\tilde{H})\tilde{\mathcal{H}}$ by Lemma 34 in Appendix C. The lemma now follows by an approximation argument using Theorem 13 iv) and the intertwining relation for Ω_+. □

We are now prepared to formulate the *main result* of this paper.

Theorem 17 (Asymptotic Completeness). *Assume that Hypotheses 0 – 2 (Sect. 3.1) are satisfied, and let* Σ *be such that* $\sup_p |\nabla\Omega(p)\chi(\Omega(p) \leq \Sigma)| < 1/3$ *(see Hypotheses 2). Then, for* $|g|$ *small enough depending of* Σ*,*

$$\mathrm{Ran}\Omega_+ \supset E_\Sigma(H_g)\mathcal{H}.$$

Remark. The assumption that $\sup_p |\nabla\Omega(p)\chi(\Omega(p) \leq \Sigma)| < 1/3$ implies that $\||\nabla\Omega|E_\Sigma(H_{g=0})\| < 1/3$, which, for small $|g|$, ensures that $\||\nabla\Omega|E_\Sigma(H_g)\| < 1/3$. This last inequality is actually what we shall make use of. Since $|g|$ must be small for reasons other than this one as well, we have chosen the above formulation of the theorem.

This result follows from Theorem 30 in Section 9, where asymptotic completeness for a modified Hamiltonian (with a modified dispersion law for the bosons) is proved, and from Lemma 21 in Section 5, where the behavior of the soft bosons in the scattering process is investigated.

In the most interesting cases of a relativistic dispersion $\Omega(p) = \sqrt{p^2 + M^2}$ and of a non-relativistic dispersion $\Omega(p) = p^2/2M$ Theorem 17 implies the following result.

Corollary 18. *Assume that Hypothesis 1 and one of the following hypotheses hold.*

1. $\Omega(P) = P^2/2M$ *and* $0 < \Sigma < M/18$,
2. $\Omega(P) = \sqrt{P^2 + M^2}$ *and* $M < \Sigma < 3M/\sqrt{8}$.

Then, for $|g|$ *small enough,*

$$\mathrm{Ran}\Omega_+ \supset E_\Sigma(H_g)\mathcal{H}.$$

Proof. Hypotheses 0 and 2 are clearly satisfied in both cases and the bounds on Σ are chosen in such a way that $\sup_p |\nabla\Omega(p)\chi(\Omega(p) \leq \Sigma)| < 1/3$. Thus the corollary follows from Theorem 17. □

5 The Modified Hamiltonian

Since the bosons in our model are massless, their number is not bounded in terms of the total energy. This, however, is an artefact, since the number of bosons with energy below σ (the IR cutoff) is conserved. To avoid technical difficulties due to the lack of a bound on the number operator, N, relative to the Hamiltonian H_g, we work with a modified Hamiltonian H_{mod} whose photon-dispersion law, $\omega(k)$, is bounded from below by a positive constant (in contrast to $|k|$).

We define

$$H_{\mathrm{mod}} = \Omega(p) + \mathrm{d}\Gamma(\omega) + g\phi(G_x),$$

and we assume that ω satisfies the following conditions.

Hypothesis 3. $\omega \in \mathcal{C}^\infty(\mathbb{R}^3)$, with $\omega(k) \geq |k|$, $\omega(k) = |k|$, for $|k| > \sigma$, $\omega(k) \geq \sigma/2$, for all $k \in \mathbb{R}^3$, $\sup_k |\nabla\omega(k)| \leq 1$, and $\nabla\omega(k) \neq 0$ unless $k = 0$. Furthermore, $\omega(k_1 + k_2) \leq \omega(k_1) + \omega(k_2)$ for all $k_1, k_2 \in \mathbb{R}^3$. Here $\sigma > 0$ is the infrared cutoff defined in Hypothesis 1.

The Hamiltonian H_{mod} shares many of the properties derived for H_g in previous sections, such as Lemma 10 and Proposition 12 (see the remarks thereafter). We now explore the similarities of H_g and H_{mod} more systematically.

The two Hamiltonians H_g and H_{mod} act identically on states of the system without soft bosons. Denoting by $\chi_i(k)$ the characteristic function of the set $\{k : |k| > \sigma\}$, the operator $\Gamma(\chi_i)$ is the orthogonal projection onto the subspace of vectors describing states without soft bosons. Since $\chi_i G_x = G_x$ it follows from Eqs. (14) and (15) that H_g and H_{mod} commute with the projection $\Gamma(\chi_i)$, and hence they leave the range of $\Gamma(\chi_i)$ invariant. Moreover, since $\chi_i \omega(k) = \chi_i\, |k|$,

$$H_g{\restriction}\mathrm{Ran}\Gamma(\chi_i) = H_{\mathrm{mod}}{\restriction}\mathrm{Ran}\Gamma(\chi_i). \tag{53}$$

Let U denote the unitary isomorphism $U : \mathcal{H} \to \mathcal{H}_i \otimes \mathcal{F}_s$ introduced in Section 3.1. Then, on the factorized Hilbert space $\mathcal{H}_i \otimes \mathcal{F}_s$, the Hamiltonians H_g and H_{mod} are given by

$$\begin{aligned} U H_g U^* &= H_i \otimes 1 + 1 \otimes \mathrm{d}\Gamma(|k|) \\ U H_{\mathrm{mod}} U^* &= H_i \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega) \quad \text{with} \\ H_i &= \Omega(p) + \mathrm{d}\Gamma(|k|) + g\phi(G_x). \end{aligned} \tag{54}$$

Again, we observe that the two Hamiltonians agree on states without soft bosons.

The modified Hamiltonian H_{mod}, like the physical Hamiltonian H_g, commutes with spatial translations of the system, i.e., $[H_{\mathrm{mod}}, P] = 0$, where $P = p + \mathrm{d}\Gamma(k)$ is the total momentum of the system. In the representation of the system on the Hilbert space $L^2(\mathbb{R}^3_P; \mathcal{F})$ the modified Hamiltonian H_{mod} is given by

$$\begin{aligned} (\Pi H_{\mathrm{mod}} \Pi^* \psi)(P) &= H_{\mathrm{mod}}(P)\psi(P), \\ H_{\mathrm{mod}}(P) &= \Omega(P - \mathrm{d}\Gamma(k)) + \mathrm{d}\Gamma(\omega) + g\phi(\kappa_\sigma), \end{aligned}$$

where $\Pi : \mathcal{H} \to L^2(\mathbb{R}^3, dP; \mathcal{F})$ has been defined in Section 3.1.

Like H_g and H_{mod}, the fiber Hamiltonians $H_g(P)$ and $H_{\mathrm{mod}}(P)$ commute with the projection $\Gamma(\chi_i)$ and agree on its range, that is

$$H_g(P){\restriction}\mathrm{Ran}\Gamma(\chi_i) = H_{\mathrm{mod}}(P){\restriction}\mathrm{Ran}\Gamma(\chi_i). \tag{55}$$

In Appendix D.1 (see Theorem 37) it is shown that, for $\Omega(P) < O_{\beta=1}$ and $|g|$ small enough,

$$\inf \sigma(H_{\mathrm{mod}}(P)) = \inf \sigma(H_g(P)) = E_g(P)$$

and that $E_g(P)$ is a simple eigenvalue of $H_g(P)$ and $H_{\mathrm{mod}}(P)$. The corresponding dressed one-electron states coincide by Theorem 37, (ii). Since the subspace $\mathcal{H}_{\mathrm{des}}$ is defined in terms of the dressed one-electron states ψ_P, it follows that vectors in $\mathcal{H}_{\mathrm{des}}$ describe dressed one-electron wave packets for the dynamics generated by the modified Hamiltonian H_{mod} as well.

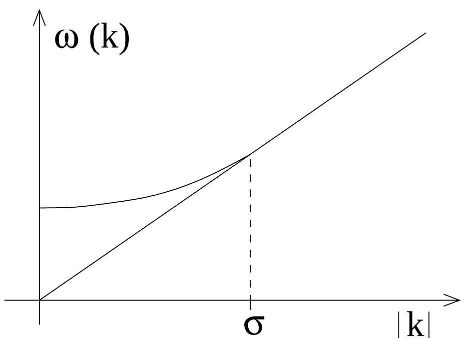

Figure 2: Typical choice of the modified photon-dispersion law $\omega(k)$.

We remark that, in view of (55), the proof of Theorem 7 shows that

$$\sigma_{\rm pp}(H_{\rm mod}(P)) \cap (-\infty, \Sigma) = \{E_g(P)\},$$

for all $P \in \mathbb{R}^3$ with $E_g(P) \le \Sigma$, and for $|g|$ sufficiently small.

Next, we consider the positive commutator discussed in Section 3.3. Thanks to Eq. (55), the inequality established in Theorem 6 continues to hold when $H_g(P)$ is replaced by $H_{\rm mod}(P)$, provided we restrict it to the range of the orthogonal projection $\Gamma(\chi_i)$. We need to rewrite this commutator estimate in terms of $H_{\rm mod}$, rather than $H_{\rm mod}(P)$, restricted to $\mathrm{Ran}\Gamma(\chi_i)$. To this end we define

$$a = \frac{1}{2}\left(\nabla\omega \cdot (y-x) + (y-x)\cdot\nabla\omega\right),$$

and we consider the conjugate operator $\mathrm{d}\Gamma(a)$. In the representation of the system on the Hilbert space $L^2(\mathbb{R}^3_P;\mathcal{F})$, the operator $\mathrm{d}\Gamma(a)$ is given by

$$(\Pi \mathrm{d}\Gamma(a)\Pi^*\psi)(P) = A\,\psi(P), \quad \text{where}$$
$$A = \frac{1}{2}\,\mathrm{d}\Gamma\left(\nabla\omega\cdot y + y\cdot\nabla\omega\right)$$

is the conjugate operator used in Theorem 6 (if restricted to states without soft bosons).

Theorem 19 (Positive Commutator). *Assume Hypotheses 0 – 3 (see Sects. 3.1 and 5) are satisfied. Let $\beta \le 1$ and choose g_0 and Σ such that $\||\nabla\Omega|E_\Sigma(H_{mod})\| \le \beta$, for all g with $|g| \le g_0$. Suppose moreover that $f \in C_0^\infty(\mathbb{R})$ and $\mathrm{supp}(f) \subset (-\infty,\Sigma)$. Then there exists a constant C, independent of the infrared cutoff σ, such that, on the range of the projector $\Gamma(\chi_i)$,*

$$f(H_{mod})[iH_{mod}, \mathrm{d}\Gamma(a)]f(H_{mod}) \ge (1-\beta)f(H_{mod})Nf(H_{mod}) - Cgf(H_{mod})^2. \tag{56}$$

for all g with $|g| \le g_0$.

Proof. Set $H \equiv H_{\rm mod}$. By definition

$$[iH, \mathrm{d}\Gamma(a)] = \mathrm{d}\Gamma(|\nabla\omega|^2) - \mathrm{d}\Gamma(\nabla\omega)\cdot\nabla\Omega - g\phi(iG_x)$$

Since $\nabla\omega(k) = k/|k|$ on the range of χ_i and since $\phi(iG_x)E_\Sigma(H)$ is bounded, it follows that

$$f(H)\Gamma(\chi_i)[iH,\mathrm{d}\Gamma(a)]\Gamma(\chi_i)f(H) \ge f(H)\Gamma(\chi_i)N\Gamma(\chi_i)f(H) - f(H)\Gamma(\chi_i)N|\nabla\Omega|\Gamma(\chi_i)f(H)$$
$$- Cgf(H)^2.$$

The assumption $\||\nabla\Omega|E_\Sigma(H)\| \le \beta$ implies

$$E_\Sigma(H)|\nabla\Omega|E_\Sigma(H) \le \beta E_\Sigma(H).$$

Using this inequality and that $[f(H), N^{1/2}]$ and $(H+i)^{1/2}[f(H), N^{1/2}]$ are of order g, uniformly in σ, we conclude that

$$\begin{aligned} f(H)N|\nabla\Omega|f(H) &= f(H)N^{1/2}|\nabla\Omega|N^{1/2}f(H) \\ &= N^{1/2}f(H)|\nabla\Omega|f(H)N^{1/2} + O(g) \\ &\le \beta f(H)Nf(H) + O(g), \end{aligned}$$

with $O(g)$ independent of σ. Since $\Gamma(\chi_i)$ commutes with $f(H)$, this proves the theorem. $\square$

Next, we discuss the scattering theory for the modified Hamiltonian. As in Theorem 15 we assume that g and $\Sigma > \inf\sigma(H_g)$ are real numbers for which $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$. Hypothesis 2 (Sect. 3.1) and Lemma 10 ensure the existence of these numbers. Then, by the assumption that $\omega(k) = |k|$ for wave vectors k of interacting bosons (cf. Hypotheses 1,3) we have that

$$\begin{aligned} e^{iH_{\text{mod}}t}a^\sharp(e^{-i\omega t}h)e^{-iH_{\text{mod}}t} &= e^{iH_gt}e^{-i\mathrm{d}\Gamma(|k|-\omega)t}a^\sharp(e^{-i\omega t}h)e^{i\mathrm{d}\Gamma(|k|-\omega)t}e^{-iH_gt} \\ &= e^{iH_gt}a^\sharp(e^{-i|k|t}h)e^{-iH_gt} \end{aligned} \tag{57}$$

for all t. It follows that the limit

$$a^\sharp_{\text{mod},+}(h)\varphi = \lim_{t\to\infty} e^{iH_{\text{mod}}t}a^\sharp(e^{-i\omega t}h)e^{-iH_{\text{mod}}t}\varphi$$

exists and that $a^\sharp_{\text{mod},+}(h)\varphi = a^\sharp_+(h)\varphi$, for all $\varphi \in \mathrm{Ran}E_\Sigma(H_{\text{mod}}) \subset \mathrm{Ran}E_\Sigma(H_g)$ and for all $h \in L^2_\omega(\mathbb{R}^3)$. This and the discussion of $\mathcal{H}_{\text{des}}$, above, show that the asymptotic states constructed with the help of the Hamiltonians H_g and H_{mod} coincide.

On the extended Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{F}$, we define the extended modified Hamiltonian

$$\tilde{H}_{\text{mod}} = H_{\text{mod}} \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega).$$

In terms of H_{mod} and $\tilde{H}_{\text{mod}}$ we also define an extended (modified) version $\tilde{\Omega}^{\text{mod}}_+$ of the wave operator Ω_+ introduced in Section 4.

Lemma 20. *Let Hypotheses 0, 2 and 3 be satisfied, and assume $\kappa_\sigma \in C_0^\infty(\mathbb{R}^3)$ ($\sigma = 0$ is allowed). For every pair of real numbers g and Σ with $\||\nabla\Omega|E_\Sigma(H_g)\| < 1$, the limit*

$$\tilde{\Omega}^{mod}_+\varphi = \lim_{t\to\infty} e^{iH_{mod}t}Ie^{-i\tilde{H}_{mod}t}\varphi \tag{58}$$

exists for all $\varphi \in E_\Sigma(\tilde{H}_{mod})\tilde{\mathcal{H}}$. The modified wave operator Ω^{mod}_+ defined by $\Omega^{mod}_+ = \tilde{\Omega}^{mod}_+(P_{des}\otimes 1)$ agrees with Ω_+ defined by Theorem 15. More precisely

$$\Omega^{mod}_+\varphi = \Omega_+\varphi, \tag{59}$$

for all $\varphi \in \mathrm{Ran}E_\Sigma(\tilde{H}_{mod}) \subset \mathrm{Ran}E_\Sigma(\tilde{H}_g)$.

Remark: Recall from the discussion above that P_{des} does not depend on whether it is constructed using H_g or H_{mod}.

Proof. Since $IE_\Sigma(\tilde{H}_{\rm mod})$ is bounded, $e^{iH_{\rm mod}t}Ie^{-i\tilde{H}_{\rm mod}t}E_\Sigma(\tilde{H}_{\rm mod})$ is bounded uniformly in $t \in \mathbb{R}$ and hence it suffices to prove existence of $\tilde{\Omega}_+^{\rm mod}$ on a dense subspace of $\mathrm{Ran}E_\Sigma(\tilde{H}_{\rm mod})$. By Lemma 34, finite linear combinations of vectors of the form

$$\varphi = \gamma \otimes a^*(h_{1,t}) \cdot \ldots \cdot a^*(h_{n,t})\Omega$$

with $\lambda + \sum_i M_i < \Sigma$, where $\gamma = E_\lambda(H_{\rm mod})\gamma$, and $M_i = \sup\{\omega(k) : h_i(k) \neq 0\}$, form such a subspace. Existence of $\tilde{\Omega}_+^{\rm mod}$ on these vectors follows from

$$e^{iH_{\rm mod}t}Ie^{-i\tilde{H}_{\rm mod}t} = e^{iH_gt}Ie^{-i\tilde{H}_gt}$$

and from Theorem 15. This also proves (59). $\square$

We shall now extend the domain of Ω_+ to include arbitrarily many soft, non-interacting bosons. As a byproduct we obtain a second proof of (59). To start with, we recall the isomorphism $U : \mathcal{F} \to \mathcal{F}_i \otimes \mathcal{F}_s$ introduced in Section 3.1 and define a unitary isomorphism $U \otimes U : \tilde{\mathcal{H}} \to \mathcal{H}_i \otimes \mathcal{F}_i \otimes \mathcal{F}_s \otimes \mathcal{F}_s$ separating interacting from soft bosons in the extended Hilbert space $\tilde{\mathcal{H}}$. With respect to this factorization the extended Hamiltonian $\tilde{H}_g$ becomes $\tilde{H}_g = \tilde{H}_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes d\Gamma(|k|) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes d\Gamma(|k|)$, where $\tilde{H}_i = H_i \otimes 1 + 1 \otimes d\Gamma(|k|)$. As an operator from $\mathcal{H}_i \otimes \mathcal{F}_i \otimes \mathcal{F}_s \otimes \mathcal{F}_s$ to $\mathcal{H}_i \otimes \mathcal{F}_s$, the wave operator Ω_+ acts as

$$U\Omega_+(U^* \otimes U^*) = \Omega_+^{\rm int} \otimes \Omega_+^{\rm soft} \tag{60}$$

where $\Omega_+^{\rm int} : \mathcal{H}_i \otimes \mathcal{F}_i \to \mathcal{H}_i$ is given by

$$\Omega_+^{\rm int} = s - \lim_{t\to\infty} e^{iH_it}Ie^{-i\tilde{H}_it}(P_{\rm des}^{\rm int} \otimes 1) \tag{61}$$

while $\Omega_+^{\rm soft} : \mathcal{F}_s \otimes \mathcal{F}_s \to \mathcal{F}_s$ is given by

$$\Omega_+^{\rm soft} = I(P_\Omega \otimes 1), \tag{62}$$

where P_Ω is the orthogonal projection onto the vacuum vector $\Omega \in \mathcal{F}_s$. In view of (60) and (61), the domain of Ω_+ can obviously be extended to $\mathrm{Ran}E_\Sigma(\tilde{H}_i) \otimes \mathcal{F}_s \otimes \mathcal{F}_s \supset \mathrm{Ran}E_\Sigma(\tilde{H}_g)$. For the modified wave operator $\Omega_+^{\rm mod} = \tilde{\Omega}_+^{\rm mod}(P_{\rm des} \otimes 1)$, we have $\Omega_+^{\rm mod} = \Omega_{+,{\rm mod}}^{\rm int} \otimes \Omega_+^{\rm soft}$, and from $H_g\restriction \mathrm{Ran}\Gamma(\chi_i) = H_{\rm mod}\restriction \mathrm{Ran}\Gamma(\chi_i)$ it follows that $\Omega_{+,{\rm mod}}^{\rm int} = \Omega_+^{\rm int}$. Consequently, also $\Omega_+^{\rm mod}$ is well defined on $\mathrm{Ran}E_\Sigma(\tilde{H}_i) \otimes \mathcal{F}_s \otimes \mathcal{F}_s$ and $\Omega_+^{\rm mod} = \Omega_+$.

We summarize the main conclusions in a lemma.

Lemma 21. *Let the assumptions of Theorem 20 be satisfied, and let* Ω_+ *be defined on* $\mathrm{Ran}E_\Sigma \otimes \mathcal{F}_s \otimes \mathcal{F}_s$ *as explained above. Then*

$$\mathrm{Ran}\Omega_+ \cong \mathrm{Ran}\Omega_+^{int} \otimes \mathcal{F}_s \tag{63}$$

with respect to the factorization $\mathcal{H} \cong \mathcal{H}_i \otimes \mathcal{F}_s$. *In particular, the following statements are equivalent:*

i) $\mathrm{Ran}\Omega_+ \supset E_\Sigma(H_g)\mathcal{H}$.

ii) $\mathrm{Ran}\Omega_+ \supset \Gamma(\chi_i) E_\Sigma(H_g)\mathcal{H}$.

iii) $\mathrm{Ran}\Omega_+ \supset E_\Sigma(H_{mod})\mathcal{H}$.

iv) $\mathrm{Ran}\Omega_+ \supset \Gamma(\chi_i) E_\Sigma(H_{mod})\mathcal{H}$.

Proof. Eq. (63) follows from (60) and (62). The equivalences i) ⇔ ii) and iii) ⇔ iv) follow directly from Eq. (63), while ii) ⇔ iv) follows from Eq. (53). □

6 Propagation Estimates for Photons

The purpose of this Section is to prove a phase-space propagation estimate (Proposition 24), which is used in the next section to establish existence of the asymptotic observable W and of the Deift-Simon wave operator W_+.

Henceforth we shall always work with the modified Hamiltonian H_{mod}, and we will use the shorthand notation

$$H \equiv H_{\mathrm{mod}} = \Omega(p) + \mathrm{d}\Gamma(\omega) + g\,\phi(G_x).$$

Moreover, for an operator b acting on the one-boson space $\mathfrak{h}$, we define the Heisenberg derivative

$$db := [i\omega(k), b] + \frac{\partial b}{\partial t},$$

while we define the Heisenberg derivatives of an operator A on $\mathcal{H}$ with respect to H and H_0, respectively, by

$$DA := i[H, A] + \frac{\partial A}{\partial t} \quad \text{and}$$
$$D_0 A := i[H_0, A] + \frac{\partial A}{\partial t}$$

We observe that

$$D_0(\mathrm{d}\Gamma(b)) = \mathrm{d}\Gamma(db).$$

The first propagation estimate is a maximal velocity propagation estimate saying that photons cannot propagate into the region $|y| > ut$, if $u = \max(1, \beta)$ (if $\beta > 1$ there will always be some photons, in the vicinity of the electron, propagating into the region $|y| > t$).

Proposition 22 (Upper bound on the velocity of bosons). *Assume Hypotheses 0–3 are satisfied. Suppose β, g and $\Sigma > \inf\sigma(H)$ are real numbers for which $\||\nabla\Omega| E_\Sigma(H)\| \leq \beta$. Let $f \in C_0^\infty(\mathbb{R})$ be real-valued with $\mathrm{supp}\, f \subset (-\infty, \Sigma)$, and suppose $F \in C_0^\infty(\mathbb{R})$, with $F \geq 0$ and $\mathrm{supp}\, F \subset (-\infty, \beta]$. Then, for each pair of real numbers λ, λ' with $\max(1, \beta) < \lambda < \lambda'$, there exists a constant $C_{\lambda,\lambda'}$ such that*

$$\int_1^\infty \frac{dt}{t} \langle \varphi_t, f \mathrm{d}\Gamma(\chi_{[\lambda,\lambda']}(|y|/t)) F(|x|/t) f \varphi_t \rangle \leq C_{\lambda,\lambda'} \|\varphi\|^2$$

for all $\varphi \in \mathcal{H}$. Here $f = f(H)$.

Remark. The lower bound, 1, in the assumption $\max(1, \beta) < \lambda$ is the upper bound on the photon propagation speed $|\nabla\omega|$ in Hypothesis 3.

Proof. Choose $\varepsilon > 0$ so small that $3\varepsilon < \lambda - \beta$ and $\lambda - \varepsilon > 1$. Without loss of generality we may assume that $F(s) = 1$ for $s \le \beta + \varepsilon$, $F(s) = 0$ for all $s \ge \beta + 2\varepsilon$, and $F' \le 0$. Choose $h \in C_0^\infty(\mathbb{R}; [0,1])$ with $h = 1$ on $[\lambda, \lambda']$ and $\mathrm{supp}(h) \subset [\lambda - \varepsilon, \lambda' + 1]$. It is important that there are gaps between $(-\infty, \beta]$ and $\mathrm{supp}(F')$, and between $\mathrm{supp}(F)$ and $\mathrm{supp}(h)$.

We define $\tilde{h}(s) = \int_0^s d\tau h^2(\tau)$ and we use the notation $h = h(|y|/t)$ and $\tilde{h} = \tilde{h}(|y|/t)$. Consider the propagation observable

$$\phi(t) = -f(H)\mathrm{d}\Gamma(\tilde{h})F(|x|/t)f(H).$$

Since $\phi(t)$ is a bounded operator, uniformly in t, the proposition follows if we show that

$$D\phi(t) \equiv \frac{\partial \phi(t)}{\partial t} + [iH, \phi(t)] \ge \frac{C}{t} f\mathrm{d}\Gamma(h^2)F(|x|/t)f + B(t), \tag{64}$$

for some operator-valued function $B(t)$ with $\int_1^\infty |\langle \varphi_t, B(t)\varphi_t\rangle| dt \le C\|\varphi\|^2$. We have that

$$\begin{aligned} \frac{\partial \phi(t)}{\partial t} &= \frac{1}{t} f\mathrm{d}\Gamma(h^2\,|y|/t)F(|x|/t)f + \frac{1}{t} f\mathrm{d}\Gamma(\tilde{h})F'(|x|/t)\,\frac{|x|}{t}\,f \\ &\ge \frac{\lambda - \varepsilon}{t} f\mathrm{d}\Gamma(h^2)Ff - \frac{\beta + 2\varepsilon}{t} f\mathrm{d}\Gamma(\tilde{h})|F'|f. \end{aligned} \tag{65}$$

The second term on the right side gives a contribution to $B(t)$ by Proposition 12. In fact, since $\mathrm{supp}(F') \subset [\beta + \varepsilon, \beta + 2\varepsilon] \subset [\beta + \varepsilon, \lambda]$ and $F' \le 0$ we have that

$$\begin{aligned} \langle \varphi_t, f\mathrm{d}\Gamma(\tilde{h})|F'|f\varphi_t\rangle &\le \|\chi_{[\beta+\varepsilon,\lambda]}(|x|/t)f\varphi_t\|\,\|\mathrm{d}\Gamma(\tilde{h})F'f\varphi_t\| \\ &\le \|\chi_{[\beta+\varepsilon,\lambda]}(|x|/t)f\varphi_t\|\,\|\mathrm{d}\Gamma(\tilde{h})(H+i)^{-1}\|\left(\|F'(H+i)f\varphi_t\| + O(t^{-1})\|\varphi\|\right) \\ &\le C\|\chi_{[\beta+\varepsilon,\lambda]}(|x|/t)f\varphi_t\|\,\|\chi_{[\beta+\varepsilon,\lambda]}(|x|/t)g(H)\varphi_t\| + O(t^{-1})\|\varphi\|^2, \end{aligned}$$

where we used that $\|[H, F']f\| = O(t^{-1})$, by Lemma 32 and Hypothesis 0, and put $g(s) := (s+i)f(s)$ and $C = \|\mathrm{d}\Gamma(\tilde{h})(H+i)^{-1}\|$ in the last line. Thus, by the Schwarz inequality and Proposition 12

$$\int_1^\infty \frac{dt}{t}\,\langle \varphi_t, f\mathrm{d}\Gamma(\tilde{h})|F'|f\varphi_t\rangle \le \mathrm{const}\,\|\varphi\|^2, \tag{66}$$

that is, the second term in (65) contributes to $B(t)$ in (64). To evaluate the commutator in (64), we use Lemma 32 and get

$$\begin{aligned} -[iH, \phi(t)] &= f[iH, \mathrm{d}\Gamma(\tilde{h})]Ff + f\mathrm{d}\Gamma(\tilde{h})[iH, F]f \\ &= f[i\mathrm{d}\Gamma(\omega), \mathrm{d}\Gamma(\tilde{h})]Ff + f[ig\phi(G_x), \mathrm{d}\Gamma(\tilde{h})]Ff + f\mathrm{d}\Gamma(\tilde{h})[i\Omega(p), F]f \\ &= \frac{1}{2t} f\mathrm{d}\Gamma\left(\nabla\omega \cdot \frac{y}{|y|}h^2 + h^2\frac{y}{|y|}\cdot\nabla\omega\right)Ff + gf\phi(i\tilde{h}G_x)Ff \\ &\quad + \frac{1}{2t} f\mathrm{d}\Gamma(\tilde{h})\left(\nabla\Omega \cdot \frac{x}{|x|}F' + F'\frac{x}{|x|}\cdot\nabla\Omega\right)f + O(t^{-2}). \end{aligned} \tag{67}$$

The term that involves F' is integrable w.r. to t, by Proposition 12 and Hypothesis 0. This is seen in the same way as the integrability of the second term of (65). Next we bound the second term of (67). By Lemma 9 part ii),

$$\|\phi(i\tilde{h}G_x)F(|x|/t)f\| \le C \sup_{|x| \le (\beta+2\varepsilon)t} \|\chi(|y| \ge (\lambda - \varepsilon)t)G_x\| \le Ct^{-\mu} \tag{68}$$

for some $\mu > 1$, because $\operatorname{supp}(F) \subset (-\infty, \beta + 2\varepsilon]$, $\operatorname{supp}(\tilde{h}) \subset [\lambda - \varepsilon, \infty)$, and $\lambda - \varepsilon > \beta + 2\varepsilon$. Finally, in the first term of (67), we commute one factor of h to the left and one to the right and conclude that

$$\begin{aligned} f[iH, \phi(t)]f &= -\frac{1}{2t} f d\Gamma\Big(h(\nabla\omega \cdot \frac{y}{|y|} + \frac{y}{|y|} \cdot \nabla\omega)h\Big)Ff + B(t) \\ &\geq -\frac{1}{t} f d\Gamma(h^2)Ff + B(t), \end{aligned}$$

where $\int_1^\infty |\langle\varphi_t, B(t)\varphi_t\rangle| dt \leq C\|\varphi\|^2$. Together with (65), (66), and $\lambda - \varepsilon > 1$ this proves Eq. (64). ∎

The following phase-space propagation estimate compares the group velocity $\nabla\omega$ with the average velocity y/t for bosons that escape from the electron in the limit $t \to \infty$ (i.e., for bosons with asymptotic velocity greater than γ). This result will be improved in Proposition 24.

Proposition 23. *Assume Hypotheses 0–3 are satisfied. Suppose β, g and $\Sigma > \inf \sigma(H)$ are real numbers for which $\||\nabla\Omega| E_\Sigma(H)\| \leq \beta$. Let $f \in C_0^\infty(\mathbb{R})$ be real-valued with $\operatorname{supp} f \subset (-\infty, \Sigma)$, and suppose $F \in C_0^\infty(\mathbb{R})$, with $F \geq 0$ and $\operatorname{supp} F \subset (-\infty, \beta]$. Then, for each pair of real numbers λ, λ' with $\beta < \lambda < \lambda'$, there exists a constant $C_{\lambda,\lambda'}$ such that*

$$\int_1^\infty \frac{dt}{t} \langle\varphi_t, f d\Gamma\left((\nabla\omega - \tfrac{y}{t})\chi_{[\lambda,\lambda']}(|y|/t)(\nabla\omega - \tfrac{y}{t})\right) F(|x|/t) f\varphi_t\rangle \leq C_{\lambda,\lambda'}\|\varphi\|^2,$$

for all $\varphi \in \mathcal{H}$. Here $f = f(H)$.

Proof. Choose $\varepsilon > 0$ so small that $3\varepsilon < \lambda - \beta$. Without loss of generality we may assume that $\lambda' > 1$. We may also assume that $F(s) = 1$ for $s \leq \beta + \varepsilon$ and $F(s) = 0$ for all $s \geq \beta + 2\varepsilon$. Pick $R \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(R) \subset \{y : \lambda - \varepsilon \leq |y| \leq \lambda' + 1\}$ and

$$R''(y) \geq \chi_{[\lambda,\lambda']}(|y|) - C\chi_{[\lambda',\lambda'+1]}(|y|).$$

It is easy to construct a function R with these properties explicitly. We work with the propagation observable

$$\phi(t) = f(H) d\Gamma(b(t)) F f(H) \tag{69}$$

where

$$b(t) = R(y/t) + \frac{1}{2}[(\nabla\omega - y/t) \cdot (\nabla R)(y/t) + (\nabla R)(y/t) \cdot (\nabla\omega - y/t)]$$

and F denotes the operator of multiplication by $F(|x|/t)$. For the reader who compares this proof with the proof of the related Proposition 11 in [FGS01] we remark that $b(t) = d(tR(y/t)) + O(t^{-1})$, and that we could work with $d(tR(y/t))$ here, too. The operator $\phi(t)$ is bounded uniformly in $t \geq 1$, because $b(t)$ is. Hence the proposition follows if we show that

$$D\phi(t) \geq \frac{C}{t}\langle\varphi_t, f d\Gamma\left((\nabla\omega - \tfrac{y}{t})\chi_{[\lambda,\lambda']}(|y|/t)(\nabla\omega - \tfrac{y}{t})\right) F f\varphi_t\rangle + B(t) \tag{70}$$

for some operator-valued function $B(t)$ with $\int_1^\infty |\langle\varphi_t, B(t)\varphi_t\rangle| dt \leq C\|\varphi\|^2$. By the Leibniz rule for the Heisenberg derivative,

$$D\phi(t) = f d\Gamma(db(t)) F f + f\phi(ib(t)G_x) F f + f d\Gamma(b(t))(DF) f. \tag{71}$$

The second and the third term contribute to the integrable part $B(t)$. For the second term this follows from Lemma 9, since the distance between the support of R and the support of F is strictly positive. The integrability of the third term follows from Proposition 12, thanks to the location of the support of F', and from boundedness of $\nabla\Omega$ w.r.to H (Hypothesis 0); (see the proof of Proposition 22 for details). The first term in (70) comes from the first term in (71). Using Lemma 32, it is straightforward to show that

$$\begin{aligned} db(t) &= \frac{1}{t}(\nabla\omega - y/t)\cdot R''(|y|/t)\,(\nabla\omega - y/t) + O(t^{-2}) \\ &\geq \frac{1}{t}(\nabla\omega - y/t)\cdot \chi_{[\lambda,\lambda']}(|y|/t)\,(\nabla\omega - y/t) \\ &\quad -\frac{C}{t}(\nabla\omega - y/t)\cdot \chi_{[\lambda',\lambda'+1]}(|y|/t)\,(\nabla\omega - y/t) + O(t^{-2}) \end{aligned}$$

where

$$(\nabla\omega - y/t)\cdot \chi_{[\lambda,\lambda+1]}(|y|/t)\,(\nabla\omega - y/t) \leq C_\eta \chi_{[\lambda'-\eta,\lambda'+\eta+1]}(|y|/t) + O(t^{-1})$$

for some $\eta > 0$ chosen so small that $\lambda' - \eta > \max(1,\beta)$; (recall that $\lambda' > \max(1,\beta)$). Hence this term contributes to $B(t)$, by Proposition 22, and (70) is proven. □

Using Proposition 23, we can establish an improved phase-space propagation estimate, which is the main result of this section. Existence of an asymptotic observable, W, and of the inverse wave operator, W_+, in Sections 7 and 8 will follow from this propagation estimate alone; (see [DG99] for a similar result). Some technical parts in the proof of Proposition 24 are stated as Lemma 25 below.

Proposition 24. *Assume Hypotheses 0–3 are satisfied. Suppose β, g and $\Sigma > \inf\sigma(H)$ are real numbers for which $\||\nabla\Omega|E_\Sigma(H)\| \leq \beta$. Let $f \in C_0^\infty(\mathbb{R})$ be real-valued with $\operatorname{supp} f \subset (-\infty,\Sigma)$, and pick $F \in C_0^\infty(\mathbb{R})$, with $F(s) \geq 0$ and $\operatorname{supp} F \subset (-\infty,\beta]$. For each pair of real numbers λ, λ' with $\max(1,\beta) < \lambda < \lambda'$ and each $J = (J_1, J_2, J_3) \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\operatorname{supp} J_l \subset \{y \in \mathbb{R}^3 : \lambda < |y| < \lambda'\}$ there exists a constant $C_{\lambda,\lambda'}$ such that*

$$\int_0^\infty \frac{dt}{t}\langle\varphi_t, f\mathrm{d}\Gamma\left(|J(y/t)\cdot(\nabla\omega - y/t) + (\nabla\omega - y/t)\cdot J(y/t)|\right) F(|x|/t) f\varphi_t\rangle \leq C_{\lambda,\lambda'}\|\varphi\|^2$$

for all $\varphi \in \mathcal{H}$. Here $f = f(H)$.

Proof. Choose $\varepsilon > 0$ so small that $2\varepsilon < \lambda - \beta$. Without loss of generality we may assume that $F(s) = 1$ for $s \leq \beta + \varepsilon$ and $F(s) = 0$ for $s \geq \beta + 2\varepsilon$.

Let $A = (y/t - \nabla\omega)^2 + t^{-\delta}$, for some $\delta \in (0,1]$, and set

$$b(t) = \tilde{J}(y/t)\cdot A^{1/2}\tilde{J}(y/t) = \sum_{i=1}^3 \tilde{J}_i(y/t)A^{1/2}\tilde{J}_i(y/t),$$

where $\tilde{J} \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ is chosen such that $\tilde{J}_i = 1$ on the support of J_i and with $\operatorname{supp}\tilde{J}_i \subset \{y \in \mathbb{R}^3 : \lambda < |y| < \lambda'\}$. Note that the operator $b(t)$ is bounded uniformly in t, because of the space cutoff J. We consider the propagation observable

$$\phi(t) = -f(H)\mathrm{d}\Gamma(b(t))F(|x|/t)f(H).$$

Because of the boundedness of $b(t)$ and the energy cutoff f, the observable $\phi(t)$ is bounded, uniformly in time. Thus, to prove the proposition, it is enough to show that

$$D\phi(t) \geq \frac{C}{t} f(H)\mathrm{d}\Gamma\left(|J(y/t)\cdot(\nabla\omega - y/t) + \mathrm{h.c}|\right) F(|x|/t) f(H) + B(t), \tag{72}$$

for some operator-valued function $B(t)$ with $\int_1^\infty dt |\langle\varphi_t, B(t)\varphi_t\rangle| \leq C\|\varphi\|^2$. The Heisenberg derivative of $\phi(t)$ is given by

$$\begin{aligned} D\phi(t) &= -f\left(D\mathrm{d}\Gamma(b(t))\right) F f - f\mathrm{d}\Gamma(b(t))(DF)f \\ &= -f\mathrm{d}\Gamma(db(t))Ff - f\phi(ib(t)G_x)Ff - f\mathrm{d}\Gamma(b(t))(DF)f. \end{aligned} \tag{73}$$

The last term, involving DF, contributes to $B(t)$. This follows from Proposition 12, since, by Lemma 32,

$$DF = \frac{1}{t}\left(F'\frac{x}{|x|}\cdot\nabla\Omega - \frac{|x|}{t}F'\right) + O(t^{-2}),$$

where F' is supported in the interval $[\beta+\varepsilon, \beta+2\varepsilon]$, and $\nabla\Omega$ is bounded w.r.to H, by Hypothesis 0 (see the proof of Proposition 22 for more details). The term with the factor $\phi(ib(t)G_x)$ also contributes to $B(t)$. This follows from Lemma 9, part ii), because the distance between the support of F and the support of $\tilde{J}$ is positive, and thus

$$\|\phi(ib(t)G_x)F(|x|/t)f\| \leq Ct^{-\mu},$$

for some $\mu > 1$. It remains to consider the contribution of the first term on the r.h.s. of (73). To this end we use that

$$\begin{aligned} db(t) &= \tilde{J}\cdot(dA^{1/2})\tilde{J} + (d\tilde{J})\cdot A^{1/2}\tilde{J} + \tilde{J}\cdot A^{1/2}(d\tilde{J}) \\ &= \tilde{J}\cdot(dA^{1/2})\tilde{J} + \sum_{i=1}^{3}\left((d\tilde{J}_i)A^{1/2}\tilde{J}_i + \tilde{J}_i A^{1/2}(d\tilde{J}_i)\right). \end{aligned} \tag{74}$$

Applying Lemma 25 below, part ii) and part iii), we find that

$$-\tilde{J}(y/t)\cdot(dA^{1/2})\tilde{J}(y/t) \geq \frac{C}{t}|J(y/t)\cdot(\nabla\omega - y/t) + (\nabla\omega - y/t)\cdot J(y/t)| + O(t^{-1-\eta/2}), \tag{75}$$

with $\eta = \min(\delta, 1-\delta/2)$. The other terms in Eq. (74) turn out to contribute to $B(t)$ in (72), (a consequence of Proposition 23). To prove this, we start with the bound

$$\pm\left(d\tilde{J}_i A^{1/2}\tilde{J}_i + \tilde{J}_i A^{1/2} d\tilde{J}_i\right) \leq t\,(d\tilde{J}_i)^2 + \frac{1}{t}\tilde{J}_i A\tilde{J}_i. \tag{76}$$

Observing that

$$d\tilde{J}_i = \frac{1}{2t}\left(\nabla\tilde{J}_i\cdot(\nabla\omega - y/t) + (\nabla\omega - y/t)\cdot\nabla\tilde{J}_i\right) + O(t^{-2})$$

we find that

$$(d\tilde{J}_i)^2 \leq \frac{C}{t^2}\,(\nabla\omega - y/t)\cdot\chi_{[\lambda,\lambda']}(|y|/t)(\nabla\omega - y/t) + O(t^{-3}).$$

To bound the second term on the r.h.s. of (76), we use that

$$\tilde{J}_i A\tilde{J}_i = \tilde{J}_i(\nabla\omega - y/t)^2\tilde{J}_i + O(t^{-\delta}) = (\nabla\omega - y/t)\tilde{J}_i^2(\nabla\omega - y/t) + O(t^{-\delta}).$$

We then find that

$$\pm\left(d\tilde{J}_iA^{1/2}\tilde{J}_i+\tilde{J}_iA^{1/2}d\tilde{J}_i\right)\le\frac{C+1}{t}(\nabla\omega-y/t)\cdot\chi_{[\lambda,\lambda']}(|y|/t)(\nabla\omega-y/t)+O(t^{-1-\delta}).$$

By (74) and (75) we thus conclude that

$$\begin{aligned}-f\mathrm{d}\Gamma(db(t))F(|x|/t)f\ge{}&\frac{C_1}{t}f\mathrm{d}\Gamma\left(|J(y/t)\cdot(\nabla\omega-y/t)+(\nabla\omega-y/t)\cdot J(y/t)|\right)Ff\\&-\frac{C_2}{t}f\mathrm{d}\Gamma\left((\nabla\omega-y/t)\chi_{[\lambda,\lambda']}(|y|/t)(\nabla\omega-y/t)\right)Ff+O(t^{-1-\eta}),\end{aligned}$$

where the second term on the right hand side is integrable by Proposition 23. This, together with (73), proves Eq. (72) and completes the proof of the proposition. □

Lemma 25. *Let $A=(y/t-\nabla\omega)^2+t^{-\delta}$, $0<\delta\le 1$, and assume that $J\in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$ (J has three components J_i, $i=1,2,3$). Then*

i) $[A^{1/2},J(y/t)]=O(t^{-1+\delta/2})$.

ii) $dA^{1/2}=-\frac{1}{t}A^{1/2}+O(t^{-1-\delta/2})$.

iii) Suppose that $\tilde{J}\in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$ with $\tilde{J}_i=1$ on the support of J_i, for $i=1,2,3$. Then

$$|J(y/t)\cdot(y/t-\nabla\omega)+(y/t-\nabla\omega)\cdot J(y/t)|\le C\tilde{J}A^{1/2}\tilde{J}+O(t^{-\eta/2}),$$

where $\eta=\min(\delta,1-\delta/2)$.

This Lemma is taken from [DG99]. For the sake of completeness its proof is included in this paper.

Proof. i) Writing $A^{1/2}=AA^{-1/2}$ and using the representation

$$A^{-1/2}=\frac{1}{\pi}\int_0^\infty\frac{ds}{\sqrt{s}}\,\frac{1}{s+A}$$

one finds that

$$[A^{1/2},J]=\frac{1}{\pi}\int_0^\infty ds\sqrt{s}\frac{1}{s+A}[A,J]\frac{1}{s+A}. \tag{77}$$

With the help of Lemma 32 it is easy to see that $[A,J]=O(t^{-1})$ and, by definition of A, $\|(s+A)^{-1}\|\le(s+t^{-\delta})^{-1}$. Hence (77) implies that

$$\|[A^{1/2},J]\|\le\frac{C}{t}\int_0^\infty ds\frac{\sqrt{s}}{(s+t^{-\delta})^2}=O(t^{-1+\delta/2}).$$

ii) The main observation is that

$$e^{it\omega(k)}A^{1/2}e^{-it\omega(k)}=\left(\frac{y^2}{t^2}+t^{-\delta}\right)^{1/2}. \tag{78}$$

On the one hand, by definition of $dA^{1/2}$,

$$\frac{d}{dt}\left(e^{it\omega(k)}A^{1/2}e^{-it\omega(k)}\right)=e^{it\omega(k)}dA^{1/2}e^{-it\omega(k)}, \tag{79}$$

and, on the other hand, by (78),

$$\frac{d}{dt}\left(e^{it\omega(k)}A^{1/2}e^{-it\omega(k)}\right)=\frac{d}{dt}\left(\frac{y^2}{t^2}+t^{-\delta}\right)^{1/2}=-\frac{1}{t}\left(\frac{y^2}{t^2}+t^{-\delta}\right)^{1/2}+O(t^{-1-\delta/2}).$$

Combining these two equations and using (78) again proves the assertion.

iii) First we note that

$$|J\cdot(\nabla\omega-y/t)+(\nabla\omega-y/t)\cdot J|^2\le\sum_{i,j}J_i(\partial_i\omega-y_i/t)(\partial_j\omega-y_j/t)J_j+O(t^{-1})$$

Using that $a_i^*a_j+a_j^*a_i\le a_i^*a_i+a_j^*a_j$ it follows that

$$\begin{aligned}|J\cdot(\nabla\omega-y/t)+(\nabla\omega-y/t)\cdot J|^2 &\le C\sum_i J_i(\partial_i\omega-y_i/t)^2J_i+O(t^{-1})\\ &\le CJAJ+O(t^{-\delta}).\end{aligned}\tag{80}$$

Furthermore, by part i), and since $\tilde{J}^4\ge J^2$ by our choice of $\tilde{J}$,

$$\begin{aligned}\left(\tilde{J}A^{1/2}\tilde{J}\right)^2 &= \sum_{i,j}\tilde{J}_iA^{1/2}\tilde{J}_i\tilde{J}_jA^{1/2}\tilde{J}_j = A^{1/2}\tilde{J}^4A^{1/2}+O(t^{-1+\delta/2})\\ &\ge A^{1/2}J^2A^{1/2}+O(t^{-1+\delta/2}) = JAJ+O(t^{-1+\delta/2}).\end{aligned}$$

Combined with (80) this shows that

$$\left(\tilde{J}A^{1/2}\tilde{J}\right)^2\ge C\,|J\cdot(\nabla\omega-y/t)+(\nabla\omega-y/t)\cdot J|^2+O(t^{-\eta}),$$

where $\eta=\min(\delta,1-\delta/2)$. The assertion now follows from the operator monotonicity of the square root. □

7 The Asymptotic Observable

Let β, g and Σ be given real numbers for which $\||\nabla\Omega|E_\Sigma(H)\|\le\beta$. Let $\gamma>\beta$ and pick $\chi_\gamma\in\mathbb{C}^\infty(\mathbb{R};[0,1])$ such that $\chi_\gamma\equiv 1$ on $[\gamma,\infty)$ and $\chi_\gamma\equiv 0$ on $(-\infty,\beta_3]$ for some $\beta_3\in(\beta,\gamma)$ (see Figure 3). Our goal, in this section, is to establish existence of the *asymptotic observable*

$$W=s-\lim_{t\to\infty}e^{iHt}f\mathrm{d}\Gamma(\chi_\gamma(|y|/t))fe^{-iHt},$$

where f is a smooth energy cutoff supported in $(-\infty,\Sigma)$. By construction of W, $\langle\psi,W\psi\rangle$ is the expectation value of the number of bosons present in $f\psi$ that propagate into the region $\{|y|\ge\gamma t\}$ as $t\to\infty$. These bosons are asymptotically free, since the energy cutoff and the assumption on $\nabla\Omega$ guarantee that the electron stays confined to $\{|x|\le\beta t\}$ (cf. Proposition 12) and since $\beta<\gamma$. As a consequence, the interaction strength between the electron and those bosons counted by W decays in t at an integrable rate. This is one of the two key ingredients for proving existence of W and of the Deift-Simon operator W_+. The other one is the propagation estimate in Proposition 24.

Theorem 26 (Existence of the asymptotic observable). *Assume that Hypotheses 0 – 3 are satisfied. Let β, g, and Σ be real numbers for which $\||\nabla\Omega|E_\Sigma(H)\| \leq \beta$. Suppose that $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \subset (-\infty, \Sigma)$. Let β, γ, and χ_γ be as defined above, and let $\chi_{\gamma,t}$ be the operator of multiplication with $\chi_\gamma(|y|/t)$. Then*

$$W = s - \lim_{t\to\infty} e^{iHt} f \mathrm{d}\Gamma(\chi_{\gamma,t}) f e^{-iHt}$$

exists, $W = W^*$ and W commutes with H. Here $f = f(H)$, (as before).

Proof. Pick $F \in C_0^\infty(\mathbb{R})$ with $0 \leq F \leq 1$, $F(s) = 1$ for $s \leq \beta_0$, and $F(s) = 0$ for $s \geq \beta_1$, where $\beta < \beta_0 < \beta_1 < \beta_3 < \gamma$ (see Figure 3, Sect. 8). We also use F to denote the operator of multiplication by $F(|x|/t)$. By Proposition 12 (ii) applied to $1 - F$, and since $e^{iHt} f \mathrm{d}\Gamma(\chi_{\gamma,t})$ is bounded, it suffices to prove the existence of

$$\lim_{t\to\infty} \varphi(t), \quad \text{where} \quad \varphi(t) = e^{iHt} f \mathrm{d}\Gamma(\chi_{\gamma,t}) F f e^{-iHt} \varphi.$$

By a variant of Cook's argument this limit will exist if there exists a constant C such that

$$\int_1^\infty |\langle \psi, \varphi'(t)\rangle|\, dt \leq C\|\psi\|$$

for all $\psi \in \mathcal{H}$. We have

$$\begin{aligned}\frac{d}{dt}\langle\psi, \varphi(t)\rangle =& \langle\psi_t, fD\Big[\mathrm{d}\Gamma(\chi_{\gamma,t})F\Big] f\varphi_t\rangle \\ =& \langle\psi_t, f\mathrm{d}\Gamma(d\chi_{\gamma,t})Ff\varphi_t\rangle + g\langle\psi_t, f\phi(i\chi_{\gamma,t}G_x)Ff\varphi_t\rangle + \langle\psi_t, f\mathrm{d}\Gamma(\chi_{\gamma,t})(DF)f\varphi_t\rangle,\end{aligned} \tag{81}$$

and we shall prove integrability of these three terms, beginning with the third one.

Since $\mathrm{supp}(F') \subset [\beta_0, \beta_1]$ and by Lemma 32,

$$\begin{aligned} DF &= \frac{1}{t} F' \left(\frac{x}{|x|}\cdot\nabla\Omega - \frac{|x|}{t}\right) + O(t^{-2}) \\ &= \frac{1}{t}\chi_{[\beta_0,\beta_1]}(|x|/t)\left(\frac{x}{|x|}\cdot\nabla\Omega - \frac{|x|}{t}\right) F' + O(t^{-2})\end{aligned}$$

and hence, using that, by Hypothesis 0, $|\nabla\Omega|$ is bounded w.r.t. H,

$$\begin{aligned}&|\langle\psi_t, f\mathrm{d}\Gamma(\chi_{\gamma,t})(DF)f\varphi_t\rangle| \\ &\quad\leq \frac{1}{t}\|\chi_{[\beta_0,\beta_1]} f\psi\|\,\|\mathrm{d}\Gamma(\chi_{\gamma,t})F'f\varphi_t\| + O(t^{-2})\|\psi\|\|\varphi\|.\end{aligned} \tag{82}$$

On the right hand side the operator $F'f$ can be replaced by $(H+i)^{-1}F'g(H)$, $g(s) = (s + i)f(s)$, at the expense of another term of order t^{-2} originating from $t^{-1}[H, F'] = O(t^{-2})$. The integrability of (82) then follows from Proposition 12.

The second term on the r.h.s. of (81) is integrable because $|x|/t \leq \beta_1$ on $\mathrm{supp}(F)$, while $|y|/t \geq \beta_3$ on $\mathrm{supp}(\chi_{\gamma,t})$, and hence, by Lemma 9,

$$\begin{aligned}|\langle\psi_t, f\phi(i\chi_{\gamma,t}G_x)Ff\varphi_t\rangle| &\leq C \sup_{|x|/t\leq\beta_1} \|\chi_{\gamma,t}G_x\|\|\psi\|\|\varphi\| \\ &\leq \text{const } t^{-\mu}\|\psi\|\,\|\varphi\|,\end{aligned}$$

with $\mu > 1$. This is integrable in t.

To bound the first term on the r.h.s. of (81), we note that

$$\begin{aligned} d\chi_{\gamma,t} &= \frac{1}{2}[(\nabla\omega - y/t)\cdot\nabla\chi_{\gamma,t} + h.c.] + O(t^{-2}) \\ &=: \frac{1}{t}P_t + O(t^{-2}), \end{aligned}$$

where $1/t$ has been factored out from $\nabla\chi_{\gamma,t} = (1/t)\chi'_\gamma(|y|/t)\, y/|y|$. It follows that

$$\begin{aligned} |\langle\psi_t, f\mathrm{d}\Gamma(d\chi_{\gamma,t})Ff\varphi_t\rangle| &\leq \frac{1}{t}\left|\langle\psi_t, fF^{1/2}\mathrm{d}\Gamma(P_t)F^{1/2}f\varphi_t\rangle\right| + O(t^{-2})\|\psi\|\|\varphi\| \\ &\leq \frac{1}{t}\langle\psi_t, fF^{1/2}\mathrm{d}\Gamma(|P_t|)F^{1/2}f\psi_t\rangle^{1/2}\langle\varphi_t, fF^{1/2}\mathrm{d}\Gamma(|P_t|)F^{1/2}f\varphi_t\rangle^{1/2} \\ &\quad + O(t^{-2})\|\psi\|\|\varphi\|. \end{aligned}$$

Since $F^{1/2}$ commutes with $\mathrm{d}\Gamma(|P_t|)$, this is integrable thanks to Proposition 24.

To prove that W commutes with H we show that $e^{-iHs}W = We^{-iHs}$ for all $s \in \mathbb{R}$. By definition of W

$$[e^{-iHs}We^{iHs} - W]\varphi = \lim_{t\to\infty} e^{iHt} f[\mathrm{d}\Gamma(\chi_{\gamma,\tau})]_{\tau=t}^{\tau=t+s} fe^{-iHt}\varphi.$$

This limit vanishes because $\partial_\tau\chi_{\gamma,\tau} = O(\tau^{-1})$ and hence $\|\mathrm{d}\Gamma(\chi_{\gamma,\tau})|_t^{t+s}(N+1)^{-1/2}\| \leq Cs/t$. □

7.1 Positivity of W

The upper bound β on the electron speed (cf. Proposition 12) could usually be chosen arbitrarily, so far. Only in our proof of the existence of the wave operator we required $\beta < 1$. To prove positivity of W, we must require that $\beta < 1/3$.

Recall that $\langle\psi, W\psi\rangle$ is the number of bosons in $f\psi$ with asymptotic speed γ or higher, while the energy cutoff f in W ensures that the speed of the electron does not exceed β. By the positive commutator estimate, Theorem 19, in a state orthogonal to $\mathcal{H}_{\mathrm{des}}$ with energy in the support of f, the photons have a speed, relative to the electron, of at least $1-\beta$. Their speed relative to the origin is thus bounded below by $1-2\beta$. By assuming $\gamma \leq 1-2\beta$ we we can ensure that these bosons are counted by W. (Their number is positive by our smallness assumption on g.) Since $\beta < \gamma$ is required for the existence of W, we need to assume that $\beta < 1/3$.

Theorem 27. *Assume Hypotheses 0 – 3 are satisfied. Given $\beta < 1/3$, pick $\Sigma < O_\beta$ and suppose that $g_\Sigma > 0$ is so small that $\sup_{|g|\leq g_\Sigma}\||\nabla\Omega|E_\Sigma(H_g)\| \leq \beta$ (cf. Hypothesis 2 and Lemma 10). Pick $\gamma \in (\beta, 1-2\beta)$, and let W be defined as in Theorem 26. Choosing g_Σ even smaller if necessary, there exists a constant $C > 0$ such that*

$$W\restriction P_{des}^{\perp}\Gamma(\chi_i)\mathcal{H} \geq Cf(H)^2.$$

for $|g| \leq g_\Sigma$. In particular, if $f = 1$ on an interval $\Delta \subset (-\infty, \Sigma)$, then

$$W\restriction E_\Delta(H)P_{des}^{\perp}\Gamma(\chi_i)\mathcal{H} \geq C > 0.$$

Remark. Our proof shows that $g_\Sigma = O(1-3\beta)$, as $(1-3\beta) \to 0$, is sufficient if $\gamma > \beta$ is chosen close to β.

Proof. Let $\mathcal{D} = D(\mathrm{d}\Gamma(a)) \cap \mathrm{Ran} P_{\mathrm{des}}^{\perp} \Gamma(\chi_i)$, where $a = 1/2(\nabla\omega \cdot (y-x) + (y-x) \cdot \nabla\omega)$. Since $\mathcal{D}$ is dense in $\mathrm{Ran} P_{\mathrm{des}}^{\perp} \Gamma(\chi_i)$ (see Lemma 46 in Appendix G), and since W is bounded, it suffices to prove that there is a constant $C > 0$ such that

$$\langle \varphi, W\varphi \rangle \equiv \lim_{t\to\infty} \langle \varphi_t, f\mathrm{d}\Gamma(\chi_{\gamma,t}) f\varphi_t \rangle \geq C\|f\varphi\|^2 \tag{83}$$

for all $\varphi \in \mathcal{D}$. In the following $\varphi \in \mathcal{D}$ is fixed. The proof of (83) is based on estimates of $\langle \varphi_t, f\mathrm{d}\Gamma(a/t) f\varphi_t \rangle$ from above and from below. The upper bound relates $\langle \varphi_t, f\mathrm{d}\Gamma(a/t) f\varphi_t \rangle$ to $\langle \varphi, W\varphi \rangle$ and the lower bound uses the positive commutator estimate, Theorem 19. We begin with the estimate from above.

Step 1. Let $\varepsilon > 0$. There exists a finite constant C such that

$$\begin{aligned} \langle \varphi_t, f\mathrm{d}\Gamma(a/t) f\varphi_t \rangle \quad &\leq \quad C\langle f\varphi_t, \mathrm{d}\Gamma(\chi_{\gamma,t}) f\varphi_t \rangle^{1/2} \|f\varphi\| \\ &\quad\quad + (\gamma+\beta+\varepsilon)\langle \varphi_t, fNf\varphi_t \rangle + o(1), \qquad t \to \infty. \end{aligned}$$

To see this, suppose $F \in C^\infty(\mathbb{R};[0,1])$, $\mathrm{supp}(F) \subset (-\infty, \beta+\varepsilon]$ and $F(s) = 1$ for $s \leq \beta$. Then

$$\begin{aligned} \chi_\gamma(|y|/t) \quad &\geq \quad \chi(|y|/t \geq \gamma) \\ &\geq \quad \chi(|x|/t \leq \beta+\varepsilon)\, \chi(|x-y|/t \geq \gamma+\beta+\varepsilon) \\ &\geq \quad F(|x|/t)\, \chi(|x-y|/t \geq \gamma+\beta+\varepsilon). \end{aligned}$$

It follows that

$$\begin{aligned} \langle \varphi_t, f\mathrm{d}\Gamma(\chi_\gamma(|y|/t)) f\varphi_t \rangle &\geq \langle \varphi_t, fF(|x|/t)\mathrm{d}\Gamma(\chi(|x-y|/t \geq \gamma+\beta+\varepsilon)) f\varphi_t \rangle \\ &= \langle \varphi_t, f\mathrm{d}\Gamma(\chi(|x-y|/t \geq \gamma+\beta+\varepsilon)) f\varphi_t \rangle + o(1), \end{aligned} \tag{84}$$

where we used Proposition 12 to get rid of the factor $F(|x|/t)$. Next we estimate the right side from below by showing that

$$\begin{aligned} \langle \varphi_t, f\mathrm{d}\Gamma(a/t) f\varphi_t \rangle \quad &\leq \quad C\langle f\varphi_t, \mathrm{d}\Gamma(\chi(|x-y|/t \geq \gamma+\beta+\varepsilon)) f\varphi_t \rangle^{1/2} \|f\varphi\| \\ &\quad\quad + (\gamma+\beta+\varepsilon)\langle \varphi_t, fNf\varphi_t \rangle + O(t^{-1}), \qquad t \to \infty, \end{aligned} \tag{85}$$

for some σ-dependent but finite constant C. Combined with (84) this will prove Step 1.

¿From now on $\lambda := \gamma+\beta+\varepsilon$, $\chi \equiv \chi(|x-y|/t \geq \lambda)$ and $\bar{\chi} \equiv 1-\chi$, for short. Using the identity $1 = \Gamma(\bar{\chi}) + (1-\Gamma(\bar{\chi}))$ we split each photon wave function into parts in- and outside of the sphere $|x-y|/t = \lambda$. We find the bound

$$\begin{aligned} \langle \varphi_t, f\mathrm{d}\Gamma(a/t) f\varphi_t \rangle =& 1/2\langle \varphi_t, f\mathrm{d}\Gamma(a/t)\Gamma(\bar{\chi}) f\varphi_t \rangle + h.c \\ &+ 1/2\langle \varphi_t, f\mathrm{d}\Gamma(a/t)(1-\Gamma(\bar{\chi})) f\varphi_t \rangle + h.c. \\ \leq& \langle \varphi_t, f\mathrm{d}\Gamma\left(\bar{\chi}, 1/2((a/t)\,\bar{\chi} + \bar{\chi}\,(a/t))\right) f\varphi_t \rangle + \|\mathrm{d}\Gamma(a/t) f\varphi_t\| \|(1-\Gamma(\bar{\chi})) f\varphi_t\|. \end{aligned} \tag{86}$$

To estimate the first term on the right hand side, note that $\mathrm{d}\Gamma(\bar{\chi}, b) \leq \mathrm{d}\Gamma(b) \leq \|b\| N$ for every symmetric one-photon operator b. Since

$$\begin{aligned} \|(a/t)\,\bar{\chi}\| &\leq 1/t\,\|\nabla\omega(k)\cdot(y-x)\,\bar{\chi}\| + 1/2t\|\Delta\omega(k)\,\bar{\chi}\| \\ &\leq \lambda + O(t^{-1}), \end{aligned}$$

one arrives at

$$\langle \varphi_t, f\mathrm{d}\Gamma[\bar{\chi}, 1/2((a/t)\,\bar{\chi} + \bar{\chi}(a/t))]f\varphi_t\rangle \leq \lambda\langle\varphi_t, fNf\varphi_t\rangle + O(t^{-1}). \tag{87}$$

The first factor in the second term of (86) is estimated by

$$\|\mathrm{d}\Gamma(a/t)f\varphi_t\| \leq C(\|f\varphi\| + 1/t\|\mathrm{d}\Gamma(a)\varphi\|), \tag{88}$$

by Lemma 44 (use $f = gf$, for a suitable $g \in C_0^\infty(\mathbb{R})$ to see this). This is finite, since $\varphi \in D(\mathrm{d}\Gamma(a))$ by assumption. For the second factor in the second term of (86) we use that

$$\|(1-\Gamma(\bar{\chi}))f\varphi_t\|^2 = \langle\varphi_t, f(1-\Gamma(\bar{\chi}))f\varphi_t\rangle \leq \langle\varphi_t, f\mathrm{d}\Gamma(\chi)f\varphi_t\rangle \tag{89}$$

since $\bar{\chi}$ and hence $(1-\Gamma(\bar{\chi}))$ is a projection. The bound $(1-\Gamma(\bar{\chi})) \leq \mathrm{d}\Gamma(\chi)$ is easily verified on each n-boson sector separately.

After inserting (87), (88) and (89) into (86) one arrives at (85), which proves Step 1.

Step 2. For each $\delta > 0$, there is a sequence $t_n \to \infty$ such that

$$\langle\varphi_{t_n}, f\mathrm{d}\Gamma(a/t_n)f\varphi_{t_n}\rangle \geq \frac{1}{1+\delta}(1-\beta)\langle\varphi_{t_n}, fNf\varphi_{t_n}\rangle - C_M g\|f\varphi\|^2 + o(1) \tag{90}$$

as $n \to \infty$.

By the positive commutator estimate, Theorem 19,

$$\langle\varphi_t, f\mathrm{d}\Gamma(a)f\varphi_t\rangle \geq \langle\varphi, f\mathrm{d}\Gamma(a)f\varphi\rangle + (1-\beta)\int_0^t ds\langle\varphi_s, fNf\varphi_s\rangle - C_M g t\|f\varphi\|^2,$$

and, after dividing both sides by t,

$$\langle\varphi_t, f\mathrm{d}\Gamma(a/t)f\varphi_t\rangle \geq (1-\beta)\,\frac{1}{t}\int_0^t ds\langle\varphi_s, fNf\varphi_s\rangle - C_M g\|f\varphi\|^2 + O(t^{-1}),$$

as $t \to \infty$. This inequality proves Step 2 thanks to the following general fact: for every bounded, continuous function $h(t) \geq 0$ and for each $\delta > 0$, there exists a sequence $t_n \to \infty$ such that

$$m(t) := \frac{1}{t}\int_0^t ds h(s) \geq \frac{1}{1+\delta}h(t)$$

for all $t \in \{t_n\}_{n\in\mathbb{N}}$. In fact, the opposite assumption that $h(t) \geq (1+\delta)m(t)$, for all $t > T_0$ and some $T_0 \in \mathbb{R}$, would imply that

$$\frac{d}{dt}\log m(t) = \frac{m'(t)}{m(t)} \geq \frac{\delta}{t}$$

for all $t > T_0$. This is impossible since $m(t)$ is bounded.

Combining Steps 1 and 2 we get

$$C\|f\varphi\|\langle\varphi_{t_n}, f\mathrm{d}\Gamma(\chi_{\gamma,t})f\varphi_{t_n}\rangle^{1/2} \geq \left\{\frac{1}{1+\delta}(1-\beta)-(\gamma+\beta+\varepsilon)\right\}\langle\varphi_{t_n}, fNf\varphi_{t_n}\rangle - C_M g\|f\varphi\|^2 + o(1), \qquad n\to\infty. \tag{91}$$

Using $(1+\delta)^{-1} \geq 1-\delta$ and the assumption on γ, one finds that $\{\ldots\} \geq (1-2\beta-\gamma-\varepsilon-\delta) \geq (1-2\beta-\gamma)/2 > 0$ for ε and δ small enough. To bound the second factor on the r.h.s of (91), we use that $N \geq 1-P_\Omega$ and that $fP_\Omega f \geq fP_{\mathrm{des}}f - D_\Sigma|g|^{1/2}f^2$, by the remark after Theorem 7 (here we use that $\operatorname{supp} f \subset (-\infty,\Sigma)$). Since $P_{\mathrm{des}}\varphi = 0$ by assumption on φ, we conclude that

$$\langle\varphi_{t_n}, f\mathrm{d}\Gamma(\chi_{\gamma,t})f\varphi_{t_n}\rangle \geq \frac{1}{C}\left\{\frac{1}{2}(1-2\beta-\gamma)(1-D_\Sigma|g|^{1/2}) - C_M|g|\right\}^2 \|f\varphi\|^2 + o(1),$$

as $n\to\infty$. For $|g|$ small enough this proves Eq. (83), because $\lim_{n\to\infty}\langle\varphi_{t_n}, f\mathrm{d}\Gamma(\chi_\gamma)f\varphi_{t_n}\rangle = \langle\varphi, W\varphi\rangle$ by Theorem 26, and the proof is complete. □

8 The Inverse of the Wave Operator

The purpose of this section is the construction of an operator $W_+ : \mathcal{H} \to \tilde{\mathcal{H}}$ inverting the extended wave operator $\tilde{\Omega}_+$ with respect to the asymptotic observable W; that is $W = \tilde{\Omega}_+ W_+$. To this end one needs to show that the dynamics of bosons that escape from the electron ballistically - if there are any - is well approximated by the free-boson dynamics. We shall prove this with the help of Proposition 24, which was established for exactly this purpose.

Many elements in the construction of W_+ are familiar from the construction of W. We recall from Section 7 that β, g, and Σ are real numbers with $\||\nabla\Omega|E_\Sigma(H)\| \leq \beta$ and that $\gamma > \beta$. Then

$$W_+ := s - \lim_{t\to\infty} e^{i\tilde{H}t}\tilde{f}\breve{\Gamma}(j_t)\mathrm{d}\Gamma(\chi_{\gamma,t})fe^{-iHt},$$

where $\tilde{f} = f(\tilde{H})$ and $f = f(H)$ are smooth energy cutoffs supported in $(-\infty,\Sigma)$. As in Section 7, $\chi_{\gamma,t}$ is the operator of multiplication with $\chi_\gamma(|y|/t)$ where $\chi_\gamma \in C^\infty(\mathbb{R};[0,1])$, $\chi_\gamma \equiv 1$ on $[\gamma,\infty)$ and $\operatorname{supp}(\chi_\gamma) \subset [\beta_3,\infty)$ for some $\beta_3 > \beta$. The purpose of $\breve{\Gamma}(j_t) : \mathcal{F} \to \mathcal{F}\otimes\mathcal{F}$ is to split each boson state into two parts, the second part being mapped to the second Fock-space of prospective asymptotically freely moving bosons. We introduce β_1 and β_2 such

$$\beta < \beta_1 < \beta_2 < \beta_3 < \gamma$$

and define $j_t : \mathfrak{h} = L^2(\mathbb{R}^3, dk) \to \mathfrak{h}\oplus\mathfrak{h}$ as follows: let $j_t h = (j_{0,t}h, j_{\infty,t}h)$, where $j_{\sharp,t}(y) = j_\sharp(|y|/t)$, $j_\sharp \in C^\infty(\mathbb{R};[0,1])$, $j_0 + j_\infty \equiv 1$, $j_0 \equiv 1$ on $(-\infty,\beta_2]$, $\operatorname{supp}(j_0) \subset (-\infty,\beta_3]$ while $j_\infty \equiv 1$ on $[\beta_3,\infty)$ and $\operatorname{supp}(j_\infty) \subset [\beta_2,\infty)$ (see Figure 3, below).

As in the last section, we work with the modified Hamiltonian $H_{\mathrm{mod}} = \Omega(p) + \mathrm{d}\Gamma(\omega) + g\phi(G_x)$ and with the extended modified Hamiltonian $\tilde{H}_{\mathrm{mod}} = H_{\mathrm{mod}}\otimes 1 + 1\otimes\mathrm{d}\Gamma(\omega)$, and we use the notation $H \equiv H_{\mathrm{mod}}$, $\tilde{H} \equiv \tilde{H}_{\mathrm{mod}}$. Moreover, as in Section 7, we use the notation DA

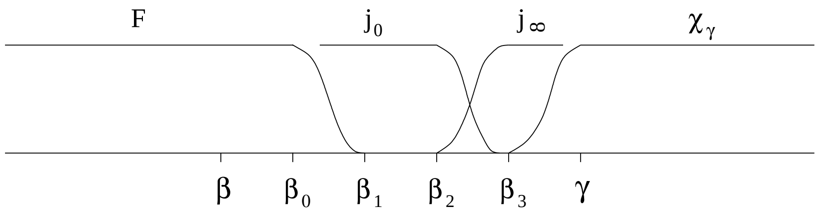

Figure 3: Typical choice of the function χ_γ, of the electron space cutoff F and of the partition in the photon space j_0, j_∞.

and $D_0 A$ to denote Heisenberg derivatives of operators A on $\mathcal{H}$. If B is an operator on the extended Hilbert space $\tilde{\mathcal{H}}$, and if C maps $\mathcal{H}$ to $\tilde{\mathcal{H}}$ we set

$$\begin{aligned} DB &:= i[\tilde{H}, B] + \frac{\partial B}{\partial t} \\ \tilde{D}C &:= i\big(\tilde{H}C - CH\big) + \frac{\partial C}{\partial t}. \end{aligned}$$

The derivatives D_0, and $\tilde{D}_0$ are defined in a similar way, using H_0 and $\tilde{H}_0$ instead of H and $\tilde{H}$. The Heisenberg derivative of an operator a on $L^2(\mathbb{R}^3)$ is denoted by $da = [i\omega(k), a] + \partial a/\partial t$. Finally, the Heisenberg derivative db of an operator b mapping the one-boson space $\mathfrak{h}$ to $\mathfrak{h} \oplus \mathfrak{h}$ is defined by

$$db = i \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} b - b\, i\omega + \frac{\partial b}{\partial t} =: \begin{pmatrix} db_0 \\ db_\infty \end{pmatrix}.$$

Theorem 28 (Existence of W_+). *Assume Hypotheses 0 – 3 are satisfied. Let β, g and Σ be real numbers for which $\||\nabla\Omega| E_\Sigma(H)\| \leq \beta$. Suppose that $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \subset (-\infty, \Sigma)$, and that β, γ and χ_γ are defined as described above. Then*

(i) The limit

$$W_+ = s - \lim_{t\to\infty} e^{i\tilde{H}t} \tilde{f} \check{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t}) f e^{-iHt}$$

exists, and $e^{-i\tilde{H}s} W_+ = W_+ e^{-iHs}$, for all $s \in \mathbb{R}$.

(ii) $(1 \otimes \chi(N = 0)) W_+ = 0$.

(iii) $W = \tilde{\Omega}_+ W_+$.

Proof. Statement (ii) follows from $(1 \otimes \chi(N = 0))\check{\Gamma}(j_t) = \check{\Gamma}(j_{0,t}, 0)$ and $j_{0,t}\chi_{\gamma,t} = 0$.

(i) Pick $F \in C_0^\infty(\mathbb{R})$ with $F(s) = 1$ for $s \leq \beta_0$ and $F(s) = 0$ for $s \geq \beta_1$, where $\beta_0 \in (\beta, \beta_1)$. We also use F to denote the operator of multiplication with $F(|x|/t)$. By Proposition 12, it suffices to prove the existence of

$$\lim_{t\to\infty} \varphi(t), \quad \text{where} \quad \varphi(t) = e^{i\tilde{H}t} \tilde{f} \check{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t}) F f e^{-iHt} \varphi$$

for all $\varphi \in \mathcal{H}$. Using Cook's argument one is led to show that

$$\int_1^\infty |\langle \psi, \varphi'(t) \rangle| dt \leq C\|\psi\|$$

for all $\psi \in \mathcal{H}$. We have

$$\begin{aligned}
\frac{d}{dt}\langle\psi,\varphi(t)\rangle &= \langle\psi_t, \tilde{f}\tilde{D}\Big[\breve{\Gamma}(j_t)\mathrm{d}\Gamma(\chi_{\gamma,t})F\Big]f\varphi_t\rangle \\
&= \langle\psi_t, \tilde{f}\mathrm{d}\breve{\Gamma}(j_t, dj_t)\mathrm{d}\Gamma(\chi_{\gamma,t})Ff\varphi_t\rangle \\
&\quad + \langle\psi_t, \tilde{f}\breve{\Gamma}(j_t)\mathrm{d}\Gamma(d\chi_{\gamma,t})Ff\varphi_t\rangle \\
&\quad + g\langle\psi_t, \tilde{f}\Big[\big(i\phi(G_x)\otimes 1\big)\breve{\Gamma}(j_t) - \breve{\Gamma}(j_t)i\phi(G_x)\Big]\mathrm{d}\Gamma(\chi_{\gamma,t})Ff\varphi_t\rangle \\
&\quad + g\langle\psi_t, \tilde{f}\breve{\Gamma}(j_t)\phi(i\chi_{\gamma,t}G_x)Ff\varphi_t\rangle \\
&\quad + \langle\psi_t, \tilde{f}\breve{\Gamma}(j_t)\mathrm{d}\Gamma(\chi_{\gamma,t})(DF)f\varphi_t\rangle.
\end{aligned} \tag{92}$$

We now prove integrability of all these terms, beginning with the last one. Since

$$DF = \frac{1}{t}\chi_{[\beta_0,\beta_1]}(|x|/t)\left(F'\frac{x}{|x|}\cdot\nabla\Omega - \frac{|x|}{t}F'\right)\chi_{[\beta_0,\beta_1]}(|x|/t) + O(t^{-2})$$

the last term on the r.h.s. of (92) is integrable, by Proposition 12 and the remarks thereafter and because $|\nabla\Omega|$ is bounded w.r.to H, by Hypothesis 0. See the proof of Proposition 22 for details.

The second but last term on the r.h.s. of (92) decays like $t^{-\mu}$ with $\mu > 1$, because $|x|/t \le \beta_1$ on $\mathrm{supp}(F)$, $|y|/t \ge \beta_3$ on $\mathrm{supp}(\chi_{\gamma,t})$ and hence $|x-y| \ge t(\beta_3 - \beta_1)$ on $\mathrm{supp}(\chi_{\gamma,t}G_xF)$; (see the proof of Proposition 22 for details). Similar remarks prove the integrability of the third term, because

$$[\phi(G_x)\otimes 1]\breve{\Gamma}(j_t) - \breve{\Gamma}(j_t)\phi(G_x) = \big[\phi((1-j_{0,t})G_x)\otimes 1 - 1\otimes\phi(j_\infty G_x)\big]\breve{\Gamma}(j_t),$$

where $1 - j_{0,t}$ and $j_{\infty,t}$ are supported in $|y|/t \ge \beta_2$, while $|x|/t \le \beta_1$ on $\mathrm{supp}(F)$, hence $|x-y| \ge t(\beta_2 - \beta_1)$.

The integrability of the first and second term on the r.h.s. of (92) will follow from the improved propagation estimate in Proposition 24. For the second term we use that $\breve{\Gamma}(j_t)\mathrm{d}\Gamma(d\chi_{\gamma,t}) = \mathrm{d}\breve{\Gamma}(j_t, j_t d\chi_{\gamma,t})$ where

$$\begin{aligned}
j_t d\chi_{\gamma,t} &= \frac{1}{2}\big[(\nabla\omega - y/t)\cdot\nabla\chi_{\gamma,t}j_t + j_t\nabla\chi_{\gamma,t}\cdot(\nabla\omega - y/t)\big] + O(t^{-2}) \\
&=: \frac{1}{t}P_t + O(t^{-2}),
\end{aligned}$$

where one power of $1/t$ has been factored out from $\nabla\chi_{\gamma,t} = (1/t)\chi'_{\beta,t}(|y|/t)y/|y|$. The error term $O(t^{-2})$ is integrable. By Lemma 2 and since $P_{0,t} = 0$, $P_{0,t}$ being the first component of $P_t = (P_{0,t}, P_{\infty,t})$,

$$\begin{aligned}
&\big|\langle\psi_t, \tilde{f}\mathrm{d}\breve{\Gamma}(j_t, P_t)Ff\varphi_t\rangle\big| \\
&\le \langle\tilde{f}\psi_t, \big[1\otimes\mathrm{d}\Gamma(|P_{\infty,t}|)\big]F\tilde{f}\psi_t\rangle^{1/2}\langle f\varphi_t, \mathrm{d}\Gamma(|P_{\infty,t}|)Ff\varphi_t\rangle^{1/2}.
\end{aligned}$$

This is integrable by Proposition 24 and the remarks thereafter.

Finally, we estimate the first term on the r.h.s. of (92). Let $K_t = 1/2((\nabla\omega - y/t)\cdot\nabla j_t + h.c.)$ and let the operator $\underline{\chi}_\gamma$ be defined by $\underline{\chi}_\gamma(h_1, h_2) = (0, \chi_{\gamma,t}h_2)$ on $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. Then $dj_t = K_t + O(t^{-2})$, $j_t\chi_{\gamma,t} = \underline{\chi}_\gamma j_t$ and $K_t\chi_{\gamma,t} = \underline{\chi}_\gamma K_t + O(t^{-2})$. Therefore

$$\mathrm{d}\breve{\Gamma}(j_t, dj_t)\mathrm{d}\Gamma(\chi_{\gamma,t}) = \big[1\otimes\mathrm{d}\Gamma(\chi_{\gamma,t})\big]\mathrm{d}\Gamma(j_t, K_t) + O(t^{-2})N^2.$$

We write

$$\big[1 \otimes \mathrm{d}\Gamma(\chi_{\gamma,t})\big]\mathrm{d}\breve{\Gamma}(j_t, K_t) = \mathrm{d}\breve{\Gamma}(j_t, \underline{\chi}_{\gamma} K_t) + UR_t \tag{93}$$

where R_t is defined by this equation and U is as in $\mathrm{d}\breve{\Gamma} = U\mathrm{d}\Gamma$. The term $\mathrm{d}\breve{\Gamma}(j_t, \underline{\chi}_{\gamma} K_t)$ is treated very much like $\mathrm{d}\breve{\Gamma}(j_t, j_t d\chi_{\gamma,t})$ above, and it leads to an integrable contribution thanks to the choice of $\mathrm{supp}(\nabla j)$ and Proposition 22. On $\otimes_s^n L^2(\mathbb{R}^3)$ the operator R_t is given by

$$\sum_{l=1}^{n} \sum_{k=1,\, k\neq l}^{n} j_t \otimes \dots \underbrace{(\underline{\chi}_{\gamma} j_t)}_{kth} \otimes \dots \underbrace{K_t}_{lth} \dots \otimes j_t.$$

¿From the defining equation (93) for R_t and from Lemma 2 it is plausible that

$$\begin{aligned}
&|\langle \psi_t, \tilde{f} U R_t F f \psi_t\rangle| \\
&\leq \langle \psi_t, \tilde{f}\big[1 \otimes \mathrm{d}\Gamma(|K_{\infty,t}|)\big] N_\infty^2 F \tilde{f} \psi_t\rangle^{1/2} \langle \varphi_t, f \mathrm{d}\Gamma(|K_{\infty,t}|) F f \varphi_t\rangle^{1/2} \\
&\quad + \langle \psi_t, \tilde{f}\big[\mathrm{d}\Gamma(|K_{0,t}|) \otimes N^2\big] F \tilde{f} \psi_t\rangle^{1/2} \langle \varphi_t, f \mathrm{d}\Gamma(|K_{0,t}|) F f \varphi_t\rangle^{1/2}.
\end{aligned} \tag{94}$$

To prove this, we return to the proofs of Lemma 1 and Lemma 2 with $K_t = r_2^* r_1$, and $r_2^* r_2 = |K_{\sharp,t}| = r_1^* r_1$. The number operators in (94) prevent us from applying Proposition 24. We choose $g \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(g) \subset (-\infty, \Sigma)$ and $gf = f$. Then

$$N_\infty \tilde{f} \psi_t = g(\tilde{H}) e^{-i\tilde{H}t} (N_\infty \tilde{f}) \psi$$

where $N_\infty \tilde{f}$ is a bounded operator. Now the integrability of (94) follows from $\mathrm{supp}(\nabla j) \subset \{\beta_2 \leq |y| \leq \beta_3\}$ and Proposition 24.

The second assertion in (i) is proved in the same way as the corresponding statement for W. By definition of W_+

$$\Big[e^{-i\tilde{H}s} W_+ e^{iHs} - W_+\Big]\varphi = \lim_{t\to\infty} e^{i\tilde{H}t} \tilde{f} \big[\breve{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t})\big]_t^{t+s} f e^{-iHt} \varphi.$$

Since $\partial_t j_t = O(t^{-1})$ and $\partial_t \chi_{\gamma,t} = O(t^{-1})$ we conclude that

$$\frac{d}{dt} \breve{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t}) f = \big[\mathrm{d}\breve{\Gamma}(j_t, \partial_t j_t) \mathrm{d}\Gamma(\chi_{\gamma,t}) + \breve{\Gamma}(j_t) \mathrm{d}\Gamma(\partial_t \chi_{\gamma,t})\big] f = O(t^{-1})$$

and hence $\| \big[\breve{\Gamma}(j_t) \mathrm{d}\Gamma(\chi_{\gamma,t})\big]_t^{t+s} f \| = O(t^{-1})$.

It remains to prove (iii). Recall from Eq. (28) that $I\Gamma(j_t) = 1$, because $j_0 + j_\infty = 1$. Furthermore

$$I \tilde{f} \breve{\Gamma}(j_t) F = fF + o(1), \qquad (t \to \infty). \tag{95}$$

as can be shown using Lemma 43 in Appendix F (see the proof of Lemma 16 in [FGS01] for details). Let $g \in C_0^\infty(\mathbb{R})$ with $gf = f$, and let $\tilde{g} = g(\tilde{H})$. By definition of W, Proposition 12, and by (95),

$$\begin{aligned}
W\varphi &= e^{iHt} f F \mathrm{d}\Gamma(\chi_{\gamma,t}) f e^{-iHt} \varphi + o(1) \\
&= e^{iHt} I \tilde{g} \Big(e^{-i\tilde{H}t} e^{i\tilde{H}t}\Big) \tilde{f} \breve{\Gamma}(j_t) F \mathrm{d}\Gamma(\chi_{\gamma,t}) f e^{-iHt} \varphi + o(1) \\
&= e^{iHt} I \tilde{g} e^{-i\tilde{H}t} W_+ \varphi + o(1),
\end{aligned}$$

where the last step uses that $I\tilde{g}$ is a bounded operator. Since $\tilde{g} W_+ = W_+$ the assertion follows. □

9 Putting It All Together: Asymptotic Completeness

As explained in the introduction, we prove asymptotic completeness by induction in the energy measured in units of $\sigma/2$, σ being the infrared cutoff. The first step is the following essentially trivial lemma. The idea is that AC on $E_\eta(H)$, as characterized by Eq. (6), implies the same property for $Ie^{-i\tilde{H}t}$ on $E_\eta(H)\otimes\mathcal{F}$, the photons from $\mathcal{F}$ merely contributing to the asymptotically free radiation.

Lemma 29. *Assume that Hypotheses 0 – 3 are satisfied. Suppose g and $\Sigma > \inf\sigma(H)$ are real numbers for which $\||\nabla\Omega(p)|E_\Sigma(H)\| < 1$. Let the wave operators $\tilde{\Omega}_+$ and Ω_+ be defined as in Lemma 20 and in Theorem 15, respectively. Suppose* $\mathrm{Ran}\Omega_+ \supset E_\eta(H)\mathcal{H}$*, for some $\eta < \Sigma$. Then, for every $\varphi \in \mathrm{Ran}E_\Sigma(\tilde{H})$, there exists a $\psi \in \mathrm{Ran}E_\Sigma(\tilde{H})$ such that*

$$\tilde{\Omega}_+(E_\eta(H)\otimes 1)\varphi = \Omega_+\psi.$$

If $\Delta \subset (-\infty,\Sigma)$ and $\varphi \in E_\Delta(\tilde{H})\tilde{\mathcal{H}}$ then $\psi \in E_\Delta(\tilde{H})\tilde{\mathcal{H}}$.

Proof. By Lemma 34 (Appendix C), every given $\varphi \in \mathrm{Ran}E_\Sigma(\tilde{H})$ can be approximated by a sequence of vectors $\varphi_n \in E_\Sigma(\tilde{H})$ which are finite linear combinations of vectors of the form

$$\gamma = \alpha\otimes a^*(h_1)\dots a^*(h_n)\Omega, \qquad \lambda + \sum_{i=1}^n M_i < \Sigma, \tag{96}$$

for some λ, where $\alpha = E_\lambda(H)\alpha$ and $M_i = \sup\{|k| : h_i(k) \neq 0\}$. Let $\gamma \in \tilde{\mathcal{H}}$ be of the form (96). Then

$$\begin{aligned} e^{iHt}Ie^{-i\tilde{H}t}(E_\eta(H)\otimes 1)\,\gamma &= e^{iHt}a^*(h_{1,t})\dots a^*(h_{n,t})\,e^{-iHt}\,E_\eta(H)\alpha \\ &= a_+^*(h_1)\dots a_+^*(h_n)E_\eta(H)\alpha + o(1), \end{aligned} \tag{97}$$

as $t\to\infty$. By assumption, $E_\eta(H)\alpha = \Omega_+\beta$, for some $\beta \in \tilde{\mathcal{H}}$, and we may assume that $\beta = E_\eta(\tilde{H})\beta$, thanks to the intertwining relation for Ω_+. From (97) it follows that

$$\begin{aligned} \tilde{\Omega}_+(E_\eta(H)\otimes 1)\gamma &= a_+^*(h_1)\dots a_+^*(h_n)\Omega_+\beta \\ &= \Omega_+(1\otimes a^*(h_1)\dots a^*(h_n))\beta, \end{aligned}$$

where, in the second equation, we have used Lemma 16. Hence, to each vector φ_n as in Eq. (96), there corresponds a vector $\psi_n \in E_\mu(H)\tilde{\mathcal{H}}$ such that $\tilde{\Omega}_+(E_\eta(H)\otimes 1)\varphi_n = \Omega_+\psi_n$. The left side converges to $\tilde{\Omega}_+(E_\eta(H)\otimes 1)\varphi$, as $n\to\infty$, and hence the right side converges as well. Since Ω_+ is isometric on $\mathcal{H}_{\mathrm{des}}\otimes\mathcal{F}$, it follows that $(P_{\mathrm{des}}\otimes 1)\psi_n$ is Cauchy and hence has a limit $\psi \in E_\mu(H)\tilde{\mathcal{H}}$. Thus $\tilde{\Omega}_+(E_\eta(H)\otimes 1)\varphi = \Omega_+\psi$ which proves the lemma. □

Theorem 30. *Assume Hypotheses 0 – 3 are satisfied. Suppose that $\Sigma > \inf\sigma(H)$ and $g_0 > 0$ are so small that $\||\nabla\Omega|E_\Sigma(H)\| < 1/3$, for all $g < g_0$. Then, if $g < g_0$ is sufficiently small (compared to $(1-3\||\nabla\Omega|E_\Sigma(H)\|)$)*

$$\mathrm{Ran}\,\Omega_+ \supset E_{(-\infty,\Sigma)}(H)\mathcal{H}.$$

Proof. The proof is by induction in energy steps of size $m = \sigma/2$. We show that

$$\mathrm{Ran}(\Omega_+) \supset E_{(-\infty,\Sigma-km)}(H)\mathcal{H}, \tag{98}$$

for $k = 0$, by proving it for all $k \in \{0, 1, 2, \ldots\}$. Since H is bounded below, (98) is certainly correct for k large enough. Assuming that (98) holds for $k = n+1$, we now prove it for $k = n$. Since $\mathrm{Ran}\,\Omega_+$ is closed, by Theorem 15, it suffices to prove that $\mathrm{Ran}\,\Omega_+ \supset E_\Delta(H)\mathcal{H}$ for all compact intervals $\Delta \subset (-\infty, \Sigma - nm)$, which is equivalent to

$$\mathrm{Ran}\,\Omega_+ \supset P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)E_\Delta(H)\mathcal{H},$$

by Lemma 21 and because $\mathrm{Ran}\,\Omega_+ \supset \mathcal{H}_{\mathrm{des}}$. Choose $f \in C_0^\infty(\mathbb{R};\mathbb{R})$ with $f = 1$ on Δ and $\mathrm{supp}(f) \subset (-\infty, \Sigma)$, and define W in terms of f as in Theorem 26. By Theorem 27, the operator $\Gamma(\chi_i)P_{\mathrm{des}}^{\perp}WP_{\mathrm{des}}^{\perp}\Gamma(\chi_i)$ is positive on $P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)E_\Delta(H)\mathcal{H}$, and hence onto, if $g < g_0$ is small enough. Given ψ in this space we can thus find $\varphi = P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)\varphi$ such that

$$P_{\mathrm{des}}^{\perp}\Gamma(\chi_i)W\varphi = \psi.$$

By Theorem 28, $W\varphi = \tilde{\Omega}_+W_+\varphi$ and $W_+\varphi = E_{\Sigma-mn}(\tilde{H})W_+\varphi$. Furthermore, by part (ii) of Theorem 28, $W_+\varphi$ has at least one boson in the outer Fock space, and thus an energy of at most $\Sigma - (n+1)m$ in the inner one. That is,

$$W_+\varphi = [E_{\Sigma-(n+1)m}(H) \otimes 1]W_+\varphi,$$

and we can now use the induction hypothesis $\mathrm{Ran}\,\Omega_+ \supset E_{\Sigma-(n+1)m}(H)\mathcal{H}$. Using Lemma 29, it follows that $\tilde{\Omega}_+W_+\varphi = \Omega_+\gamma$ for some $\gamma \in E_\Delta(\tilde{H})\mathcal{H}$. We conclude that

$$\begin{aligned}
\psi &= \Gamma(\chi_i)P_{\mathrm{des}}^{\perp}\Omega_+\gamma \\
&= \Gamma(\chi_i)\Omega_+(1 \otimes P_\Omega^{\perp})\gamma \\
&= \Omega_+(\Gamma(\chi_i) \otimes \Gamma(\chi_i)P_\Omega^{\perp})\gamma,
\end{aligned}$$

where $P_\Omega^{\perp}$ is the projection onto the orthogonal complement of the vacuum. This proves the theorem. □

10 Outlook

It is clear that the infrared cutoff $\sigma > 0$ has played an unpleasantly crucial role in our proof of AC for Compton scattering. We do not know how to remove this cutoff in several key estimates used in our proof; see Sect. 8.

However, the construction of a suitable Møller wave operator in the limit $\sigma \to 0$ has been accomplished by Pizzo [Piz00], using results of [Frö73] and of [Che01].

In the presence of an infrared cutoff we are also able to construct Møller wave operators for the scattering theory of $N \geq 2$ conserved electrons interacting with scalar bosons or photons. The proof follows arguments used in Haag–Ruelle scattering theory; see [Jos65] and refs. given there. However, because the models studied here are neither Galilei–, nor Lorentz covariant, in particular, because the dispersion law $E_g(P)$ of dressed one–electron states does not reflect any symmetries other than Euclidian motions and hence the center of mass motion of bound clusters does not factor out, there are no methods known to us enabling one to attack the problem of proving AC for the scattering of many electrons.

By combining the methods developed in this paper with those in [FGS01] and with elements of Mourre theory for Schrödinger operators, we expect to be able to extend the results of this paper to a model, where the electron moves under the influence of a screened electrostatic force generated by some static nuclei. We thus expect to be able to describe scattering processes corresponding to ionization of an atom and electron capture by a nucleus (Bremsstrahlung).

A Functional Calculus

The Helffer-Sjöstrand Functional Calculus is a useful tool in the computation of commutators of functions of self adjoint operators. Suppose that $f \in C_0^\infty(\mathbb{R};\mathbb{C})$ and that A is a self adjoint operator. A convenient representation for $f(A)$, which is often used in this paper, is then given by

$$f(A) = -\frac{1}{\pi}\int dxdy\, \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\,(z-A)^{-1}, \qquad z = x+iy,$$

which holds for any extension $\tilde{f} \in C_0^\infty(\mathbb{R}^2;\mathbb{C})$ of f with $|\partial_{\bar{z}}\tilde{f}| \le C|y|$,

$$\tilde{f}(z) = f(z), \qquad \text{and} \qquad \frac{\partial \tilde{f}}{\partial \bar{z}}(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(z) = 0, \quad \text{for all} \quad z \in \mathbb{R}. \tag{99}$$

Such a function $\tilde{f}$ is called an *almost analytic extension* of f. A simple example is given by $\tilde{f}(z) = (f(x) + iyf'(x))\,\chi(z)$ where $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\chi = 1$ on some complex neighborhood of $\operatorname{supp} f$. Sometimes we need faster decay of $|\partial_{\bar{z}}\tilde{f}|$, as $|y| \to 0$; namely $|\partial_{\bar{z}}\tilde{f}| \le C|y|^n$. We then work with the almost analytic extension

$$\tilde{f}(z) = \left(\sum_{k=0}^{n} f^{(k)}(x)\frac{(iy)^k}{k!}\right)\chi(z),$$

with χ as above. We call this an almost analytic extension *of order n*. For more details and extensions of this functional calculus the reader is referred to [HS00] or [Dav95].

To estimate commutators involving $\Omega(p) = \sqrt{p^2+M^2}$ we will use the following lemma.

Lemma 31. *Let B be an operator on $\mathcal{H}$. Then*

$$[\Omega(p), B] = \frac{1}{\pi}\int_{M^2}^{\infty} dy\, \frac{\sqrt{y-M^2}}{y+p^2}\,[p^2, B]\,\frac{1}{y+p^2}. \tag{100}$$

B Pseudo Differential Calculus

In order to compute commutators of functions of the momentum-coordinates with functions of the position-coordinates the following lemma is very useful.

Lemma 32. *Suppose* $f \in \mathcal{S}(\mathbb{R}^d)$*,* $g \in C^n(\mathbb{R}^d)$ *and* $\sup_{|\alpha|=n} \|\partial^\alpha g\|_\infty < \infty$*. Let* $p = -i\nabla$*. Then*

$$\begin{aligned} i[g(p), f(x)] &= i \sum_{1 \le |\alpha| \le n-1} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial^\alpha f)(x)(\partial^\alpha g)(p) + R_{1,n} \\ &= (-i) \sum_{1 \le |\alpha| \le n-1} \frac{i^{|\alpha|}}{\alpha!} (\partial^\alpha g)(p)(\partial^\alpha f)(x) + R_{2,n} \end{aligned}$$

where

$$\|R_{j,n}\| \le C_n \sup_{|\alpha|=n} \|\partial^\alpha g\|_\infty \int dk\, |k|^n |\hat{f}(k)|.$$

In particular, and most importantly, if $n = 2$ *then*

$$\begin{aligned} i[g(p), f(\varepsilon x)] &= \varepsilon \nabla g(p) \cdot \nabla f(\varepsilon x) + O(\varepsilon^2) \\ &= \varepsilon \nabla f(\varepsilon x) \cdot \nabla g(p) + O(\varepsilon^2), \end{aligned}$$

as $\varepsilon \to 0$*.*

For the proof of this lemma see [FGS01].

C Representation of States in $\chi(\tilde{H} \le c)\tilde{\mathcal{H}}$

The representation of states in $\operatorname{Ran}\chi(\tilde{H} \le c)$ proved in this section is used in Section 4 to prove the existence of the wave operator and in Lemma 29. See [FGS01] for the proofs.

Lemma 33. *Suppose* $\omega(k) = |k|$ *or that* ω *satisfies Hypothesis 3, and let* $c > 0$*. Then the space of linear combinations of vectors of the form* $a^*(h_1) \dots a^*(h_n)\Omega$ *with* $h_i \in L^2(\mathbb{R}^d)$ *and* $\sum_{i=1}^n \sup\{\omega(k) : k \in \operatorname{supp}(h_i)\} \le c$ *is dense in* $\chi(\mathrm{d}\Gamma(\omega) \le c)\mathcal{F}$*.*

Lemma 34. *Suppose that* ω *satisfies Hypothesis 3, set* $H = H_{mod}$ *and* $\tilde{H} = \tilde{H}_{mod}$*, where* H_{mod} *and* $\tilde{H}_{mod}$ *are the Hamiltonians on* $\mathcal{H}$ *and on* $\tilde{\mathcal{H}}$ *introduced in Section 5. Let* $c > 0$*. Then the set of all linear combinations of vectors of the form*

$$\varphi \otimes a^*(h_1) \dots a^*(h_n)\Omega, \qquad \lambda + \sum_{i=1}^N M_i \le c \tag{101}$$

where $\varphi = \chi(H \le \lambda)\varphi$ *for some* $\lambda \le c$*,* $n \in \mathbb{N}$ *and* $M_i = \sup\{\omega(k) : h_i(k) \ne 0\}$*, is dense in* $\chi(\tilde{H} \le c)\tilde{\mathcal{H}}$*.*

D Spectral Results

In the first subsection of this appendix we prove the existence of ground state vectors for $H_g(P)$, which are used in Section 3.2 to construct the dressed electron states (DES). In the second subsection we prove a version of the Virial Theorem for the modified Hamiltonian $H_{\mathrm{mod}}(P)$ introduced in Section 5, which together with the positive commutator discussed in Section 3.3 allows us to prove the absence of eigenvalues of $H_g(P)$ above its ground state energy.

D.1 Existence of DES

Our proof that $E_g(P) = \inf \sigma(H_g(P))$ is an eigenvalue of the Hamiltonian $H_g(P)$ for $\sigma > 0$ relies on the Lipshitz property

$$\inf_{|k|\geq\varepsilon} \left\{E_g(P-k) + |k| - E_g(P)\right\} > 0 \tag{102}$$

valid whenever $\varepsilon > 0$, $\Omega(P) < O_{\beta=1}$, and $|g|$ is small enough. To prove Eq. (102), we argue by way of perturbation theory and we use that

$$(1-\alpha)E_0(P) - \frac{g^2}{\alpha}\int \frac{|\kappa(k)|^2}{|k|}\, dk \leq E_g(P) \leq \Omega(P) \tag{103}$$

for all $P \in \mathbb{R}^3, g \in \mathbb{R}$ and $\alpha \in (0,1]$. The upper bound in (103) follows from $\langle\Omega, \phi(\kappa_\sigma)\Omega\rangle = 0$ (Rayleigh–Ritz principle) and the lower bound from $H_g(P) \geq (1-\alpha)H_0(P) + \alpha \mathrm{d}\Gamma(|k|) + g\phi(\kappa_\sigma)$ and from Lemma 8. Note that the lower bound is independent of the IR cutoff σ, because, by Hypothesis 1, $\kappa_\sigma(k) = \kappa(k)\chi(|k|/\sigma)$, and $0 \leq \chi \leq 1$.

Lemma 35. *Assume Hypotheses 0 – 2 and define* $B := \sup_P \|\partial^2\Omega(P)\| < \infty$ *and* $C := \int |\kappa(k)|^2/|k|\, dk < \infty$. *If* $\beta < 1$, $\Omega(P) \leq O_\beta$, *and*

$$|g| < g_\beta := \min\left(1, \frac{(1-\beta)^{3/2}}{3(BC)^{1/2}}, \frac{(1-\beta)^2}{3B(C+O_\beta)}\right)$$

then, for all $\varepsilon > 0$, *Eq.* (102) *holds true.*

Proof. For shortness we write P_f and H_f instead of $\mathrm{d}\Gamma(k)$ and $\mathrm{d}\Gamma(|k|)$ in the following. Let $P \in \mathbb{R}^3$ with $\Omega(P) \leq O_\beta$ be fixed. Given $\delta > 0$ and $k \in \mathbb{R}^3$ pick $\psi_\delta \in D(H_g(P-k))$ with $\|\psi_\delta\| = 1$ and

$$\langle\psi_\delta, H_g(P-k)\psi_\delta\rangle \leq E_g(P-k) + \delta \tag{104}$$

Since $\langle\psi_\delta, H_g(P)\psi_\delta\rangle \geq E_g(P)$, it follows that

$$\begin{aligned} E_g(P-k) - E_g(P) &\geq \langle\psi_\delta, [H_g(P-k) - H_g(P)]\psi_\delta\rangle - \delta \\ &= \langle\psi_\delta, [\Omega(P-k-P_f) - \Omega(P-P_f)]\psi_\delta\rangle - \delta. \end{aligned} \tag{105}$$

¿From the formula

$$\begin{aligned} &\Omega(P-k-q) - \Omega(P-q) \\ &\quad= \Omega(P-k) - \Omega(P) + \int_0^1 dt \int_0^1 ds \sum_{i,j} (\partial_i\partial_j\Omega)(P-sk-tq)k_i q_j, \end{aligned}$$

the assumptions and (30), we obtain the estimate

$$\Omega(P-k-q) - \Omega(P-q) \geq -\beta|k| - B|k||q| \tag{106}$$

valid for all $k, q \in \mathbb{R}^3$. Since $|P_f| \leq H_f$, Eq. (106) leads to the operator bound

$$\Omega(P-k-P_f) - \Omega(P-P_f) \geq -\beta|k| - B|k|H_f. \tag{107}$$

In conjunction with (105) this proves that

$$E_g(P-k) - E_g(P) \geq -\beta|k| - B|k|\langle\psi_\delta, H_f\psi_\delta\rangle - \delta \tag{108}$$

and hence we need a bound on $\langle\psi_\delta, H_f\psi_\delta\rangle$ from above.

¿From the bound (104) characterizing ψ_δ we see that

$$\begin{aligned}\Omega(P-k) + \delta \geq E_g(P-k) + \delta &\geq \langle\psi_\delta, H_g(P-k)\psi_\delta\rangle \\ &= \langle\psi_\delta, [\Omega(P-k-P_f) + H_f + g\phi(\kappa_\sigma)]\psi_\delta\rangle\end{aligned}$$

which we estimate from below using the operator bounds

$$\begin{aligned}\Omega(P-k-P_f) &\geq \Omega(P-k) - (\beta + B|k|)H_f \\ g\phi &\geq -\alpha H_f - \frac{g^2 C}{\alpha},\end{aligned}$$

obtained from (106) with q and k interchanged, and Lemma 8, respectively . We conclude that

$$\delta \geq (1-\beta-B|k|-\alpha)\langle\psi_\delta, H_f\psi_\delta\rangle - \frac{g^2}{\alpha}C. \tag{109}$$

Inserting this bound on $\langle\psi_\delta, H_f\psi_\delta\rangle$ in (108) and letting $\delta \to 0$ leads to

$$\begin{aligned}E_g(P-k) + |k| - E_g(P) &\geq \left(1 - \beta - \frac{g^2 BC/\alpha}{1-\beta-B|k|-\alpha}\right)|k| \\ &\geq \left(1-\beta-g^2\frac{9BC}{(1-\beta)^2}\right)\varepsilon\end{aligned}$$

for $\alpha = (1-\beta)/3$ and $\varepsilon \leq |k| \leq (1-\beta)/(3B)$. This is positive under our assumption on g. It remains to estimate the left hand side from below when $|k| \geq (1-\beta)/(3B)$.

To this end we note that for $g = 0$

$$E_0(P-k) + |k| - E_0(P) \geq (1-\beta)|k| \tag{110}$$

while, by (103) with $\alpha = |g|$,

$$\begin{aligned}E_g(P-k) &\geq (1-|g|)E_0(P-k) - C|g| && (111)\\ E_g(P) &\leq \Omega(P) = E_0(P). && (112)\end{aligned}$$

Eq. (110) follows from $E_0(P-k) = \inf_q(\Omega(P-k-q)+|q|) \geq \Omega(P) - \beta|k+q| + |q| \geq \Omega(P) - \beta|k| \geq E_0(P) - \beta|k|$. By (110), (111), (112), and $E_0(P) = \Omega(P) \leq O_\beta$,

$$\begin{aligned}&E_g(p-k) + |k| - E_g(P) \\ &\geq (1-|g|)(E_0(P-k) - E_0(P)) - C|g| + |k| - |g|E_0(P) \\ &\geq (1-\beta)^2/3B - |g|(C+O_\beta) > 0\end{aligned}$$

where $|k| \geq (1-\beta)/(3B)$ and $|g| < (1-\beta)^2/(3B(C+O_\beta))$ was used in the last line. □

To prove that $E_g(P) = \inf \sigma(H_g(P))$ is an eigenvalue of the Hamiltonian $H_g(P)$ we first show the corresponding result for the modified Hamiltonian

$$H_{\rm mod}(P) = \Omega(P - \mathrm{d}\Gamma(k)) + \mathrm{d}\Gamma(\omega) + g\phi(\kappa_\sigma)$$

introduced in Section 5.

Lemma 36. *Assume Hypotheses 0, 1, and 3. Let $E_{mod}(P) := \inf \sigma(H_{mod}(P))$, and $\Delta(P) := \inf_k(E_{mod}(P - k) + \omega(k) - E_{mod}(P))$. Then*

$$\inf \sigma_{\rm ess}(H_{mod}(P)) \geq E_{mod}(P) + \Delta(P).$$

In particular, if $\Delta(P) > 0$ then $E_{mod}(P)$ is an eigenvalue of $H_{mod}(P)$.

Remark. The assumption that $\Delta(P) > 0$ will be derived from Hypothesis 3 in the proof of Theorem 37 below.

Proof. Let $\lambda \in \sigma_{ess}(H_{\rm mod}(P))$. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D(H_{\rm mod}(P))$, $\|\varphi_n\| = 1$, such that $\|(H_{\rm mod}(P) - \lambda)\varphi_n\| \to 0$ and $\varphi_n \rightharpoonup 0$ (weakly) as $n \to \infty$. Hence

$$\lambda = \lim_{n\to\infty} \langle \varphi_n, H_{\rm mod}(P)\varphi_n\rangle.$$

To estimate $\langle \varphi_n, H_{\rm mod}(P)\varphi_n\rangle$ from below, we need to localize the photons. Pick $j_0, j_\infty \in C^\infty(\mathbb{R}^3)$ with $j_0^2 + j_\infty^2 = 1$, $j_0(y) = 1$ for $|y| \leq 1$ and $j_0(y) = 0$ for $|y| \geq 2$. Given $R > 0$ set $j_{\sharp,R}(y) = j_\sharp(y/R)$ where $\sharp = 0$ or ∞. Let $j_R : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ be defined by $h \mapsto (j_{0,R}h, j_{\infty,R}h)$ and let $j_{x,R}$ be defined in a similar way with $j_\sharp(y)$ replaced by $j_\sharp(y - x)$. By Lemma 42

$$\begin{aligned} &\mathrm{esssup}_P \|[H_{\rm mod}(P) - \breve{\Gamma}(j_R)^*\tilde{H}_{\rm mod}(P)\breve{\Gamma}(j_R)](N+1)^{-1}\| \\ &\qquad = \|[H - \breve{\Gamma}(j_{x,R})^*\tilde{H}\breve{\Gamma}(j_{x,R})](N+1)^{-1}\| = O(R^{-1}) \qquad \text{as } R \to \infty, \end{aligned} \tag{113}$$

where

$$\tilde{H}_{\rm mod}(P) = \Omega(P - \mathrm{d}\Gamma(k) \otimes 1 - 1 \otimes \mathrm{d}\Gamma(k)) + \mathrm{d}\Gamma(\omega) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega) + g\phi(\kappa_\sigma) \otimes 1.$$

In (113) we may replace "essup$_P$" by "sup$_P$" because $\|[H_{\rm mod}(P) - \breve{\Gamma}(j_R)^*\tilde{H}_{\rm mod}(P)\breve{\Gamma}(j_R)](N + 1)^{-1}\|$ is continuous as a function of P. Using that $\sum_{i=1}^N \omega(k_i) \geq \omega(\sum_{i=1}^N k_i)$, by Hypothesis 3, and the definition of $\Delta(P)$, we arrive at the lower bound

$$\tilde{H}_{\rm mod}(P) \geq E_{\rm mod}(P) + \Delta(P) - \Delta(P)E_{\{0\}}(N_\infty),$$

which, in conjunction with (113) and $\breve{\Gamma}(j_R)^*E_{\{0\}}(N_\infty)\breve{\Gamma}(j_R) = \Gamma(j_{0,R}^2)$, shows that

$$\begin{aligned} \langle \varphi_n, H_{\rm mod}(P)\varphi_n\rangle &= \langle \varphi_n, \breve{\Gamma}(j_R)^*\tilde{H}_{\rm mod}(P)\breve{\Gamma}(j_R)\varphi_n\rangle + O(R^{-1}) \\ &\geq E_{\rm mod}(P) + \Delta(P) - \langle \varphi_n, \Gamma(j_{0,R}^2)\varphi_n\rangle\Delta(P) + O(R^{-1}) \end{aligned}$$

where $O(R^{-1})$ is independent of n. Now let $n \to \infty$ and observe that $\Gamma(j_{0,R}^2)(H_{\rm mod}(P) + i)^{-1}$ is compact to get

$$\lambda \geq E_{\rm mod}(P) + \Delta(P) + O(R^{-1}) \qquad \text{for all } R > 0.$$

Letting $R \to \infty$ this proves the theorem. □

Theorem 37. *Assume Hypotheses 0 – 3. Suppose $\beta < 1$ and $|g| < g_\beta$, with g_β defined by Lemma 35.*

i) If $\Omega(P) \leq O_\beta$ then $E_g(P) = E_{mod}(P)$ and $E_g(P)$ is an eigenvalue of $H_g(P)$.

ii) Suppose $\Omega(P) \leq O_\beta$. If $\psi_P \in \mathcal{F}$ is a ground state of $H_g(P)$ or of $H_{mod}(P)$, then it belongs to $\mathrm{Ran}\Gamma(\chi_i)$. In particular, by i), ψ_P is ground state of $H_g(P)$ if and only if it is a ground state of $H_{mod}(P)$.

iii) The mapping $P \mapsto E_g(P)$ is twice continuously differentiable on $\{P \in \mathbb{R}^3 | \Omega(P) \leq O_\beta\}$.

Proof. Recall from the proof of Theorem 7, that $\mathcal{F} \cong \oplus_{n\geq 0}\mathcal{F}_{s,n}$ where each subspace $\mathcal{F}_{s,n}$ is invariant under $H_g(P)$ and that on $\mathcal{F}_{s,n} = L^2_s(B_\sigma(0)^{\times n}, dk_1 \ldots dk_n; \mathcal{F}_i)$ the operator $H_g(P)$ is given by

$$(H_g(P)\psi)(k_1, \ldots, k_n) = H_P(k_1, \ldots, k_n)\psi(k_1, \ldots, k_n),$$

where

$$\begin{aligned} H_P(k_1, \ldots, k_n) &= H_g(P - k_1 \ldots - k_n) + |k_1| + \ldots + |k_n| \\ &> H_g(P) \qquad \text{if } (k_1, \ldots, k_n) \neq (0, \ldots, 0) \end{aligned} \tag{114}$$

as an operator inequality on $\mathcal{F}_i$. In the last equation we used that $\Omega(P-k) + |k| > \Omega(P)$ by assumption and Hypothesis 2.

i) Inequality (114) proves that

$$\inf \sigma(H_g(P)\lceil \mathcal{F}_{s,n}) \geq \inf \sigma(H_g(P)\lceil \mathcal{F}_i) = \inf \sigma(H_{\mathrm{mod}}(P)\lceil \mathcal{F}_i) \geq E_{\mathrm{mod}}(P)$$

for each $n \in \mathbb{N}$. This shows that $E_g(P) \geq E_{\mathrm{mod}}(P)$ and hence that $E_g(P) = E_{\mathrm{mod}}(P)$. We next verify that $\Delta(P) > 0$ in Lemma 36. In fact, $\inf_{|k|\geq\sigma/4}(E_{\mathrm{mod}}(P-k) + \omega(k) - E_{\mathrm{mod}}(P)) \geq \inf_{|k|\geq\sigma/4}(E_g(P-k) + |k| - E_g(P)) > 0$ by Lemma 35 while, for $|k| \leq \sigma/4$, by (114), $E_{\mathrm{mod}}(P-k) + \omega(k) - E_{\mathrm{mod}}(P) \geq \sigma/2 - |k| \geq \sigma/4$. Hence, by Lemma 36, $E_{\mathrm{mod}}(P)$ is an eigenvalue of $H_{\mathrm{mod}}(P)$, and that $E_g(P)$ is an eigenvalue will now follow from ii) because $H_{\mathrm{mod}}(P) = H_g(P)$ on $\mathrm{Ran}\Gamma(\chi_i)$.

ii) By (114), $H_P(k_1, \ldots, k_n) > E_g(P)$ if $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$. This shows that any hypothetical eigenvector of $H_g(P)$ with eigenvalue $E_g(P)$ belongs to $\mathrm{Ran}\Gamma(\chi_i)$. The corresponding result for $H_{\mathrm{mod}}(P)$ follows from an inequality similar to (114) for $H_{\mathrm{mod}}(P)$.

iii) This statement follows by analytic perturbation theory, because $E_g(P) = E_{\mathrm{mod}}(P)$, and because $E_{\mathrm{mod}}(P)$ is an isolated eigenvalue of $H_{\mathrm{mod}}(P)$.

□

Lemma 38. *Assume Hypotheses 0–2 are satisfied. Suppose that $\Omega(P) \leq O_\beta$ for some $\beta < 1$ (see Hypothesis 2 for the definition of O_β) and that $E_g(P) = \inf \sigma(H_g(P))$ is an eigenvalue of $H_g(P)$. Then $E_g(P)$ is a simple eigenvalue.*

Proof. If $g = 0$ (or if $\kappa_\sigma(k) = 0$ a.e.) the lemma is true, under our assumptions, because the only ground state of $H_{g=0}(P)$ is the vacuum. In fact, in this case $H_g(P)$ commutes with N and the absence of ground state vectors in the n particle sector, for any $n > 0$, can easily be proven using the equation

$$\Omega(P - k_1 - \dots k_n) \geq \Omega(P) - \beta|k_1| - \dots - \beta|k_n|$$

with $\beta < 1$ (see the remark after Hypothesis 2). Thus, without loss of generality we can assume that $g \neq 0$ and that the set $\{k \in \mathbb{R}^3 : \kappa_\sigma(k) \neq 0\}$ has positive measure. We consider here the case $g > 0$. The proof for $g < 0$ is then similar. Suppose that $\psi = \{f^{(n)}(k_1, \dots, k_n)\}_{n=0}^\infty \in \mathcal{F}$ is an eigenvector of $H_g(P)$ corresponding to the eigenvalue $E_g(P)$. Then we have

$$\begin{aligned}\langle \psi, H_g(P)\psi\rangle = &\sum_{n=0}^\infty \int dk_1 \dots dk_n |f^{(n)}(k_1, \dots, k_n)|^2 \left\{\Omega(P - \sum_{i=1}^n k_i) + \sum_{i=1}^n |k_i|\right\} \\ &+ 2g\,\mathrm{Re} \sum_{n=0}^\infty \sqrt{n+1} \int dk_1 \dots dk_n\, \overline{f^{(n)}(k_1, \dots k_n)} \int dk\, \kappa_\sigma(k) f^{(n+1)}(k, k_1, \dots, k_n).\end{aligned}$$

Now define

$$g^{(n)}(k_1, \dots k_n) = (-1)^n |f^{(n)}(k_1, \dots k_n)|$$

and set $\tilde\psi = \{g^{(n)}(k_1, \dots, k_n)\}_{n=0}^\infty$. Then $\|\tilde\psi\| = \|\psi\|$ and since $\kappa_\sigma \geq 0$ we have

$$\begin{aligned}\langle \tilde\psi, H_g(P)\tilde\psi\rangle = &\sum_{n=0}^\infty \int dk_1 \dots dk_n |f^{(n)}(k_1, \dots, k_n)|^2 \left\{\Omega(P - \sum_{i=1}^n k_i) + \sum_{i=1}^n |k_i|\right\} \\ &- 2g\,\mathrm{Re} \sum_{n=0}^\infty \sqrt{n+1} \int dk_1 \dots dk_n\, |f^{(n)}(k_1, \dots k_n)| \\ &\times \int dk\, \kappa_\sigma(k) |f^{(n+1)}(k, k_1, \dots, k_n)| \\ \leq &\ \langle \psi, H_g(P)\psi\rangle,\end{aligned}$$

where the equality holds if and only if there is some real θ with

$$g^{(n)}(k_1, \dots, k_n) = e^{i\theta} f^{(n)}(k_1, \dots, k_n), \quad \text{for all} \quad n \geq 0. \tag{115}$$

Since ψ is a ground state vector for $H_g(P)$, Eq. (115) has to be satisfied.

Now suppose that $\psi_1 = \{f_1^{(n)}(k_1, \dots, k_n)\}_{n=0}^\infty$ and $\psi_2 = \{f_2^{(n)}(k_1, \dots, k_n)\}_{n=0}^\infty$ are two orthonormal ground state vectors of $H_g(P)$. Then, by (115),

$$\begin{aligned}&f_1^{(n)}(k_1, \dots, k_n) = e^{i\theta_1} (-1)^n |f_1^{(n)}(k_1, \dots, k_n)| \quad \text{and} \\ &f_2^{(n)}(k_1, \dots, k_n) = e^{i\theta_2} (-1)^n |f_2^{(n)}(k_1, \dots, k_n)|,\end{aligned}$$

for some constants θ_1, θ_2 and thus

$$\begin{aligned}0 = \langle \psi_1, \psi_2\rangle &= \sum_{n=0}^\infty \int dk_1 \dots dk_n \overline{f_1^{(n)}(k_1, \dots, k_n)} f_2^{(n)}(k_1, \dots, k_n) \\ &= e^{i(\theta_2 - \theta_1)} \sum_{n=0}^\infty \int dk_1 \dots dk_n |f_1^{(n)}(k_1, \dots, k_n)|\, |f_2^{(n)}(k_1, \dots, k_n)|.\end{aligned} \tag{116}$$

This implies, in particular, that $f_1^{(0)} \cdot f_2^{(0)} = 0$. We claim that this is not possible. In fact, let $\psi = \{f^{(n)}(k_1, \ldots, k_n)\}_{n=0}^{\infty}$ be an eigenvector of $H_g(P)$, and suppose that $f^{(n)} = 0$ for all $n < n_0$ for some $n_0 > 0$, and that $f^{(n_0)} \neq 0$, that is, $f^{(n_0)}(k_1, \ldots k_{n_0}) \neq 0$ on a set G of positive measure. Since $f^{(n_0)}(k_1, \ldots k_n) = 0$ unless $k_i \in \operatorname{supp} \kappa_\sigma$, for all $i = 1, \ldots n_0$ (this can be proved in the same way as the absence of soft bosons in the ground state, see Theorem 37), the set G must (essentially) belong to $(\operatorname{supp} \kappa_\sigma)^{\times n_0}$. Using that $\kappa_\sigma(k) \geq 0$ and that $f^{(n_0)}(k_1, \ldots, k_{n_0}) = (-1)^{n_0} e^{i\theta} |f^{(n_0)}(k_1, \ldots, k_{n_0})|$ it follows that

$$\begin{aligned}(H_g(P)\psi)^{(n_0-1)}(k_1, \ldots k_{n_0-1}) &= (ga(\kappa_\sigma)\psi)^{(n_0-1)}(k_1, \ldots k_{n_0-1}) \\ &= g\sqrt{n_0} \int dk\, \kappa_\sigma(k) f^{(n_0)}(k, k_1, \ldots, k_{n_0-1}) \neq 0,\end{aligned}$$

which is in contradiction with $(H_g(P)\psi)^{(n_0-1)} = Ef^{(n_0-1)} = 0$. Hence $n_0 = 0$ and $f^{(0)} \neq 0$. Thus Eq. (116) cannot be true. $\square$

The following Lemma is needed to apply Theorem 37 in cases where an upper bound on $E_g(P)$, rather than $\Omega(P)$, is given.

Lemma 39. *Suppose $\beta \leq 1$ and $\Sigma < O_\beta$. If $|g| \leq (O_\beta - \Sigma)/(O_\beta + C)$ and $E_g(P) \leq \Sigma$, then $\Omega(P) \leq O_\beta$.*

Proof. Recall from (103) that

$$E_g(P) \geq (1 - |g|)E_0(P) - C|g|$$

for all $P \in \mathbb{R}^3$ and all g. Hence $E_g(P) \leq \Sigma$ and $|g| \leq (O_\beta - \Sigma)/(O_\beta + C) < 1$ imply that

$$E_0(P) \leq \frac{\Sigma + C|g|}{1 - |g|} \leq O_\beta$$

It remains to prove that $E_0(P) \leq O_\beta$ implies that $\Omega(P) \leq O_\beta$ for $\beta \leq 1$. This is fairly obvious from $E_0(P) = \inf_k(\Omega(P - k) + |k|)$ and a sketch of $E_0(P)$ for a typical Ω. We nevertheless give an analytical proof. Since $\Omega(P) \leq O_{\beta=1}$ implies that $E_0(P) = \Omega(P)$ it suffices to consider the case $\beta = 1$. Let $A := \{P : \Omega(P) \leq O_{\beta=1}\} \neq \emptyset$. We derive a contradictions from the two assumptions $P \notin A$ and $E_0(P) \leq O_{\beta=1}$. Let $d := \operatorname{dist}(P, A) > 0$, let k be any vector with $P - k \in A$ and choose a point P' on the intersection of ∂A and the line segment from $P - k$ to P. Then $\Omega(P') = O_{\beta=1}$ and hence

$$\begin{aligned}\Omega(P - k) &\geq \Omega(P') - |P' - (P - k)| \\ &= O_{\beta=1} - (|k| - |P - P'|) \\ &\geq E_0(P) - |k| + d.\end{aligned}$$

Using again that $E_0(P) \leq O_{\beta=1}$ and the above inequality we get

$$\begin{aligned}E_0(P) &= \min_k(\Omega(P - k) + |k|) \\ &= \min_{k:(P-k)\in A}(\Omega(P - k) + |k|) \geq E_0(P) + d,\end{aligned}$$

a contradiction. $\square$

D.2 Virial Theorem for the modified Hamiltonian

Let $A_{\text{mod}} = \mathrm{d}\Gamma(a)$ where $a = 1/2(\nabla\omega \cdot y + y \cdot \nabla\omega)$ and define the commutator $[iH_{\text{mod}}(P), A_{\text{mod}}]$ by the quadratic form

$$\langle\varphi, [iH_{\text{mod}}(P), A_{\text{mod}}]\varphi\rangle := \langle\varphi, \mathrm{d}\Gamma(|\nabla\omega|^2)\varphi\rangle - \langle\nabla\Omega(P - \mathrm{d}\Gamma(k))\varphi, \mathrm{d}\Gamma(\nabla\omega)\varphi\rangle - \langle\varphi, \phi(ia\kappa_\sigma)\varphi\rangle$$

for $\varphi \in D(H_{\text{mod}}(P))$.

Lemma 40 (Virial theorem). *Let Hypothesis 0 be satisfied. If φ is an eigenvector of $H_{mod}(P)$ then*

$$\langle\varphi, [iH_{mod}(P), A_{mod}]\varphi\rangle = 0.$$

Proof. We adapt the strategy used to prove Lemma 3 in [FGS01] to the present situation. Let $\varepsilon > 0$ and define $y_\varepsilon = y/(1 + \varepsilon y^2)$, $a_\varepsilon = 1/2(\nabla\omega \cdot y_\varepsilon + y_\varepsilon \cdot \nabla\omega)$ and $A_\varepsilon = \mathrm{d}\Gamma(a_\varepsilon)$. The subspace $\mathcal{D} = \{\varphi \in \mathcal{F}_0 : \varphi_n \in C_0^\infty(\mathbb{R}^{3n}, dk_1 \dots dk_n)\}$ is a core of $\Omega(P - P_f) + \mathrm{d}\Gamma(\omega)$, and hence it is also a core of $H_{\text{mod}}(P)$. On $\mathcal{D}$

$$i\langle H_{\text{mod}}(P)\varphi, A_\varepsilon\varphi\rangle - i\langle A_\varepsilon\varphi, H_{\text{mod}}(P)\varphi\rangle = \langle\varphi, \big\{[i\Omega(P - P_f), A_\varepsilon] + \mathrm{d}\Gamma(i[\omega, a_\varepsilon]) - \phi(a_\varepsilon\kappa_\sigma)\big\}\varphi\rangle \tag{117}$$

where

$$2i[\omega, a_\varepsilon] = |\nabla\omega|^2 \frac{1}{1 + \varepsilon y^2} - (\nabla\omega \cdot y)\frac{\varepsilon}{1 + \varepsilon y^2}(y \cdot \nabla\omega + \nabla\omega \cdot y)\frac{1}{1 + \varepsilon y^2} + \text{h.c.}$$

and, on $\otimes_s^n L^2(\mathbb{R}^3, dk)$,

$$\begin{aligned} 2i[\Omega(P - P_f), A_\varepsilon] \;=\; & -\sum_{i=1}^n \nabla\omega(k_i) \cdot \nabla\Omega(P - P_f)\frac{1}{1 + \varepsilon y_i^2} \\ & + \nabla\omega(k_i) \cdot y_i \frac{\varepsilon}{1 + \varepsilon y_i^2}(y_i \cdot \nabla\Omega(P - P_f) + \nabla\Omega(P - P_f) \cdot y_i)\frac{1}{1 + \varepsilon y_i^2} \\ & + \text{h.c.} \end{aligned}$$

Since $\mathcal{D}$ is a core of $H_{\text{mod}}(P)$, since A_ε is bounded w.r.to $H_{\text{mod}}(P)$ and the quadratic for on the right side of (117) is form bounded with respect to $H_{\text{mod}}(P)^2$, this equation carries over to all $\varphi \in D(H_{\text{mod}}(P))$. If φ is an eigenvector of $H_{\text{mod}}(P)$ then the left side vanishes because A_ε is symmetric, and thus it remains to show that the right side converges to $[iH_{\text{mod}}(P), A_{\text{mod}}]$ as $\varepsilon \to 0$. This is done by repeated application of Lebesque's dominated convergence theorem, see [FGS01] for more details. □

E Number–Energy Estimates.

In this section we consider the modified Hamiltonian

$$H_{\text{mod}} = \Omega(p) + \mathrm{d}\Gamma(\omega) + g\phi(G_x)$$

introduced in Section 5, where the dispersion relation ω satisfies Hypothesis 3. We use the notation $H \equiv H_{\text{mod}}$. Thanks to the lower bound $\omega(k) \geq \sigma/2 > 0$, one has the operator inequality

$$N \leq aH + b, \tag{118}$$

for some constants a and b. The purpose of this section is to prove that also higher powers of N are bounded with respect to the same powers of H. This easily follows from (118) if the commutator $[N, H]$ is zero, that is, for vanishing interaction. Otherwise it follow from the boundedness of $ad_N^k(H)(H+i)^{-1}$ for all k.

Lemma 41. *Assume the Hypotheses 0, 1 and 3 are satisfied and suppose* $m \in \mathbb{N} \cup \{0\}$.

i) Then uniformly in z, *for* z *in a compact subset of* $\mathbb{C}$,

$$\|(N+1)^{-m}(z-H)^{-1}(N+1)^{m+1}\| = O(|\operatorname{Im} z|^{-m}).$$

ii) $(N+1)^m(H+i)^{-m}$ *is a bounded operator. In particular* $(N+1)^m\chi(H)$ *is bounded, for all* $m \in \mathbb{N}$, *if* $\chi \in C_0^\infty(\mathbb{R})$.

Proof. This lemma follows from Lemma 31 i) and ii) in [FGS01], where it is proved for a class of Hamiltonians which is larger than the one we consider here. Note that Hypothesis 3 in this paper implies Hypothesis (H1) in [FGS01], and that Hypothesis (H1) in [FGS01] is sufficient to prove parts i) and ii) of Lemma 31 in [FGS01]. □

F Commutator Estimates

In this section we consider the modified Hamiltonians $H_{\text{mod}} = \Omega(p) + \mathrm{d}\Gamma(\omega) + g\phi(G_x)$ and $\tilde{H}_{\text{mod}} = H_{\text{mod}} \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega)$ introduced in Section 5. We use the notation $H = H_{\text{mod}}$ and $\tilde{H} = \tilde{H}_{\text{mod}}$.

Let $j_0,\ j_\infty \in C^\infty(\mathbb{R}^d)$ be real-valued with $j_0^2 + j_\infty^2 \le 1$, $j_0(y) = 1$ for $|y| \le 1$ and $j_0(y) = 0$ for $|y| \ge 2$. Given $R > 0$ set $j_{\#,R} = j_\#((x-y)/R)$ and let $j_{R,x} = (j_{0,R}; j_{\infty,R})$ ($j_{R,x}$ is an operator from $L^2(\mathbb{R}^3) \otimes \mathfrak{h}$ to $L^2(\mathbb{R}^3) \otimes (\mathfrak{h} \oplus \mathfrak{h})$).

Lemma 42. *Assume Hypotheses 0,1 and 3 are satisfied. Suppose* $m \in \mathbb{N} \cup \{0\}$, *and* $j_{R,x}$ *is as above. Suppose also that* $\chi, \chi' \in C_0^\infty(\mathbb{R})$. *Then, for* $R \to \infty$,

i) $(N_0 + N_\infty + 1)^m \left(\check{\Gamma}(j_{R,x})H - \tilde{H}\check{\Gamma}(j_{R,x})\right)\chi' = O(R^{-1})$,

ii) $(N_0 + N_\infty + 1)^m \left(\chi(\tilde{H})\check{\Gamma}(j_{R,x}) - \check{\Gamma}(j_{R,x})\chi(H)\right)\chi'(H) = O(R^{-1})$.

Remark: this Lemma also holds if we replace the modified Hamiltonian $H \equiv H_{\text{mod}}$ with the original Hamiltonian H_g and if we restrict the equality to states with no soft bosons, that is to states in the range of the orthogonal projection $\Gamma(\chi_i)$.

Proof. i) From the intertwining relations (25), and (26) we have that

$$\begin{aligned}\check{\Gamma}(j_{R,x})H - \tilde{H}\check{\Gamma}(j_{R,x}) =& \mathrm{d}\check{\Gamma}(j_{R,x}, [j_{R,x}, \omega(k) + \Omega(p)]) \\ &+ [\phi((j_{0,R}-1)G_x) \otimes 1 + 1 \otimes \phi(j_{\infty,R}G_x)]\check{\Gamma}(j_{R,x}).\end{aligned}$$

By Lemma 32, and because of Hypothesis 0 (which guarantees that $\nabla\Omega$ is bounded with respect to H), we have

$$(N_0 + N_\infty + 1)^m \mathrm{d}\check{\Gamma}(j_{R,x}, [j_{R,x}, \omega(k) + \Omega(p)])\chi'(H) = O(R^{-1}).$$

To see that the other two terms lead to contributions of order $O(R^{-1})$ write

$$\begin{aligned}
(N_0+&N_\infty+1)^m[\phi((j_{0,R}-1)G)\otimes 1 + 1\otimes\phi(j_{\infty,R}G)]\\
&=[\phi((j_{0,R}-1)G)\otimes 1 + 1\otimes\phi(j_{\infty,R}G)](N_0+N_\infty+1)^m\\
&\quad+\sum_{l=1}^{m}\binom{m}{l}(-i)^l[\phi(i^l(j_{0,R}-1)G)\otimes 1 + 1\otimes\phi(i^l j_{\infty,R}G)](N_0+N_\infty+1)^{m-l},
\end{aligned}$$

and then use $(N_0+N_\infty+1)^{m-l}\check{\Gamma}(j_{R,x})=\check{\Gamma}(j_{R,x})(N+1)^{m-l}$, the fact that $(N+1)^{m-l}\chi'(H)$ is bounded (see Lemma 41) and Lemma 9.

ii) Let $\tilde{\chi}$ be an almost analytic extension of χ of order m, as defined in Appendix A. Then we have

$$\begin{aligned}
&(N_0+N_\infty+1)^m(\chi(\tilde{H})\check{\Gamma}(j_{R,x})-\check{\Gamma}(j_{R,x})\chi(H))\chi'(H)\\
&\qquad= -\frac{1}{\pi}\int dxdy\,\partial_{\bar{z}}\tilde{\chi}(N_0+N_\infty+1)^m(z-\tilde{H})^{-1}(\tilde{H}\check{\Gamma}(j_{R,x})-\check{\Gamma}(j_{R,x})H)\\
&\qquad\qquad\times\chi'(H)(z-H)^{-1}.
\end{aligned}$$

Then the statement follows by i) because

$$(N_0+N_\infty+1)^m(z-\tilde{H})^{-1}(N_0+N_\infty+1)^{-m+1}=O(|\operatorname{Im} z|^{-m}). \tag{119}$$

□

Now suppose that $j_0, j_\infty\in C^\infty(\mathbb{R}^3)$, with $j_0^2+j_\infty^2\le 1$, $j_0\in C_0^\infty(\mathbb{R}^3)$ and with $j_0(y)=1$ for $|y|<\lambda_0$, for some $\lambda_0>0$. Set $j_{\sharp,R}=j_\sharp(y/R)$ and $j_R=(j_{0,R},j_{\infty,R})$ (note that here the operator j_R does not depend on the electron position x). Suppose moreover that $F\in C_0^\infty(\mathbb{R})$ with $F(s)=0$ for $s>\lambda_1$, for some $\lambda_1<\lambda_0$.

Lemma 43. *Assume that Hypotheses 0, 1 and 3 are satisfied. Suppose that $m\in\mathbb{N}$ and that j_R and F are defined as above and that $f, f'\in C_0^\infty(\mathbb{R})$. Then, if $R\to\infty$,*

i) $(N_0+N_\infty+1)^m\left(\check{\Gamma}(j_R)H-\tilde{H}\check{\Gamma}(j_R)\right)F(|x|/R)(N+1)^{-m-1}=O(R^{-1})$,

ii) $(N_0+N_\infty+1)^m\left(f(\tilde{H})\check{\Gamma}(j_R)-\check{\Gamma}(j_R)f(H)\right)F(|x|/R)f'(H)=O(R^{-1})$.

The proof of the last lemma is very similar to the proof of Lemma 42. The only difference is that now, in order to bound the commutator with the interaction $\phi(G_x)$ we use the space cutoff $F(|x|/t)$ and part ii) of Lemma 9.

G Invariance of Domains

In this section the invariance of the domain of $\mathrm{d}\Gamma(\nabla\omega\cdot(y-x)+(y-x)\cdot\nabla\omega)$ with respect to the action of $f(H)$ for smooth functions f is proven. Here H denotes the modified Hamiltonian $H_{\mathrm{mod}}=\Omega(p)+\mathrm{d}\Gamma(\omega)+g\phi(G_x)$ introduced in Section 5. Moreover we prove in Lemma 44 that the norm of $\mathrm{d}\Gamma(a)f(H)e^{-iHt}\varphi$ can only grow linearly in t if $\varphi\in D(\mathrm{d}\Gamma(a))$. All these results are only used in Section 7.1 to prove the positivity of the asymptotic observable W.

In the following we use the notation $a=1/2\,(\nabla\omega\cdot(y-x)+(y-x)\cdot\nabla\omega)$.

Lemma 44. *Assume Hypotheses 0,1 and 3 are satisfied and let* $f \in C_0^\infty(\mathbb{R})$. *Then* $f(H)D(\mathrm{d}\Gamma(a)) \subset D(\mathrm{d}\Gamma(a))$ *and*

$$\|\mathrm{d}\Gamma(a)e^{-iHt}f(H)\varphi\| \leq C(\|\mathrm{d}\Gamma(a)\varphi\| + t\,\|\varphi\|),$$

for all $t \geq 0$ *and for all* $\varphi \in D(\mathrm{d}\Gamma(a))$.

Proof. First we note, that

$$\begin{aligned} e^{iHt}\mathrm{d}\Gamma(a)e^{-iHt}f(H) - \mathrm{d}\Gamma(a)f(H) &= \int_0^t ds\, e^{iHs}\,[iH, \mathrm{d}\Gamma(a)]\, f(H)e^{-iHs} \\ &= \int_0^t ds\, e^{iHs}\,(\mathrm{d}\Gamma(\nabla\omega\cdot(\nabla\omega - \nabla\Omega)) - \phi(iaG_x))\, f(H)e^{-iHs}. \end{aligned}$$

Since the operator in the integral on the r.h.s. of the last equation is bounded (because of the energy cutoff $f(H)$ and because, by Hypothesis 0, $\nabla\Omega$ is bounded w.r.t. H) it follows that

$$\|\mathrm{d}\Gamma(a)e^{-iHt}f(H)\varphi\| \leq C\left(\|\mathrm{d}\Gamma(a)f(H)\varphi\| + t\|\varphi\|\right). \tag{120}$$

Now we have

$$\mathrm{d}\Gamma(a)f(H)\varphi = f(H)\mathrm{d}\Gamma(a)\varphi + [\mathrm{d}\Gamma(a), f(H)]\,\varphi. \tag{121}$$

To compute the commutator in the last equation we choose an almost analytic extension $\tilde{f}$ of f, and we expand $f(H)$ in an Helffer-Sjöstrand integral (see Appendix A).

$$\begin{aligned} [\mathrm{d}\Gamma(a), f(H)] &= \frac{-1}{\pi}\int dxdy\partial_{\bar z}\tilde{f}\,(z-H)^{-1}[\mathrm{d}\Gamma(a), H](z-H)^{-1} \\ &= \frac{-i}{\pi}\int dxdy\partial_{\bar z}\tilde{f}\,(z-H)^{-1}\mathrm{d}\Gamma(\nabla\omega\cdot(\nabla\omega - \nabla\Omega))(z-H)^{-1} \\ &\quad + \frac{i}{\pi}\int dxdy\partial_{\bar z}\tilde{f}\,(z-H)^{-1}\phi(iaG_x)(z-H)^{-1}. \end{aligned}$$

Both integral on the r.h.s. of the last equation are bounded (because, by Hypothesis 0, $\nabla\Omega$ is bounded w.r.t. H). This together with (121) and (120) completes the proof of the lemma. □

In the following lemma we prove the invariance of the domain of $\mathrm{d}\Gamma(a+1)$ with respect to the action of operators like $\Gamma(\chi(k))$, where χ is a smooth function. This result is used below, in the proof of Lemma 46.

Lemma 45. *Assume Hypothesis 3 is satisfied. Suppose moreover that* $\varphi \in D(\mathrm{d}\Gamma(a+1))$ *and that* $\chi \in C_0^\infty(\mathbb{R}^3)$ *with* $\chi(k) \leq 1$ *for all* $k \in \mathbb{R}^3$. *Then*

$$\|\mathrm{d}\Gamma(a)\Gamma(\chi(k))\varphi\| \leq C\|\mathrm{d}\Gamma(a+1)\varphi\|$$

Proof. For $\varphi \in D(\mathrm{d}\Gamma(a))$ we have

$$\mathrm{d}\Gamma(a)\Gamma(\chi(k))\varphi = \Gamma(\chi(k))\mathrm{d}\Gamma(a)\varphi + \mathrm{d}\Gamma(\chi(k), [a, \chi(k)])\varphi.$$

The lemma follows because

$$[a, \chi(k)] = i\nabla\omega(k)\cdot\nabla\chi(k)$$

is a bounded operator (and thus the operator $\mathrm{d}\Gamma(\chi(k), [a, \chi(k)])$ can be estimated by the number-operator N). □

Next, using Lemma 45, we prove that vectors in the domain of $d\Gamma(a+1)$ are dense in the range of $\Gamma(\chi_i)$, the orthogonal projection onto the subspace of vectors without soft bosons. This is used in the proof of Theorem 27, where the positivity of the asymptotic observable W is proven.

Lemma 46. *Suppose Hypothesis 3 is satisfied and that χ_i is the characteristic function of the set $\{k \in \mathbb{R}^3 : |k| \geq \sigma\}$. Let $\mathcal{D} := D(d\Gamma(a+1))$ and $\mathcal{H}_i = \mathrm{Ran}\Gamma(\chi_i)$. Then the linear space $\mathcal{H}_i \cap \mathcal{D}$ is a dense subspace of $\mathcal{H}_i$.*

Proof. First, we note that $\mathcal{H}_i \cap D(N)$ is dense in $\mathcal{H}_i$. This is clear, since $[N, \Gamma(\chi_i)] = 0$. The lemma follows if we show that $\mathcal{H}_i \cap \mathcal{D}$ is dense in $\mathcal{H}_i \cap D(N)$. To this end choose an arbitrary $\varphi \in \mathcal{H}_i \cap D(N)$. Then, since $\mathcal{D}$ is dense in $\mathcal{H}$, we find a sequence $\varphi_n \in \mathcal{D}$ with $\varphi_n \to \varphi$, as $n \to \infty$. Moreover we find functions $f_n \in \mathcal{C}^\infty(\mathbb{R}^3)$ with $f_n(k) = 0$, if $|k| < \sigma$, and with $f_n \to \chi_i$, as $n \to \infty$, pointwise. Then we define $\psi_n := \Gamma(f_n)\varphi_n$. On the one hand, by Lemma 45, $\psi_n \in \mathcal{H}_i \cap \mathcal{D}$ for all $n \in \mathbb{N}$. On the other hand

$$\begin{aligned}\|\psi_n - \varphi\| = \|\Gamma(f_n)\varphi_n - \varphi\| &\leq \|\Gamma(f_n)(\varphi_n - \varphi)\| + \|(\Gamma(f_n) - \Gamma(\chi_i))\varphi\| \\ &\leq \mathrm{const}\|\varphi_n - \varphi\| + \|(\Gamma(f_n) - \Gamma(\chi_i))\varphi\| \to 0\end{aligned}$$

for $n \to \infty$. In the last step we used that, by assumption, $\varphi \in \mathcal{H}_i \cap D(N)$. □

References

[Amm00] Zied Ammari. Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model. *Math. Phys. Anal. Geom.*, 3(3):217–285, 2000.

[Ara83] Asao Arai. A note on scattering theory in nonrelativistic quantum electrodynamics. *J. Phys. A*, 16(1):49–69, 1983.

[BFS98] V. Bach, J. Fröhlich, and I.M. Sigal. Quantum electrodynamics of confined nonrelativistic particles. *Adv. Math.*, 137(2):299–395, 1998.

[BKZ01] V. Bach, F. Klopp, and H. Zenk. Mathematical analysis of the photoelectric effect. preprint, mp-arc/01-316, 2001.

[BN37] F. Bloch and A. Nordsieck. Note on the radiation field of the electron. *Phys. Rev.*, 52:54–59, 1937.

[Che01] T. Chen. Operator-theoretic infrared renormalization and construction of dressed 1–particle states. preprint, mp-arc/01-310, 2001.

[Dav95] E.B. Davies. The functional calculus. *J. London Math. Soc. (2)*, 52(1):166–176, 1995.

[DG99] J. Dereziński and C. Gérard. Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, 11(4):383–450, 1999.

[DG00] J. Dereziński and C. Gérard. Spectral and scattering theory of spatially cut-off $P(\phi)_2$ Hamiltonians. *Comm. Math. Phys.*, 213(1):39–125, 2000.

[FGS00] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field. *Adv. Math.*, Vol. 164(2): 349–398, 2001.

[FGS01] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic completeness for Rayleigh scattering. 2001. *Ann. Henri Poincaré*, 3: 107–170, 2002.

[Frö73] J. Fröhlich. On the infrared problem in a model of scalar electrons and massless, scalar bosons. *Ann. Inst. H. Poincaré, Sect. A*, XIX(1):1–103, 1973.

[Frö74] J. Fröhlich. Existence of dressed one-electron states in a class of persistent models. *Fortschr. Phys.*, 22:159–198, 1974.

[Gér02] C. Gérard. On the scattering theory of massless Nelson models. *Rev. Math. Phys.*, 14:1165–1280, 2002.

[HS00] W. Hunziker and I.M. Sigal. The quantum N–body problem. *J. Math. Phys.*, 41(6):3448–3510, 2000.

[Jos65] R. Jost. *The general theory of quantized fields*, volume IV. Mark Kac editor. Lectures in Applied Mathematics (Proceedings of the Summer Seminar, Boulder, Colorado, 1960), 1965.

[Nel64] Edward Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Mathematical Phys.*, 5:1190–1197, 1964.

[PF38] W. Pauli and M. Fierz. Zur Theorie der Emission langwelliger Lichtquanten. *Nuovo Cimento*, 15:167–188, 1938.

[Piz00] A. Pizzo. One particle (improper) states and scattering states in Nelson's massless model. 2000. preprint, Los Alamos Preprint Archive, math-ph/0010043.

[Spo97] H. Spohn. Asymptotic completeness for Rayleigh scattering. *J. Math. Phys.*, 38(5):2281–2296, 1997.

[Spo88] H. Spohn. The polaron model at large momentum. *J. Phys. A: Math. Gen.*, 21:1199–1211, 1988.

[SS88] I.M. Sigal and A. Soffer. Local decay and propagation estimates for time–dependent and time–independent Hamiltonians. Princeton University preprint, 1988.

[RS79] M. Reed and B. Simon. Methods of modern mathematical physics: Scattering Theory. Volume 3. 1979. *Academic Press.*

[DG97] J. Dereziński and C. Gérard. Scattering theory of classical and quantum N-particle systems. 1997. *Springer.*

[Ka66] T. Kato Perturbation theory for linear operators. 1966. *Springer-Verlag New York.*

[YFS61] D. Yennie, S. Frautschi, and H. Suura. The infrared divergence phenomena and high–energy processes. *Ann. Phys.*, 13:379–452, 1961.

[SZ76] T. Spencer, and F. Zirilli. Scattering states and bound states in $\lambda\mathcal{P}(\phi)_2$. *Comm. Math. Phys.*, 49(1):1–16, 1976.

[Spe74] T. Spencer The decay of the Bethe-Salpeter kernel in $P(\phi)_2$ quantum field models. *Comm. Math. Phys.*, 44(2):143–164, 1975.