LOCALIZATION LENGTHS AND BOLTZMANN LIMIT FOR THE ANDERSON MODEL AT SMALL DISORDERS IN DIMENSION 3

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Abstract. We prove lower bounds on the localization length of eigenfunctions in the three-dimensional Anderson model at weak disorders. Our results are similar to those obtained by Shubin, Schlag and Wolff, [8], for dimensions one and two. We prove that with probability one, most eigenfunctions have localization lengths bounded from below by $O(\frac{\lambda^{-2}}{\log\frac{1}{\lambda}})$, where λ is the disorder strength. This is achieved by time-dependent methods which generalize those developed by Erdös and Yau [3] to the lattice and non-Gaussian case. In addition, we show that the macroscopic limit of the corresponding lattice random Schrödinger dynamics is governed by the linear Boltzmann equations.

1. Introduction

The Anderson model in dimension d is defined by the discrete random Schrödinger operator

$$(H_\omega\psi)(x) = -\frac{1}{2}(\Delta\psi)(x) + \lambda\omega(x)\psi(x) \ ,$$

acting on $\ell^2(\mathbb{Z}^d)$, where λ is a small coupling constant,

$$(\Delta\psi)(x) := 2d\psi(x) - \sum_{|x-y|=1}\psi(y)$$

is the nearest neighbor lattice Laplacian, and $\omega(x)$ are, for $x \in \mathbb{Z}^d$, bounded, i.i.d. random variables. In the present paper, we study the case $d = 3$, and prove that with probability one, most eigenfunctions of H_ω have localization lengths bounded from below by $O(\frac{\lambda^{-2}}{\log\frac{1}{\lambda}})$. In contrast to $d = 1, 2$, we note that there are no restrictions on the energy range for this result to hold. Furthermore, we derive the macroscopic limit of the quantum dynamics in this system, and prove that it is governed by the linear Boltzmann equations.

The present paper is closely related to work of L. Erdös and H.-T. Yau in [3], where the weak coupling and hydrodynamic limit is derived for a random Schrödinger equation in the continuum $\mathbb{R}^d$, $d = 2, 3$, for a Gaussian random potential. For macroscopic time and space variables (T, X), microscopic variables (t, x), and the scaling $(X, T) = \lambda^2(x, t)$, where λ is the coupling constant in the continuum analogue of H_ω, they established in the limit $\lambda \to 0$ that the macroscopic dynamics is governed by the *linear Boltzmann equations*, and thus ballistic, for all $T > 0$. We note that the corresponding result for sufficiently small values of T was first proved by H. Spohn [9]. For larger time scales, it has very recently been established that the macroscopic dynamics in $d = 3$ is determined by a diffusion equation, [4].

Also closely related to the present paper is a recent work of C. Shubin, W. Schlag and T. Wolff, [8], who have, by techniques of harmonic analysis, established for the Anderson model at small disorders in $d = 1, 2$, that with probability one, most eigenstates are in frequency space concentrated on shells of thickness $\leq \lambda^2$ in $d = 1$, and $\leq \lambda^{2-\delta}$ in $d = 2$. The eigenenergies are required to be bounded away from the edges of the spectrum of $-\frac{1}{2}\Delta_{\mathbb{Z}^d}$, and in $d = 2$, also away from its center. By the uncertainty principle, this implies lower bounds of order $O(\lambda^{-2})$ in $d = 1$, and and $O(\lambda^{-2+\delta})$ in $d = 2$, on the localization lengths in position space. Closely related to their work are the papers [5, 6] by J. Magnen, G. Poirot, V. Rivasseau, and [7] by G. Poirot, which address properties of the Greens functions associated to H_ω.

The proof of our main results uses an extension of the time-dependent techniques of L. Erdös and H.-T. Yau in [3] to the lattice, and to non-Gaussian random potentials. Higher correlations, which are now abundant, are shown to have an insignificant effect, hence the character of our results does not differ from that obtained in the Gaussian case. We have adapted part of our notation and nomenclature to [3], in order to facilitate the referencing of results.

The link between the lower bounds on the localization lengths of eigenfunctions, and the Schrödinger dynamics generated by H_ω is a joint result with L. Erdös and H.-T. Yau included in this paper. The author is deeply grateful to them for their support and generosity.

2. Definition of the Model and the Main Theorem

We consider the discrete random Schrödinger operator

$$H_\omega = -\frac{1}{2}\Delta + \lambda V_\omega$$

acting on $\psi \in \ell^2(\mathbb{Z}^3)$. The impurity potential is given by

$$V_\omega(x) = \sum_{y \in \Lambda_L} \omega_y \delta(x - y) \ , \tag{1}$$

where ω_y are bounded, independent, identically distributed random variables. For each $x \in \mathbb{Z}^3$, ω_x is a random variable on a single site probability space (J, F, μ), where J is a bounded Borel subset of $\mathbb{R}$, F is the σ-algebra of Borel subsets of J, and μ is a probability measure on F, with $\mathrm{supp}(\mu) = J$. V_ω is a random field over $\mathbb{Z}^3$ realized on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = \times_{\mathbb{Z}^3} J$, where $\mathcal{F}$ is the σ-algebra generated by the cylinder sets induced by F, and the probability measure $\mathbb{P}$ is given by $\times_{\mathbb{Z}^3}\mu$. For simplicity, we assume μ to be even, $\mu(I) = \mu(-I)$, for all $I \subset F$. Then, $\mathbb{E}[\omega_x^{2m+1}] = 0 \ \forall x \in \mathbb{Z}^3$, $\forall m \geq 0$. This will help to reduce some of the notation in our analysis, but for our methods to apply, only $\mathbb{E}[\omega_x] = 0$ is necessary. In addition, we assume the uniform moment bounds

$$\mathbb{E}[\omega_x^{2m}] =: \tilde{c}_{2m} \leq c_\omega \ \ , \ \ \tilde{c}_2 = 1 \ , \ \forall x \in \mathbb{Z}^3 \ , \ \forall m \geq 1 \ , \tag{2}$$

where the constant $c_\omega < \infty$ is independent of m. H_ω is a selfadjoint linear operator on $\ell^2(\mathbb{Z}^3)$ for every realization of V_ω.

The Fourier transform and its inverse are given by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^3} e^{-2\pi i k\cdot x} f(x) \quad , \quad \check{g}(x) = \int_{\mathbb{T}^3} dk \; g(k) e^{2\pi i k\cdot x} \; .$$

We note that

$$\frac{1}{2}(\Delta f)\hat{}(k) = -e(k)\hat{f}(k) \; ,$$

where

$$e(k) = \sum_{i=1}^{3}\big(1-\cos(2\pi k_i)\big) = 2\sum_{i=1}^{3}\sin^2(\pi k_i) \tag{3}$$

is the expression for the kinetic energy in frequency space.

Let $L\in\mathbb{N}$ with $L\gg\lambda^{-2}$, and $\Lambda_L = \{-L,-L+1,\dots,-1,0,1,\dots,L-1,L\}^3\subset\mathbb{Z}^3$. To formulate our main result, we shall introduce approximate characteristic functions for shells centered around lattice sites $x\in\Lambda_L$. Let, for $m,\ell\in\mathbb{N}_0$, with $m\leq\ell\ll L$,

$$h_\ell(m) := \begin{cases} 1 & 0\leq m\leq \ell/2 \\ 2-\frac{2m}{\ell} & \ell/2 < m\leq\ell \end{cases} \; , \tag{4}$$

and $h_\ell(m)=0$ for $m>\ell$.

Then, let $K_\ell(x) := \prod_{j=1}^3 h_\ell(|x_j|)$, and it is clear that $\hat{K}_\ell$ is a product of differences of Fejér kernels. For $x\in\Lambda_L$, and $\delta>0$, let

$$R_{x,\delta,\ell}(y) := K_{\ell_2}(x-y) - K_{\ell_1}(x-y)$$

with

$$\ell_1 := \delta\ell \quad , \quad \ell_2 := \ell \; . \tag{5}$$

$R_{x,\delta,\ell}$ is an approximate characteristic function for a cubical shell of side length $2\ell_2$ centered at x, and thickness $\ell_2-\ell_1$.

The following observation, for which the author thanks H.-T. Yau and L. Erdös, is the key to linking the localization length of eigenvectors to the dynamics generated by H_ω. Let $\{\psi_\alpha^{(L)}\}$ denote an orthonormal basis in $\ell^2(\Lambda_L)$ of eigenfunctions of H_ω restricted to Λ_L. That is,

$$(H_\omega - e_\alpha^{(L)})\psi_\alpha^{(L)} = 0 \text{ on } \Lambda_L \;\text{ and }\; \psi_\alpha^{(L)} = 0 \text{ on } \partial\Lambda_L := \Lambda_{L+1}\setminus\Lambda_L \; , \tag{6}$$

for $\alpha\in\mathcal{A}_L := \{1,\dots,|\Lambda_L|\}$, and $e_\alpha^{(L)}\in\mathbb{R}$. For ε small, the subset of eigenvectors labeled by

$$\mathcal{A}_{L,\varepsilon,\delta,\ell} := \Big\{\alpha\in\mathcal{A}_L\Big|\sum_x |\psi_\alpha^{(L)}(x)|\big\|R_{x,\delta,\ell}\psi_\alpha^{(L)}\big\|_{\ell^2(\Lambda_L)} < \varepsilon\Big\}$$

contains the class of exponentially localized states concentrated in balls of radius $\sim\frac{\delta\ell}{\log\ell}$ or smaller, where δ is notably independent of ℓ. The additional factor $\log\ell$ in the denominator compensates a volume factor $\ell^{3/2}$, which arises due to the fact that $|\psi_\alpha^{(L)}(x)|$ appears only linearly, and not quadratically in the sum. Our main result is the following theorem.

Theorem 2.1. *Assume for $L \gg \lambda^{-2}$, that $\{\psi_\alpha^{(L)}\}$ is an orthonormal H_ω-eigenbasis in $\ell^2(\Lambda_L)$, satisfying (6) with $\alpha \in \mathcal{A}_L$, and $e_\alpha^{(L)} \in \mathbb{R}$. Then, for $\lambda^{\frac{14}{15}} < \delta < 1$, $\varepsilon_\delta := \delta^{\frac{3}{7}}$,*

$$\mathbb{E}\left[\frac{|\mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon_\delta,\delta,\lambda^{-2}}|}{|\mathcal{A}_L|}\right] \geq 1 - c\delta^{\frac{3}{14}} - \frac{c(\ell)}{L} \; ,$$

for a constant $c < \infty$ independent of L, δ, λ. Furthermore,

$$\mathbb{P}\left[\liminf_{L \to \infty} \frac{|\mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon_\delta,\delta,\lambda^{-2}}|}{|\mathcal{A}_L|} \geq 1 - c\delta^{\frac{3}{14}}\right] = 1$$

for $\lambda > 0$ sufficiently small, and a constant $c < \infty$ that is uniform in λ and δ.

We note that in contrast to lower dimensional cases, there is no condition in dimension 3 that requires $e_\alpha^{(L)}$ to be bounded away from the edges or center of $\mathrm{spec}\{-\frac{1}{2}\Delta\}$.

3. Proof of Theorem 2.1

The following lemma relates the localization length of eigenvectors of H_ω to the dynamics generated by H_ω.

Lemma 3.1. *(Joint with L. Erdös and H.-T. Yau) Let $\{\psi_\alpha^{(L)}\}$ denote an orthonormal basis in $\ell^2(\Lambda_L)$, consisting of eigenvectors of H_ω satisfying (6), and assume that $1 \ll \ell \ll L$. Let*

$$\mathcal{A}^c_{L,\varepsilon,\delta,\ell} := \mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon,\delta,\ell} \; ,$$

and suppose that for all $x \in \mathbb{Z}^3$,

$$\mathbb{E}\left[\left\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \right\|^2_{\ell^2(\mathbb{Z}^3)}\right] \geq 1 - \varepsilon \tag{7}$$

is satisfied for some $\varepsilon = \varepsilon(\delta, \ell, t) > 0$. Then,

$$\mathbb{E}\left[\frac{|\mathcal{A}^c_{L,\varepsilon,\delta,\ell}|}{|\mathcal{A}_L|}\right] \geq 1 - 2\varepsilon^{1/2} - \frac{c(\ell)}{L} \; ,$$

where $c(\ell)$ is a constant that only depends on ℓ.

Proof. We have

$$\begin{aligned} \delta_x &= \sum_\alpha a_x^\alpha \psi_\alpha^{(L)} \\ a_x^\alpha &= \left\langle \delta_x \, , \, \psi_\alpha^{(L)} \right\rangle = \psi_\alpha^{(L)}(x) \; . \end{aligned}$$

By the Schwarz inequality, we get

$$\begin{aligned} \left\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \right\|^2_{\ell^2(\Lambda_L)} &\leq (1 + \kappa^{-1}) \Big\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} \\ &+ (1 + \kappa) \Big\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \mathcal{A}^c_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} \; . \end{aligned} \tag{8}$$

For the first term on the r.h.s., we find

$$\begin{aligned}\Big\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} &\leq \Big\| R_{x,\delta,\ell} \sum_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} e^{-ite_\alpha^{(L)}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} \\ &\leq \Big\| R_{x,\delta,\ell} \sum_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} e^{-ite_\alpha^{(L)}} a_x^\alpha \psi_\alpha^{(L)} \Big\|_{\ell^2(\Lambda_L)} \\ &\leq \sum_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} |\psi_\alpha^{(L)}(x)| \big\| R_{x,\delta,\ell} \psi_\alpha^{(L)} \big\|_{\ell^2(\Lambda_L)} ,\end{aligned}$$

where we have used $\|R_{x,\delta,\ell}\| = 1$, and unitarity of the time evolution in the step preceding the second line. For the second term on the r.h.s. of (8),

$$\begin{aligned}\Big\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha\in\mathcal{A}^c_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} &\leq \Big\| e^{-itH_\omega} \sum_{\alpha\in\mathcal{A}^c_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} \\ &= \Big\| \sum_{\alpha\in\mathcal{A}^c_{L,\varepsilon,\delta,\ell}} a_x^\alpha \psi_\alpha^{(L)} \Big\|^2_{\ell^2(\Lambda_L)} \\ &= \sum_{\alpha\in\mathcal{A}^c_{L,\varepsilon,\delta,\ell}} |\psi_\alpha^{(L)}(x)|^2 .\end{aligned}$$

Summing over $x \in \Lambda_L$, we have

$$\begin{aligned}&\frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \big\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \big\|^2_{\ell^2(\Lambda_L)} \\ &\quad\leq (1+\kappa) \frac{1}{|\Lambda_L|} \sum_{\alpha\in\mathcal{A}^c_{L,\varepsilon,\delta,\ell}} \sum_{x\in\Lambda_L} |\psi_\alpha^{(L)}(x)|^2 \\ &\qquad + (1+\kappa^{-1}) \frac{1}{|\Lambda_L|} \sum_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} \sum_{x\in\Lambda_L} |\psi_\alpha^{(L)}(x)| \big\| R_{x,\delta,\ell} \psi_\alpha^{(L)} \big\|_{\ell^2(\Lambda_L)} \\ &\quad\leq (1+\kappa) \frac{1}{|\Lambda_L|} \big|\mathcal{A}^c_{L,\varepsilon,\delta,\ell}\big| + (1+\kappa^{-1}) Av_{\alpha\in\mathcal{A}_{L,\varepsilon,\delta,\ell}} \sum_{x\in\Lambda_L} |\psi_\alpha^{(L)}(x)| \big\| R_{x,\delta,\ell} \psi_\alpha^{(L)} \big\|_{\ell^2(\Lambda_L)} . \qquad (9)\end{aligned}$$

Let

$$S_{2\ell,L} = \big\{ x \,\big|\, \inf_{y\in\partial\Lambda_L} |x-y| \leq 2\ell \big\}$$

and $\tilde{\Lambda}_L := \Lambda_L \setminus S_{2\ell,L}$, such that $R_{x,\delta,\ell} \cap \partial\Lambda_L = \emptyset \ \forall x \in \tilde{\Lambda}_L$. Then,

$$\frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \big\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \big\|^2_{\ell^2(\Lambda_L)} = \frac{1}{|\tilde{\Lambda}_L|} \sum_{x\in\tilde{\Lambda}_L} \big\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \big\|^2_{\ell^2(\mathbb{Z}^3)} + O(\frac{1}{L}) ,$$

by compactness of the support of $R_{x,\delta,\ell}$. By definition of $\mathcal{A}_{L,\varepsilon,\delta,\ell}$, the last term in (9) is bounded by $(1+\kappa^{-1})\varepsilon$, and $|\Lambda_L| = |\mathcal{A}_L|$. Thus,

$$\frac{|\mathcal{A}^c_{L,\varepsilon,\delta,\ell}|}{|\mathcal{A}_L|} \geq \frac{1}{1+\kappa} \frac{1}{|\tilde{\Lambda}_L|} \sum_{x\in\tilde{\Lambda}_L} \big\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \big\|^2_{\ell^2(\mathbb{Z}^3)} - \frac{1+\kappa^{-1}}{1+\kappa} \varepsilon - \frac{c(\ell)}{L} . \qquad (10)$$

Taking expectations, using (7), and choosing $\kappa = \varepsilon^{\frac{1}{2}}$, the claim follows. $\square$

Lemma 3.2. *Under the same assumptions as in Lemma 3.1,*

$$\mathbb{P}\left[\liminf_{L\to\infty} \frac{|\mathcal{A}^c_{L,\varepsilon,\delta,\ell}|}{|\mathcal{A}_L|} \geq 1 - 2\varepsilon^{1/2}\right] = 1 \, .$$

Proof. We consider the family of translation operators $\tau_x : \omega_y \mapsto \omega_{x+y}$, for $x \in \mathbb{Z}^3$, which acts ergodically on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, [1].

Let U_{τ_x} denote the unitary translation operator $(U_{\tau_x}\phi)(y) = \phi(x+y)$ on $\ell^2(\mathbb{Z}^3)$. Then, clearly,

$$U^*_{\tau_x} H_\omega U_{\tau_x} = -\frac{1}{2}\Delta + \lambda V_{\tau_{-x}\omega} = H_{\tau_{-x}\omega}$$

with $V_{\tau_x\omega}(y) = V_\omega(x+y)$, and

$$\begin{aligned}\frac{1}{|\tilde{\Lambda}_L|}\sum_{x\in\tilde{\Lambda}_L} \left\| R_{x,\delta,\ell} e^{-itH_\omega}\delta_x \right\|^2_{\ell^2(\mathbb{Z}^3)} &= \frac{1}{|\tilde{\Lambda}_L|}\sum_{x\in\tilde{\Lambda}_L} \left\| (U^*_{\tau_x} R_{x,\delta,\ell} U_{\tau_x})(U^*_{\tau_x} e^{-itH_\omega} U_{\tau_x})\delta_0 \right\|^2_{\ell^2(\mathbb{Z}^3)} \\ &= \frac{1}{|\tilde{\Lambda}_L|}\sum_{x\in\tilde{\Lambda}_L} \left\| R_{0,\delta,\ell} e^{-itH_{\tau_{-x}\omega}}\delta_0 \right\|^2_{\ell^2(\mathbb{Z}^3)} \, ,\end{aligned}$$

by unitarity of U_{τ_x}. By the Birkhoff-Khinchin ergodic theorem, applied to the random variable $f(\omega) := \| R_{0,\delta,\ell} e^{-itH_\omega}\delta_0 \|^2_{\ell^2(\mathbb{Z}^3)}$, we obtain

$$\liminf_{L\to\infty} \frac{1}{|\tilde{\Lambda}_L|}\sum_{x\in\tilde{\Lambda}_L} \left\| R_{0,\delta,\ell} e^{-itH_{\tau_{-x}\omega}}\delta_0 \right\|^2_{\ell^2(\mathbb{Z}^3)} = \mathbb{E}\left[\left\| R_{0,\delta,\ell} e^{-itH_\omega}\delta_0 \right\|^2_{\ell^2(\mathbb{Z}^3)} \right] \tag{11}$$

with probability one. We note here that clearly, the left hand side of (7) is independent of $x \in \mathbb{Z}^3$. Therefore, (7), (10) and (11) imply

$$\liminf_{L\to\infty} \frac{|\mathcal{A}^c_{L,\varepsilon,\delta,\ell}|}{|\mathcal{A}_L|} \geq 1 - \frac{\kappa}{1+\kappa} - \frac{2+\kappa^{-1}}{1+\kappa}\varepsilon$$

with probability one, and choosing $\kappa = \varepsilon^{\frac{1}{2}}$, the claim follows. $\square$

Henceforth, we will write $\|\cdot\|_2 \equiv \|\cdot\|_{\ell^2(\mathbb{Z}^3)}$ for brevity. The key to the conclusion of the proof of Theorem 2.1 is Lemma 3.3 below, which implies that for the choice $t = \delta^{\frac{6}{7}}\lambda^{-2}$,

$$\mathbb{E}\left[\left\| R_{x,\delta,\frac{1}{\lambda^2}} e^{-i\delta^{\frac{6}{7}}\lambda^{-2}H_\omega}\delta_x \right\|_2 \right] \geq 1 - c\delta^{\frac{3}{7}} \tag{12}$$

holds for $\lambda^{\frac{14}{15}} < \delta < 1$, and some constant c that is independent of x, λ and δ. Thus, with the choice $\varepsilon = \delta^{\frac{3}{7}}$, (12) immediately implies the assertion of Theorem 2.1.

Lemma 3.3. *Let $t = \delta^{\frac{6}{7}}\lambda^{-2}$, and $H_0 := -\frac{1}{2}\Delta$. Then, for λ sufficiently small, $0 < \delta < 1$, and all $x \in \mathbb{Z}^3$, the free evolution term satisfies*

$$\left\| R_{x,\delta,\frac{1}{\lambda^2}} e^{-itH_0}\delta_x \right\|_2 \geq 1 - c\delta^{\frac{3}{7}} \, , \tag{13}$$

while the sum over collision histories yields

$$\mathbb{E}\left[\left\| R_{x,\delta,\frac{1}{\lambda^2}} \left(e^{-itH} - e^{-itH_0}\right)\delta_x \right\|^2_2 \right] \leq c'\delta^{\frac{6}{7}} + t^{-\frac{1}{3}} \, , \tag{14}$$

for positive constants $c, c' < \infty$ that are independent of x, λ and δ.

Proof. We may assume that $x = 0$. Since $K_{\ell_1} \geq K_{\ell_1}^2$, we have $(R_{0,\delta,\ell})^2 \geq K_{\ell_2}^2 - K_{\ell_1}$. To bound $\|K_{\ell_2} e^{-itH_0}\delta_0\|_2$, we note that

$$\begin{aligned} \left| e^{-ite(p)} - \int_{\mathbb{T}^3} dk \hat{K}_{\ell_2}(p-k) e^{-ite(k)} \right| &\leq c \sup_{|p-k|\leq\gamma} \left| e^{-ite(p)} - e^{-ite(k)} \right| \\ &\quad +2 \int_{|p-k|\geq\gamma} dk |\hat{K}_{\ell_2}(k)| \\ &\leq c\Big(t\gamma + \frac{1}{1+\gamma\ell_2} \Big) , \end{aligned} \tag{15}$$

since by basic properties of the Fejér kernel,

$$|\hat{K}_{\ell_2}(k)| \leq \prod_{j=1}^{3} \frac{c\ell_2}{1 + \|k_j\|_{\mathbb{Z}}^2 \ell_2^2} , \quad \int_{\mathbb{T}^3} dk \hat{K}_{\ell_2}(k) = 1 ,$$

where $\|r\|_{\mathbb{Z}} :=$dist$(r, \mathbb{Z})$ for $r \in \mathbb{R}$. Thus, with $\gamma = (t\ell_2)^{-1/2}$, $\ell_2 = \lambda^{-2}$, and $t = \delta^\alpha \lambda^{-2}$, we find (15) $\leq c\delta^{\alpha/2}$. Hence,

$$\begin{aligned} \big\| K_{\ell_2} e^{-itH_0} \delta_0 \big\|_2^2 &= \int_{\mathbb{T}^3} dp \Big| e^{-ite(p)} + O(\delta^{\alpha/2}) \Big|^2 \\ &\geq 1 - c\delta^{\alpha/2} \end{aligned} \tag{16}$$

follows immediately.

To bound $\||K_{\ell_1}|^{1/2} e^{-itH_0}\delta_0\|_2$, we derive the usual propagation estimates for the free time evolution operator in the lattice case. Clearly,

$$\big\| |K_{\ell_1}|^{1/2} e^{-itH_0} \delta_0 \big\|_2^2 = \sum_{y\in\mathbb{Z}^3} |K_{\ell_1}(y)| \, \big| (e^{-itH_0}\delta_0)(y) \big|^2 , \tag{17}$$

where $(e^{-itH_0}\delta_0)(y) = \int_{\mathbb{T}^3} dk e^{-ite(k)} e^{2\pi iky}$. The kinetic energy $e(\,\cdot\,) : \mathbb{T}^3 \to [0, 6]$ is a real analytic Morse function with 8 critical points in the corners of the embedded subcube $[0, \frac{1}{2}]^3$ of $\mathbb{T}^3 = [0, 1]^3$. Each of the remaining critical points in $[0, 1]^3 \setminus [0, \frac{1}{2}]^3$ is identified with one of the latter by symmetry. The Hessians are diagonal in the present coordinate system, and have entries $\pm(2\pi)^2$.

We introduce a suitable smooth partition of unity ϕ_j on $[0, 1]^3$ with $j = 1, \ldots, 8$ and $\sum \phi_i = 1$, continued over the boundary of the unit cube by periodicity, and such that every ϕ_i contains precisely one critical point p_{crit} of $e(\,\cdot\,)$ in its interior, yielding

$$\begin{aligned} \big| (e^{-itH_0}\delta_0)(y) \big| &= \Big| \sum_{j=1}^{8} \int_{\mathbb{T}^3} dk \phi_j(k) e^{-ite(k)} e^{2\pi iky} \Big| \\ &\leq c\Big(t^{-3/2} + t^{-5/2} \sup_{y\in\text{supp}\{K_{\ell_1}\}} \{|y|\} \Big) , \end{aligned} \tag{18}$$

for $y \in \text{supp}\{K_{\ell_1}\}$ by a standard stationary phase estimate. Since $\sup_{y\in\text{supp}\{K_{\ell_1}\}}\{|y|\} \leq \ell_1$, we find for $\ell_1 = \delta\lambda^{-2}$, and $t = \delta^\alpha\lambda^{-2}$ that

$$|(17)| \leq c\frac{\ell_1^3}{t^3} = c\delta^{3(1-\alpha)} \; . \tag{19}$$

Matching the sizes of the error terms, we must have $3(1-\alpha) = \alpha/2$, that is, $\alpha = \frac{6}{7}$.

In order to establish (14), we use a modification of the methods of L. Erdös and H.-T. Yau from [3]. That is, we invoke a Duhamel expansion with remainder term, and control the expectation by classifying all contraction types occurring in the products of the random potential. To bound the remainder term, we shall exploit the rarity of the event that a large number of collisions occurs in a small time interval. The result is

$$\mathbb{E}\left[\left\|R_{x,\delta,\frac{1}{\lambda^2}}\big(e^{-itH} - e^{-itH_0}\big)\delta_x\right\|_2^2\right] \leq c'\lambda^2 t + t^{-\frac{1}{3}} \; , \tag{20}$$

for some constant c' that is independent of x, λ and δ. One thus immediately obtains (14) for the asserted choice of t. The proof of (20) will occupy the rest of section 3. $\square$

3.1. **Expectation of products of random potentials.** We shall next discuss the expectation of products of the random potential V_ω in more detail. The pair correlator is given by the Kronecker delta

$$\mathbb{E}\big[\omega_{x_1}\omega_{x_2}\big] = \delta_{x_1,x_2} \; ,$$

while for the four-point correlator, we have

$$\begin{aligned}\mathbb{E}\big[\omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4)\big] &= (1-\delta_{x_1,x_3})\delta_{x_1,x_2}\delta_{x_3,x_4} + (1-\delta_{x_1,x_2})\delta_{x_1,x_3}\delta_{x_2,x_4} \\ &+ (1-\delta_{x_1,x_3})\delta_{x_1,x_4}\delta_{x_2,x_3} + \tilde{c}_4\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3} \\ &= \delta_{x_1,x_2}\delta_{x_3,x_4} + \delta_{x_1,x_3}\delta_{x_2,x_4} + \delta_{x_1,x_4}\delta_{x_2,x_3} \\ &+ (\tilde{c}_4 - 3)\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3} \; .\end{aligned}$$

We will refer to the operation applied in passing to the expression after the second equality sign as Wick ordering. Its effect is that by a renormalization of the fourth order moment of ω_x,

$$\tilde{c}_4 \to c_4 := \tilde{c}_4 - 3\tilde{c}_2^2 \; ,$$

(where $\tilde{c}_2 = 1$) the products of pair correlators become independent.

The relevance of Wick ordering can be seen as follows. Let $\hat{\omega}(k) := \sum_x \omega_x e^{2\pi i k x}$. One easily verifies that

$$\begin{aligned}\mathbb{E}\big[\hat{\omega}(k_1)\hat{\omega}(k_2)\hat{\omega}(k_3)\hat{\omega}(k_4)\big] &= \delta(k_1+k_2)\delta(k_3+k_4) \\ &+ \delta(k_1+k_3)\delta(k_2+k_4) + \delta(k_1+k_4)\delta(k_2+k_3) \\ &+ c_4\delta(k_1+k_2+k_3+k_4) \; .\end{aligned}$$

That is, for the Wick ordered expression, one obtains exact Dirac deltas. However, we note that this is not true for the individual summands of the expression prior to Wick ordering.

Along these lines, we shall next determine Wick ordered products of an arbitrary even number of random potentials. We introduce, for $n, n' \in \mathbb{N}$ with $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$, the set

$$\mathcal{V}_{n,n'} := \Big\{1, \ldots, n, n+2, \ldots, n+n'+1\Big\} \; .$$

In our later discussion, $\mathcal{V}_{n,n'}$ labels a linearly ordered set of $n+n'$ random potentials that are, in frequency space, subdivided into a group of n' copies of $\hat{V}_\omega$, and a group of n copies of $\overline{\hat{V}_\omega}$ (the complex conjugate). The label $n+1$ excluded here is reserved for a distinguished point that does not belong to a random potential. We note again that the case of $n+n' \in 2\mathbb{N}_0+1$ is a priori trivial by the assumption (2).

Definition 3.1. *For $\bar{n} = \frac{n+n'}{2} \in \mathbb{N}$, let*

$$\Pi_{n,n'} := \bigcup_{m=1}^{\bar{n}} \left\{ \{S_j\}_{j=1}^m \Big| |S_j| \in 2\mathbb{N} \; ; \; \mathcal{V}_{n,n'} = \cup_{j=1}^m S_j \; ; \; S_j \cap S_{j'} = \emptyset \text{ if } j \neq j' \right\} \Big/ \mathfrak{S}_m$$

denote the set of partitions of $\mathcal{V}_{n,n'}$ into disjoint subsets S_j (referred to as blocks) of size $|S_j| \in 2\mathbb{N}$, where $\mathfrak{S}_m$ is the m-th symmetric group. Two partitions $\pi = \{S_j\}_{j=1}^m$, $\pi' = \{S_j'\}_{j=1}^m$, are equivalent, $\pi = \pi'$, if $\exists \sigma \in \mathfrak{S}_m$ such that $S_j = S'_{\sigma(j)}$ for all $j \in \{1, \dots, m\}$. A partition $\pi \in \Pi_{n,n'}$ will also be referred to as a contraction (corresponding to contractions among random potentials).

The number of $\pi \in \Pi_{n,n'}$ consisting of m blocks is given by

$$\begin{aligned} B_{\bar{n}}(m) \; &:= \; \sharp \left\{ \{S_j\}_{j=1}^m \Big| \cup_{j=1}^m S_j = \mathcal{V}_{n,n'} \, ; \, |S_j| \in 2\mathbb{N} \, ; S_i \cap S_j = \emptyset \text{ if } i \neq j \right\} \Big/ \mathfrak{S}_m \\ &= \; \sum_{r=1}^{\bar{n}} \sum_{1 \le j_1, \cdots, j_r \le \bar{n}} \sum_{1 \le l_1 < \cdots < l_r \le \bar{n}} \delta_{m, |\underline{j}|} \delta_{\bar{n}, \langle \underline{j}, \underline{l} \rangle} \frac{(2\bar{n})!}{((2l_1)!)^{j_1} \cdots ((2l_r)!)^{j_r}} \frac{1}{(j_1!) \cdots (j_r!)} \, , \end{aligned} \tag{21}$$

where $\underline{j} := (j_1, \dots, j_r)$, $|\underline{j}| := \sum_{i=1}^r j_i$, and $\langle \underline{j}, \underline{l} \rangle := \sum_{i=1}^r j_i l_i$ for every r. Here, j_i is the number of blocks of size $2l_i$. The factor $\frac{1}{j_i!}$ arises because the order is irrelevant, according to which blocks of the same size are counted. We note that the number of partitions into products of pair correlators, corresponding to the case $r = 1$, $j = \bar{n}$, $l = 1$, is

$$B_{\bar{n}}(\bar{n}) = 1 \cdot 3 \cdots (2\bar{n} - 1) < 2^{\bar{n}} (\bar{n}!) \, .$$

On the other hand, it is clear that

$$B_{\bar{n}}(m) < \sum_{0 \le s_1, \dots, s_m \le 2\bar{n}} \delta_{2\bar{n}, \sum_{i=1}^m s_i} \frac{(2\bar{n})}{(s_1!) \cdots (s_m!)} = m^{2\bar{n}} \, ,$$

hence for non-pairing contractions, i.e. $m < \bar{n}$,

$$\sum_{m=1}^{\bar{n}-1} B_{\bar{n}}(m) < \bar{n}^{2\bar{n}+1} \, .$$

This trivial estimate will suffice for our purposes.

For $S \subset \mathcal{V}_{n,n'}$, with $|S| \in 2\mathbb{N}$, we define

$$\delta(x_S) := \sum_{y \in \mathbb{Z}^3} \prod_{j \in S} \delta_{x_j, y} \, ,$$

where $x_S := (x_j)_{j\in S}$. Then,

$$(22)\qquad \mathbb{E}\Big[\prod_{j\in\mathcal{V}_{n,n'}} V_\omega(x_j)\Big] = \sum_{m=1}^{\bar n} \sum_{\substack{\pi\in\Pi_{n,n'} \\ \pi=\{S_j\}_{j=1}^m}} \Big(\prod_{j=1}^m \tilde c_{|S_j|}\delta(x_{S_j})\Big) \prod_{1\le i<j\le m}(1-\delta_{x_{\mu(i)},x_{\mu(j)}})$$

where for definiteness, $\mu(i) := \min\big\{q\big| q\in S_i\big\}$ (clearly, one could choose any arbitrary element of S_i). Due to the second product, the factors in $\prod c_{|S_j|}\delta_{S_j}$ are not independent. We note that

$$(23)\qquad \delta(x_{S_i})\delta(x_{S_j})\delta_{x_{\mu(i)},x_{\mu(j)}} = \delta(x_{S_i\cup S_j})\ ,$$

where of course, $|S_i\cup S_j| = |S_i| + |S_j|$. Therefore, expanding $\prod(1-\delta_{x_{\mu(i)},x_{\mu(j)}})$ in (22), using (23) recursively, and collecting all terms belonging to the same blocks, we find

$$(24)\qquad \mathbb{E}\Big[\prod_{j\in\mathcal{V}_{n,n'}} V_\omega(x_j)\Big] = \sum_{m=1}^{\bar n} \sum_{\substack{\pi\in\Pi_{n,n'} \\ \pi=\{S_j\}_{j=1}^m}} \prod_{j=1}^m c_{|S_j|}\delta(x_{S_j})\ ,$$

where

$$c_{2k} \;=\; \sum_{m=1}^k\sum_{r=1}^k \sum_{1\le j_1,\cdots,j_r\le k}\ \sum_{1\le l_1<\cdots<l_r\le k} \delta_{m,|\underline{j}|}\delta_{k,\langle\underline{j},\underline{l}\rangle} \frac{(-1)^{m-1}(2k)!}{((2l_1)!)^{j_1}\cdots((2l_r)!)^{j_r}} \frac{\tilde c_{2l_1}^{j_1}\cdots\tilde c_{2l_r}^{j_r}}{(j_1!)\cdots(j_r!)}$$

are the renormalized moments of ω_x. Thus, (24) decomposes the expectation value into the sum of all possible products of correlators, which are now mutually independent. It is easy to see that

$$(25)\qquad |c_{2k}| \le (c_\omega k)^{2k+1}\ ,$$

where $c_\omega \ge 1$ is the constant defined in (2). This simple bound suffices for our purposes.

Then, the expectation of the full product of random potentials decomposes into

$$(26)\qquad \mathbb{E}\Big[\prod_{j\in\mathcal{V}_{n,n'}} \hat V_\omega(k_j)\Big] = \sum_{m=1}^{\bar n} \sum_{\substack{\pi\in\Pi_{n,n'} \\ \pi=\{S_j\}_{j=1}^m}} \Big(\prod_{j=1}^m c_{|S_j|}\Big)\delta\Big(\sum_{i\in S_j} k_i\Big)$$

in momentum space, where $c_{|S_j|}$ are the renormalized moments of ω_x.

3.2. **Duhamel Expansion.** As previously noted, our proof of (20) comprises a modification of the methods developed in [3] to cover the lattice case, together with a generalization that allows to control non-Gaussian distributed random potentials. To this end, we invoke the Duhamel expansion of $\phi_t = e^{-itH_\omega}\delta_x$, then classify and estimate all contractions occurring in (14). In this analysis, $\varepsilon = t^{-1}$ and λ will be the small parameters of the theory, ultimately to be related by $\varepsilon = c\lambda^2$.

For some large $N \in \mathbb{N}$, which will be chosen as a function of ε later, we consider the trucated Duhamel expansion

$$\begin{aligned}\Big(\big(e^{-itH_\omega} - e^{-itH_0}\big)\delta_x\Big)(y) &= \Big((-i\lambda)\int_0^t ds e^{-i(t-s)H_0} V_\omega e^{-isH_\omega}\delta_x\Big)(y) \\ &= \sum_{n=1}^{N} \hat{\phi}_{n,t}(y) + R_{N,t}(y) \,.\end{aligned}$$

The explicit formula for the remainder term $R_{N,t}$ is given in (65) below. Defining

$$\Big[\prod_{j=0}^{n} ds_j\Big]_t := ds_0 \cdots ds_n \, \delta(\sum_{j=0}^{n} s_j - t) \,,$$

the Fourier transform of $\phi_{n,t}$ is given by

$$\begin{aligned}\hat{\phi}_{n,t}(k_0) &= (-i\lambda)^n \int_{\mathbb{R}_+^{n+1}} \Big[\prod_{j=0}^{n} ds_j\Big]_t \int_{(\mathbb{T}^3)^n} e^{2\pi i k_n\cdot x} e^{-i\sum_{j=0}^{n} s_j e(k_j)} \prod_{j=1}^{n} \hat{V}_\omega(k_j - k_{j-1}) dk_j \\ &= e^{\varepsilon t}(-i\lambda)^n \int_{\mathbb{R}} d\alpha e^{-i\alpha t} \int_{(\mathbb{T}^3)^n} e^{2\pi i k_n\cdot x} \Big[\prod_{j=0}^{n} \frac{1}{e(k_j) - \alpha - i\varepsilon}\Big] \prod_{l=1}^{n} \hat{V}_\omega(k_l - k_{l-1}) dk_l \,,\end{aligned}$$

which is true for arbitrary $\varepsilon > 0$. In this work, we will always choose $\varepsilon = t^{-1}$, to keep the exponential factor $e^{\varepsilon t}$ bounded for large t. The multiplication operators $\frac{1}{e(k_j)-\alpha-i\varepsilon}$ corresponding to the Fourier transformed resolvents of $-\frac{1}{2}\Delta$ will also be referred to as particle propagators.

Let $\mathbb{H}_- := \{z \in \mathbb{C} \Big| Im(z) \leq 0\}$. By analyticity of the integrand with respect to the variable α, and noting its exponential decay as $Im(z) \to -\infty$, the path of the α-integration can, for any fixed $n \in \mathbb{N}$, be deformed away from $\mathbb{R}$ into the closed contour

$$I = I_{\mathbb{R}} \cup I_{\mathbb{H}_-} \tag{27}$$

with

$$\begin{aligned}I_{\mathbb{R}} &:= [-1, 7] \\ I_{\mathbb{H}_-} &:= ([-1,7] - i) \cup (-1 - i(0,1]) \cup (7 - i(0,1]) \subset \mathbb{H}_- \,,\end{aligned}$$

which is the boundary of a rectangle whose interior contains $\mathrm{spec}\big\{-\frac{1}{2}\Delta - i\varepsilon\big\} = [0,6] - i\varepsilon$. Consequently,

$$\begin{aligned}\hat{\phi}_{n,t}(k_0) = e^{\varepsilon t}(-i\lambda)^n \int_I d\alpha e^{-i\alpha t} \int_{(\mathbb{T}^3)^n} dk_1 \cdots dk_n \, e^{2\pi i k_n\cdot x} \\ \cdot \Big[\prod_{j=0}^{n} \frac{1}{e(k_j) - \alpha - i\varepsilon}\Big] \prod_{l=1}^{n} \hat{V}_\omega(k_l - k_{l-1}) \,,\end{aligned}$$

where the loop I is taken in clockwise direction.

By the Schwarz inequality,

$$\text{l.h.s. of (20)} \le 2\sum_{n=1}^{N}\mathbb{E}\left[\left\|\phi_{n,t}\right\|_2^2\right] + 2\mathbb{E}\left[\left\|R^{\varepsilon}_{N,t}\right\|_2^2\right] \ .$$

For $1 \le n \le N$, we have

$$\begin{aligned}
\mathbb{E}\left[\|\phi_{n,t}\|_2^2\right] &= e^{2\varepsilon t}\lambda^{2n}\int_{I\times\bar I} d\alpha d\beta e^{-it(\alpha-\beta)}\int_{(\mathbb{T}^3)^{2n+2}}\prod_{l=0}^{n} dk_l d\tilde k_l e^{-2\pi i(k_n-\tilde k_n)\cdot x}\\
&\quad\cdot\,\delta(k_0-\tilde k_0)\prod_{j=0}^{n}\frac{1}{e(k_j)-\alpha-i\varepsilon}\frac{1}{e(\tilde k_j)-\beta+i\varepsilon}\\
&\quad\cdot\,\mathbb{E}\Big[\prod_{i=1}^{n}\hat V_\omega(k_i-k_{i-1})\bar{\hat V}_\omega(\tilde k_i-\tilde k_{i-1})\Big]\ , \qquad (28)
\end{aligned}$$

where $\bar I$ is the complex conjugate of I, and taken in the counterclockwise direction by the variable β. Introducing the variables

$$\begin{aligned}
\underline{p} &= (p_0,\dots,p_n,p_{n+1},\dots,p_{2n+1})\\
&:= (\tilde k_n,\dots,\tilde k_0,k_0,\dots,k_n)\\
(\alpha_j,\sigma_j) &= \begin{cases}(\alpha,1) & 0\le j\le n\\ (\beta,-1) & n<j\le 2n+1\ ,\end{cases}
\end{aligned}$$

one can write more compactly

$$\begin{aligned}
\mathbb{E}[\|\phi_{n,t}\|_2^2] &= e^{2\varepsilon t}\lambda^{2n}\int_{I\times\bar I} d\alpha d\beta e^{-it(\alpha-\beta)}\int_{(\mathbb{T}^3)^{2n+2}} d\underline{p}\,\delta(p_n-p_{n+1})\\
&\quad\cdot\, e^{-2\pi i(p_0-p_{2n+1})\cdot x}\prod_{j=0}^{2n+1}\frac{1}{e(p_j)-\alpha_j-i\sigma_j\varepsilon}\\
&\quad\cdot\,\mathbb{E}\Big[\prod_{\substack{i=1\\ i\neq n+1}}^{2n+1}\hat V_\omega(p_i-p_{i-1})\Big]\ , \qquad (29)
\end{aligned}$$

where $\bar{\hat V}(k)=\hat V(-k)$.

Recalling (26), we consider a partition $\pi=\{S_j\}_{j=1}^m\in\Pi_{n,n}$, and define

$$\delta_{S_j}(\underline{p}) := \delta\Big(\sum_{i\in S_j}(p_i-p_{i-1})\Big)\ .$$

The contribution to (29) corresponding to π is given by

$$
\begin{aligned}
C_\pi \;:=\; & e^{2\varepsilon t}\lambda^{2n}\int_{I\times\bar I} d\alpha d\beta e^{-it(\alpha-\beta)}\int_{(\mathbb{T}^3)^{2n+2}} d\underline{p}\delta(p_n-p_{n+1}) \\
& \cdot e^{-2\pi i(p_0-p_{2n+1})\cdot x}\Big(\prod_{l=1}^{m} c_{|S_l|}\Big)\delta_\pi(\underline{p}) \\
& \cdot \prod_{j=0}^{2n+1}\frac{1}{e(p_j)-\alpha_j-i\sigma_j\varepsilon}\,, \qquad (30)
\end{aligned}
$$

where $\delta_\pi(\underline{p}) := \prod_{j=1}^m \delta_{S_j}(\underline{p})$. The expectation (29) is thus obtained from summing the quantities C_π over all partitions $\pi\in\Pi_{n,n}$.

3.3. **The Graph Representation of Contractions.** For the combinatorial problem of classifying the possible classes of integrals C_π, it is natural to represent the delta distributions in (30) that correspond to various contractions by graphs. We will use the following prescription, cf. Figure 1. We draw two parallel solid 'particle lines', joined together at one end, accounting for $\delta(p_n-p_{n+1})$, containing n, respectively n' vertices, where $n+n'\in 2\mathbb{N}$. Every pairing contraction is depicted by a dashed line joining the respective vertices. The higher correlation contractions corresponding to δ_{S_j} are represented by $|S_j|\in 2\mathbb{N}$ dashed lines connecting the corresponding vertices to one mutual vertex that is disjoint from the particle lines. Any edge in this graph that lies on a particle line corresponds to a particle propagator.

Let G_π denote the graph associated to a partition $\pi=\{S_j\}_{j=1}^m\in\Pi_{n,n'}$. The set of vertices of the graph G_π is denoted by $V(G_\pi)$, and the set of edges as $E(G_\pi)$. $V(G_\pi)$ is the union $V(G_\pi)=V_p(G_\pi)\cup V_h(G_\pi)$, where $V_p(G_\pi)=\mathcal{V}_{n,n'}$ is the $n+n'$-subset of vertices on the particle line, and $V_{hc}(G_\pi)$ is the subset of vertices disjoint from the particle lines, which are associated to higher correlations. In the product of Kronecker deltas (24) in the position space picture, the elements of $V_p(G_\pi)$ correspond to the sites x_i of random potentials, while the elements of $V_{hc}(G_\pi)$ correspond to the dummy summation variables y_j.

A spanning tree T of G_π is a connected tree graph that contains $V(G_\pi)$. We denote the set of edges contained in T by E_T, and refer to the corresponding momenta as tree momenta. The momenta corresponding to the edges in the complement $E_L=E_T'$ are referred to as loop momenta. Adding any edge of E_L to the spanning tree T produces a loop.

Definition 3.2. *A contraction $\pi=\{S_j\}\in\Pi_{n,n'}$ is called a pairing contraction if $m=\bar n$, from which follows that $|S_j|=2$ for all j. Otherwise, it is called a higher correlation contraction, or a type III contraction (cf. Definition 3.3 below).*

Let us further classify the families of contractions as follows, cf. [3].

Definition 3.3. *The delta distributions associated to partitions $\pi\in\Pi_{n,n'}$ of the set $\mathcal{V}_{n,n'}$ are classified into the following types, according to the graph structure of the corresponding contraction among random potentials.*
A delta function $\delta_S(\underline{p})=\delta(p_i-p_{i-1}+p_j-p_{j-1})$ associated to a pairing S with $|S|=2$ is of Type I *if $i,j\le n$.*

TYPE I' *if* $i, j > n$.
TYPE II *if* $i \leq n$, *but* $j \geq n+2$.
A delta function δ_S *is of*
TYPE III *if* $|S| \geq 4$, *that is, if it is not associated to a pairing contraction.*

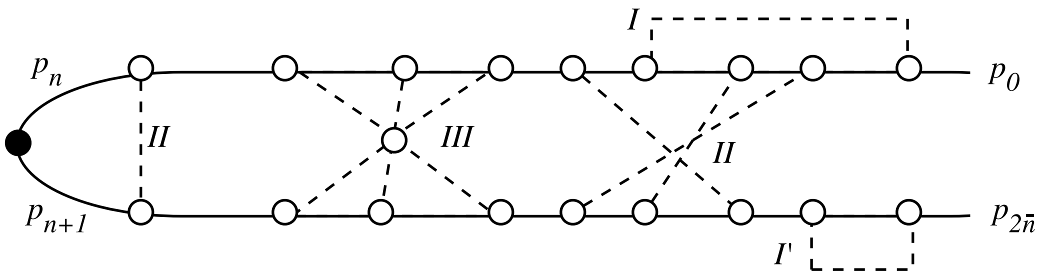

Figure 1.A graph containing type I, I', II, and III contractions.

Hence, a partition of $\mathcal{V}_{n,n'}$ is of type III if it contains a type III delta distribution.

Definition 3.4. *A pairing contraction* $\pi \in \Pi_{n,n'}$ *is called crossing if* δ_π *contains two delta distributions* $\delta(p_{i_1} - p_{i_1-1} + p_{j_1} - p_{j_1-1})$ *and* $\delta(p_{i_2} - p_{i_2-1} + p_{j_2} - p_{j_2-1})$, *with* $j_r > i_r$, *such that* $i_1 - i_2$ *and* $j_1 - j_2$ *have the same signs.*

Definition 3.5. *A non-crossing pairing contraction* $\pi \in \Pi_{n,n'}$ *is called nested if* δ_π *contains two delta distributions* $\delta(p_{i_1} - p_{i_1-1} + p_{j_1} - p_{j_1-1})$ *and* $\delta(p_{i_2} - p_{i_2-1} + p_{j_2} - p_{j_2-1})$, *with* $j_r > i_r$, *both either of type I or of type I', such that* $i_1 - i_2$ *and* $j_1 - j_2$ *have opposite signs.*

Definition 3.6. *A non-crossing and non-nested pairing contraction is called simple. A simple pairing contraction is called a ladder graph if all of its associated delta functions are of type II.*

Definition 3.7. *A spanning tree* T *of* G_π *with* $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n'}$ *is called complete if it contains the edge corresponding to the momentum* p_n, *but not the one corresponding to the momentum* p_{n+1}.

3.4. **Simple Pairing Contractions.** For each type of contractions $\pi \in \Pi_{n,n}$ listed above, we will next derive bounds on the corresponding Feynman amplitudes C_π. It turns out that as in [3], all dominant contributions stem from simple pairings, even with type III contractions included. We will proceed by first discussing simple pairings, then crossing and nested pairings, and finally type III contractions.

3.4.1. *The Ladder Graph.* For each fixed n, the simplest member in the class of simple pairings in $\Pi_{n,n}$ is the ladder graph. It corresponds to the pairing $\pi = \{S_j\}_{j=1}^n \in \Pi_{n,n}$, with $S_j = \{j, 2n+2-j\}$, such that $|S_j| = 2$, and

$$\begin{aligned} C_\pi &= e^{2\varepsilon t}\lambda^{2n} \int_{I\times\bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \int_{(\mathbb{T}^3)^{2n+2}} d\underline{p}\delta(p_n - p_{n+1}) e^{-2\pi i(p_0 - p_{2n+1})\cdot x} \\ &\quad \cdot \Big[\prod_{l=0}^{2n+1} \frac{1}{e(p_l) - \alpha_l - i\sigma_l\varepsilon} \Big] \prod_{j=1}^{n} \delta\Big((p_j - p_{j-1}) + (p_{2n+2-j} - p_{2n+1-j}) \Big) , \end{aligned} \tag{31}$$

cf. (30). The following three obvious estimates will be used extensively in the sequel.

Lemma 3.4. *Let $0 < \varepsilon \ll 1$. Then,*

$$\Big|\frac{1}{e(p)-\alpha-i\varepsilon}\Big| \;\leq\; \varepsilon^{-1} \tag{32}$$

$$\sup_{\alpha\in I}\int_{\mathbb{T}^3} dp\Big|\frac{1}{e(p)-\alpha-i\varepsilon}\Big| \;,\; \sup_{p\in\mathbb{T}^3}\int_I |d\alpha|\Big|\frac{1}{e(p)-\alpha-i\varepsilon}\Big| \;\leq\; c_1|\log\varepsilon| \tag{33}$$

for a constant c_1 that is uniform in ε.

Proof. Recalling that by definition of I, $\inf_{p\in\mathbb{T}^3}\mathrm{dist}(e(p)-i\varepsilon, I) = \varepsilon$, and that $|I|$ is finite, (32) and the second estimate in (33) are evident.

To prove the first estimate in (33), we first show that the measure of the isoenergy surface

$$\Sigma_\alpha := \Big\{p\in\mathbb{T}^3\Big|e(p)=\alpha\Big\}$$

is uniformly bounded in $\alpha \in I\cap\mathbb{R}$. We note that the case $Im(\alpha)\neq 0$, the asserted bound is trivial. To this end, for $p=(p_1,p_2,p_3)\in\mathbb{T}^3$, let $e_2(p_1,p_2) := \sum_{j=1}^2(1-\cos 2\pi p_j)$ denote the Fourier transform of the 2-dimensional lattice Laplacian, and

$$A_2(r) := \mathrm{meas}\Big\{(p_1,p_2)\in\mathbb{T}^2\Big|e_2(p_1,p_2)=r\Big\}$$

the length of the corresponding isoenergy curve. It is easy to see that $\|A_2\|_\infty < c$. Clearly,

$$\begin{aligned}
\mathrm{meas}\{\Sigma_\alpha\} &= \int_{[0,1]^3} dp_1\,dp_2\,dp_3\,\delta(e(p)-\alpha)\\
&= \int_{[0,1]} dp_3\int_0^\alpha dr\,A_2(r)\delta(r+(1-\cos 2\pi p_3)-\alpha)\\
&= \int_{[0,1]} dp_3\,A_2(\alpha-(1-\cos 2\pi p_3))\chi(1-\cos 2\pi p_3-\alpha\geq 0)\\
&\leq \|A_2\|_\infty < c\;,
\end{aligned}$$

uniformly in α. Thus, defining

$$R_j(\alpha,\varepsilon) := \Big\{p\in\mathbb{T}^3\Big|p\in\Sigma_{\alpha'}\;,\;2^j\varepsilon<|\alpha'-\alpha|\leq 2^{j+1}\varepsilon\Big\}\;,$$

we have $\mathrm{meas}\{R_j(\alpha,\varepsilon)\}\leq c2^j\varepsilon$ for a constant c that is independent of j, ε, and α.

Hence, by a dyadic decomposition of $\mathbb{T}^3$ centered around Σ_α, we find

$$\begin{aligned}
\int_{\mathbb{T}^3} dp\Big|\frac{1}{e(p)-\alpha-i\varepsilon}\Big| &\leq C+\sum_{j=0}^{c\log\frac{1}{\varepsilon}}\int_{R_j(\alpha,\varepsilon)} dp\Big|\frac{1}{e(p)-\alpha-i\varepsilon}\Big|\\
&\leq C\Big(1+\sum_{j=0}^{c\log\frac{1}{\varepsilon}}\mathrm{meas}\{R_j(\alpha,\varepsilon)\}\frac{1}{2^j\varepsilon}\Big)\\
&\leq c_1\log\frac{1}{\varepsilon}\;,
\end{aligned}$$

for some constant c_1 that is independent of ε and α, as claimed. $\square$

Lemma 3.5. *Let*

$$
\begin{aligned}
K^{(n)}(p_0,\dots,p_n;t) &:= \int ds_0\dots ds_n \delta\Big(t-\sum_{r=0}^{n} s_r\Big) e^{-i\sum_{j=0}^{n} s_j e(p_j)} \\
&= e^{\varepsilon t}\int_I d\alpha e^{-i\alpha t}\prod_{j=0}^{n}\frac{1}{e(p_j)-\alpha-i\varepsilon}\,.
\end{aligned}
$$

Then, for every $0<\mu<1$*, there exists a constant* $c_\mu<\infty$*, such that*

$$
\big\|K^{(n)}(\,\cdot\,;t)\big\|^2_{L^2((\mathbb{T}^3)^{n+1})}\le\frac{(c_\mu t)^n}{(n!)^\mu}\,.
$$

Proof. Clearly,

$$
\begin{aligned}
\big\|K^{(n)}(\,\cdot\,;t)\big\|_{L^\infty((\mathbb{T}^3)^{n+1})} &\le \int_{\mathbb{R}_+^{n+1}} ds_0\dots ds_n\delta(t-\sum s_r)\\
&= \frac{t^n}{n!}\,.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\big\|K^{(n)}(\,\cdot\,;t)\big\|^{2-\mu}_{L^{2-\mu}((\mathbb{T}^3)^{n+1})} &\le \int_{(\mathbb{T}^3)^{n+2}} d\underline{p}\delta(p_n-p_{n+1})\Big(e\int_I|d\alpha|\prod_{j=0}^{n}\Big|\frac{1}{e(p_j)-\alpha-i\varepsilon}\Big|\Big)^{2-\mu}\\
&\le (e|I|)^2\int_{(\mathbb{T}^3)^{n+2}} d\underline{p}\delta(p_n-p_{n+1})\int_I|d\alpha|\prod_{j=0}^{n}\Big|\frac{1}{e(p_j)-\alpha-i\varepsilon}\Big|^{2-\mu}\\
&\le c_\mu^n\, t^{n(1-\mu)}\,, \qquad (34)
\end{aligned}
$$

for some constant $c_\mu<\infty$.

The claim thus follows immediately from interpolation. We note that the proof in the lattice case is much simpler than in the continuum case, [3], owing to $|I|<\infty$ and $|\mathbb{T}^3|=1<\infty$. $\square$

The ladder contribution can thus be estimated by

$$
\begin{aligned}
\lambda^{2n}\int_{(\mathbb{T}^3)^{2n+2}} & d\underline{p}K^{(n)}(p_0,\dots,p_n;t)\bar{K}^{(n)}(p_{n+1},\dots,p_{2n+1};t)\prod_{i=1}^{n}\delta(p_i-p_{2n+1-i})\\
&\le\lambda^{2n}\big\|K^{(n)}(\cdots;t)\big\|^2_{L^2((\mathbb{T}^3)^{n+1})}\\
&\le\frac{(c_\mu\lambda^2 t)^n}{(n!)^\mu} \qquad (35)
\end{aligned}
$$

for $0<\mu<1$. Clearly, the product of delta distributions here is equivalent to that of (31).

3.4.2. *Immediate Recollisions.* We next estimate general simple pairings which include all possible combinations of type I and I' contractions. Any type I or type I' delta function $\delta(p_i - p_{i-1} + p_j - p_{j-1})$, where $j > i$, necessarily exhibits $i = j - 1$. Otherwise, either a crossing or a nesting pairing occurs. Hence, any type I or I' delta function in a simple pairing reduces to $\delta(p_{i+1} - p_{i-1})$, for some i.

Definition 3.8. *A type I or type I' pairing of the form $\delta(p_{i+1} - p_{i-1})$ is called an immediate recollision.*

The subintegral in C_π corresponding to an immediate recollision is either

$$\Theta(\alpha, \varepsilon) = \int_{\mathbb{T}^3} \frac{dq}{e(q) - \alpha - i\varepsilon} ,$$

or $\Theta(\beta, -\varepsilon)$. It contributes to a renormalization of the particle propagator, cf. [3], and satisfies the following estimates, which will be used extensively in the sequel.

Lemma 3.6. *Let $\alpha \in I$, and recall that $\varepsilon = t^{-1}$. Then, for constants c_2, c_3 uniform in $\varepsilon, \alpha, \alpha'$,*

$$|\Theta(\alpha, \varepsilon)| \; < \; c_2 ,$$

$$|\Theta(\alpha, \varepsilon) - \Theta(\alpha', \varepsilon)| \; \leq \; c_3 \varepsilon^{-1/2} |\alpha - \alpha'| \tag{36}$$

$$\left| \partial_\alpha^m \Theta(\alpha, \varepsilon) \right| \; \leq \; c'(m!) \varepsilon^{-(m-1/2)} , \tag{37}$$

and for $m \in \mathbb{N}_0$, where c' is independent of m, ε, α.

Proof. We recall that $\alpha \in I = I_{\mathbb{R}} \cup I_{\mathbb{H}_-}$ from (27). The case $\alpha \in I_{\mathbb{H}_-}$ is trivial.

For $\alpha \in I_{\mathbb{R}} = [-1, 7]$, we consider

$$\Theta(\alpha, \varepsilon) = \int_{\mathbb{R}_+} ds \int_{\mathbb{T}^3} dp e^{-is(e(p) - \alpha - i\varepsilon)} .$$

As discussed in more detail in the appendix, $e(\,\cdot\,) : \mathbb{T}^3 \to [0, 6]$ is a real analytic Morse function with 8 critical points. We choose a smooth partition of unity $1 = \sum_{j=1}^{8} \phi_j$ on $[0, 1]^3$, such that the support of each ϕ_i contains precisely one critical point $p_{crit,i}$ of $e(\,\cdot\,)$ at its center. Applying a stationary phase estimate to

$$\Theta(\alpha, \varepsilon) = \sum_{j=1}^{8} \int_{\mathbb{R}_+} ds \int_{\mathbb{T}^3} dp \phi_j(p) e^{-is(e(p) - \alpha - i\varepsilon)} ,$$

one gets

$$\Theta(\alpha, \varepsilon) = \int_{\mathbb{R}^+} ds \sum_j \left(\frac{I(p_{crit,j})}{s^{3/2}} e^{-is(e(p_{crit,j}) - \alpha)} e^{-\varepsilon s} + O(s^{-5/3} e^{-\varepsilon s}) \right) ,$$

where $I(p_{crit,j}) \in \mathbb{C}$ are uniform in ε and s. Thus, $|\Theta(\alpha, \varepsilon)| < c_2$, for some constant c_2 that is independent of ε and α.

Furthermore, since

$$\partial_\alpha^m \Theta(\alpha, \varepsilon) = \int_{\mathbb{R}^+} ds \, (is)^m \sum_j \left(\frac{I(p_{crit,j})}{s^{3/2}} e^{-is(e(p_{crit,j}) - \alpha)} e^{-\varepsilon s} + O(s^{-5/3} e^{-\varepsilon s}) \right) ,$$

(37) straightforwardly follows for any $m \geq 1$. Clearly, for $m = 1$, this also implies the second inequality in (36). $\square$

3.4.3. *General Simple Pairings.* In contrast to the ladder diagram, simple pairings do in general comprise progressions of neighboring immediate recollisions on each particle line before and after each type II contraction. Let us assume the case of q neighboring delta functions of type I', starting at the particle propagator carrying the momentum p_i. Then, C_π contains the corresponding subintegral

$$\int_{(\mathbb{T}^3)^{2q}} dp_{i+1} \cdots dp_{i+2q} \Big[\prod_{l=i}^{i+2q} \frac{1}{e(p_l) - \alpha - i\varepsilon} \Big] \prod_{j=1}^{q} \delta(p_{i+2j+1} - p_{i+2j-1}) = \frac{(\Theta(\alpha, \varepsilon))^q}{(e(p_i) - \alpha - i\varepsilon)^{q+1}} \ .$$

The analogous expression for a progression of q neighboring delta functions of type I is evidently obtained from $\alpha \to \beta$ and $\varepsilon \to -\varepsilon$.

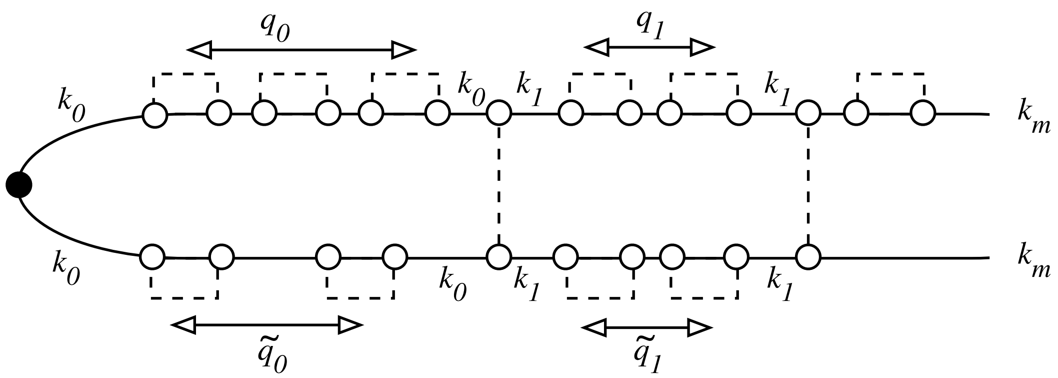

Figure 2. A simple pairing contraction graph.

Let us consider a simple pairing $\pi \in \Pi_{n,n}$ which contains m type II contractions, and introduce multiindices

$$\begin{aligned} q^{(m+1)} &:= (q_0, q_1, \ldots, q_m) \in \mathbb{N}^{m+1} \\ |q^{(m+1)}| &:= \sum_{i=0}^{m} q_i \ . \end{aligned}$$

The set of all simple pairings with fixed n gives (after reindexing the momentum variables)

$$\begin{aligned} \sum_{\substack{\pi \in \Pi_{n,n} \\ \pi \text{ simple}}} C_\pi &= \sum_{0 \leq m \in n - 2\mathbb{N}_0} \ \sum_{|q^{(m+1)}| = |\tilde{q}^{(m+1)}| = \frac{n-m}{2}} \lambda^{2m} e^{2\varepsilon t} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha - \beta)} \\ &\quad \cdot \prod_{i=0}^{m} \int_{\mathbb{T}^3} dp_i \frac{(\lambda^2 \Theta(\alpha, \varepsilon))^{q_i}}{(e(p_i) - \alpha - i\varepsilon)^{q_i + 1}} \frac{(\lambda^2 \Theta(\beta, -\varepsilon))^{\tilde{q}_i}}{(e(p_i) - \beta + i\varepsilon)^{\tilde{q}_i + 1}} \ . \end{aligned} \tag{38}$$

Let us comment on this expression, cf. Figure 2. For $i = 1, \ldots, m$, p_{i-1} is the momentum preceding, and p_i the momentum following the i-th type II pairing. Notably, a direct recollision conserves the momentum. For $1 \leq i \leq m$, q_i and $\tilde{q}_i$ are the numbers of neighboring type

I and I' pairings after the i-th type II contraction. q_0 and $\tilde{q}_0$ are the number of neighboring type I and I' pairings before the 1-st type II pairing.

Clearly, all $n-m$ random potentials on each particle line not involved in type II contractions are part of type I, respectively type I' pairings (immediate recollisions). Since each immediate recollision contracts precisely two random potentials, the sum over m can only take steps of size 2, that is, $0 \le m \in n - 2\mathbb{N}_0$ (the case $m = n$ of course corresponds to the ladder graph). Therefore,

$$|q^{(m+1)}| = |\tilde{q}^{(m+1)}| = \frac{n-m}{2}$$

is clear.

Lemma 3.7. *For fixed n, the contribution of the sum of all simple pairings is bounded by*

$$\sum_{\pi \in \Pi_{n,n} \text{ simple}} |C_\pi| \le \frac{(c_0\lambda^2 t)^n}{(n!)^{1/2}} + n^2 t^{-1/2} \log t \big(ct\lambda^2 \log t \big)^n , \tag{39}$$

and for $\lambda^2 t < 1$,

$$\sum_{n=1}^{N} \sum_{\pi \in \Pi_{n,n} \text{ simple}} |C_\pi| \le c_4 \lambda^2 t + t^{-1/2} N^3 \big(c\lambda^2 t \log t \big)^N ,$$

where c, c_0, c_4 are uniform in N and t, and where c_0 and c_4 are defined in (49).

Proof. Let us assume for fixed n that $\pi \in \Pi_{n,n}$ is simple, and contains m type II pairings. Let

$$C_\pi = C_\pi^{main} + C_\pi^{error} , \tag{40}$$

where

$$\begin{aligned} C_\pi^{main} \; := & \sum_{|q^{(m+1)}|=|\tilde{q}^{(m+1)}|=\frac{n-m}{2}} e^{2\varepsilon t} \lambda^{2m} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ & \cdot \prod_{i=0}^{m} \int_{\mathbb{T}^3} dp_i \frac{(\lambda^2 \Theta(e(p_0), \varepsilon))^{q_i}}{(e(p_i) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Theta(e(p_0), -\varepsilon))^{\tilde{q}_i}}{(e(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}} . \end{aligned} \tag{41}$$

Summing over all possible values of m,

$$\begin{aligned} \sum_{\pi \in \Pi_{n,n} \text{ simple}} C_\pi^{main} \; = & \sum_{0 \le m \in n - 2\mathbb{N}_0} \sum_{|q^{(m+1)}|=|\tilde{q}^{(m+1)}|=\frac{n-m}{2}} e^{2\varepsilon t} \lambda^{2m} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ & \cdot \prod_{i=0}^{m} \int_{\mathbb{T}^3} dp_i \frac{(\lambda^2 \Theta(e(p_0), \varepsilon))^{q_i}}{(e(p_i) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Theta(e(p_0), -\varepsilon))^{\tilde{q}_i}}{(e(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}} . \end{aligned} \tag{42}$$

Let $p_j^{(q_j)} = (p_j, \dots, p_j)$ (q_j copies), and $d\underline{p}^{(m+1)} := dp_0 \cdots dp_m$. By the Schwarz inequality,

$$\begin{aligned}\sum_{\pi\in\Pi_{n,n}\text{ simple}} |C_\pi^{main}| &\leq \sum_{0\leq m\in n-2\mathbb{N}_0} \sum_{|q^{(m+1)}|=\frac{n-m}{2}} 2\lambda^{2m} \int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} \\ &\quad \cdot \Big| K^{(\frac{m+n}{2})}(p_0^{(q_0+1)}, \dots, p_m^{(q_m+1)}; t) \Big|^2 \Big| \lambda^2 \Theta(p_0; \varepsilon) \Big|^{n-m} . \end{aligned} \tag{43}$$

Clearly,

$$\int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} \Big| K^{(\frac{m+n}{2})}(p_0^{(q_0+1)}, \dots, p_m^{(q_m+1)}; t) \Big|^2 \leq \frac{(c_\mu t)^{\frac{m+n}{2}}}{((\frac{m+n}{2})!)^\mu} . \tag{44}$$

By

$$\sum_{|q^{(m+1)}|=\frac{n-m}{2}} 1 < c^{n-m} \tag{45}$$

and Lemma 3.6,

$$\begin{aligned}\sum_{\pi\in\Pi_{n,n}\text{ simple}} |C_\pi^{main}| &\leq \sum_{0\leq m\in n-2\mathbb{N}_0} \lambda^{2m} (c\lambda^2)^{n-m} \frac{(c_\mu t)^{\frac{m+n}{2}}}{((\frac{m+n}{2})!)^\mu} \\ &\leq \frac{(c_\mu \lambda^2 t)^n}{(n!)^\mu} + n t^{-1} (c\lambda^2 t)^n , \end{aligned} \tag{46}$$

for a constant $c < \infty$ that is independent of ε.

Next, we consider

$$\begin{aligned}\sum_{\pi\in\Pi_{n,n}\text{ simple}} C_\pi^{error} &= \sum_{0\leq m\in n-2\mathbb{N}_0} \sum_{|q^{(m+1)}|=|\tilde{q}^{(m+1)}|=\frac{n-m}{2}} e^{2\varepsilon t} \lambda^{2m+2\sum_{j=0}^m (q_j+\tilde{q}_j)} \\ &\quad \cdot \int_{I\times\bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ &\quad \cdot \int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} \prod_{i=0}^m \Big[\frac{1}{e(p_i)-\alpha-i\varepsilon} \Big]^{q_i+1} \Big[\frac{1}{e(p_i)-\beta+i\varepsilon} \Big]^{\tilde{q}_i+1} \\ &\quad \cdot \Big[\Big(\Theta(\alpha,\varepsilon)\Theta(\beta,-\varepsilon) \Big)^{\frac{n-m}{2}} - \Big(\Theta(e(p_0),\varepsilon)\Theta(e(p_0),-\varepsilon) \Big)^{\frac{n-m}{2}} \Big] . \end{aligned} \tag{47}$$

Lemma 3.6 implies that the last term in $[\,\cdot\,]$ is bounded by

$$\frac{n-m}{2} \Big(|e(p_0)-\alpha| + |e(p_0)-\beta| \Big) \varepsilon^{-1/2} c^{n-m-1} ,$$

for some constant c that is independent of ε. Thus, we arrive at

$$\begin{aligned} |(\,47)| &\leq n\varepsilon^{1/2} \sum_{0\leq m\in n-2\mathbb{N}_0} (c\lambda^2\varepsilon^{-1}|\log\varepsilon|)^m (c\lambda^2\varepsilon^{-1}|\log\varepsilon|)^{n-m} \\ &\leq n^2 \varepsilon^{1/2} \Big(c\lambda^2 \varepsilon^{-1} |\log\varepsilon| \Big)^n . \end{aligned}$$

again using (45).

Summarizing, we have

$$\Big|\sum_{\pi\in\Pi_{n,n}\text{ simple}} C_\pi\Big| \;\le\; \frac{(c_\mu\lambda^2\varepsilon^{-1})^n}{(n!)^\mu} + n\varepsilon\big(c\lambda^2\varepsilon^{-1}\big)^n + n^2\varepsilon^{1/2}\big(c\lambda^2\varepsilon^{-1}|\log\varepsilon|\big)^n \,. \tag{48}$$

We choose $\mu=\frac{1}{2}$, and define

$$\begin{aligned} c_0 &:= c_{\mu=\frac{1}{2}} \\ c_4 &:= \sum_{n=1}^\infty \frac{c_0^n}{(n!)^{\frac{1}{2}}} \,, \end{aligned} \tag{49}$$

thus,

$$\sum_{n=1}^N |(\,38)| \le c_4\lambda^2\varepsilon^{-1} + \varepsilon^{1/2}N^3\big(c\lambda^2\varepsilon^{-1}|\log\varepsilon|\big)^N \,.$$

Clearly, c_0, c_4, c are uniform in N, λ, ε. □

We remark that for the case of arbitrary $T:=\lambda^2 t>0$, the constant c_4 in the above estimate would be replaced by c^T, for some constant c that is uniform in λ and t.

3.5. **Crossing and nested pairings.** We shall next prove that for all $\pi\in\Pi_{n,n}$ which contain a crossing or nested pairing contraction, $|C_\pi|$ is a factor $O(t^{-1/2})$ smaller than the bound for the ladder graph in $\Pi_{n,n}$. This is sufficient to compensate the factor $n!$ from the number of pairing contractions.

Lemma 3.8. *The sum of all crossing and nested pairing contractions in* $\Pi_{n,n}$ *is bounded by*

$$\sum_{\pi\in\Pi_{n,n}\text{ crossing or nested}} |C_\pi| \le (n!)t^{-1/2}(\log t)^3(c\lambda^2 t\log t)^n \,.$$

Proof. By lemmata 3.9 and 3.10 below, every pairing contraction of crossing or nesting type can be bounded by

$$(c\lambda^2 t\log t)^n t^{-1/2}(\log t)^3 \,,$$

and clearly, there are at most $2^n n!$ such graphs. □

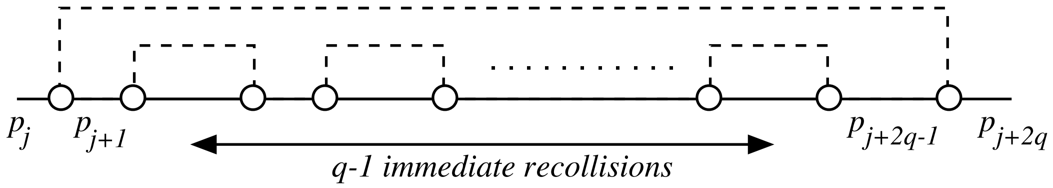

Figure 3. A simple nest.

Lemma 3.9. *If* $\pi\in\Pi_{n,n}$ *corresponds to a pairing contraction that contains at least one crossing, then*

$$|C_\pi| \le t^{-1/2}(c\lambda^2 t\log t)^n(\log t)^4 \,.$$

Proof. Let T denote a complete spanning tree for the graph G_π, and T^c its complement. All momenta supported on T can be expressed as linear combinations of loop momenta supported on T^c. If there exists a crossing pairing, it is shown in [3] that there is a tree momentum p_r in T that depends on at least two loop momenta p_j, p_l in T^c,

$$p_r = \pm p_j \pm p_l \pm w$$

where $w \in \mathbb{T}^3$ is a linear combination of momenta not depending on p_j, p_l. Thus, C_π contains a subintegral

$$\begin{aligned} A_\varepsilon(w,\alpha,\beta) &:= \int_{(\mathbb{T}^3)^2} \frac{dp_j dp_l}{(e(p_j)-\alpha_j \pm i\varepsilon)\,(e(p_l)-\alpha_l \pm i\varepsilon)} \\ &\qquad \cdot \frac{1}{(e(p_j \pm p_l + w) - \alpha_r \pm i\varepsilon)} \ . \end{aligned} \tag{50}$$

Writing $p = (p_1, p_2, p_3) \equiv p_j$ and $p' = (p'_1, p'_2, p'_3) \equiv p_l$ for brevity, we have

$$(\ 50) \le (I) \cdot (II) \cdot (III) \cdot (IV)\ , \tag{51}$$

where

$$\begin{aligned} (I) &:= \sup_{p',p_1,p_2,w} \int dp_3 \Big| \frac{1}{e(p \pm p' \pm w) - \alpha_r \pm i\varepsilon} \Big|^{\frac12}\ , \\ (II) &:= \sup_{p'_1,p'_2,p,w} \int dp'_3 \Big| \frac{1}{e(p \pm p' \pm w) - \alpha_r \pm i\varepsilon} \Big|^{\frac12} \\ (III) &:= \sup_{p_3} \int dp_1 dp_2 \Big| \frac{1}{e(p) - \alpha_j \pm i\varepsilon} \Big| \\ (IV) &:= \sup_{p'_3} \int dp'_1 dp'_2 \Big| \frac{1}{e(p') - \alpha_l \pm i\varepsilon} \Big|\ . \end{aligned}$$

We have

$$\begin{aligned} (I),(II) &\le \sup_{a\in\mathbb{R}} \int_0^1 dp_3 \Big| \frac{1}{\cos 2\pi p_3 + a \pm i\varepsilon} \Big|^{\frac12} \\ &\le C \sup_{a\in\mathbb{R}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \Big| \frac{1}{x + a \pm i\varepsilon} \Big|^{\frac12} \\ &\le C \log \frac{1}{\varepsilon}\ , \end{aligned} \tag{52}$$

and using similar arguments as in the proof of Lemma 3.4,

$$(III),(IV) \ \le\ C \log \frac{1}{\varepsilon}\ , \tag{53}$$

such that

$$\big| A_\varepsilon(w,\alpha,\beta) \big| \le (C|\log\varepsilon|)^4\ .$$

For the remaining part of C_π, excluding the propagators corresponding to the indices n and $n+1$, L^∞-bounds on propagators in T, and L^1-bounds on propagators in T^c, produce a factor $(c\lambda^2 t \log t)^{n-1}$. The propagators corresponding to the indices n and $n+1$ contribute a factor $(c \log t)^2$, as in (60) below. A detailed exposition is given in [2, 3]. $\square$

Lemma 3.10. *Let $\pi \in \Pi_{n,n}$ correspond to a non-crossing pairing contraction that contains at least one nested subgraph. Then,*

$$|C_\pi| \leq t^{-1/2}(c\lambda^2 t \log t)^n$$

Proof. In this case, π comprises a nested subgraph of length $1 < q \leq n-2$, and C_π thus contains a subintegral

$$\begin{aligned} N_q(\alpha,\varepsilon)\delta(p_{j+2q}-p_j) &= \int_{(\mathbb{T}^3)^{2q}} dp_j \cdots dp_{j+2q-1} \prod_{l=j+1}^{j+2q-2} \frac{1}{e(p_l)-\alpha-i\varepsilon} \\ &\quad \cdot \prod_{k=1}^{q-1} \lambda^2 \delta(p_{j+2k+2}-p_{j+2k}) \\ &\quad \cdot \lambda^2 \delta\Big(p_{j+1}-p_j+p_{j+2q}-p_{j+2q-1}\Big) \ , \end{aligned} \tag{54}$$

where

$$N_q(\alpha,\varepsilon) := \lambda^2 \int_{\mathbb{T}^3} dp \frac{(\lambda^2\Theta(\alpha,\varepsilon))^{q-1}}{(e(p)-\alpha-i\varepsilon)^q} \ . \tag{55}$$

We refer to this nested subgraph as a simple nest, since it does not contain further nested subgraphs in its interior, cf. Figure 3. We note that p_0, p_n, p_{n+1}, and p_{2n+1} can never appear in the interior of a nested subgraph. It is clear that

$$\begin{aligned} \Big|N_q(\alpha,\varepsilon)\Big| &\leq \lambda^2 \Big| \int_{\mathbb{T}^3} dp \frac{1}{e(p)-\alpha-i\varepsilon} \Big[\lambda^2\Theta(\alpha,\varepsilon)\frac{1}{e(p)-\alpha-i\varepsilon}\Big]^{q-1} \Big| \\ &= \lambda^2 \Big|(\lambda^2\Theta(\alpha,\varepsilon))^{q-1}\Big| \Big| \frac{1}{(q-1)!}\partial_\alpha^{q-1} \int_{\mathbb{T}^3} dp \frac{1}{e(p)-\alpha-i\varepsilon} \Big| \\ &\leq \lambda^2 (c\lambda^2)^{q-1} \frac{1}{(q-1)!} \int_{\mathbb{R}_+} ds\, s^{q-1} \frac{1}{s^{3/2}} e^{-\varepsilon s} \\ &\leq (c\lambda^2\varepsilon^{-1})^q \varepsilon^{3/2} \ , \end{aligned} \tag{56}$$

where we have used (37).

Without any loss of generality, let us assume that the simple nest has length q (that is, it contains q immediate recollisions), and that the momentum with largest label preceding it is p_j, with $j+2q < n$, so that the expression corresponding to (54) is $N_q\delta(p_{j+2q}-p_j)$. The associated subintegral in C_π is given by

$$\begin{aligned} &e^{2\varepsilon t} \quad \lambda^{2n} \int_{I\times\bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} N_q(\alpha,\varepsilon) \\ &\cdot \int_{\mathbb{T}^{2(n+1-q)}} \prod_{r=0}^{j} \prod_{r=j+2q+1}^{n} \prod_{s=n+1}^{2n+1} \frac{dp_r dp_s \delta(p_n-p_{n+1}) \prod_{j=1}^{n-q} \delta_{S_j}(\underline{p}')}{(e(p_r)-\alpha-i\varepsilon)^{\mu_j(r)}(e(p_s)-\beta+i\varepsilon)} \ , \end{aligned} \tag{57}$$

where $|S_j| = 2$ for all j. Here, $\mu_l(r) := 1+\delta_{l,r}$, and where $\underline{p}'$ is the $2(n+1-q)$-tuple of momenta in the exterior of the nest (in this expression, $\delta(p_{j+2q}-p_j)$ has been integrated out). The product of delta functions corresponds to pairing contractions which are strictly non-crossing.

The product of delta functions accounting for the pairing contractions outside of the simple nest has the following structure. Let

$$I_{j,q;n} := \begin{cases} \{0,\dots,2n+1\} \setminus \{j,\dots j+2q-1, n, n+1\} & \text{if } j < n \\ \{0,\dots,2n+1\} \setminus \{j+1,\dots j+2q, n, n+1\} & \text{if } j > n+1 \end{cases}$$

denote the index set accounting for all $2(n-q)$ momenta p_i not contained in the nest, apart from the special indices n, $n+1$. One can find an index subset $A_{n-q} \subset I_{j,q;n}$ of size $n-q$, with the property that there exists a bijection $\tau : A_{n-q} \to A^c_{n-q}$ (the complement of A_{n-q} in $I_{j,q;n}$), such that the product of delta functions has the structure

$$\prod_{j=1}^{n-q} \delta_{S_j}(\underline{p}') = \prod_{r\in A_{n-q}} \delta\Big(p_r - p_{r-1} + p_{\tau(r)} - p_{\tau(r)-1}\Big) \,. \tag{58}$$

This is obtained as follows. For each pairing contraction outside of the nest $N_q(\alpha,\varepsilon)$, let $p_j - p_{j-1}$ and $p_{j'} - p_{j'-1}$ be the arguments of the random potentials at the adjacing vertices. If $j \le n$, then we let $j-1 \in A_{n-q}$, and if $j > n+1$, then $j \in A_{n-q}$. Furthermore, if $j' \le n$, we let $\tau(j) = j'-1$, snd if $j' > n+1$, then $\tau(j) = j'$. Clearly, j and j' are interchangeable, and $I_{j,q;n} = A_{n-q} \cup \tau(A_{n-q})$. Thus, the pairings outside of the nest $N_q(\alpha,\varepsilon)$ can be written in the form

$$\begin{aligned}&\int_{(\mathbb{T}^3)^{2(n-q)}} \prod_{r\in A_{n-q}} \frac{dp_r dp_{\tau(r)} \delta(p_r - p_{r-1} + p_{\tau(r)} - p_{\tau(r)-1})}{(e(p_r) - \alpha_r - i\sigma_r\varepsilon)^{\mu_j(r)} (e(p_{\tau(r)}) - \alpha_{\tau(r)} - i\sigma_{\tau(r)}\varepsilon)} \\ &= \int_{(\mathbb{T}^3)^{n-q}} \prod_{r\in A_{n-q}} \frac{dp_r}{(e(p_r) - \alpha_r - i\sigma_r\varepsilon)^{\mu_j(r)} (e(p_r + w_r) - \alpha_{\tau(r)} - i\sigma_{\tau(r)}\varepsilon)} \,,\end{aligned}$$

where for each $r \in A_{n-q}$, $w_r \in \mathbb{T}^3$ is a linear combination of momenta not containing p_r.

It is then easy to see that $|(57)|$ is bounded by

$$\begin{aligned}&\Big(\sup_{\alpha\in I} |N_q(\alpha,\varepsilon)|\Big) \lambda^{2n} \int_{I\times\bar{I}} |d\alpha|\,|d\beta| \int_{\mathbb{T}^3} \frac{dp_n}{|e(p_n) - \alpha - i\varepsilon|^{\mu_j(n)} |e(p_n) - \beta + i\varepsilon|} \\ &\cdot \prod_{p_r\in A_{n-q}} \sup_{w_r\in\mathbb{T}^3} \int \frac{dp_r}{|e(p_r) - \alpha_r - i\sigma_r\varepsilon|^{\mu_j(r)}} \frac{1}{|e(p_r + w_r) - \alpha_{\tau(r)} - i\sigma_{\tau(r)}\varepsilon|} \,.\end{aligned} \tag{59}$$

Since

$$\sup_{w_r\in\mathbb{T}^3} \int \frac{dp_r}{|e(p_r) - \alpha_r - i\sigma_r\varepsilon|^{\mu_j(r)} |e(p_r + w_r) - \alpha_{\tau(r)} - i\sigma_{\tau(r)}\varepsilon|} \le c_1 \varepsilon^{-\mu_j(r)} |\log\varepsilon|$$

and

$$\int_{I\times\bar{I}} |d\alpha|\,|d\beta| \int_{\mathbb{T}^3} \frac{dp_n}{|e(p_n) - \alpha - i\varepsilon|^{\mu_j(n)} |e(p_n) - \beta + i\varepsilon|} \le c_1^2 \varepsilon^{1-\mu_j(n)} |\log\varepsilon|^2, \tag{60}$$

we find

$$\begin{aligned}|(57)| &\le \lambda^{2n} (c|\log\varepsilon|)^{n-q+2} \varepsilon^{1-(\mu_j(n)+\sum_{r\in A_{n-q}} \mu_j(r))} (c\varepsilon^{-1})^q \varepsilon^{3/2} \\ &\le (c\lambda^2\varepsilon^{-1} |\log\varepsilon|)^n \varepsilon^{1/2} \,,\end{aligned} \tag{61}$$

due to

$$\mu_j(n) + \sum_{r \in A_{n-q}} \mu_j(r) = n - q + 2 \, ,$$

where $q \geq 2$. This proves the lemma. $\square$

3.6. **Type III Contractions.** We shall next derive bounds on C_π, where $\pi \in \Pi_{n,n}$ contains type III contractions. We recall that the number of $\pi \in \Pi_{n,n}$ of type III is bounded by n^{2n}. The following lemma shows that the superfactorially large number of such graphs is compensated by the very small size of the associated integrals C_π.

Lemma 3.11. *For any $1 \leq m < n$, and $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n}$ of type III,*

$$|C_\pi| \leq (cn)^{3n-1} \lambda^{2n} t^m (c \log t)^{2n-2m+2} \, .$$

Proof. After integrating out $\delta(p_n - p_{n+1})$,

$$\begin{aligned} |C_\pi| \;\; &\leq \;\; \lambda^{2n} e \int_{I \times \bar{I}} |d\alpha| \, |d\beta| \int_{\mathbb{T}^3} dp_n \Big| \frac{1}{e(p_n) - \alpha - i\varepsilon} \Big| \, \Big| \frac{1}{e(p_n) - \beta - i\varepsilon} \Big| \\ & \quad \cdot \Big[\prod_{j=1}^m \int_{(\mathbb{T}^3)^{|S_j|}} \Big(\prod_{r \in S_j} \Big| \frac{dp_r}{e(p_r) - \alpha_r - i\sigma_r \varepsilon} \Big| \Big) |c_{|S_j|}| \delta_{S_j}(\underline{p}) \Big] \\ &\leq \;\; \lambda^{2n} e \int_{\mathbb{T}^3} dp_n \int_{I \times \bar{I}} |d\alpha| \, |d\beta| \Big| \frac{1}{e(p_n) - \alpha - i\varepsilon} \Big| \, \Big| \frac{1}{e(p_n) - \beta + i\varepsilon} \Big| \\ & \quad \cdot \sup_{\alpha, \bar{\beta} \in I} \Big[\prod_{j=1}^m \int_{(\mathbb{T}^3)^{|S_j|}} \Big(\prod_{r \in S_j} \Big| \frac{dp_r}{e(p_r) - \alpha_r - i\sigma_r \varepsilon} \Big| \Big) |c_{|S_j|}| \delta_{S_j}(\underline{p}) \Big] \, . \end{aligned} \tag{62}$$

First of all,

$$\int_{\mathbb{T}^3} dp_n \int_{I \times \bar{I}} |d\alpha| \, |d\beta| \Big| \frac{1}{e(p_n) - \alpha - i\varepsilon} \Big| \, \Big| \frac{1}{e(p_n) - \beta + i\varepsilon} \Big| \leq (c_1 |\log \varepsilon|)^2 \, ,$$

by (33).

For $1 \leq j \leq m$,

$$\begin{aligned} \sup_{\alpha, \beta \in I} \sup_{p_l \in \mathbb{T}^3; l \notin S_j} \;\; & \Big[\int_{(\mathbb{T}^3)^{|S_j|}} \Big(\prod_{r \in S_j} \Big| \frac{dp_r}{e(p_r) - \alpha_r - i\sigma_r \varepsilon} \Big| \Big) |c_{|S_j|}| \delta_{S_j}(\underline{p}) \Big] \\ & < \;\; |c_{|S_j|}| \Big(\sup_{\alpha \in I} \int_{\mathbb{T}^3} dp \Big| \frac{1}{e(p) - \alpha - i\varepsilon} \Big| \Big)^{|S_j| - 2} \\ & \quad \cdot \sup_{\alpha, \alpha' \in I} \sup_{p'} \Big[\int_{\mathbb{T}^3} dp \Big| \frac{1}{e(p) - \alpha - i\varepsilon} \Big| \Big| \frac{1}{e(p + p') - \alpha' - i\varepsilon} \Big| \Big] \\ & < \;\; (c_\omega |S_j|/2)^{2|S_j| + 1} \varepsilon^{-1} (c_1 |\log \varepsilon|)^{|S_j| - 1} \, , \end{aligned} \tag{63}$$

by $|S_j| - 1$ bounds of type $L^1 - L^\infty$, using (32) and (33). Furthermore, we have recalled (25), and again used

$$\sup_{\alpha, \alpha' \in I} \sup_{p'} \Big[\int_{\mathbb{T}^3} dp \Big| \frac{1}{e(p) - \alpha - i\varepsilon} \Big| \Big| \frac{1}{e(p + p') - \alpha' - i\varepsilon} \Big| \Big] \leq c_1 \varepsilon^{-1} |\log \varepsilon| \, ,$$

cf. (32) and (33).

From

$$\prod_{j=1}^{m}(c_\omega|S_j|/2)^{2|S_j|+1} < (c_\omega n)^{3n-1} ,$$

if $m < n$, we find

$$|C_\pi| \leq (cn)^{3n-1}(c\lambda^2)^n \varepsilon^{-m}|\log\varepsilon|^{2n-2m+2} ,$$

which proves the lemma. □

Proposition 3.1. *For fixed n, the sum of all contributions to the expectation (29) that comprise type III contractions is bounded by*

$$\sum_{\pi\in\Pi_{n,n}\text{ type III}} |C_\pi| \leq (c\lambda^2 t\log^2 t)^n \Big(n^4(n!)t^{-1} + n^{5n}t^{-2}\Big) .$$

Proof. We note that the total number of graphs for $1 \leq m \leq n-2$ is bounded by

$$\sum_{m=1}^{n-2} B_n(m) < n^{2n+1} ,$$

cf. the discussion of (21). In the case $m = n-1$, we find

$$B_n(n-1) < 2^n(n!)n^4 ,$$

since we have $n-1$ pair correlations, and one correlation of order 4.

Application of Lemma 3.11 to the cases $m = n-1$ and $m < n-1$ separately immediately implies the claim. □

3.7. **The Remainder Term.** In this section, we control the remainder term $R_{N,t}$ in the Duhamel series. We use the partial time integration method developed in [3], according to which we subdivide $[0,t]$ into much smaller subintervals, and exploit the rarity of the event in which the quantum particle experiences a large number of collisions in a small time interval. Here we will again see that crossing-, nesting-, and type III contractions only give rise to small errors.

The remainder term is defined by

$$R_{N,t} = (-i)\int_0^t ds\, e^{-i(t-s)H_\omega}V_\omega\phi_{N,s} . \tag{64}$$

Introducing some large number $\kappa \in \mathbb{N}$, to be determined as a function of ε below, and $\kappa+1$ equidistant points $\{\theta_0,\dots,\theta_\kappa\}$ on $[0,t]$ with $t_0 = 0$ and $\theta_\kappa = t$, we subdivide the time integration into κ intervals,

$$R_{N,t} = (-i)\sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega}\int_{\theta_j}^{\theta_{j+1}} ds\, e^{-i(\theta_{j+1}-s)H_\omega}V_\omega\phi_{N,s} . \tag{65}$$

Furthermore, we define

$$\hat\phi_{m,n,\theta}(s) := \int_\theta^s ds' D^{(m-n)}_{s-s'}\hat\phi_{m,s'} ,$$

where

$$
\begin{aligned}
(D_t^{(m)}\hat{\phi})(p_0) \;&:=\; (-i\lambda)^m \int_{\mathbb{R}^{n+1}} \Big[\prod_{j=0}^{m} ds_j\Big]_t e^{-is_0 e(p_0)} \\
&\quad \cdot \int_{(\mathbb{T}^3)^m} dp_1 \cdots dp_m \Big(\prod_{j=1}^{m} e^{-is_j e(p_j)} \hat{V}_\omega(p_j - p_{j-1})\Big) \hat{\phi}_{s_m}(p_m) \, . \qquad (66)
\end{aligned}
$$

Then, let

$$
\begin{aligned}
R_{N,t} \;&=\; R_1(t) + R_2(t) \, , \\
R_1(t) \;&:=\; \sum_{N \le n < 4N} \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} V_\omega \phi_{n,N,\theta_j}(\theta_{j+1}) \, , \\
R_2(t) \;&:=\; (-i) \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds \, e^{-i(\theta_{j+1}-s)H_\omega} V_\omega \phi_{4N,N,\theta_j}(s) \, . \qquad (67)
\end{aligned}
$$

We intend to prove that $\mathbb{E}[\|R_i(t)\|^2] = O(\varepsilon^\delta)$ for some $\delta > 0$, where $i = 1, 2$. While the character of the estimates required to control $\mathbb{E}[\|R_1(t)\|^2]$ is essentially the same as for $n \le N$, our strategy to bound $\mathbb{E}[\|R_2(t)\|^2]$ exploits the rarity of events comprising large collision numbers - that is, of order $O(N)$ - in the time intervals $[\theta_j, \theta_{j-1})$, which are much shorter than $[0, t]$. The following lemma is the main result in this context.

Lemma 3.12. *The remainder term satisfies the bounds*

$$
\begin{aligned}
\mathbb{E}\Big[\|R_1(t)\|^2\Big] \;&\le\; \frac{N^2\kappa^2(c\lambda^2 t)^{4N}}{(N!)^{1/2}} \\
&\quad + \; N^2\kappa^2(c\lambda^2 t\log t)^{4N}(\log t)^3\Big(t^{-1/2}(4N)! + t^{-2}(4N)^{20N}\Big) \\
\mathbb{E}\Big[\|R_2(t)\|^2\Big] \;&\le\; t^2\frac{(c\lambda^2 t\log t)^{4N}(\log t)^3}{\kappa^N}\Big((4N)! + t^{-2}(4N)^{20N}\Big) \, ,
\end{aligned}
$$

where the constant c is uniform in t and N.

Proof. By the Schwarz inequality, and by the unitarity of e^{-itH_ω},

$$
\begin{aligned}
\mathbb{E}\Big[\|R_1(t)\|^2\Big] \;&\le\; (3N)^2\kappa^2 \sup_{N<n\le 4N} \sup_{0\le j<\kappa} \mathbb{E}\Big[\|\phi_{n,N,\theta_j}(\theta_{j+1})\|^2\Big] \\
\mathbb{E}\Big[\|R_2(t)\|^2\Big] \;&\le\; t^2 \sup_{\substack{0\le j<\kappa \\ s\in[\theta_j,\theta_{j+1}]}} \mathbb{E}\Big[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|^2\Big] \qquad (68)
\end{aligned}
$$

are clear.

Let us first address the estimates on $\|R_2(t)\|$.

Each pairing contraction occurring in $\mathbb{E}[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|^2]$ can be bounded by

$$
\frac{(\log t)^3(c\lambda^2 t\log t)^n}{\kappa^N} \, . \qquad (69)
$$

This is proved in the same manner as demonstrated previously, but noting that $2(4N-N)=6N$ propagators have a denominator with imaginary part $\kappa\varepsilon \gg \varepsilon$ instead of ε. Since there are $< 2^n(n!)$ pairing contractions, their sum is bounded by

$$\frac{(4N)!(\log t)^3(c\lambda^2 t\log t)^{4N}}{\kappa^N} \, .$$

This bound is of particular use if $n=O(N)$. Indeed, for the estimate on $\|R_2(t)\|$, where $n=4N$,

$$\begin{aligned} \mathbb{E}\Big[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|^2\Big] \;&\leq\; \frac{(4N)!(\log t)^3(c\lambda^2 t\log t)^{4N}}{\kappa^N} \\ &+\; \frac{(\log t)^3(c\lambda^2 t\log^2 t)^{4N}}{\kappa^N}\Big((4N)^4((4N)!)t^{-1}+(4N)^{20N}t^{-2}\Big) \, , \end{aligned}$$

where the first term on the r.h.s. stems from all pairing contractions, and where the second term accounts for all type III contractions. Again, the factor κ^{-N} appears because in $6N$ of the $2(4N+1)$ free resolvents, we have $\kappa\varepsilon$ instead of ε (ε appears only in $2N+2$ of the free resolvents). This proves the asserted estimate on $\mathbb{E}[\|R_2(t)\|^2]$.

The bound on $\mathbb{E}[\|R_1(t)\|^2]$ follows from Lemmata 3.13, 3.14, 3.15, and 3.16 below. □

3.7.1. *Pairing Contractions.* Let us to begin with estimate the contributions to $\mathbb{E}[\|R_1(t)\|^2]$ stemming from pairing contractions. For simple and crossing pairings, the necessary bounds for the present case $N<n\leq 4N$ are obtained precisely in the same manner as for $n\leq N$. The discussion of nested pairing contractions is slightly more involved, due to the fact that particle propagators with different imaginary parts $\pm i\varepsilon$ and $\pm i\kappa\varepsilon$ can appear in the same simple nest.

Lemma 3.13. *Let $N<n\leq 4N$, and $\lambda^2 t<1$. The contribution of the sum of all simple pairings is bounded by*

$$|(\ 38)| \leq \frac{(c_0\lambda^2 t)^n}{(n!)^{1/2}} + n t^{-1/2}(\log t)^3(ct\lambda^2|\log t|)^n \, , \tag{70}$$

where c_0 is defined in (49).

Proof. The argument is the same as in Lemma 3.7, while noting that due to $\kappa>1$,

$$\Big|\frac{1}{e(p_j)-\alpha_j-i\sigma_j\kappa\varepsilon}\Big| \leq \Big|\frac{1}{e(p_j)-\alpha_j-i\sigma_j\varepsilon}\Big| \tag{71}$$

for all j in the estimates on $|C_\pi|$. (It is not necessary to exploit the factors κ^{-N} here.) □

The remark after the proof of Lemma 3.7 about the case of arbitrary $T=\lambda^2 t>0$ also applies to the present situation.

Lemma 3.14. *Let $N<n<4N$, and let $\pi\in\Pi_{n,n'}$ correspond to a pairing contraction that contains at least one crossing. Then,*

$$|C_\pi| \leq t^{-1/2}(c\lambda^2 t\log t)^n(\log t)^4 \, .$$

Proof. This result is proved in the precise same way as Lemma 3.9, and using (71). □

Lemma 3.15. *Let $N < n < 4N$, and let $\pi \in \Pi_{n,n'}$ represent a non-crossing pairing contraction that contains at least one nested subgraph. Then,*

$$|C_\pi| \leq t^{-1/2}(\log t)^3 (c\lambda^2 t \log t)^n \, .$$

Proof. In the case $N < n < 4N$, particle resolvents with imaginary parts $i\varepsilon$ and $i\kappa\varepsilon$ in the denominator can appear simultaneously in the same nested pairing subgraph. If so, C_π contains a subintegral corresponding to a nest of the form

$$\begin{aligned} N_{q_1,q_2}(\alpha,\varepsilon,\kappa)\delta(p_{i+2q-1}-p_{i-1}) := \lambda^{2q}\int_{(\mathbb{T}^3)^{2q}} dp_i \cdots dp_{i+2q-2} \\ \cdot\, \delta(p_i - p_{i-1} + p_{i+2q-1} - p_{i+2q-2}) \prod_{j=1}^{q-1} \delta(p_{i+2j+1} - p_{i+2j-1}) \\ \cdot \Big[\prod_{l=i}^{N} \frac{1}{e(p_l) - \alpha - i\varepsilon} \Big] \prod_{k=N+1}^{i+2q-2} \frac{1}{e(p_k) - \alpha - i\kappa\varepsilon} \, , \end{aligned} \tag{72}$$

where for $q_1 + q_2 = q-1$, and $q_1, q_2 \geq 1$,

$$N_{q_1,q_2}(\alpha,\varepsilon,\kappa) = \lambda^2 \int_{\mathbb{T}^3} dp_i \frac{(\lambda^2\Theta(\alpha,\varepsilon))^{q_1}}{(e(p_i)-\alpha-i\varepsilon)^{q_1+\zeta}} \frac{(\lambda^2\Theta(\alpha,\kappa\varepsilon))^{q_2}}{(e(p_i)-\alpha-i\kappa\varepsilon)^{q_2+1-\zeta}} \, , \tag{73}$$

with $\zeta = 0$ or 1. Let us assume that $\zeta = 0$, the case $\zeta = 1$ is completely analogous. Then,

$$\begin{aligned} (\ 73) &= \lambda^2 (\lambda^2\Theta(\alpha,\varepsilon))^{q_1} (\lambda^2\Theta(\alpha,\kappa\varepsilon))^{q_2} \\ &\quad \cdot \int_{\mathbb{T}^3} dp_i \Big[\frac{1}{(q_1-1)!} \partial_\alpha^{q_1-1} \int_{\mathbb{R}_+} ds_1 e^{-is_1(e(p_i)-\alpha-i\varepsilon)} \\ &\qquad \cdot \frac{1}{(q_2)!} \partial_\alpha^{q_2} \int_{\mathbb{R}_+} ds_2 e^{-is_2(e(p_i)-\alpha-i\kappa\varepsilon)} \Big] \\ &= \lambda^2 \frac{1}{(q_1-1)!} \frac{1}{(q_2)!} (\lambda^2\Theta(\alpha,\varepsilon))^{q_1} (\lambda^2\Theta(\alpha,\kappa\varepsilon))^{q_2} \\ &\quad \cdot \int_{\mathbb{R}_+^2} ds_1 ds_2 (is_1)^{q_1-1} (is_2)^{q_2} \int_{\mathbb{T}^3} dp_i e^{-i(s_1+s_2)(e(p_i)-\alpha)} e^{-\varepsilon s_1 - \kappa\varepsilon s_2} \, . \end{aligned} \tag{74}$$

Using (37), the integral on the last line is bounded by

$$\begin{aligned} &\int_{\mathbb{R}_+^2} ds_1 ds_2 s_1^{q_1-1} s_2^{q_2} \frac{c^{q_1+q_2}}{(s_1+s_2)^{3/2}} e^{-\varepsilon s_1 - \kappa\varepsilon s_2} \\ &\leq \frac{\varepsilon^{-(q_1+q_2-1/2)}}{\kappa^{q_2}} \int_{\mathbb{R}_+^2} ds_1 ds_2 s_1^{q_1-1} s_2^{q_2} \frac{c^{q_1+q_2}}{s_1^{3/2}} e^{-s_1-s_2} \\ &< \frac{\varepsilon^{-(q-3/2)}}{\kappa^{q_2}} c^{q_1+q_2} ((q_1-1)!)(q_2!) \, . \end{aligned}$$

Therefore,

$$\Big| N_{q_1,q_2}(\alpha,\varepsilon,\kappa) \Big| < \frac{\varepsilon^{3/2}(c\lambda^2\varepsilon^{-1})^q}{\kappa^{q_2}} \, , \tag{75}$$

where (37) has been used. Furthermore, we note that in the special case $q_1 = 0$, $q_2 = q-1$, this is replaced by

$$|N_q(\alpha,\kappa\varepsilon)| \leq \frac{(c\varepsilon^{-1}\lambda^2)^q\varepsilon^{-3/2}}{\kappa^{q-3/2}} \,, \tag{76}$$

cf. (56).

Let us assume that G_π possesses a simple nested subgraph on the particle line with momenta $\{p_0,\dots,p_N\}$. Then,

$$\begin{aligned} C_\pi &= e^{2\varepsilon t}\lambda^{2k}\int_{I\times\bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} N_{q_1,q_2}(\alpha,\varepsilon,\kappa) \\ &\quad\cdot\int_{(\mathbb{T}^3)^{k+1}}\prod_{r=0}^{j}\prod_{r=j+2q-1}^{n}\prod_{s=n+1}^{2n+1}\frac{dp_r dp_s}{(e(p_r)-\alpha-i\kappa^{\nu_r}\varepsilon)^{\mu_j(r)}(e(p_s)-\beta+i\kappa^{\nu_s}\varepsilon)} \\ &\quad\quad\cdot\delta(p_n-p_{n+1})\prod_{j=1}^{n-q}\delta_{S_j}(\underline{p}') \,, \end{aligned} \tag{77}$$

where, by the assumption of a pairing contraction, $|S_j| = 2$ for all j. Moreover, let again $\mu_l(r) := 1+\delta_{l,r}$, and let $\underline{p}'$ denote the $2(n+1-q)$-tuple of momenta in the exterior of the nest. Furthermore, let

$$\nu_r := \chi(0\leq r<N)+\chi(n+N<r\leq 2n+1) \,.$$

By definition of a nested pairing, there are no crossings in this product of delta functions.

For the contractions outside of the nest, we recall the definition of $I_{j,q;n}$ from the proof of Lemma 3.10 (where in (72), $i := j+1$). As in (58), the product of delta functions has the structure

$$\prod_{j=1}^{n-q}\delta_{S_j}(\underline{p}') = \prod_{r\in A_{n-q}}\delta\Big(p_r-p_{r-1}+p_{\tau(r)}-p_{\tau(r)-1}\Big) \,, \tag{78}$$

for an index subset $A_{n-q}\subset I_{j,q;n}$ of size $n-q$, where $\tau : A_{n-q}\to A^c_{n-q}$ (the complement of A_{n-q} in $I_{j,q;n}$) is bijective. Thus, we have pairings of the form

$$\begin{aligned} &\int_{(\mathbb{T}^3)^{2(n-q)}}\prod_{r\in A_{n-q}}\frac{dp_r dp_{\tau(r)}\delta(p_r-p_{r-1}+p_{\tau(r)}-p_{\tau(r)-1})}{(e(p_r)-\alpha_r-i\sigma_r\kappa^{\nu_r}\varepsilon)^{\mu_j(r)}(e(p_{\tau(r)})-\alpha_{\tau(r)}-i\sigma_{\tau(r)}\kappa^{\nu_{\tau(r)}}\varepsilon)} \\ &\quad=\int_{(\mathbb{T}^3)^{n-q}}\prod_{r\in A_{n-q}}\frac{dp_r}{(e(p_r)-\alpha_r-i\sigma_r\kappa^{\nu_r}\varepsilon)^{\mu_j(r)}} \\ &\quad\quad\cdot\frac{1}{(e(p_r+w_r)-\alpha_{\tau(r)}-i\sigma_{\tau(r)}\kappa^{\nu_{\tau(r)}}\varepsilon)} \,, \end{aligned} \tag{79}$$

where $w_r\in\mathbb{T}^3$ is, for each $r\in A_{n-q}$, a linear combination of momenta independent of p_r.

Therefore, it follows that $|(77)|$ is bounded by

$$\Big(\sup_{\alpha\in I}|N_{q_1,q_2}(\alpha,\varepsilon,\kappa)|\Big)\int_{I\times\bar I}|d\alpha|\,|d\beta|\int_{\mathbb{T}^3}\frac{dp_n}{|e(p_n)-\alpha-i\varepsilon|^{\mu_j(n)}|e(p_n)-\beta+i\varepsilon|}$$
$$\cdot\lambda^{2(n-q)}\prod_{r\in A_{n-q}}\sup_{w_r\in\mathbb{T}^3}\int\frac{dp_r}{|e(p_r)-\alpha_r-i\sigma_r\kappa^{\nu_r}\varepsilon|^{\mu_j(r)}}$$
$$\cdot\frac{1}{|e(p_r+w_r)-\alpha_{\tau(r)}-i\sigma_{\tau(r)}\kappa^{\nu_{\tau(r)}}\varepsilon|}\,. \tag{80}$$

Since

$$\sup_{w_r\in\mathbb{T}^3}\int\frac{dp_r}{|e(p_r)-\alpha_r-i\sigma_r\kappa^{\nu_r}\varepsilon|^{\mu_j(r)}|e(p_r+w_r)-\alpha_{\tau(r)}-i\sigma_{\tau(r)}\kappa^{\nu_{\tau(r)}}\varepsilon|}$$
$$\le c_1\frac{\varepsilon^{-\mu_j(r)}|\log\varepsilon|}{\kappa^{\max\{\nu_r,\nu_{\tau(r)}\}}}$$

and

$$\int_{I\times\bar I}|d\alpha|\,|d\beta|\int_{\mathbb{T}^3}\frac{dp_n}{|e(p_n)-\alpha-i\varepsilon|^{\mu_j(n)}|e(p_n)-\beta+i\varepsilon|}\le c_1^2\varepsilon^{1-\mu_j(n)}|\log\varepsilon|^2\,,$$

we find

$$\begin{aligned}|(77)| &\le (c\varepsilon^{-1})^q\varepsilon^{3/2}\lambda^{2(n-q)}(c_1|\log\varepsilon|)^{n-q+2}\frac{\varepsilon^{1-(\mu_j(n)+\sum_{r\in A_{n-q}}\mu_j(r))}}{\kappa^{\sum_{r\in A_{n-q}}\max\{\nu_r,\nu_{\tau(r)}\}}}\\ &\le |\log\varepsilon|^4(c\lambda^2\varepsilon^{-1}|\log\varepsilon|)^n\varepsilon^{1/2}\,.\end{aligned}\tag{81}$$

Here,

$$\sum_{r\in A_{n-q}}\max\{\nu_r,\nu_{\tau(r)}\}\ge\frac{n-N}{2}+\frac{n-N-2q}{2}\,,$$

since by bijectivity of $\tau:A_{n,q}\to A^c_{n,q}$, we have $\{N+1,\dots,n\}\cap I_{j,q;n}\ge n-N-2q$, while $\{n+N+1,2n+1\}\cap I_{j,q;n}=n-N$. This proves the lemma. □

3.7.2. *Type III Contractions.* The estimates on type III contractions necessary for $n>N$ are the same as for $n\le N$.

Lemma 3.16. *Let $N<n\le 4N$, and let $\pi\in\Pi_{n,n}$ correspond to a type III contraction. Then,*

$$\sum_{\pi\in\Pi_{n,n}\text{ type III}}|C_\pi|\le(c\lambda^2t\log t)^n\Big((n!)t^{-1}+n^{5n}t^{-2}\Big)\,,$$

where the constant c is uniform in t, λ, and n.

Proof. This is proved in the exact same way as Lemma 3.11. We remark that subfactorial factors n^4, 4^n, etc. have here been absorbed into the multiplicative constant. □

3.8. **Completing the Proof of Lemma 3.3.** Collecting all error estimates derived so far, and recalling that $\varepsilon = t^{-1}$, we can finally prove the main bound (14). From Lemmata 3.7, 3.9, 3.10 we find (after collecting terms, and redefining the constants)

$$\begin{aligned} |\text{l.h.s. of (14)}| &\leq c_4\lambda^2\varepsilon^{-1} \\ &+ (4N\kappa)^2|\log\varepsilon|^4(c\lambda^2\varepsilon^{-1}|\log\varepsilon|)^{4N}\Big((N!)\varepsilon^{1/2}+\varepsilon^2(4N)^{20N}\Big) \\ &+ \frac{|\log\varepsilon|^3(c\lambda^2\varepsilon^{-1}|\log\varepsilon|^2)^{4N}}{\varepsilon^2\kappa^N}\Big((4N)!+\varepsilon^2(4N)^{20N}\Big) \end{aligned} \tag{82}$$

(where some subexponential factors, such as N^4, etc., have been absorbed into the multiplicative constant c in c^{4N}). We choose

$$\varepsilon^{-1} = t = \delta^{\frac{6}{7}}\lambda^{-2}$$

for the given $0 < \delta < 1$. Furthermore, we let

$$\begin{aligned} N(\varepsilon) &\sim \frac{|\log\varepsilon|}{40\log|\log\varepsilon|}\,, \\ \kappa(\varepsilon) &\sim |\log\varepsilon|^{120}\,. \end{aligned} \tag{83}$$

One then easily verifies that

$$\begin{aligned} (4N(\varepsilon))^{20N(\varepsilon)} &< \varepsilon^{-1/2} \\ \kappa(\varepsilon)^{N(\varepsilon)} &\sim \varepsilon^{-3}\,, \end{aligned} \tag{84}$$

such that

$$\frac{(4N(\varepsilon))!}{\varepsilon^2\kappa(\varepsilon)^{N(\varepsilon)}} < \varepsilon^{1/2}\,. \tag{85}$$

Hence,

$$|\text{l.h.s. of (14)}| \leq c_4\lambda^2\varepsilon^{-1} + \varepsilon^{\frac{1}{3}}\,. \tag{86}$$

This completes the proof of Lemma 3.3.

4. Linear Boltzmann Equations

In this section, we study the macroscopic limit of the Schrödinger dynamics of the random lattice model discussed above, and demonstrate that it is governed by the linear Boltzmann equations. Because many arguments here will be very close to those presented in [3] for the continuum case, the exposition will be very brief at various places.

Let $\phi_t \in \ell^2(\mathbb{Z}^3)$ denote the solution of the random Schrödinger equation

$$i\partial_t\phi_t = H_\omega\phi_t\,, \tag{87}$$

with initial condition $\phi_0 \in \ell^2(\mathbb{Z}^3)$, and for a fixed realization of the random potential. We define its Wigner transform $W_{\phi_t} : \mathbb{Z}^3 \times \mathbb{T}^3 \to \mathbb{C}$ by

$$W_{\phi_t}(x,v) = \sum_{y\in\mathbb{Z}^3}\overline{\phi_t(x+y)}\phi_t(x-y)e^{2\pi iyv}\,. \tag{88}$$

We note that this convention is different from that used in [3], where the arguments of ϕ_t and $\overline{\phi_t}$ are given by $x \pm \frac{y}{2}$. Here, the latter would in half of all cases not correspond to lattice points. This convention is the origin of the factor 2 in front of $\sin 2\pi V$ in the linear Boltzmann equations (94).

Fourier transformation with respect to the variable x yields

$$\hat{W}_{\phi_t}(\xi, v) = \overline{\hat{\phi}_t(v-\xi)}\hat{\phi}_t(v+\xi) \ , \tag{89}$$

where $\xi \in \mathbb{T}^3$.

The Wigner transform is our key tool for deriving the macroscopic limit for the given quantum system. Let $J \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{T}^3)$ be a Schwartz class function. We introduce macroscopic variables $T := \epsilon t$, $X := \epsilon x$, and consider the rescaled Wigner transform of ϕ_t

$$W^{\varepsilon}_{\phi_t}(X, V) := \varepsilon^{-3} W_{\phi_t}(X/\varepsilon, V) \tag{90}$$

with $X \in (\varepsilon\mathbb{Z})^3$, and $V \in \mathbb{T}^3$. Then, let

$$\langle J, W^{\varepsilon}_{\phi_t} \rangle \ = \ \sum_{X \in (\varepsilon\mathbb{Z})^3} \int_{\mathbb{T}^3} dV \overline{J(X,V)} W^{\varepsilon}_{\phi_t}(X, V) \ , \tag{91}$$

where $J_\varepsilon(x, v) := J(\varepsilon x, v)$. In momentum space,

$$(91) = \langle \hat{J}, \hat{W}^{\varepsilon}_{\phi_t} \rangle = \int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv \overline{\hat{J}_\varepsilon(\xi, v)} \hat{W}_{\phi_t}(\xi, v) \ . \tag{92}$$

Here we note that $\hat{J}_\varepsilon(\xi, v) = \varepsilon^{-3} \hat{J}(\xi/\varepsilon, v)$.

Given $w \in \mathbb{T}^3$, we will use the notation $\sin 2\pi w \in [-1, 1]^3$ for the 3-vector with components $\sin 2\pi w_j$, $j = 1, 2, 3$, in the rest of this section.

Theorem 4.1. *Let $\varepsilon = \lambda^2$, and let ϕ^ε_t be a solution of the random Schrödinger equation (87) with initial condition*

$$\phi^\varepsilon_0(x) = \varepsilon^{3/2} h(\varepsilon x) e^{iS(\varepsilon x)/\varepsilon} \ , \tag{93}$$

where $h, S \in S(\mathbb{R}^3)$. Then, for any $T > 0$,

$$\mathbb{E}\big[W^{\varepsilon}_{\phi^\varepsilon_{T/\varepsilon}}\big] \to F_T(X, V) \ ,$$

weakly as $\varepsilon \to 0$, for $X \in \mathbb{R}^3$, $V \in \mathbb{T}^3$, where $F_T(X, V)$ solves the linear Boltzmann equation

$$\begin{aligned} &\partial_T F_T(X, V) + 2 \sin 2\pi V \times \nabla_X F_T(X, V) \\ &\quad = \int_{\mathbb{T}^3} dU \sigma(U, V) \left[F_T(X, U) - F_T(X, V)\right] \ , \end{aligned} \tag{94}$$

with collision kernel

$$\sigma(U, V) = 4\pi\delta(e(U) - e(V)) \ ,$$

and for the initial condition

$$W^{\varepsilon}_{\phi^\varepsilon_0} \to |h(X)|^2 \delta(V - \nabla S(X)) =: F_0(X, V) \ , \tag{95}$$

weakly as $\varepsilon \to 0$.

For (95) and details of the proof, cf. the closely related case analyzed in [3]. Using our analysis in previous sections, it is straightforward to infer that the non-zero contributions to $F_T(X,V)$ in the macroscopic limit stem from type I and II contractions among the random potentials in the Duhamel expansion of $W_{\phi_t^\varepsilon}^\varepsilon(X,V)$.

Let

$$\phi_t^{main} := \sum_{n=0}^{N} \phi_{n,t} \ ,$$

and

$$\begin{aligned} \mathbb{E}\Big[\int_{\mathbb{T}^3\times\mathbb{T}^3} d\xi dv \overline{\hat{J}_\varepsilon(\xi,v)} \hat{W}_{\phi_t^{main}}(\xi,v)\Big] &= \sum_{n,n'=0}^{N} U_{n,n'}^{\hat{J}_\varepsilon} \\ &= \sum_{n,n'=0}^{N} \sum_{\pi\in\Pi_{n,n'}} C_\pi^{\hat{J}_\varepsilon} \ , \end{aligned} \tag{96}$$

where

$$U_{n,n'}^{\hat{J}_\varepsilon} = \mathbb{E}\Big[\int_{\mathbb{T}^3\times\mathbb{T}^3} d\xi dv \overline{\hat{J}_\varepsilon(\xi,v)\hat{\phi}_{n,t}(v-\xi)} \hat{\phi}_{n',t}(v+\xi)\Big] \ .$$

$C_\pi^{\hat{J}_\varepsilon}$ denotes the value of the integral corresponding to the contraction $\pi \in \Pi_{n,n'}$, .

Lemma 4.1. *Let $\pi\in\Pi_{n,n'}$, and $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$. Then,*

$$\begin{aligned} U_{n,n'}^{\hat{J}_\varepsilon} &= \sum_{\pi\in\Pi_{n,n'} \text{ simple}} C_\pi^{\hat{J}_\varepsilon} \\ &\quad + O\Big((c\lambda^2 t \log t)^{\bar{n}} (\log t)^3 \big(t^{-1/2} \bar{n}! + t^{-2} \bar{n}^{5\bar{n}} \big) \Big) \ , \end{aligned} \tag{97}$$

and for any simple pairing π,

$$|C_\pi^{\hat{J}_\varepsilon}| \leq (c\lambda^2 t)^{\bar{n}} \ . \tag{98}$$

Proof. The difference here to the discussion in the previous sections is the presence of J_ε, and $n \neq n'$. We note that for the choice $J_\varepsilon = \delta(\xi)$, (96) reduces to the previous case $\mathbb{E}[\|\phi_t^{main}\|_{\ell^2(\mathbb{Z}^3)}^2]$. The necessary modifications are straightforward, and the same estimates on the Feynman amplitudes of families of graphs enter as in the situation discussed before, and as expressed in (97). For a detailed account on this issue, which can be straightforwardly applied to the problem at hand, we refer to [3]. □

Similarly as in the proof of Lemma 3.7, we decompose $C_\pi^{\hat{J}_\varepsilon}$, for π simple, into a main part $C_\pi^{\hat{J}_\varepsilon,main}$, and an error part, where $C_\pi^{\hat{J}_\varepsilon,main}$ is obtained by replacing the recollision terms $\Theta(\alpha,\varepsilon)$ and $\Theta(\beta,-\varepsilon)$ in $C_\pi^{\hat{J}_\varepsilon}$ by $\Theta(e(v_0),\varepsilon)$ and $\Theta(e(v_0),-\varepsilon)$, respectively. Thus, assuming for π that $C_\pi^{\hat{J}_\varepsilon}$ contains m type II contractions, where we index the immediate recollisions by $(q_0, \ldots, q_m)$ and $(\tilde{q}_0, \ldots, \tilde{q}_m)$, respectively, as in (38), we have

$$
\begin{aligned}
C_\pi^{\hat{J}_\varepsilon, main} &= \int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv_0 \overline{\hat{J}_\varepsilon(\xi, v_0)} \lambda^{2m} e^{2\varepsilon t} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha - \beta)} \\
&\quad \cdot \int_{(\mathbb{T}^3)^m} dv_1 \cdots dv_m \overline{\phi_0^\varepsilon(v_n - \xi)} \phi_0^\varepsilon(v_n + \xi) \\
&\quad \cdot \prod_{i=0}^{m} \frac{(\lambda^2 \Theta(e(v_0), \varepsilon))^{q_i}}{(e(v_i + \xi) - \alpha - i\varepsilon)^{q_i + 1}} \frac{(\lambda^2 \Theta(e(v_0), -\varepsilon))^{\tilde{q}_i}}{(e(v_i - \xi) - \beta + i\varepsilon)^{\tilde{q}_i + 1}} \, .
\end{aligned}
\tag{99}
$$

The error term is controlled by the following lemma.

Lemma 4.2. *Let $\pi \in \Pi_{n,n'}$ be a simple pairing. Then,*

$$
\begin{aligned}
C_\pi^{\hat{J}_\varepsilon} &= C_\pi^{\hat{J}_\varepsilon, main} + O((c\lambda^2 t)^{\bar{n}} t^{-1/2}) \\
|C_\pi^{\hat{J}_\varepsilon, main}| &\leq \frac{(c\lambda^2 t)^{\bar{n}}}{(\bar{n}!)^{1/2}} \, .
\end{aligned}
\tag{100}
$$

Proof. The proof is the same as for lemma 3.7, with the appropriate modifications included to cover the case $n \neq n'$. Since this generalization is straightforward, and also treated in detail in [3], we will not repeat the argument here. □

We perform the contour integral with respect to the variables α and β, and evaluate the sum over $n, n' \in \{0, \ldots, N\}$ by first summing over all $q_i, \tilde{q}_i$, where $i = 1, \ldots, m$, for fixed m, and sum over the indices m at the end. We then obtain

$$
\begin{aligned}
\lim_{N \to 0} \sum_{n,n'=0}^{N} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi \text{ simple}}} C_\pi^{\hat{J}_\varepsilon, main} &= \int dv_0 d\xi \overline{J_\varepsilon(\xi, v_0)} \sum_{m=0}^{\infty} \lambda^{2m} \\
&\quad \cdot \int \Big[\prod_{j=0}^{m} ds_j \Big]_t \Big[\prod_{j=0}^{m} d\tilde{s}_j \Big]_t \int_{(\mathbb{T}^3)^m} dv_1 \cdots dv_m \hat{W}_{\phi_0^\varepsilon}(\xi, v_m) \\
&\quad \cdot \; e^{2t\lambda^2 Im(\Theta(v_0, \varepsilon))} \prod_{i=0}^{m} e^{-is_i e(v_i + \xi) + i\tilde{s}_i e(v_i - \xi)}
\end{aligned}
\tag{101}
$$

where

$$
\begin{aligned}
\hat{W}_{\phi_0^\varepsilon}(\xi, v) &= \overline{\hat{\phi}_0^\varepsilon(v - \xi)} \hat{\phi}_0^\varepsilon(v + \xi) \, , \\
Im(\Theta(v_0, \varepsilon)) &= \frac{1}{2} \big(\Theta(v_0, \varepsilon) - \Theta(v_0, -\varepsilon) \big) \, .
\end{aligned}
$$

To demonstrate that in the macroscopic scaling limit, the contributions from the main term tend to a solution of the linear Boltzmann equations, we introduce the new time variables

$$
\begin{aligned}
a_j &:= \frac{s_j + \tilde{s}_j}{2} \\
b_j &:= \frac{s_j - \tilde{s}_j}{2} \, ,
\end{aligned}
$$

with $a_j \geq 0$ and $\sum_{j=0}^n a_j = t$, and $b_j \in [-a_j, a_j]$. so that

$$\begin{aligned} s_i e(v_i - \xi) - \tilde{s}_i e(v_i + \xi) &= a_i\big(e(v_i + \xi) - e(v_i - \xi)\big) \\ &+ b_i\big(e(v_i - \xi) + e(v_i + \xi)\big) . \end{aligned} \tag{102}$$

Furthermore, we introduce macroscopic variables

$$T := \varepsilon t \ , \ \ \tau_j := \varepsilon a_j \ , \ \ \zeta := \xi/\varepsilon \ ,$$

where by the compactness of the support of $\hat{J}_\varepsilon$, we have $|\zeta| \leq O(1)$. For all finite τ_j,

$$\prod_{j=1}^n \int_{-\tau_j/\varepsilon}^{\tau_j/\varepsilon} db_j \, e^{2ib_j(e(v_j)-e(v_0)+O(\varepsilon))} \to \prod_{j=1}^n 2\pi\delta\big(e(v_j) - e(v_0)\big) \ ,$$

weakly as $\varepsilon \to 0$.

By the same arguments as in [3], we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbb{E}\big[\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t^\varepsilon}\rangle\big] &= \sum_{n\geq 0} \int_{(\mathbb{T}^3)^{n+1}} dv_0 \cdots dv_n e^{2TIm(\Theta(v_0))} \\ &\quad \cdot \int \Big[\prod_{j=0}^n d\tau_j\Big]_T \prod_{j=1}^n 4\pi\delta\big(e(v_j) - e(v_0)\big) \\ &\quad \cdot \lim_{\varepsilon\to 0} \int_{\mathbb{T}^3_\varepsilon} d\zeta \overline{\hat{J}(\zeta, v_0)} e^{2\pi i \sum_{j=0}^n \tau_j \zeta \cdot 2\sin 2\pi v_j} \hat{W}_{\phi_0^\varepsilon}(\varepsilon\zeta, v_n) \ , \end{aligned} \tag{103}$$

where $\Theta(v) := \lim_{\varepsilon\to 0} \Theta(v, \varepsilon)$. The factor ε^{-n}, which emerges from rescaling a_i, has notably eliminated λ^{2n}.

Moreover, recalling (95), we have, for the given choice of ϕ_0^ε,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{T}^3_\varepsilon} d\zeta \overline{\hat{J}(\zeta, v_0)} e^{2\pi i \sum_{j=0}^n \tau_j \zeta \cdot 2\sin 2\pi v_j} \hat{W}_{\phi_0^\varepsilon}(\varepsilon\zeta, v_n) \\ &\qquad = \int_{\mathbb{R}^3} dX \overline{J(X, v_0)} F_0\Big(X - \sum_{j=0}^n \tau_j 2\sin 2\pi v_j, v_n\Big) \ . \end{aligned} \tag{104}$$

Notably, the microscopic and macroscopic velocities are equal, $v_j = V_j$, for $j = 0, \ldots, n$.

Since the test function J is arbitrary, one obtains

$$\lim_{N\to\infty} \mathbb{E}\big[W^\varepsilon_{\phi^{main}_{T/\varepsilon,N}}(X, V)\big] \to F_T(X, V) \tag{105}$$

weakly as $\varepsilon \to 0$, where

$$\begin{aligned} F_T(X, V) &= e^{2T\sigma(V)} \sum_{n\geq 0} \int d\tau_0 \cdots d\tau_n \delta\Big(\sum_{j=0}^n \tau_j - T\Big) \\ &\quad \cdot \int dV_1 \cdots dV_n \sigma(V, V_1) \cdots \sigma(V_{n-1}, V_n) F_0\Big(X - \sum_{j=0}^n 2\tau_j \sin 2\pi V_j, V_n\Big) \ , \end{aligned}$$

with $V = V_0$, and differential cross-section

$$\sigma(V, U) := 4\pi\delta\big(e(V) - e(U)\big) \ . \tag{106}$$

The total scattering cross-section is given by

$$\sigma(V) := \int dU \sigma(V, U) = -2Im(\Theta(V)) \,. \tag{107}$$

In particular, $F_T(X, V)$ satisfies the linear Boltzmann equations (94). This concludes our proof of Theorem 4.1.

Acknowledgements. The author is profoundly grateful to L. Erdös, and especially H.-T. Yau, for their support, guidance, and generosity. He has benefitted immensely from very numerous discussions with H.-T. Yau, without whom this work would not have been possible. He also thanks A. Elgart and B. Schlein for discussions. The author was supported by a Courant Instructorship, and in part by a grant from the NYU Research Challenge Fund Program.

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