Infrared Catastrophe for Nelson's Model

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Abstract

We study the infrared catastrophe for Nelson's Hamiltonian, which has the external potential in a general class. We show that both, the divergence of the total number of soft photons and non-existence of any ground state, are due to the common reason under general conditions.

1 Introduction

The purpose of this study is to investigate the infrared catastrophe for Nelson's Hamiltonian. This Hamiltonian is of the so-called Nelson's model describing a system of a quantum particle, which moves in the 3-dimensional Euclidean space $\mathbb{R}^3$ under the influence of an external potential, and which interacts with a massless scalar Bose field. Betz *et al.* showed in [8, (6.7)] that the total number of soft bosons for Nelson's Hamiltonian diverges under the infrared singularity condition which is defined concretely later. In their work it is assumed that the external potential is in Kato class. Lőrinczi *et al.* showed in [23, Theorem 4.5] that in the case of spatial dimension 3 there is no ground state of Nelson's Hamiltonian, even under the assumption of a strongly confining external potential. The results in both, [8] and [23], are proved by means of functional integrals. As to operator theoretical method, in [11] Gérard and Deresiński showed non-existence of the ground state for Nelson's Hamiltonian with the strongly confining external potential by amazing simple way. However, following the methods by Lőrinczi *et al.* and Gérard and Deresiński, we are required to invent more suitable technique when we apply the methods to Nelson's Hamiltonian with the Coulomb potential, which has not been achieved yet. Thus, the present paper looks at the problem from a different angle from theirs, namely we use a method of operator theory given by combining Griesemer, Lieb, and Loss' method [16] and Arai, Hiroshima, and the author's [6]. Then, we can improve the result by Lőrinczi *et al.* and Gérard and Deresiński so that our result extends to mathematically general external potentials including the Coulomb potential. As is well known, the potential in quantum electrodynamics (QED) to imply the electromagnetic force as one of theories of four forces is nothing but the Coulomb potential. Thus, when we regard Nelson's model as a model in nonrelativistic QED, it is important that we extends the result about non-existence to the case of Coulomb potential. As a matter of fact, several mathematical techniques were developed in [7, 16] for the treatment of the Coulomb potential in nonrelativistic QED as well as in the theory of Schrödinger operators. We also give an explicit estimate of how the number of soft bosons diverges. The brief statements appears in [20, §2] and we give exact proof of them in this paper. This is our main purpose.

In physics, we expect that such non-existence of the ground state occurs from divergence of the total number of soft bosons. So, there is no ground state in the standard state space constructed from Fock space. Then, another representation for the state space is required, which is known as the Bloch-Nordsieck theorem in physics [9]. In a mathematical treatment this was achieved for a fermion-boson model related to Nelson's by Fröhlich [14]. For its Hamiltonian he proves the non-existence of ground state and constructs the renormalized Hamiltonian which has a ground state, by constructing the Wightman distributions and applying the Wightman reconstruction theorem to them. It is worthy of note that Pizzo developed Fröhlich's work [Pi]. We are interested in proving mathematically that the divergence of the number of soft bosons implies non-existence of the ground state in the standard state space. We tried this in [6] from an operator-theoretical point of view, based on [14, Theorem D] and [1, Theorem 3.3], while we studied a mathematical mechanism of existence of ground states in [4]. However, because we treated general models, so-called generalized spin-boson model, in [6], we could not entirely achieve this goal. In this paper we show that there is the same factor as in the case of the van Hove model, which results in the divergence of the total number of soft bosons and the non-existence of any ground state for Nelson's Hamiltonian under the infrared singularity condition, of which image is explained in the following section. To see it, we invent an identity from the pull-through formula in §5.

2 Overview of Infrared Problem in Nonrelativistic Quantum Electrodynamics

Before we state and prove our main results, a good place to start from is to explain briefly the idea we use in this paper. Here we only explain the idea for non-existence of ground state for Nelson's model. The detailed explanation of mathematical mechanism to show the existence or non-existence of ground state for some models around the Pauli-Fierz model appears in [20, §1.1] from the point of the view of the velocity of the quantized particle. Namely, as is well known, there is apparently the so-called infrared catastrophe for the cross section of a charged particle when it passes through a force field and is deflected at a given angle. For a given energy loss lying the interval $(E\,,\; E+dE)$, the cross section is seemingly proportional to dE/E for small E following a rough estimate of QED. So, its integration of this cross section over a finite interval $(E\,,\; E')$ yields a logarithmic divergence at $E=0$. However, Pauli and Fierz performed a more precise physical investigation and showed that the cross section for the nonrelativistic model in QED is actually proportional to $E^C dE/E$ with a positive number C which is about $\alpha(\Delta v)^2/c^2$ by estimating the velocity of the quantized particle with classical mechanics [27], where α is the fine structure constant, Δv the difference of the initial and final velocities of the particle before and after the interaction of the particle and bosons (photons), respectively, c the speed of the light. Although the word of "velocity" does not appear explicitly in [7, 16], the authors make good use of the concept of the velocity and prove the infrared catastrophe does not occur for the full Hamiltonian of the Pauli-Fierz model, which is explained in [20, §1.1]. We can obtain the similar effect to this C in the total number of soft bosons in order to remove both infrared and ultraviolet cutoffs from Gross-transformed Nelson's Hamiltonian, where we note that it has the same form as the Wick-ordered full Hamiltonian of the Pauli-Fierz model has and that it is *not* unitary equivalent to the original Nelson's Hamiltonian after removing the infrared cutoff. The effect appears in [19, Lemma 7.2 and Corollary 7.4]. We succeeded in deriving the effect from the velocity of the particle. Thus, binding energy for Gross-transformed Nelson's Hamiltonian without both cutoffs can be also investigated in [18]. The method to derive δ can be available to the full Hamiltonian of the Pauli-Fierz model [20, §4]. In a sense, this situation represents that the velocity of the quantized particle for the full Hamiltonian has the gauge potential in it so that the particle feels the Lorentz force, which is also explained in [20, §1.1].

On the other hand, the velocity of the quantized particle for Nelson's Hamiltonian lose the gauge potential in it, so that it cannot feel the Lorentz force though it is a charged particle. This is one of the reasons why Nelson's Hamiltonian meets an infrared catastrophe. Incidentally, as mentioned in [16] a little and written in [20, §1.1], we have a physical difference in the infrared problem if we employ an approximation which breaks the gauge invariance of the energy in the level of quantum mechanics, though we have to fix a gauge actually when we do the second quantization. It could be mathematically interesting to investigate the relation between the infrared problem and the gauge invariance of the energy in the level of quantum mechanics.

More precisely, the Hamiltonian H_κ^{N} of the *Nelson's model* is given by replacing the quantized vector field by the scalar Bose field $\phi_\kappa(x)$ in the linearized Hamiltonian of the Pauli-Fierz model, and moreover by omitting the electron momentum in front of the vector potential in cross terms:

$$H_\kappa^{\mathrm{N}} = \frac{1}{2m}p^2 + V + H_{\mathrm{f}} + q\phi_\kappa(x), \tag{1}$$

where m is the mass of the quantized particle, V an external potential of the particle, and H_{f} and $\phi_\kappa(x)$ the free energy and the field of the scalar bosons given by

$$\begin{aligned} H_{\mathrm{f}} &= \int_{\mathbb{R}^3} \omega(k)a^\dagger(k)a(k)d^3k, & (2)\\ \phi_\kappa(x) &= \int_{\mathbb{R}^3} \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\left(e^{ikx}a(k) + e^{-ikx}a^\dagger(k)\right)d^3k & (3)\end{aligned}$$

with the characteristic function χ_κ of $\kappa \le |k| \le \Lambda$, where κ and Λ are infrared and ultraviolet cutoffs, respectively, and we fix Λ in this paper. In this case then, the velocity is given by $v = p/m$ missing the gauge potential, so Newton's equation of motion, $mi\,[H_\kappa^{\mathrm{N}}, v]$, does not include the Lorentz force. Thus, in passing, since the quantized particle described by Nelson's Hamiltonian does not have the electromagnetic mass [13, §28-3] coming from Lorentz force, we can expect that we do not need mass renormalization for Nelson's model when we remove ultraviolet cutoff, as indeed it is proved in [17, 25] that we need the only self-energy renormalization.

Considering the pull-through formula, we get

$$a(k)\psi_\kappa = -q\frac{\chi_\kappa(k)}{\sqrt{2|k|}}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + |k|\right)^{-1}e^{-ikx}\psi_\kappa \tag{4}$$

for the ground state ψ_κ of H_κ^{N} with $\kappa > 0$. Basically, we can see the same reason of the infrared catastrophe in (4) as for the van Hove Hamiltonian through the approximation $e^{-ikx} \approx 1 - ikx$ and the dipole approximation. Let us now apply the dipole approximation to H_κ^{N}. Then, the dipole approximation implies the decomposition of the Schrödinger operator $H_{\mathrm{at}} := p^2/2m + V$ and the van Hove Hamiltonian H_κ^{vH}. Namely, we obtain the Hamiltonian $H_\kappa^{\mathrm{N,da}} = H_{\mathrm{at}} + H_\kappa^{\mathrm{vH}}$, where $H_\kappa^{\mathrm{vH}} = H_{\mathrm{f}} + q\phi_\kappa(0)$, the Hamiltonian of the van Hove model (or fixed source model). As is well known, the following two conditions are equivalent:

(vH1) H_0^{vH} has no ground state in $\mathcal{F}$,

(vH2) $\langle N_{\mathrm{f}}\rangle \equiv \lim_{\kappa\to 0}\langle \psi_\kappa^{\mathrm{vH}}, N_{\mathrm{f}}\psi_\kappa^{\mathrm{vH}}\rangle = \infty$,

where $\psi_\kappa^{\mathrm{vH}}$ is a normalized ground state of H_κ^{vH}, $\kappa > 0$, and N_{f} the number operator of bosons defined by replacing $\omega(k)$ by 1 in (2). See §8 below.

As a matter of fact, in view of using the pull-through formula, for a normalized ground state $\psi_\kappa^{\mathrm{vH}}$ and the ground state energy E_κ^{vH} of H_κ^{vH}, $\kappa > 0$, we get

$$a(k)\psi_\kappa^{\mathrm{vH}} = -q\frac{\chi_\kappa(k)}{\sqrt{2|k|}}\left(H_\kappa^{\mathrm{vH}} - E_\kappa^{\mathrm{vH}} + |k|\right)^{-1}\psi_\kappa^{\mathrm{vH}} = -q\frac{\chi_\kappa(k)}{\sqrt{2|k|}\,|k|}\psi_\kappa^{\mathrm{vH}}, \tag{5}$$

which implies

$$\langle N_{\mathrm{f}}\rangle_\kappa \equiv \langle \psi_\kappa^{\mathrm{vH}}, N_{\mathrm{f}}\psi_\kappa^{\mathrm{vH}}\rangle = \frac{q^2}{4\pi^2}\log\frac{\Lambda}{\kappa}$$

for the boson number operator N_{f}. Insert the approximation $e^{-ikx} \approx 1 - ikx$ in (4) and compare it with the first equation in (5), then we can see the same reason for the infrared catastrophe. We show the infrared catastrophe for Nelson's model following this idea. The identity from the pull-through formula in §5 is to achieve this idea. Incidentally, we can, of course, apply the operator theoretical method introduced in this paper to H_κ^{vH} and prove that *there exists no ground state for all* $q \neq 0$, though it is the well-known fact without saying here.

3 Main results

The position of the quantum particle is denoted by x, momentum by $p = -i\nabla_x$. We set $\hbar = 1, c = 1$ and $m = 1$ throughout. The particle is coupled with a massless scalar Bose field. In momentum representation the creation and annihilation operators of the field satisfy the standard CCR,

$$[a(k), a^\dagger(k')] = \delta(k - k'),$$
$$[a(k), a(k')] = 0, \quad [a^\dagger(k), a^\dagger(k')] = 0, \quad k, k' \in \mathbb{R}^3.$$

The field energy and cutoff Bose field are given by (2) and (3), respectively, as an operator acting on the symmetric Fock space $\mathcal{F}$ over $L^2 = L^2(\mathbb{R}^3)$. It is well known that $\phi_\kappa(x)$ is a self-adjoint operator on $\mathcal{F}$ for every $x \in \mathbb{R}^3$. We set

$$\lambda_{\kappa,x}(k) = \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\,e^{-ikx} \qquad \text{for } \forall k, x \in \mathbb{R}^3 \text{ and } \kappa \geq 0.$$

Then,

$$\phi_\kappa(x) = a^\dagger(\lambda_{\kappa,x}) + a(\lambda_{\kappa,x}),$$

where

$$a^\dagger(f) = \int a^\dagger(k)f(k)d^3k, \qquad a(f) = \int a(k)f(k)^* d^3k$$

for every $f \in L^2(\mathbb{R}^3)$. $H_{\mathrm{N}} \equiv H_0^{\mathrm{N}}$ has an *infrared singularity* at $k = 0$, i.e.,

$$\lim_{|k|\to 0}\frac{\lambda_{0,x}(k)}{\omega(k)} = \infty \quad \text{and} \quad \frac{\lambda_{0,x}}{\omega} \notin L^2(\mathbb{R}^3),$$

which is called *infrared singularity* condition in [5] (see also [6, (3.5)]). In our situation, we have $\lambda_{\kappa,x}/\omega \in L^2(\mathbb{R}^3)$ for $\kappa > 0$, which is called *infrared regularity* condition. The reason why we operated with the notions of infrared singularity and regularity is explained in §8 below.

Our external potentials include strongly confining ones which are in the same class as the one treated in [2], long and short range potentials which are basically the same ones as those treated in [16]. Thus, we assume either classes below for the external potential V:

(Case 1) (*strongly confining potential*)

(C1-1) The Schrödinger operator $p^2/2 + V$ is self-adjoint on $D(p^2) \cap D(V)$ and bounded from below,

(C1-2) for all $R > 0$,

$$\int_{|x| \le R} |V(x)|^2 d^3x < \infty,$$

(C1-3) there exist positive constants c_1 and c_2 such that

$$|x|^2 \le c_1 V(x) + c_2 \quad \text{for a.e. } x \in \mathbb{R}^3.$$

or

(Case 2) (*long & short range potential*)

(C2-1) For a.e. $x \in \mathbb{R}^3$, $V(x) \le 0$, and for all $R > 0$,

$$\int_{|x| \le R} |V(x)|^2 d^3x < \infty,$$

(C2-2) there exists a positive δ such that $V(x) = \mathcal{O}(|x|^{-\delta})$ as $|x| \to \infty$.

(C2-3) the Schrödinger operator $p^2/2 + V$ has a normalized ground state $\psi_{\rm at}$ with negative ground state energy such that $\psi_{\rm at}(x) > 0$ for a.e. $x \in \mathbb{R}^3$.

We denote the Schrödinger operator with external potential and its ground state energy by $H_{\rm at}$ and $E_{\rm at}$, respectively, i.e.,

$$H_{\rm at} = \frac{1}{2}p^2 + V \;\text{ and }\; E_{\rm at} = \inf \sigma(H_{\rm at}),$$

where $\sigma(T)$ denotes the spectrum of an operator T.

Proposition 3.1 *Assume (Case 2). Then:*

(i) $H_{\rm at}$ *is self-adjoint on* $D(p^2)$.

(ii) V *is infinitesimally* p^2*-compact.*

Proof: By (C2-1) and (C2-2) we can decompose V such that $V = V_{1,\varepsilon} + V_{2,\varepsilon}$ with $V_{1,\varepsilon} \in L^2(\mathbb{R}^3)$, $V_{2,\varepsilon} \in L^\infty(\mathbb{R}^3)$, and $\|V_{2,\varepsilon}\|_{L^\infty} < \varepsilon$. So, (i) and (ii) follow from [28, Theorem X.15] and the famous theorem, [21, Theorem 3.9], respectively. □

The cutoff Nelson Hamiltonian is given by

$$H_\kappa^{\rm N} \;=\; \frac{1}{2}p^2 \otimes 1 + V \otimes 1 + 1 \otimes H_{\rm f} + q\phi_\kappa(x) \qquad (6)$$
$$\text{for } 0 \le \forall\kappa < 1 < \Lambda, \;\; \forall q \in \mathbb{R},$$

acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

We set

$$H_{\rm N} = H_0^{\rm N} \equiv H_\kappa^{\rm N}\lceil_{\kappa=0}$$

and denote ground state energy of $H_\kappa^{\rm N}$ and $H_{\rm N}$ by $E_\kappa^{\rm N}$ and $E_{\rm N}$, respectively, i.e.,

$$E_\kappa^{\rm N} = \inf \sigma(H_\kappa^{\rm N}) \ \text{ and } \ E_{\rm N} = \inf \sigma(H_{\rm N}).$$

In the following we mostly omit the tensor notation $\otimes$.

We define a non-negative free Hamiltonian by

$$H_0 = H_{\rm at} - E_{\rm at} + H_{\rm f}.$$

Applying [2, Proposition 2.1] to $H_\kappa^{\rm N}$, by (C1-1) and Proposition 3.1, we obtain

Proposition 3.2 *H_κ^N, $0 \le \kappa \le \Lambda$, is self-adjoint with $D(H_\kappa^N) = D(H_0) \equiv D(p^2) \cap D(V) \cap D(H_{\rm f})$ under (Case 1), and $D(H_\kappa^N) = D(H_0) \equiv D(p^2) \cap D(H_{\rm f})$ under (Case 2). H_κ^N, $0 \le \kappa \le \Lambda$, is bounded from below for arbitrary values of q. Moreover, H_κ^N, $0 \le \kappa \le \Lambda$, is essentially self-adjoint on each core for H_0.*

For (Case 2), we define a positive constant q_Λ by

$$-E_{\rm at} = \frac{q_\Lambda^2}{2} \int |\lambda_{0,0}(k)|^2 \frac{k^2}{\omega(k) + k^2/2} d^3k.$$

We set $q_\Lambda = \infty$ for (Case 1). Note that q_Λ is independent of κ.

By [29, Theorem 1], we have

Proposition 3.3 *Let us fix $\Lambda > 0$. Then, H_κ^N has a unique ground state ψ_κ for every κ, q with $0 < \kappa < \Lambda$ and $|q| < q_\Lambda$.*

Denote the number operator of bosons by $N_{\rm f}$,

$$N_{\rm f} = \int a^\dagger(k) a(k) d^3k.$$

Theorem 3.4 *(soft-boson divergence). For every q with $0 < |q| < q_\Lambda$, there exists positive constants K_0 and M such that*

$$\begin{aligned} &\left\{ \frac{q^2}{8\pi^2} \left(\log \frac{\Lambda}{\kappa} \right) - M \right\} \|\psi_\kappa\|^2_{\mathcal{H}} \\ &\qquad \le \ \langle \psi_\kappa \, , \, N_{\rm f} \psi_\kappa \rangle_{\mathcal{H}} \ \le \left\{ \frac{q^2}{2\pi^2} \left(\log \frac{\Lambda}{\kappa} \right) + 2M \right\} \|\psi_\kappa\|^2_{\mathcal{H}} \end{aligned} \tag{7}$$

for $\kappa < K_0$.

Theorem 3.5 *(non-existence of ground state for $\kappa = 0$). For every q with $q \neq 0$, $H_N = H_0^N$ has no ground state in $\mathcal{H}$.*

Remark. Recently, C. Gérard and J. Dereziński also succeeded in proving the non-existence of ground state for Nelson's Hamiltonian with (strongly) confining external potentials under the infrared singularity condition by an amazingly simple way [11].

We prove Theorem 3.4 and Theorem 3.5 in Sections 6 and Section 7, respectively.

4 Preliminaries

In (Case 1), as proved in [2, (2.43)], by applying the well-known fact (see e.g., [22]) to the closed operators p^2, $H_{\rm at}$, there exists a non-negative constant $c_{\rm at,1}$ such that

$$\|(\frac{1}{2}p^2+1)\psi\|_{\mathcal{H}} \le c_{\rm at,1}\,(\|\,(H_{\rm at}-E_{\rm at})\,\psi\|_{\mathcal{H}}+\|\psi\|_{\mathcal{H}}) \quad \text{for } \forall\psi\in D(H_{\rm at}).$$

On the other hand, in (Case 2), by Proposition 3.1, p^2 and $H_{\rm at}$ are closed with $D(p^2)=D(H_{\rm at})$. So, in the same way as above, there exists a non-negative constant $c_{\rm at,2}$ such that

$$\|(\frac{1}{2}p^2+1)\psi\|_{\mathcal{H}} \le c_{\rm at,2}\,(\|\,(H_{\rm at}-E_{\rm at})\,\psi\|_{\mathcal{H}}+\|\psi\|_{\mathcal{H}}) \quad \text{for } \forall\psi\in D(H_{\rm at}).$$

Thus, we obtain the following lemma.

Lemma 4.1 *There exists a positive constant $C_{\rm at}$ such that*

$$\|(\frac{1}{2}p^2+1)(H_{\rm at}-E_{\rm at}+1)^{-1}\| \le C_{\rm at}.$$

Moreover, by using Schwarz's inequality, for all $\psi\in D(p^2)$

$$\|p\psi\|^2_{\mathcal{H}} = \langle\psi\,,\,p^2\psi\rangle_{\mathcal{H}} \le 2\|\psi\|_{\mathcal{H}}\|\frac{1}{2}p^2\psi\|_{\mathcal{H}} \le \|\frac{1}{2}p^2\psi\|^2_{\mathcal{H}}+\|\psi\|^2_{\mathcal{H}} \le \|(\frac{1}{2}p^2+1)\psi\|^2_{\mathcal{H}},$$

where we used $p^2\ge 0$, which implies that

$$\|p(\frac{1}{2}p^2+1)^{-1}\| \le 1. \tag{8}$$

Similarly, by Proposition 3.2, H_0 and $H_{\rm N}$ are closed with $D(H_0)=D(H_{\rm N})$. So, for $\kappa=0$ and $q\neq 0$, there exists a non-negative constant c_q such that

$$\|(H_0+1)\psi\|_{\mathcal{H}} \le c_q\,(\|\,(H_{\rm N}-E_{\rm N})\,\psi\|_{\mathcal{H}}+\|\psi\|_{\mathcal{H}}) \quad \text{for } \forall\psi\in D(H_{\rm N}). \tag{9}$$

Thus, we obtain the following lemma.

Lemma 4.2 (i) *For $\kappa=0$ and $q\neq 0$, there exists a positive constant C_q such that*

$$\|(H_0+1)(H_N-E_N+1)^{-1}\| \le \frac{C_q}{2}.$$

(ii) *There exists a constant K_0 with $0<K_0<\Lambda$ such that*

$$\sup_{0\le\kappa\le K_0}\|\,(H_0+1)\,(H^N_\kappa-E^N_\kappa+1)^{-1}\,\| \le C_q. \tag{10}$$

Proof: (i) follows from (9) directly. It is easy to check that there exists a real number z satisfying $z<\sup_{0\le\kappa<\Lambda}E^{\rm N}_\kappa$. By the second resolvent equation (e.g., [22]), we have

$$\begin{aligned}&(H_{\rm N}-z+1)^{-1}-(H^{\rm N}_\kappa-z+1)^{-1}\\&=-q\,(H_{\rm N}-z+1)^{-1}\,\phi_{\le\kappa}(x)\,(H^{\rm N}_\kappa-z+1)^{-1}\,,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\phi_{\le\kappa}(x)&=a^\dagger(\lambda_{\le\kappa,x})+a(\lambda_{\le\kappa,x}),\\ \lambda_{\le\kappa,x}(k)&=\frac{\chi_0(k)-\chi_\kappa(k)}{\sqrt{2\omega(k)}}\,e^{-ikx}.\end{aligned}$$

Set

$$
\begin{aligned}
H_{\mathrm{f}}^{\leq\kappa} &= \int_{|k|\leq\kappa} \omega(k)a^{\dagger}(k)a(k)d^3k, \\
H_{\mathrm{f}}^{>\kappa} &= \int_{|k|>\kappa} \omega(k)a^{\dagger}(k)a(k)d^3k, \\
H_{\mathrm{N}}^{>\kappa} &= H_{\mathrm{at}} + H_{\mathrm{f}}^{>\kappa} + q\phi_{\kappa}(x).
\end{aligned}
$$

Then, it is well known that H_{κ}^{N} is unitary equivalent to $H_{\mathrm{f}}^{\leq\kappa}\otimes I + I\otimes H_{\mathrm{N}}^{>\kappa}$. In the argument below we also omit the tensor notation $\otimes$. Note that

$$
E_{\kappa}^{\mathrm{N}} = \inf\sigma\left(H_{\mathrm{f}}^{\leq\kappa}\right) + \inf\sigma\left(H_{\mathrm{N}}^{>\kappa}\right) = \inf\sigma\left(H_{\mathrm{N}}^{>\kappa}\right). \tag{12}
$$

By (11) and (12) we get

$$
\begin{aligned}
&\|\,(H_{\mathrm{N}} - z + 1)^{-1} - (H_{\kappa}^{\mathrm{N}} - z + 1)^{-1}\,\| \\
&\leq \frac{|q|}{|E_{\mathrm{N}} - z + 1|}\|\phi_{\leq\kappa}(x)(H_{\mathrm{f}}^{\leq\kappa}+1)^{-1}(H_{\mathrm{f}}^{\leq\kappa}+1)(H_{\kappa}^{\mathrm{N}} - z + 1)^{-1}\| \\
&\leq \frac{|q|}{|E_{\mathrm{N}} - z + 1|}\left(2\|\lambda_{\leq\kappa,x}/\sqrt{\omega}\|_{L^2} + \|\lambda_{\leq\kappa,x}\|_{L^2}\right) \\
&\qquad\qquad \times\left\|\left(H_{\mathrm{f}}^{\leq\kappa}+1\right)\left(H_{\mathrm{f}}^{\leq\kappa}+1+H_{\mathrm{N}}^{>\kappa} - z\right)^{-1}\right\| \\
&\leq \frac{|q|}{|E_{\mathrm{N}} - z + 1|}\left(\frac{\sqrt{\kappa}}{\pi} + \frac{\kappa}{2\sqrt{2}\,\pi}\right).
\end{aligned}
$$

Thus, $H_{\kappa}^{\mathrm{N}} \to H_{\mathrm{N}}$ as $\kappa \to 0$ in the norm-resolvent sense, which implies that

$$
\begin{aligned}
&(E_{\kappa}^{\mathrm{N}} - z + 1)^{-1} = \|\,(H_{\kappa}^{\mathrm{N}} - z + 1)^{-1}\,\| \\
&\longrightarrow \|\,(H_{\mathrm{N}} - z + 1)^{-1}\,\| = (E_{\mathrm{N}} - z + 1)^{-1} \quad \text{as } \kappa \to 0.
\end{aligned}
$$

Thus, we have

$$
\lim_{\kappa\to 0} E_{\kappa}^{\mathrm{N}} = E_{\mathrm{N}}. \tag{13}
$$

In the same way as in (11), we have

$$
\begin{aligned}
&(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1} - (H_{\kappa}^{\mathrm{N}} - E_{\kappa}^{\mathrm{N}} + 1)^{-1} \\
&= -\,(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1}\left\{(E_{\mathrm{N}} - E_{\kappa}^{\mathrm{N}}) + q\phi_{\leq\kappa}(x)\right\}(H_{\kappa}^{\mathrm{N}} - E_{\kappa}^{\mathrm{N}} + 1)^{-1},
\end{aligned}
$$

which implies by (i), (12), and (13) that

$$
\begin{aligned}
&\|\,(H_0+1)\,(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1} - (H_0+1)\,(H_{\kappa}^{\mathrm{N}} - E_{\kappa}^{\mathrm{N}} + 1)^{-1}\,\| \\
&\leq \|\,(H_0+1)\,(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1}\,\| \\
&\qquad\qquad \times\left\{|E_{\mathrm{N}} - E_{\kappa}^{\mathrm{N}}| + |q|\left\|\phi_{\leq\kappa}(x)\,(H_{\kappa}^{\mathrm{N}} - E_{\kappa}^{\mathrm{N}} + 1)^{-1}\right\|\right\} \\
&\leq \frac{C_q}{2}\left\{|E_{\mathrm{N}} - E_{\kappa}^{\mathrm{N}}| + |q|\left\|\phi_{\leq\kappa}(x)\left(H_0^{\leq\kappa}+1\right)^{-1}\right\|\right. \\
&\qquad\qquad \left.\times\left\|\left(H_0^{\leq\kappa}+1\right)\left(H_{\mathrm{f}}^{\leq\kappa}+1+H_{\mathrm{N}}^{>\kappa} - \inf\sigma\left(H_{\mathrm{N}}^{>\kappa}\right)\right)^{-1}\right\|\right\} \\
&\leq \frac{C_q}{2}\left\{|E_{\mathrm{N}} - E_{\kappa}^{\mathrm{N}}| + |q|\left(\frac{\sqrt{\kappa}}{\pi} + \frac{\kappa}{2\sqrt{2}\,\pi}\right)\right\}.
\end{aligned}
$$

Thus, we have

$$\lim_{\kappa\to 0} \|\,(H_0+1)\,(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)^{-1}\,\| = \|\,(H_0+1)\,(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1}\,\|,$$

which implies that there exists a positive constant K_0' such that

$$\|\,(H_0+1)\,(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)^{-1}\,\| \le \|\,(H_0+1)\,(H_{\mathrm{N}} - E_{\mathrm{N}} + 1)^{-1}\,\| + \frac{C_q}{2}$$

for $\kappa < K_0'$. So, by (i) we have $\|\,(H_0+1)\,(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)^{-1}\,\| \le C_q$ for $\kappa < K_0'$. Thus, define K_0 such that $K_0 < \min\{K_0',\, \Lambda\}$. Then we obtain (ii). $\square$

We obtain the following lemma from Lemma 4.2.

Lemma 4.3 *For every $g \neq 0$ and arbitrary κ, ϵ with $0 < \epsilon$ and $0 \le \kappa < K_0$,*

$$\|(H_0+1)(H_\kappa^N - E_\kappa^N + \epsilon)^{-1}\| \le \frac{C_q}{\min\{\epsilon,\ 1\}}. \tag{14}$$

Proof: (14) follows from (10) and

$$\|(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \epsilon)^{-1}\| \le \begin{cases} \epsilon^{-1} & \text{if } \epsilon < 1, \\ 1 & \text{if } \epsilon \ge 1. \end{cases}$$

$\square$

Lemma 4.4 *Assume (Case 1). Let ψ be in $D(H_0^{1/2})$. Then, $\psi \in D(|x|)$ and*

$$\||x|\psi\|_{\mathcal{H}}^2 \le c_1\|H_0^{1/2}\psi\|_{\mathcal{H}}^2 + c_2\|\psi\|_{\mathcal{H}}^2 \le c_1\|H_0\psi\|_{\mathcal{H}}^2 + (c_1+c_2)\|\psi\|_{\mathcal{H}}^2. \tag{15}$$

In particular, for $0 \le \kappa \le K_0$

$$\||x|\psi_\kappa\|_{\mathcal{H}}^2 \le \left(c_1 C_g^2 + c_1 + c_2\right)\|\psi_\kappa\|_{\mathcal{H}}^2. \tag{16}$$

Proof: We obtain the first statement, $\psi \in D(|x|)$, and the first inequality of (15) in the same way as in [2, Lemma 4.6]. Since by Schwarz' inequality we get

$$\begin{aligned} &\|H_0^{1/2}\psi\|_{\mathcal{H}}^2 = \langle\psi\,,\, H_0\psi\rangle_{\mathcal{H}} \le \|\psi\|_{\mathcal{H}}\|H_0\psi\|_{\mathcal{H}} \le \|H_0\psi\|_{\mathcal{H}}^2 + \frac{1}{4}\|\psi\|_{\mathcal{H}}^2 \\ &\le \|H_0\psi\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2 \end{aligned}$$

for $\psi \in D(H_0)$, we obtain the second inequality in (15). By (15) and Lemma 4.3 we have

$$\begin{aligned} &\||x|\psi_\kappa\|_{\mathcal{H}}^2 \\ &\le c_1\|(H_0+1)(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)^{-1}(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)\psi_\kappa\|_{\mathcal{H}}^2 \\ &\quad + (c_1+c_2)\|\psi_\kappa\|_{\mathcal{H}}^2 \\ &= c_1\|(H_0+1)(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + 1)^{-1}\psi_\kappa\|_{\mathcal{H}}^2 + (c_1+c_2)\|\psi_\kappa\|_{\mathcal{H}}^2, \end{aligned}$$

which implies (16). $\square$

Let $E_\kappa^{\mathrm{N}\,V=0} = \inf \sigma\left(H_\kappa^{\mathrm{N}\,V=0}\right)$, where the superscript means that in (6) the external potential V is omitted. The (positive) binding energy is defined by

$$E_\kappa^{\mathrm{bin}} = E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}.$$

Following the proof of [16, Theorem 3.1], we show that $E_\kappa^{\mathrm{bin}} > 0$ in the same way as in [19, Proposition 4.4].

Proposition 4.5 *(strict positivity of binding energy). Assume (Case 2). Fix κ with $0 \le \kappa < \Lambda$. Suppose that H_κ^N has a ground state ψ_κ. Then,*

$$E_\kappa^{\mathrm{bin}} \ge -E_{\mathrm{at}} > 0.$$

Proof: Let ψ_{at} be the ground state of H_{at}, $\psi_{\mathrm{at}} > 0$ and $\|\psi_{\mathrm{at}}\|_{L^2} = 1$. For every $\epsilon > 0$ there exists a vector $F \in D(H_\kappa^{\mathrm{N}\,V=0}) = D(H_0)$ such that $\langle F\,,\, H_\kappa^{\mathrm{N}\,V=0} F\rangle_{\mathcal{H}} < E_\kappa^{\mathrm{N}\,V=0} + \epsilon$. Then

$$\begin{aligned}
&\langle \psi_{\mathrm{at}} F\,,\, \left\{ H_\kappa^{\mathrm{N}} - \left(H_\kappa^{\mathrm{N}\,V=0} + \epsilon + E_{\mathrm{at}} \right) \right\} \psi_{\mathrm{at}} F\rangle_{\mathcal{H}} \\
&= \langle \psi_{\mathrm{at}} F\,,\, \psi_{\mathrm{at}} \left\{ H_\kappa^{\mathrm{N}\,V=0} - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon \right) \right\} F\rangle_{\mathcal{H}} \\
&\quad + \langle \psi_{\mathrm{at}} F\,,\, \left(H_{\mathrm{at}} \psi_{\mathrm{at}} - E_{\mathrm{at}} \psi_{\mathrm{at}} \right) F\rangle_{\mathcal{H}} + \langle \psi_{\mathrm{at}} F\,,\, (p\psi_{\mathrm{at}})(pF)\rangle_{\mathcal{H}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle \psi_{\mathrm{at}} F\,,\, \left\{ H_\kappa^{\mathrm{N}} - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon + E_{\mathrm{at}} \right) \right\} \psi_{\mathrm{at}} F\rangle_{\mathcal{H}} \\
&= \int \left\{ \langle F\,,\, H_\kappa^{\mathrm{N}\,V=0} F\rangle_{\mathcal{F}}(x) - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon \right) \|F\|_{\mathcal{F}}^2(x) \right\} \psi_{\mathrm{at}}(x)^2 d^3x \\
&\quad + \int \langle F\,,\, pF\rangle_{\mathcal{F}}(x) \psi_{\mathrm{at}}(x) (p\psi_{\mathrm{at}})(x) d^3x.
\end{aligned} \tag{17}$$

For arbitrary $y \in \mathbb{R}^3$ there exists a translated vector $F_y \in \mathcal{H}$ so that $\langle F\,,\, H_\kappa^{\mathrm{N}\,V=0} F\rangle_{\mathcal{F}}(x) \to \langle F_y\,,\, H_\kappa^{\mathrm{N}\,V=0} F_y\rangle_{\mathcal{F}}(x) = \langle F\,,\, H_\kappa^{\mathrm{N}\,V=0} F\rangle_{\mathcal{F}}(x+y)$ and $\|F\|_{\mathcal{F}}^2(x) \to \|F_y\|_{\mathcal{F}}^2(x) = \|F\|_{\mathcal{F}}^2(x+y)$, where we used translation invariance of $H_\kappa^{\mathrm{N}\,V=0}$. Set

$$\begin{aligned}
\Omega_y &= \langle \psi_{\mathrm{at}} F_y\,,\, \left\{ H_\kappa^{\mathrm{N}} - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon + E_{\mathrm{at}} \right) \right\} \psi_{\mathrm{at}} F_y\rangle_{\mathcal{H}} \\
&= \int \left\{ \langle F\,,\, H_\kappa^{\mathrm{N}\,V=0} F\rangle_{\mathcal{F}}(x) - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon \right) \|F\|_{\mathcal{F}}^2(x) \right\} \psi_{\mathrm{at}}(x-y)^2 d^3x \\
&\quad + \int \langle F\,,\, pF\rangle_{\mathcal{F}}(x) \psi_{\mathrm{at}}(x-y) (p\psi_{\mathrm{at}})(x-y) d^3x,
\end{aligned}$$

where we used (17). Then we have

$$\int \Omega_y d^3y = \int\int \langle F\,,\, \left\{ H_\kappa^{\mathrm{N}\,V=0} - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon \right) \right\} F\rangle_{\mathcal{F}}(x) \psi_{\mathrm{at}}(x-y)^2 d^3x d^3y,$$

since $\displaystyle\int \psi_{\mathrm{at}}(x-y)(p\psi_{\mathrm{at}})(x-y) d^3y = 0$, and

$$\int \Omega_y d^3y = \langle F\,,\, \left\{ H_\kappa^{\mathrm{N}\,V=0} - \left(E_\kappa^{\mathrm{N}\,V=0} + \epsilon \right) \right\} F\rangle_{\mathcal{H}} < 0,$$

which implies that there exists $y_0 \in \mathbb{R}^3$ such that $\Omega_{y_0} < 0$. We conclude that $E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}} \geq -E_{\mathrm{at}} - \epsilon$. Since ϵ is arbitrary, taking $\epsilon \searrow 0$, we obtain $E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}} \geq -E_{\mathrm{at}}$. □

Lemma 4.6 *Fix δ and κ with $0 < \delta < 2$ and $0 \leq \kappa < \Lambda$, respectively. Assume (Case 2) and that H_κ^N has a ground state ψ_κ. Let G be in $C^\infty(\mathbb{R}^3)$, non-negative function with $\sup_x |\nabla G(x)|$, $\sup_x |V(x)| G(x)^2 < \infty$. Then, $\psi_\kappa \in D(G)$ and*

$$\|G\psi_\kappa\|_{\mathcal{H}}^2 \leq \left\{ \sup_x |\nabla G(x)|^2 + 2 \sup_x |V(x)| G(x)^2 \right\} \|\psi_\kappa\|_{\mathcal{H}}^2. \tag{18}$$

Proof: We can take $1\!\!1_n \in C_0^\infty(\mathbb{R})$ for each $n \in \mathbb{N}$ satisfying $1\!\!1_n(r) = 1$ for $|r| \leq n$; $= 0$ for $|r| \geq 3n$, and $|d1\!\!1_n(r)/dr| \leq n^{-1}$. We set $G_n(x) = 1\!\!1_n(|x|)G(x)$. Then, $G_n \in C_0^\infty(\mathbb{R}^3)$, $G_n(x) \leq G_{n+1}(x) \leq G(x)$ for a.e. $x \in \mathbb{R}^3$ and each $n \in \mathbb{N}$, and $G_n(x) \to G(x)$ as $n \to \infty$ for a.e. $x \in \mathbb{R}^3$.

First we prove (18) for G_n. Then, we have $\psi_\kappa \in D(G_n)$ since G_n is a bounded operator. Moreover, from direct computations, $G_n\psi_\kappa, G_n^2\psi_\kappa \in D(H_\kappa^{\mathrm{N}})$. Since

$$\begin{aligned}
&[\,[H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}\,,\, G_n]\,,\, G_n] \\
&= (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}) G_n^2 - 2G_n (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}) G_n + G_n^2 (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}),
\end{aligned}$$

we have

$$\langle G_n\psi_\kappa\,,\, (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}) G_n\psi_\kappa \rangle_{\mathcal{H}} = \langle \psi_\kappa\,,\, \frac{1}{2} |\nabla G_n|^2 \psi_\kappa \rangle_{\mathcal{H}}. \tag{19}$$

On the other hand, we have

$$\begin{aligned}
&\langle G_n\psi_\kappa\,,\, (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}})\, G_n\psi_\kappa \rangle_{\mathcal{H}} \\
&= \left\langle G_n\psi_\kappa\,,\, \left(H_\kappa^{\mathrm{N}\,V=0} + V - E_\kappa^{\mathrm{N}} \right) G_n\psi_\kappa \right\rangle_{\mathcal{H}} \\
&\geq (E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) \|G_n\psi_\kappa\|_{\mathcal{H}}^2 + \langle \psi_\kappa\,,\, V G_n^2 \psi_\kappa \rangle_{\mathcal{H}}.
\end{aligned} \tag{20}$$

We have by (19) and (20)

$$(E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) \|G_n\psi_\kappa\|_{\mathcal{H}}^2 \leq \langle \psi_\kappa\,,\, \frac{1}{2} |\nabla G_n|^2 \psi_\kappa \rangle_{\mathcal{H}} - \langle \psi_\kappa\,,\, V G_n^2 \psi_\kappa \rangle_{\mathcal{H}}. \tag{21}$$

The assertion (18) for G_n follows from Proposition 4.5 and (21). Before proving (18) for G, we note that

$$\sup_x |V(x)| G_n(x)^2 \leq \sup_x |V(x)| G(x)^2. \tag{22}$$

By (18) for G_n and (22), we have

$$\|G_n\psi_\kappa\|_{\mathcal{H}}^2 \leq \left\{ \sup_x |\nabla G_n(x)|^2 + 2 \sup_x |V(x)| G(x)^2 \right\} \|\psi_\kappa\|_{\mathcal{H}}^2. \tag{23}$$

On the other hand, we denote the right hand side of (22) by M_G, i.e., $M_G = \sup_x |V(x)| G(x)^2$. Then, we have

$$G(x) \leq \sqrt{M_G} |V(x)|^{-1/2} \quad \text{for } \forall x \in \mathbb{R}^3. \tag{24}$$

By using Leibniz's formula, Minkowski's inequality and (24),

$$\begin{aligned} |\nabla G_n(x)| &\le n^{-1} \sup_{n\le|x|\le 3n} G(x) + \sup_{|x|\le 3n} |\nabla G(x)| \\ &\le \sqrt{M_G}\, n^{-1} \sup_{n\le|x|\le 3n} |V(x)|^{-1/2} + \sup_x |\nabla G(x)|. \end{aligned} \tag{25}$$

Here we note that it follows from (C2-2) that there exists a positive constant c such that

$$\sup_{R\le|x|} |V(x)| \le cR^{-\delta}. \tag{26}$$

So, by (25) and (26)

$$\sup_x |\nabla G_n(x)| \le c\sqrt{3^\delta M_G}\, n^{-1} n^{\delta/2} + \sup_x |\nabla G(x)|$$

for sufficiently large $n \in \mathbb{N}$, which implies that

$$\lim_{n\to\infty} \sup_x |\nabla G_n(x)| \le \sup_x |\nabla G(x)|. \tag{27}$$

Then, by Lebesgue's monotone convergence theorem, (23), and (27), we have

$$\begin{aligned} \int \lim_{n\to\infty} |G_n(x)|^2 \|\psi_\kappa\|^2_{\mathcal{F}}(x) &= \lim_{n\to\infty} \int |G_n(x)|^2 \|\psi_\kappa\|^2_{\mathcal{F}}(x) \\ &\le \left\{ \sup_x |\nabla G(x)|^2 + 2\sup_x |V(x)| G(x)^2 \right\} \|\psi_\kappa\|^2_{\mathcal{H}}. \end{aligned}$$

Thus, $\psi_\kappa \in D(G)$ and (18) holds for G. □

To have a more concrete estimate we set

$$G_R(x) = \chi_R(|x|) g(x). \tag{28}$$

Here $\chi_R(r) = 0$ for $r < R/2$ and $\chi_R(r) = 1$ for $r > R$ with linear interpolation. The parameter $R > 0$ serves as a variation which will be optimized at the end. $g(x)$ is twice differentiable satisfying

$$\sup_{R/2<|x|} |\nabla g(x)|^2 < \infty. \tag{29}$$

Then, noting that χ_R is absolutely continuous, we have

$$\begin{aligned} \sup_x |\nabla G_R|^2 &\le 4R^{-2} \sup_{R/2<|x|<R} |g(x)|^2 + 4R^{-1} \sup_{R/2<|x|<R} |g(x)||\nabla g(x)| \\ &\quad + \sup_{R/2<|x|} |\nabla g(x)|^2. \end{aligned}$$

For example, if $g(x) = |x|^{1/2}$, then

$$\sup_x |\nabla G_R|^2 \le 7R^{-1}, \tag{30}$$

and if $g(x) = |x|$, then

$$\sup_x |\nabla G_R|^2 \le 9. \tag{31}$$

By Lemma 4.6 we have the following.

Lemma 4.7 *(spatial localization I) Fix κ with $0 \le \kappa < \Lambda$. Assume (Case 2) and that H_κ^N has a ground state ψ_κ. Let g be differentiable and non-negative, satisfy (29), and $G_R(x)$ be defined in (28). If $\sup_{R<|x|} |V(x)|\, |g(x)|^2 < \infty$, then $\psi_\kappa \in D(g)$ and*

$$\begin{aligned} \|g\psi_\kappa\|_{\mathcal{H}}^2 \quad \le \quad & \Big\{ \sup_{|x|\le R} |g(x)|^2 + \sup_x |\nabla G_R(x)|^2 \\ & + 2\sup_x |V(x)|\, G_R(x)^2 \Big\} \|\psi_\kappa\|_{\mathcal{H}}^2 \end{aligned} \tag{32}$$

Proof: First we note that $(1-\chi_R)g$ is a bounded operator, and $D(g) = G(G_R)$ holds. So, (32) follows from Lemma 4.6 and

$$\|g\psi_\kappa\|_{\mathcal{H}}^2 = \|\,(1-\chi_R+\chi_R)\,g\psi_\kappa\|_{\mathcal{H}}^2 \le \sup_{|x|\le R} |g(x)|^2 \|\psi_\kappa\|_{\mathcal{H}}^2 + \|G_R\psi_\kappa\|_{\mathcal{H}}^2. \tag{33}$$

□

The following lemma follows from Lemma 4.7 and (30) immediately.

Lemma 4.8 *Assume (Case 2). Fix κ with $0 \le \kappa < \Lambda$. If H_κ^N has a ground state ψ_κ then $\psi_\kappa \in D(|x|)$ and*

$$\langle \psi_\kappa\,,\, |x|\psi_\kappa\rangle_{\mathcal{H}} \le 7\|\psi_\kappa\|_{\mathcal{H}}^2. \tag{34}$$

Proof: It will be proven in Lemma 4.10 that $\psi_\kappa \in D(|x|)$. Take $R = 4$. Then, since $\langle \psi_\kappa\,,\, |x|\psi_\kappa\rangle_{\mathcal{H}} = \||x|\psi_\kappa\|_{\mathcal{H}}^2$, (34) is the special case of (32) as $g(x) = |x|^{1/2}$. □

In case without $\sup_{R<|x|} |V(x)|\,|g(x)|^2 < \infty$, we have the following lemma, though the upper bound gets worse than Lemma 4.7.

Lemma 4.9 *(spatial localization II). Fix κ with $0 \le \kappa < \Lambda$. Assume (Case 2) and that H_κ^N has a ground state ψ_κ. Let g be differentiable and non-negative, satisfy (29), and $G_R(x)$ be defined in (28). Then, $\psi_\kappa \in D(g)$*

$$\begin{aligned} \|g\psi_\kappa\|_{\mathcal{H}}^2 \quad \le \quad & \Big\{ \sup_{|x|\le R} |g(x)|^2 \\ & + \left(|E_{\mathrm{at}}| - \sup_{R<|x|} |V(x)| \right)^{-1} \sup_x |\nabla G_R(x)|^2 \Big\} \|\psi_\kappa\|_{\mathcal{H}}^2 \end{aligned} \tag{35}$$

for sufficiently large R so that $|E_{\mathrm{at}}| > \sup_{R<|x|} |V(x)|$.

Proof: We first note (33) also holds in this case. Let $1\!\!1_n$ be the same as given in the proof of Lemma 4.7. We set $G_n(x) = 1\!\!1_n(|x|)G_R(x)$. Then, $G_n \in C_0^2(\mathbb{R}^3)$. By using Leibniz's formula and Minkowski's inequality, we have

$$|\nabla G_n(x)| \le n^{-1} G_n(x) + 1\!\!1_n(|x|)|\nabla G_R(x)|,$$

which implies

$$|\nabla G_n(x)|^2 \le 2n^{-2} G_n(x)^2 + 2\sup_x |\nabla G_{R(x)}|^2. \tag{36}$$

In the same way as getting (21), we have by (36)

$$\left\{ (E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) - \sup_{R<|x|} |V(x)| \right\} \|G_n\psi_\kappa\|_{\mathcal{H}}^2 \le \frac{1}{2}\langle \psi_\kappa\,,\, |\nabla G_n|^2 \psi_\kappa\rangle_{\mathcal{H}}$$
$$\le n^{-2}\|G_n\psi_\kappa\|_{\mathcal{H}}^2 + \sup_x |\nabla G_R(x)|^2 \|\psi_\kappa\|_{\mathcal{H}}^2,$$

which implies

$$\left\{ (E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) - \sup_{R<|x|} |V(x)| - n^{-2} \right\} \|G_n\psi_\kappa\|_{\mathcal{H}}^2 \le \sup_x |\nabla G_R(x)|^2 \|\psi_\kappa\|_{\mathcal{H}}^2.$$

Thus, we have

$$\|G_n\psi_\kappa\|_{\mathcal{H}}^2 \le \left\{ (E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) - \sup_{R<|x|} |V(x)| - n^{-2} \right\}^{-1} \sup_x |\nabla G_R(x)|^2 \|\psi_\kappa\|_{\mathcal{H}}^2.$$

By applying Lebesgue's monotone convergence theorem to the above inequality, we have $\psi_\kappa \in D(G_R)$ and

$$\|G_R\psi_\kappa\|_{\mathcal{H}}^2 \le \left\{ (E_\kappa^{\mathrm{N}\,V=0} - E_\kappa^{\mathrm{N}}) - \sup_{R<|x|} |V(x)| \right\}^{-1} \sup_x |\nabla G_R(x)|^2 \|\psi_\kappa\|_{\mathcal{H}}^2. \qquad (37)$$

(35) follows from (33) and (37). □

The following lemma follows from Lemma 4.9 and (31) immediately.

Lemma 4.10 *Assume (Case 2). Fix κ with $0 \le \kappa < \Lambda$. If H_κ^N has a ground state ψ_κ then $\psi_\kappa \in D(|x|)$ and*

$$\||x|\psi_\kappa\|_{\mathcal{H}}^2 \le \left\{ R^2 + 9\left(|E_{\mathrm{at}}| - \sup_{R<|x|} |V(x)| \right)^{-1} \right\} \|\psi_\kappa\|_{\mathcal{H}}^2 \qquad (38)$$

for sufficiently large so that $|E_{\mathrm{at}}| > \sup_{R<|x|} |V(x)|$.

Proof: (38) is the special case of (35) for $g(x) = |x|$. □

5 An identity from pull-through formula

Let us fix $0 \le \kappa < \Lambda$. We suppose now that H_κ^{N} has a ground state ψ_κ. Then, in the same way as the derivation of [15, (IV.1)] from [15, Proposition III.4], we get

$$a(k)\psi_\kappa = -q\frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1} e^{-ikx}\psi_\kappa \qquad (39)$$

as an identity on $L^2(\mathbb{R}^3, d^3k; \mathcal{H})$ by the standard pull-through formula. From this (39) we have the following identity.

Lemma 5.1 *Let us fix $0 \le \kappa < \Lambda$, and suppose that H_κ^N has a ground state ψ_κ. Then,*

$$a(k)\psi_\kappa = \sum_{j=1}^{5} I_j(k)\psi_\kappa \tag{40}$$

as an identity in $L^2(\mathbb{R}^3, d^3k; \mathcal{H})$ with

$$\begin{aligned}
I_1(k) &= -q\Xi_1(k)e^{-ikx}, \\
\Xi_1(k) &= \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}\,\omega(k)} : \mathbb{R}^3 \to \mathbb{R}, \\
I_2(k) &= -iq\Xi_2(k)\left(H_\kappa^N - E_\kappa^N + \omega(k)\right)^{-1}\left(H_\kappa^N - E_\kappa^N\right)e^{-ikx}x, \\
\Xi_2(k) &= \frac{\chi_\kappa(k)k}{\sqrt{2\omega(k)}\,\omega(k)} : \mathbb{R}^3 \to \mathbb{R}^3, \\
I_3(k) &= -iq\Xi_3(k)\left(H_\kappa^N - E_\kappa^N + \omega(k)\right)^{-1}pe^{-ikx}x, \\
\Xi_3(k) &= \frac{\chi_\kappa(k)k^2}{\sqrt{2\omega(k)}\,\omega(k)} : \mathbb{R}^3 \to \mathbb{R}, \\
I_4(k) &= -iq\Xi_4(k)\left(H_\kappa^N - E_\kappa^N + \omega(k)\right)^{-1}e^{-ikx}x, \\
\Xi_4(k) &= \frac{\chi_\kappa(k)k^2k}{2\sqrt{2\omega(k)}\,\omega(k)} : \mathbb{R}^3 \to \mathbb{R}^3, \\
I_5(k) &= q\Xi_5(k)\left(H_\kappa^N - E_\kappa^N + \omega(k)\right)^{-1}e^{-ikx}, \\
\Xi_5(k) &= \frac{\chi_\kappa(k)k^2}{2\sqrt{2\omega(k)}\,\omega(k)} : \mathbb{R}^3 \to \mathbb{R}.
\end{aligned}$$

Proof: By the pull-through formula, we have

$$a(k)\psi_\kappa = -q\frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1}e^{-ikx}\psi_\kappa. \tag{41}$$

Noting

$$\left[H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\,,\, e^{-ikx}\right] = \left[H_\kappa^{\mathrm{N}}\,,\, e^{-ikx}\right],$$

we get

$$\begin{aligned}
&\left[e^{-ikx}\,,\,\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1}\right] \\
&= \left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1}\left[H_\kappa^{\mathrm{N}}\,,\, e^{-ikx}\right] \\
&\quad\times\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1}.
\end{aligned} \tag{42}$$

On the other hand, we get

$$\left[H_\kappa^{\mathrm{N}}\,,\, e^{-ikx}\right] = -ke^{-ikx}p + \frac{k^2}{2}e^{-ikx}, \tag{43}$$

So, by (41), (42), and (43), we have

$$\begin{aligned}
a(k)\psi_\kappa &= I_1\psi_\kappa - q\frac{\chi_\kappa(k)k}{\sqrt{2\omega(k)}\,\omega(k)}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1}e^{-ikx}p\psi_\kappa \\
&\quad+I_5\psi_\kappa.
\end{aligned} \tag{44}$$

Here we note that

$$i\,[H_\kappa^{\mathrm{N}}\,,\,x] = p, \tag{45}$$

and

$$e^{-ikx}\,[H_\kappa^{\mathrm{N}}\,,\,x] = \left[H_\kappa^{\mathrm{N}}\,,\,e^{-ikx}x\right] - \left[H_\kappa^{\mathrm{N}}\,,\,e^{-ikx}\right]x. \tag{46}$$

Thus, by (43), (45), and (46), we get

$$\begin{aligned}
&-q\frac{\chi_\kappa(k)k}{\sqrt{2\omega(k)}\,\omega(k)}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1} e^{-ikx}p\psi_\kappa \\
&= I_2\psi_\kappa - iq\frac{\chi_\kappa(k)k^2}{\sqrt{2\omega(k)}\,\omega(k)}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1} e^{-ikx}px\psi_\kappa - I_4\psi_\kappa
\end{aligned} \tag{47}$$

Since $e^{-ikx}p = pe^{-ikx} + ke^{-ikx}$, we have

$$-iq\frac{\chi_\kappa(k)k^2}{\sqrt{2\omega(k)}\,\omega(k)}\left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1} e^{-ikx}px\psi_\kappa = I_3\psi_\kappa + 2I_4\psi_\kappa. \tag{48}$$

Therefore, (40) follows from (44), (47), and (48). □

Lemma 5.2 *Let us fix $\kappa = 0$ and $\Lambda > 0$, and suppose that $H_N = H_0^N$ has a ground state ψ_0. Then, $I_1(k)\psi_0$ is an infrared catastrophic term, i.e.,*

$$\int_{0\le|k|\le\Lambda} \|I_1(k)\psi_0\|_{\mathcal{H}}^2 d^3k = \infty,$$

and for $j = 2, \cdots, 5$, $I_j(k)\psi_0$ is an infrared safe term, i.e.,

$$\int_{0\le|k|\le\Lambda} \|I_j(k)\psi_0\|_{\mathcal{H}}^2 d^3k < \infty.$$

Moreover, there exists a positive constant M such that

$$\sum_{j=2}^{5}\int \|I_j(k)\psi_0\|_{\mathcal{H}}^2 d^3k \le M\|\psi_0\|_{\mathcal{H}}^2, \tag{49}$$

where

$$M = \frac{q^2}{64\pi^2}\max\{1\,,\,M_0\}\left\{10\Lambda^2 + \left(8C_{\mathrm{at}}^2C_{\mathrm{q}}^2 + 1\right)\Lambda^4\right\} \tag{50}$$

with

$$M_0 = \begin{cases} c_1C_{\mathrm{q}}^2 + c_1 + c_2, & \textit{(Case 1)}, \\ \min\left\{R > 0 \,\middle|\, R^2 + 9(|E_{\mathrm{at}}| - \sup_{R<|x|}|V(x)|)^{-1}\right\}, & \textit{(Case 2)}. \end{cases}$$

Here C_{q} is independent of κ for $\kappa < K_0$.

Proof: It is easy to see that

$$\int \|I_1(k)\psi_0\|^2_{\mathcal{H}} d^3k = \frac{q^2}{4\pi^2}\|\psi_0\|^2_{\mathcal{H}} \lim_{\kappa\to 0} \log\frac{\Lambda}{\kappa} = \infty. \tag{51}$$

By Lemmas 4.4 and 4.10, we get

$$\int \|I_2(k)\psi_0\|^2_{\mathcal{H}} d^3k \le \frac{q^2}{4\pi^2}\||x|\psi_0\|^2_{\mathcal{H}} \int_0^\Lambda r dr \le \frac{q^2}{8\pi^2}\Lambda^2 M_0\|\psi_0\|^2_{\mathcal{H}}. \tag{52}$$

By (8) and Lemmas 4.1, 4.3, 4.4, and 4.10, we get

$$\begin{aligned}
&\int \|I_3(k)\psi_0\|^2_{\mathcal{H}} d^3k \\
&\le \frac{q^2}{4\pi^2}\||x|\psi_0\|^2_{\mathcal{H}}\|(H_0+1)^{-1}(H_{\mathrm{at}}-E_{\mathrm{at}}+1)\|^2 \\
&\qquad \times\|(H_{\mathrm{at}}-E_{\mathrm{at}}+1)^{-1}(\frac{p^2}{2}+1)\|^2\|(\frac{p^2}{2}+1)^{-1}p\|^2 \\
&\qquad \times\int_0^\Lambda r^3\|(H_{\mathrm{N}}-E_{\mathrm{N}}+r)^{-1}(H_0+1)\|^2 \\
&\le \frac{q^2}{4\pi^2}M_0C_{\mathrm{q}}^2C_{\mathrm{at}}^2\|\psi_0\|^2_{\mathcal{H}}\left(\int_0^1 r dr + \int_1^\Lambda r^3 dr\right) \\
&= \frac{q^2}{8\pi^2}M_0C_{\mathrm{q}}^2C_{\mathrm{at}}^2\Lambda^4\|\psi_0\|^2_{\mathcal{H}}.
\end{aligned} \tag{53}$$

It is easy to estimate $I_4(k)\psi_0$ and $I_5(k)\psi_0$.

$$\int \|I_4(k)\psi_0\|^2_{\mathcal{H}} d^3k \le \frac{q^2}{16\pi^2}\||x|\psi_0\|^2_{\mathcal{H}} \int_0^\Lambda r^3 dr = \frac{q^2}{64\pi^2}M_0\Lambda^4\|\psi_0\|^2_{\mathcal{H}} \tag{54}$$

by Lemmas 4.4 and 4.10, and

$$\int \|I_5(k)\psi_0\|^2_{\mathcal{H}} d^3k \le \frac{q^2}{16\pi^2}\|\psi_0\|^2_{\mathcal{H}} \int_0^\Lambda r dr = \frac{q^2}{32\pi^2}\Lambda^2\|\psi_0\|^2_{\mathcal{H}}. \tag{55}$$

Thus, (49) follows from (52)–(55). □

6 Explicit estimate of soft photon bound

In this section, we prove Theorem 3.4. We set

$$K(k) = \sum_{j=2}^{5} I(k)\psi_\kappa \in \mathcal{H}.$$

Applying the trigonometrical inequality to (40), we get

$$\begin{aligned}
&\int_{\kappa\le|k|\le\Lambda} \|a(k)\psi_\kappa\|^2_{\mathcal{H}} d^3k \\
&\le 2\int_{\kappa\le|k|\le\Lambda} \|I_1(k)\psi_\kappa\|^2_{\mathcal{H}} d^3k + 2\int_{0\le|k|\le\Lambda} \|K(k)\|^2_{\mathcal{H}} d^3k.
\end{aligned} \tag{56}$$

On the other hand, we get

$$\int_{\kappa\le|k|\le\Lambda} \|I_1(k)\psi_\kappa\|^2_{\mathcal{H}} d^3k = \frac{q^2}{4\pi^2}\left(\log\frac{\Lambda}{\kappa}\right)\|\psi_\kappa\|^2_{\mathcal{H}}. \tag{57}$$

By (49), (56), and (57), we have

$$\langle\psi_\kappa\,,\,N_{\rm f}\psi_\kappa\rangle_{\mathcal{H}} \le 2\left\{\frac{q^2}{4\pi^2}\left(\log\frac{\Lambda}{\kappa}\right)+M\right\}\|\psi_\kappa\|^2_{\mathcal{H}}. \tag{58}$$

By (40) again, we get

$$I_1(k)\psi_\kappa = a(k)\psi_\kappa - K(k),$$

which implies that

$$\begin{aligned}&\int_{\kappa\le|k|\le\Lambda} \|I_1(k)\psi_\kappa\|^2_{\mathcal{H}} d^3\\ &\le 2\int_{\kappa\le|k|\le\Lambda}\|a(k)\psi_\kappa\|^2_{\mathcal{H}}d^3k + 2\int_{0\le|k|\le\Lambda}\|K(k)\|^2_{\mathcal{H}}d^3k.\end{aligned} \tag{59}$$

By (49), (57), and (59), we have

$$\frac{q^2}{4\pi^2}\left(\log\frac{\Lambda}{\kappa}\right)\|\psi_\kappa\|^2_{\mathcal{H}} \le 2\langle\psi_\kappa\,,\,N_{\rm f}\psi_\kappa\rangle_{\mathcal{H}} + 2M\|\psi_\kappa\|^2_{\mathcal{H}},$$

which implies

$$\left\{\frac{q^2}{8\pi^2}\left(\log\frac{\Lambda}{\kappa}\right)-M\right\}\|\psi_\kappa\|^2_{\mathcal{H}} \le \langle\psi_\kappa\,,\,N_{\rm f}\psi_\kappa\rangle_{\mathcal{H}}. \tag{60}$$

Therefore, (7) follows from (58) and (60).

7 Non-existence of ground state for $\kappa = 0$

In [6] we proved that the ground state is outside $D(N_{\rm f}^{1/2})$. Here, by employing the representation (40), we progress from the limitations in [6, Theorem 3.1] and remove the ground state of $H_{\rm N}$ from $\mathcal{H}$.

Proof of Theorem 3.5:

We use reductio ad absurdum to prove Theorem 3.5. Suppose that $H_{\rm N} = H_0^{\rm N}$ has a ground state ψ_0 in $\mathcal{H}$. We can normalize the ground state ψ_0 without loss of generality, i.e.,

$$\|\psi_0\|_{\mathcal{H}} = 1. \tag{61}$$

For every $\phi\in D(N_{\rm f}^{1/2})$, define a function by

$$F_{\phi,\psi_0}(k) = \sum_{j=1}^{5}\langle\phi\,,\,I_j(k)\psi_0\rangle_{\mathcal{H}}. \tag{62}$$

We now repeat the proof of [6, Lemma 5.2(i)] so that it fits our case. We set

$$Y = \bigcup_{n=1}^{3} Y_n, \qquad Y_n = \left\{(k_1,k_2,k_3)\in\mathbb{R}^3\,|\,k_n = 0\right\}.$$

Then, for $f \in C_0^\infty(\mathbb{R}^3 \backslash Y)$, we define an anti-linear functional $T_{\phi,\psi_0} : C_0^\infty(\mathbb{R}^3 \backslash Y) \to \mathbb{C}$ by

$$T_{\phi,\psi_0}(f) = \langle a^\dagger(f)\phi\,,\,\psi_0\rangle_{\mathcal{H}} \quad \text{for } \forall\phi \in D(N_{\mathrm{f}}^{1/2}).$$

Since

$$|T_{\phi,\psi_0}(f)| \leq \|(N_{\mathrm{f}}+1)^{1/2}\phi\|_{\mathcal{H}}\|f\|_{L^2} \tag{63}$$

and $C_0^\infty(\mathbb{R}^3 \setminus Y)$ is dense in $L^2(\mathbb{R}^3)$, we have a unique extension of T_{ϕ,ψ_0} and it is a bounded anti-linear functional. We denote the extension by the same symbol, i.e., $T_{\phi,\psi_0} : L^2(\mathbb{R}^3) \to \mathbb{C}$ with (63) for $f \in L^2(\mathbb{R}^3)$. So, by Riesz's lemma, there exists a unique $F \in L^2(\mathbb{R}^3)$ such that $T_{\phi,\psi_0}(f) = \langle f\,,\,F\rangle_{L^2}$ for every $f \in L^2(\mathbb{R}^3)$. On the other hand, we note that $\psi_0 \in D(H_{\mathrm{N}}) = D(H_0) \subset D(H_0^{1/2}) \subset D(H_{\mathrm{f}}^{1/2}) \subset D(a(\lambda))$ for every $\lambda \in L^2(\mathbb{R}^3)$ with $\lambda/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Since for $f \in C_0^\infty(\mathbb{R}^3 \setminus Y)$ we have $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, we obtain $\langle f\,,\,F_{\phi,\psi_0}\rangle_{L^2} = \langle \phi\,,\,a(f)\psi_0\rangle_{\mathcal{H}} = T_{\phi,\psi_0}(f)$. Thus, we have

$$F_{\phi,\psi_0} = F \in L^2(\mathbb{R}^3) \quad \text{for } \forall\phi \in D(N_{\mathrm{f}}^{1/2}). \tag{64}$$

By Lemma 5.1 and (62), we get

$$\begin{aligned} -q\Xi_1(k)\langle \phi\,,\,e^{-ikx}\psi_0\rangle_{\mathcal{H}} &= \langle \phi\,,\,I_1(k)\psi_0\rangle_{\mathcal{H}} \\ &= F_{\phi,\psi_0}(k) - \sum_{j=2}^{5}\langle \phi\,,\,I_j(k)\psi_0\rangle_{\mathcal{H}} \end{aligned} \tag{65}$$

as a $L^2(\mathbb{R}^3)$-function of k. So, by Lemma 5.2 and (64), the right hand side of (65) is in $L^2(\mathbb{R}^3)$. Thus, we have

$$\Xi_1(\cdot)\langle \phi\,,\,e^{-i\cdot x}\psi_0\rangle_{\mathcal{H}} \in L^2(\mathbb{R}^3). \tag{66}$$

On the other hand, since $\langle \phi\,,\,e^{-ikx}\psi_0\rangle_{\mathcal{H}}$ is a continuous function of $k \in \mathbb{R}^3$, we get

$$\lim_{k\to 0}|\langle \phi\,,\,e^{-ikx}\psi_0\rangle_{\mathcal{H}}|^2 = |\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2.$$

Namely, for arbitrary $\varepsilon > 0$ there exists a positive constant K_ε such that

$$|\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2 - \varepsilon < |\langle \phi\,,\,e^{-ikx}\psi_0\rangle_{\mathcal{H}}|^2 < |\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2 + \varepsilon \quad \text{for } \forall k \text{ with } |k| < K_\varepsilon.$$

Thus, we get by (66)

$$\big(|\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2 - \varepsilon\big)\int_{|k|<K_\varepsilon}|\Xi_1(k)|^2 d^3k \leq \int |\Xi_1(k)\langle \phi\,,\,e^{-ikx}\psi_0\rangle_{\mathcal{H}}|^2 d^3k < \infty.$$

Thus, $|\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2 - \varepsilon$ should be non-positive because $\Xi_1 \notin L^2(\{k \in \mathbb{R}^3\,|\,|k| < K_\varepsilon\})$ by direct computation. Namely, $|\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}}|^2 \leq \varepsilon$. Since ε is arbitrary, we have finally

$$\langle \phi\,,\,\psi_0\rangle_{\mathcal{H}} = 0 \quad \text{for } \forall\phi \in D(N_{\mathrm{f}}^{1/2}) \text{ and } \forall g \neq 0.$$

So, we have

$$\psi_0 \in D(N_{\mathrm{f}}^{1/2})^\perp \quad \text{for } \forall g \neq 0, \tag{67}$$

where $D(N_{\mathrm{f}}^{1/2})^\perp$ is the orthogonal complement of $D(N_{\mathrm{f}}^{1/2})$. Since $D(N_{\mathrm{f}}^{1/2})$ is dense in $L^2(\mathbb{R}^3)$, we get

$$D(N_{\mathrm{f}}^{1/2})^\perp = \{0\}. \tag{68}$$

By (67) and (68), we have $\psi_0 = 0$, which contradicts (61). Therefore, we obtain Theorem 3.5.

8 Appendix: van Hove Model

As written in §2, the van Hove model has an infrared catastrophe. In this appendix we list some mathematical results for H_0^{vH}.

Let $\omega(k)$ be a non-negative, Borel measurable function of $k \in \mathbb{R}^3$ satisfying $\lim_{|k|\to\infty} \omega(k) = \infty$ and $\inf_k \omega(k) = 0$. With a cutoff function $\varrho(x)$ we set

$$\lambda(k) := -\frac{\widehat{\varrho}(k)}{\sqrt{2\omega(k)}}.$$

In this section we assume

$$\omega^{-1/2}\lambda \in L^2(\mathbb{R}^3). \tag{69}$$

We define the Hamiltonian of the van Hove (fixed source) model by

$$H_\omega^{\mathrm{vH}}(\lambda) := H_{\mathrm{f}} + q\int_{\mathbb{R}^3} \lambda(k)\left(a^\dagger(k) + a(k)\right) d^3k. \tag{70}$$

Then $H_\omega^{\mathrm{vH}}(\lambda)$ *is self-adjoint on* $D(H_{\mathrm{f}})$, *and the spectrum* $\sigma(H_\omega^{\mathrm{vH}}(\lambda))$ *of* $H_\omega^{\mathrm{vH}}(\lambda)$ *is given by*

$$\sigma(H_\omega^{\mathrm{vH}}(\lambda)) = [\,E_\omega(\lambda)\,,\,\infty)\,,$$

where

$$E_\omega(\lambda) = -q^2\int_{\mathbb{R}^3} \frac{\lambda(k)^2}{\omega(k)} d^3k.$$

Set

$$\Pi_{\omega,\lambda} := iq\int_{\mathbb{R}^3} \frac{\lambda(k)}{\omega(k)}\left(a^\dagger(k) - a(k)\right) d^3k.$$

As is well known, if $\omega^{-1}\lambda \in L^2(\mathbb{R}^3)$, then

(i) $e^{-i\Pi_{\omega,\lambda}} H_\omega^{\mathrm{vH}}(\lambda) e^{i\Pi_{\omega,\lambda}} = H_{\mathrm{f}} + E_\omega(\lambda)$.

(ii) $\sigma_{\mathrm{pp}}(H_\omega^{\mathrm{vH}}(\lambda)) = \{E_\omega(\lambda)\}$.

(iii) $H_\omega^{\mathrm{vH}}(\lambda)$ has a unique ground state $\psi_\omega(\lambda) := e^{i\Pi_{\omega,\lambda}}\Omega$, where Ω is the Fock vacuum.

We take $\lambda_\nu \in L^2(\mathbb{R}^3)$, $\nu > 0$, satisfying $\omega^{-1}\lambda_\nu \in L^2(\mathbb{R}^3)$. By (iii) above we have a unique normalized ground state $\psi_\omega(\lambda_\nu)$ of $H_\omega^{\mathrm{vH}}(\lambda_\nu)$. Then, we can prove the following in the same way as in Theorem 8.3 below.

Theorem 8.1 *The weak limit*

$$\text{w-}\lim_{\nu\to\infty} \psi_\omega(\lambda_\nu) = 0$$

if and only if

$$\lim_{\nu\to\infty}\int_{\mathbb{R}^3} \frac{|\lambda_\nu(k)|^2}{\omega(k)^2} d^3k = \infty.$$

This theorem is stated and called the *van Hove-Miyatake phenomenon* by Arai in [3, Theorems 12-14 and 12-17] though it is primarily for ultraviolet catastrophe [12]. Actually, the original phenomenon was already proved by Miyatake for a model of mesons and nucleon [24], but he employed the state space constructed by von Neumann [26], which has a different construction from our standard Fock space. Then, Arai translated the phenomenon for the case of the van Hove model in our terms.

In Theorem 8.1 we can regard ν as either an infrared cutoff or an ultraviolet cutoff. So, the physical meaning of the van Hove-Miyatake phenomenon is that, after removing either the infrared cutoff or the ultraviolet cutoff, we cannot expect the existence of a ground state for the van Hove model any longer. Especially, as for the ultraviolet problem, we meet a difficulty even for the definition of '$\lim_{\nu\to\infty} H_{\omega}^{\mathrm{vH}}(\lambda_{\nu})$.' The way to clear this difficulty is shown in [10, §1.1 and Theorem 3.4].

In order to consider the infrared catastrophe for the van Hove model, we set $\lambda_{\kappa}(k) := \lambda(k)$ for $|k| > \kappa$; $:= 0$ for $|k| \le \kappa$. Then, the *infrared* van Hove-Miyatake phenomenon for λ_{κ} says that $\psi_{\omega}(\lambda_{\kappa})$ *converges weakly to* 0 *as* $\kappa \to 0$*, i.e.,* w- $\lim_{\kappa\to 0} \psi_{\omega}(\lambda_{\kappa}) = 0$ *if and only if the infrared condition holds for* λ*, i.e.,* $\lambda/\omega \notin L^2(\mathbb{R}^3)$.

Moreover, it is well known that

Theorem 8.2 *The following three conditions are equivalent:*

(i) *The infrared singularity condition holds for* λ.

(ii) $H_{\omega}^{\mathrm{vH}}(\lambda)$ *has no ground state in* $\mathcal{F}$.

(iii) $\langle N_{\mathrm{f}}\rangle \equiv \lim_{\kappa\to 0}\langle\psi_{\omega}(\lambda_{\kappa})\,,\, N_{\mathrm{f}}\psi_{\omega}(\lambda_{\kappa})\rangle_{\mathcal{F}} = \mathrm{q}^2\int_{\mathbb{R}^3}\frac{|\lambda(k)|^2}{\omega(k)^2}d^3k = \infty.$

(see, for example, [3, Theorems 12, 13 and Proposition 12-15]). This is the mathematical reason why we introduced the notion of the infrared singularity condition in [5]. The notion of the infrared regularity condition comes from the contraposition of the above statement. We can see that the infrared van Hove-Miyatake phenomenon implies directly the non-existence of a ground state considering Dereziński's proof of [10, Proposition 3.13], though Arai proves it with a different method in [3, Theorem 12-13] and we can also prove it with the same way as in §7. Here, in order to prove the non-existence of the ground state for $H_{\omega}^{\mathrm{vH}}(\lambda)$ *from the infrared van Hove-Miyatake phenomenon*, we regard $H_{\omega}^{\mathrm{vH}}(\lambda)$ as

$$H_{\omega}^{\mathrm{vH}}(\lambda) = H_{\omega_{\le\kappa}}^{\mathrm{vH}}(\lambda_{\le\kappa})\otimes 1 + 1\otimes H_{\omega_{\kappa}}^{\mathrm{vH}}(\lambda_{\kappa}), \tag{71}$$

where $\omega_{\kappa}(k) := \omega(k)$ for $|k| > \kappa$; $:= 0$ for $|k| \le \kappa$, $\omega_{\le\kappa}(k) := 1 - \omega_{\kappa}(k)$, and $\lambda_{\le\kappa}(k) := 1 - \lambda_{\kappa}(k)$.

By (iii) above again, $H_{\omega_{\kappa}}^{\mathrm{vH}}(\lambda_{\kappa})$ has a unique normalized ground state ψ_{κ}. Then, we have the infrared van Hove-Miyatake phenomenon for ψ_{κ}:

Theorem 8.3

$$\frac{\lambda}{\omega} \notin L^2(\mathbb{R}^3) \quad \textit{iff} \quad \text{w-}\lim_{\kappa\to 0}\psi_{\kappa} = 0. \tag{72}$$

Proof: For $f_1, \cdots, f_n \in \mathcal{F}$, we get

$$\langle a^{\dagger}(f_1)\cdots a^{\dagger}(f_n)\Omega\,,\,\psi_{\kappa}\rangle_{\mathcal{F}} = \frac{(-1)^n}{2^{n/2}}e^{-\mathrm{q}^2\|\omega_{\kappa}^{-1}\lambda_{\kappa}\|_{L^2}^2/4}\prod_{\ell=1}^{n}\langle f_{\ell}\,,\,\omega_{\kappa}^{-1}\lambda_{\kappa}\rangle_{L^2}. \tag{73}$$

Assume $\text{w-lim}_{\kappa\to 0}\,\psi_\kappa = 0$ now. Then, by taking $n=0$ in (73), we have $\lim_{\kappa\to 0}\|\omega_\kappa^{-1}\lambda_\kappa\|_{L^2}^2 = 0$, which implies $\omega^{-1}\lambda \notin L^2(\mathbb{R}^3)$. Conversely, assume $\omega^{-1}\lambda \notin L^2(\mathbb{R}^3)$ now. Then, by taking $n=0$ in (73) again, we have

$$|\langle a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega\,,\,\psi_\kappa\rangle_{\mathcal{F}}| \le \frac{1}{2^{n/2}} e^{-\,q^2\|\omega_\kappa^{-1}\lambda_\kappa\|_{L^2}^2/4}\|\omega_\kappa^{-1}\lambda_\kappa\|_{L^2}^n \prod_{\ell=1}^n \|f_\ell\|_{L^2} \to 0$$

as $\kappa\to 0$, which implies that $\text{w-lim}_{\kappa\to 0}\,\psi_\kappa = 0$. $\square$

Let $\lambda/\omega \notin L^2(\mathbb{R}^3)$ now. As the assumption of reductio ad absurdum, we suppose that $H_\omega^{\mathrm{vH}}(\lambda)$ has a normalized ground state $\psi_\omega(\lambda)$. Then, by (71) there exists a normalized vector φ_κ such that $\psi_\omega(\lambda) = \varphi_\kappa \otimes \psi_\kappa$. So, for all $\phi = \phi'\otimes\phi'' \in \mathcal{F}$ we get

$$|\langle \phi\,,\,\psi_\omega(\lambda)\rangle_{\mathcal{F}}| = |\langle \phi'\,,\,\varphi_\kappa\rangle_{\mathcal{F}}\langle \phi''\,,\,\psi_\kappa\rangle_{\mathcal{F}}| \le \|\phi'\|_{\mathcal{F}}|\langle \phi''\,,\,\psi_\kappa\rangle_{\mathcal{F}}| \to 0$$

by (72). Thus, $\psi_\omega(\lambda)=0$, which is a contradiction. Therefore, applying the way of Dereziński's proof [10, Proposition 3.13] to our case, *the infrared van Hove-Miyatake phenomenon implies directly that $H_\omega^{\mathrm{vH}}(\lambda)$ has no ground state in $\mathcal{F}$.*

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