Exponential splitting of bound states in a waveguide with a pair of distant windows

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We consider Laplacian in a straight planar strip with Dirichlet boundary which has two Neumann "windows" of the same length the centers of which are $2l$ apart, and study the asymptotic behaviour of the discrete spectrum as $l \to \infty$. It is shown that there are pairs of eigenvalues around each isolated eigenvalue of a single-window strip and their distances vanish exponentially in the limit $l \to \infty$. We derive an asymptotic expansion also in the case where a single window gives rise to a threshold resonance which the presence of the other window turns into a single isolated eigenvalue.

1 Introduction

Geometrically induced bound states in waveguide systems have attracted a lot of attention recently. The main reason is that they represent an interesting physical effect with important applications in nanophysical devices, but also in flat electromagnetic waveguides – cf. [LCM] and references therein. At the same time, such a discrete spectrum poses many interesting mathematical questions.

One of the simplest systems of this kind is a straight hard-wall strip in the plane with a "window" or several "windows" in its boundary modeled by switching the Dirichlet boundary condition to Neumann in the Laplace operator which will be the Hamiltonian of our system. By an easy symmetry argument it represents the nontrivial part of the problem for a pair of adjacent parallel waveguides coupled by a window or several windows in the common boundary [EŠTV]; this explains the name we use for the Neumann segments.

The discrete spectrum of such a system is nonempty once a Neumann window is present. Various properties of these bound states were analyzed including their number and behaviour with respect to parameters. Recently we discussed the way in which the eigenvalues emerge from the continuous spectrum as the window width is increasing [BEG]; we refer to this paper for references to earlier work. Here we address a different question: we consider a strip with a pair of identical Neumann windows at the same side of the boundary and ask about the behaviour of the discrete spectrum as the distance between them grows.

There is a natural analogy with the multiple-well problem in the usual Schrödinger operator theory – see [BCD] and references therein or [Da, Sec. 8.6] – even if the nature of the effect is different. Recall that in waveguides of the considered type there are no classically closed trajectories apart of the trivial set of measure zero, and likewise, there are no classically forbidden regions. Hence the semiclassical analysis does not apply here, in particular, there is no Agmon metric to gauge the distance of the windows which replace potential wells in our situation.

Nevertheless, the picture we obtain is similar to double-well Schrödinger operators. If the half-distance l between the windows is large, there is pair of eigenvalues, above and below each isolated eigenvalue of the corresponding single-window strip. We will derive an asymptotic expansion which shows that the pair splitting vanishes exponentially as $l \to \infty$ together with the appropriate expansion for the eigenfunctions. On the other hand, the analogy a double-well Schrödinger operator can be misleading. This is illustrated by the case when the single-window strip has a threshold resonance, which turns into a (single) isolated eigenvalue under influence of the other window. We derive the asymptotic expansion as $l \to \infty$ for this case too; it appears that it is exponential again with the power determined by the term coming from the second transverse mode present in the expansion of the resonance wavefunction.

Let us describe briefly the contents of the paper. In the next section we formulate the problem precisely and state two theorems which express our main results. In Section 3 we collect general properties of the involved operator. Before coming to the proper proofs, we analyze in Sections 4 and 5 strip with a single window, in particular, we show how the original question stated in PDE terms can be reformulated as a pair Fredholm problem, the second being obtained from the first one as a perturbation. Finally, in Sections 6 and 7 we prove Thms. 2.1 and 2.2.

2 Formulation of the problem and the main results

Let $x = (x_1, x_2)$ be Cartesian coordinates and suppose that Π is a horizontal strip of a width d, i.e. $\Pi := \{x : 0 < x_2 < d\}$. In the lower boundary of the strip we select two segments of the same length $2a$. The distance between these segments, denoted as $2l$, will be large playing the role of parameter in our

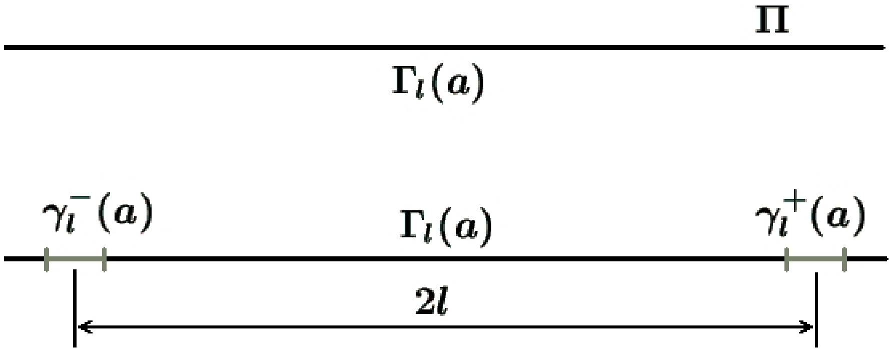

Figure 1: Waveguide with two Neumann segments

asymptotic expansions. We will employ the symbol $\gamma_l(a)$ for the union of these segments, $\gamma_l(a) := \gamma_l^+(a) \cup \gamma_l^-(a)$, where $\gamma_l^\pm(a) = \{x : |x_1 \mp l| < a,\, x_2 = 0\}$. The remaining part of the boundary of Π will be indicated by $\Gamma_l(a)$ (cf. Figure 1). The main object of our interest are discrete eigenvalues of the Laplacian in Π with Dirichlet boundary condition on $\Gamma_l(a)$ and Neumann one on γ_l. We denote such an operator by $H_l(a)$ and look what happens if $l \to \infty$.

In order to formulate main results of this paper we need some additional notations and preliminary results concerning a single-window strip. Denote $\gamma(a) := \{x : |x_1| < a,\, x_2 = 0\}$, where $\Gamma(a) := \partial\Pi \setminus \gamma(a)$. It was proven in [EŠTV] that the Laplacian in Π with Dirichlet condition on $\Gamma(a)$ and Neumann one on $\gamma(a)$ has (simple) eigenvalues below the threshold of the continuous spectrum for any $a > 0$; their number is finite and depends on a. We will indicate the operator in question and its eigenvalues by $H(a)$ and $\lambda_j(a)$, $j = 1, \dots, n$, respectively, with the natural ordering, $\lambda_1(a) < \lambda_2(a) < \ldots < \lambda_n(a) < \frac{\pi^2}{d^2}$, supposing that the corresponding eigenfunctions ψ_j are normalized in $L^2(\Pi)$. Furthermore, it was shown in [EŠTV] that there are critical values of size of Neumann segment, $0 = a_0 < a_1 < a_2 < \cdots$, for which the system has in addition a threshold resonance, i.e. the equation $(H(a_n) + 1)\psi = 0$ has a nontrivial solution $\psi^n(x)$ unique up to a multiplicative constant. This solution and eigenfunction ψ_j mentioned above have a definite parity with respect to x_1 and behave in the limit $x_1 \to +\infty$ as

$$\psi^n(x) = \sqrt{\frac{2}{d}}\,\sin\left(\frac{\pi x_2}{d}\right) + \beta_n\,\mathrm{e}^{-\frac{\pi\sqrt{3}}{d}x_1}\sin\left(\frac{2\pi x_2}{d}\right) + \mathcal{O}\left(\mathrm{e}^{-\frac{\pi\sqrt{8}}{d}x_1}\right), \tag{2.1}$$

$$\psi_j(x) = \alpha_j\,\mathrm{e}^{-\sqrt{\frac{\pi^2}{d^2}-\lambda_0}\,x_1}\sin\left(\frac{2\pi x_2}{d}\right) + \mathcal{O}\left(\mathrm{e}^{-\sqrt{\frac{4\pi^2}{d^2}-\lambda_0}\,x_1}\right), \tag{2.2}$$

with some constants α_j, β_n; it is clear that $\alpha_j = \alpha_j(a)$. While normalization of ψ_j is natural, the normalization of ψ^n can be arbitrary, of course; we choose it in such a way that asymptotically the function coincides with the first normalized transverse mode. Needless to say, when the window size is made larger than the critical value, the threshold resonance turns into a true eigenvalue.

Now we are ready to formulate the main results.

Theorem 2.1. *Let the window length be non-critical, i.e. $a \in (a_{n-1}, a_n)$ for some $n \in \mathbb{N}$. Then the operator $H_l(a)$ has for any l large enough exactly $2n$ eigenvalues $\lambda_j^{\pm}(l,a)$, $j = 1,\ldots,n$, situated in the interval $(\frac{\pi^2}{4d^2}, \frac{\pi^2}{d^2})$. Each of them is simple and has the asymptotic expansions*

$$\lambda_j^{\pm}(l,a) = \lambda_j(a) \mp \mu_j(a)\,\mathrm{e}^{-2l\sqrt{\frac{\pi^2}{d^2}-\lambda_j(a)}} + \mathcal{O}\left(\mathrm{e}^{-(4\sqrt{\frac{\pi^2}{d^2}-\lambda_j(a)}-\sigma)l}\right), \tag{2.3}$$

as $l \to \infty$ for $j = 1,\ldots,n$, where σ is an arbitrary fixed positive number. The coefficient μ_j is given by

$$\mu_j(a) := \alpha_j(a)^2 d\sqrt{\frac{\pi^2}{d^2} - \lambda_0}\,, \tag{2.4}$$

or alternatively by

$$\mu_j(a) := \frac{\pi^2}{d^3\sqrt{\frac{\pi^2}{d^2} - \lambda_j(a)}}\left(\int\limits_{\gamma(a)} \psi_j(x)\,\mathrm{e}^{\sqrt{\frac{\pi^2}{d^2}-\lambda_j(a)}\,x_1}\,dx_1\right)^2. \tag{2.5}$$

The eigenfunctions $\psi_j^{\pm}(x)$ associated with eigenvalues $\lambda_j^{\pm}(l,a)$, $j = 1,\ldots,n$, have a definite parity being even for $\lambda_j^{+}(l,a)$ and odd for $\lambda_j^{-}(l,a)$. Furthermore, in the halfstrips $\Pi^{\pm} := \{x : \pm x_1 > 0,\ 0 < x_2 < d\}$ they can be approximated by

$$\psi_j^{+}(x) = \psi_j(x_1 \mp l, x_2) + \mathcal{O}\left(\mathrm{e}^{-(2\sqrt{\frac{\pi^2}{d^2}-\lambda_j(a)}-\sigma)l}\right),$$
$$\psi_j^{-}(x) = \pm\psi_j(x_1 \mp l, x_2) + \mathcal{O}\left(\mathrm{e}^{-(2\sqrt{\frac{\pi^2}{d^2}-\lambda_j(a)}-\sigma)l}\right),$$

in $W_2^1(\Pi^{\pm})$ as $l \to \infty$.

Theorem 2.2. *Let the Neumann segment have a critical size, $a = a_n$. Then the operator $H_l(a)$ has $2n+1$ eigenvalues in $(\frac{\pi^2}{4d^2}, \frac{\pi^2}{d^2})$ for l large enough. The first $2n$ of them together with the associated eigenfunctions behave according to Theorem 2.1, while the last one, $\lambda_{n+1}^{+}(l,a_n)$, exhibits the asymptotics*

$$\lambda_{n+1}^{+}(l,a_n) = \frac{\pi^2}{d^2} - \mu\,\mathrm{e}^{-\frac{4\sqrt{3}\pi}{d}\,l} + \mathcal{O}\left(\mathrm{e}^{-\frac{2(\sqrt{8}+\sqrt{3})\pi}{d}\,l}\right), \tag{2.6}$$

where

$$\mu := 3\beta_n^4 d^2\,, \tag{2.7}$$

or alternatively,

$$\mu := \frac{16}{3d^2} \left(\int\limits_{\gamma(a_n)} \psi^n(x)\, \mathrm{e}^{\frac{\pi\sqrt{3}}{d}\,x_1}\, dx_1 \right)^4 . \tag{2.8}$$

The associated eigenfunction ψ^+_{n+1} is even w.r.t. x_1 and for any R in the rectangles $\{x : |x_1 \mp l| < R\} \cap \Pi$ it can be approximated for large values of l as

$$\psi^+_{n+1}(x) = \psi^n(x_1 \mp l, x_2) + \mathcal{O}\left(\mathrm{e}^{-\frac{2\sqrt{3}\pi}{d}\,l}\right)$$

in W^1_2-norm. In addition it behaves in the limits $x_1 \to \pm\infty$ as

$$\psi^+_{n+1}(x) = \sqrt{\frac{2}{d}}\,\mathrm{e}^{-\varkappa|x_1|} \sin\frac{\pi x_2}{d} + \mathcal{O}\left(\mathrm{e}^{-\frac{\pi\sqrt{3}}{d}|x_1|}\right) ,$$

$$\varkappa := \sqrt{\frac{\pi^2}{d^2} - \lambda_{n+1}} = \sqrt{\mu}\,\mathrm{e}^{-2\frac{\sqrt{3}\pi}{d}l} + \mathcal{O}\left(\mathrm{e}^{-\frac{2\sqrt{8}\pi}{d}\,l}\right) .$$

Before proceeding further, let us recall what we have said in the introduction about the analogy with the multi-well problem for Schrödinger operators. As in that case a (simple) eigenvalue of the single window problem gives rise to a pair of eigenvalues (corresponding to eigenfunctions of different parities) which are exponentially close to each other with respect to the window distance, and moreover, in the generic case the splitting is determined by the eigenvalue distance from the threshold. At a glance the multiplicity is doubled by the perturbation, however, in reality the problem decomposes due to mirror symmetry into a pair of problems with definite parities whose eigenvalues tend to the same limit (see below and Sec. 5). On the other hand, the asymptotics (2.6) in the critical case differs from what the Schrödinger operator analogy would suggest being determined by the distance from the second transverse eigenvalue.

Let us now describe our way to prove Theorems 2.1 and 2.2. The main idea is to reduce the eigenvalue problem at hand to a Fredholm operator equation of the second kind with a regular perturbation. Investigating this problem we will get the result both in the generic situation described in Theorem 2.1 and for perturbation of a threshold resonance, just the analysis in the latter case is more subtle.

Our task can be simplified by taking into account the symmetry of the problem with respect to reflections, $x_1 \to -x_1$, which means that the operator decomposes into orthogonal sum of parts of a definite parity which can be considered separately. This allows us to cut the strip Π into a pair of halfstrips $\Pi^\pm$ and to consider the Laplacian in Π^+ with Dirichlet condition everywhere at the horizontal boundaries of the halfstrip except for $\gamma^+_l(a)$, where the boundary condition is Neumann. According to the chosen parity of an eigenfunction ψ we impose at that Dirichlet condition for odd eigenfunctions of the original problem at the vertical part of the boundary, $x_1 = 0$, or Neumann for the even ones. Moreover, it is convenient to shift the halfstrip by $x_1 \to x_1 - l$ in order to fix position of the Neumann segment of the boundary. As a result, we arrive at the following pair of eigenvalues problems,

$$
\begin{gathered}
-\Delta\psi = \lambda\psi\,, \quad x \in \Pi^l\,, \\
\psi = 0\,, \;\; x \in \Gamma(a)\,, \quad \frac{\partial\psi}{\partial x_2} = 0\,, \;\; x \in \gamma(a)\,, \quad hu = 0\,, \;\; x_1 = -l\,.
\end{gathered} \tag{2.9}
$$

Here $\Pi^l := \{x \in \Pi : \; x_1 > -l\}$ is the shifted halfstrip and h is the boundary operator which acts as $hu = u$ or $hu = \frac{\partial u}{\partial x_1}$ in the odd and even case, respectively. Eigenvalues of (2.9) obviously coincide with those of $H_l(a)$ and by the even/odd extension one gets the eigenfunctions of the original problem.

Finally, we remark that the problem has a simple behaviour with respect to scaling transformations which allows us to perform the proofs for $d = \pi$ only.

3 Preliminaries

Let us collect first some general properties of the spectrum of our operators.

Proposition 3.1. *The discrete spectrum of the operator $H_l(a)$ is non-empty for any $l > a > 0$. It consists of a finite number of simple eigenvalues contained in the interval $\left(\frac{1}{4}, 1\right)$ for $d = \pi$ which depend continuously on l and a; for a fixed a those corresponding to even and odd eigenfunctions are increasing and decreasing, respectively, as functions of the window separation parameter l. All eigenvalues of $H_l(a)$ which remain separated from the continuum converge to those of $H(a)$ as $l \to +\infty$, and to each eigenvalue of $H(a)$ there exists a pair of eigenvalues of $H_l(a)$ associated with eigenfunctions of opposite parities converging to that eigenvalue of $H_l(a)$. If the Neumann segment has a critical width, $a = a_n$, then there is a unique eigenvalue (corresponding to an even eigenfunction) which tends to one as $l \to +\infty$.*

Proof. By the minimax principle and an elementary bracketing estimate the eigenvalues of $H_l(a)$ can be squeezed between those of $H(l+a)$ and $H(a)$. The essential spectrum of all the three operators is the same being equal to $[1, \infty)$; this fact in combination with the results of [EŠTV] shows that $\sigma_{\mathrm{disc}}(H_l(a))$ is non-empty, finite, and contained in $\left(\frac{1}{4}, 1\right)$. A similar bracketing argument shows that the eigenvalues $\lambda_j^{\pm}(l,a)$ of the problem (2.9) with Neumann and Dirichlet boundary condition at $x_1 = -l$, respectively, satisfy $\lambda_j^{+}(l,a) \le \lambda_j(a) \le \lambda_j^{-}(l,a)$ for $j = 1, \dots, n$, where the upper bound is replaced by one if the Dirichlet problem has less than j eigenvalues. In fact, bracketing implies also the stated monotonous behaviour with respect to l, i.e.

$$
\lambda_j^{+}(l',a) \le \lambda_j^{+}(l,a) \le \lambda_j(a) \le \lambda_j^{-}(l,a) \le \lambda_j^{-}(l',a) \tag{3.1}
$$

for $l' \ge l$ with the same convention as above; it is sufficient to write $\Pi_{l'}$ as a union of Π_l and a rectangle separated by an additional Neumann or Dirichlet boundary condition and to realize that in neither of these cases the rectangle can contribute to the spectrum below the continuum threshold, because it has Dirichlet condition at the horizontal part of the boundary. In addition, the standard domain-changing argument [K, Sec. VII.6.5] shows that the functions $\lambda_j^{\pm}(\cdot, a)$ are continuous. In view of the monotonicity mentioned above their limits as $l \to \infty$ exist; it remains to check that $\lambda_j^{\pm}(l, a) \to \lambda_j(a)$.

Take $\psi \in D(H(a))$ and a function $g \in C_0^{\infty}$ such that $g(x) = 0$ for $x \le 0$ and $g(x) = 1$ for $x \ge 1$. Denoting $h_l(x, y) := g(2(x + l)/l)$ we can construct a family $\{\psi_l\}$ by $\psi_l(x, y) := \psi(x, y)h_l(x, y)$; by construction the function ψ_l belongs to the domain of $H_l^-(a)$ which is the Laplacian with the boundary condition as in (2.9) for $hu = u$. Using the fact that $\|\nabla h_l\|^2 = 2\|g'\|^2 l^{-1}$ and $\|\Delta h_l\|^2 = 8\|g''\|^2 l^{-3}$ one can check easily that $\psi_l \to \psi$ and $H_l^-(a)\psi_l \to H(a)\psi$ as $l \to \infty$, so $H_l^-(a) \to H(a)$ in the strong-graph sense. By [RS, Thm VIII.26] this is equivalent to the strong resolvent convergence, hence to each $\lambda_j(a)$ there is a family of $\lambda_j^-(a)$ converging to that value. Since the spectrum of $H_l(a)$ in $(\frac{1}{4}, 1)$ is discrete, simple, finite, and depends monotonously on l, we get the desired result. In a similar way one can check that $\lambda_j^+(l, a) \to \lambda_j(a)$ as $l \to \infty$.

The continuity w.r.t. a is proved as the a-continuity in case of a single Neumann window. We expand the solution inside and outside the window regions with respect to the appropriate transverse bases and match the Ansätze smoothly at the window edges. This yields an infinite family of linear equations for the coefficients of the expansions, which can be regarded as a search for the kernel of a certain operator in the ℓ^2 space of the coefficients with a properly chosen weight. One has to check that this operator is Hilbert-Schmidt and continuous with respect to the parameters in the Hilbert-Schmidt norm. The argument is analogous to that from the proof of Proposition 2.1 of [BEG], so we skip the details; the only difference is that due to the lack of symmetry the matching has to be performed at each window separately and the coefficient space is "twice as large".

It remains to check the last claim. Using bracketing once more we see that if the presence of the other window turns a threshold resonance into an eigenvalue, the corresponding eigenfunction must be symmetric; in view of the proved monotonicity it is sufficient to show that this happens for l large enough. Since this part of the proposition is not used in the proof of the claim of Theorem 2.2 concerning the first $2n$ eigenvalues $\lambda_j^{\pm}$, we may assume that it is already proven and that we thus know that for large l the operator $H_l(a_n)$ posesses $2n$ eigenvalues $\lambda_j^{\pm} \in \left(\frac{1}{4}, 1\right)$ corresponding to eigenfunctions $\psi_j^{\pm}$. We seek a $(2n+1)$-dimensional subspace such that for any ψ from it we have $(\psi, H_l(a_n)\psi) - \|\psi\|^2 < 0$. To this aim we employ a Goldstone-Jaffe-type argument inspired by [EŠTV] and choose

$$\psi = c_0(\chi_{L,\varsigma}\psi^n + \varepsilon p) + \sum_{j=1}^{n} c_j^{\pm}\psi_j^{\pm} \,,$$

where ψ^n is the resonance function (2.1), $\chi_{L,\varsigma} : \mathbb{R} \to (0, 1]$ equals one in $(-L, L)$ for some $L > l + a$ and $\chi_{L,\varsigma} = \exp(-\varsigma(|x| - L))$ otherwise, and p is a C_0^{∞} function supported in the other window region. In view of the asymptotic behaviour (2.1) such functions span a subspace of the needed dimension. Evaluating the energy form $(\psi, H_l(a_n)\psi) - \|\psi\|^2$ we see that if some of the coefficients $c_j^\pm$ is nonzero, it is negative even with $\varepsilon = 0$. In the opposite case we use the fact that in the leading term we have, as in [EŠTV], two competing terms, one linear in ε and the other positive coming from the tails of ψ controlled by the parameters L and ς; we can choose them in such a way that the form is negative again.

Hence $H_l(a_n)$ has for l sufficiently large at least $2n+1$ eigenvalues. In fact, it has exactly this number, because its symmetric and antisymmetric parts have for large l enough $l+1$ and l eigenvalues, respectively, otherwise we would have an contradiction with the monotonicity and continuity properties stated above. In particular, the largest eigenvalue is increasing w.r.t. l since it corresponds to an even eigenfunction. In view of (3.1) and the fact that a_n is the critical width, we conclude that $\lambda_{n+1}^+ \to 1-$ as $l \to \infty$. ∎

4 Analysis of the limiting operator

After these preliminaries let us pass to the proper subject of the paper. First we are going to discuss the limiting, i.e. one-window operator which means to analyze the following boundary value problem,

$$-(\Delta + \lambda)u = f\,, \quad x \in \Pi\,, \qquad u = 0\,, \quad x \in \Gamma(a)\,, \qquad \frac{\partial u}{\partial x_2} = 0\,, \quad x \in \gamma(a)\,. \tag{4.1}$$

The right hand side f is here assumed to be finite and to belong to $L^2(\Pi)$; our aim is to discuss the existence and uniqueness of the solution to (4.1) as well as its dependence on λ. The method we use is to reduce (4.1) to a Fredholm operator equation. Then our task will be reduced to analysis of operator families, in particular their holomorphic dependence on the spectral parameter λ (one need not specify at that the topology – cf. [RS, Sec. VI.3]). The reduction will follow a general scheme proposed by Sanchez-Palencia [SP] and it will be analogous to the treatment of a similar problem in [BEG, Sec. 3.1].

We will use the symbol $\mathcal{D}_\delta$ to indicate the open subset $\{\lambda : \operatorname{Re}\lambda < \delta\}$ of the complex plane. The structure of the solution to the problem (4.1) for λ close to 1 and for λ separated from 1 is different. This is the reason why we will consider these two cases separately. We suppose first that $\lambda \in \mathcal{D}_\delta$, where $\lambda_n(a) < \delta < 1$. In this situation it is sufficient to consider solutions of the problem (4.1) in the class of functions which behave as $\mathcal{O}(\mathrm{e}^{-\sqrt{1-\lambda}|x_1|})$ in the limit $|x_1| \to \infty$.

Since the function f is finite by assumption, its support lies inside the rectangle $\Pi_b := \Pi \cap \{x : |x_1| < b\}$ for some $b > 0$. Consider two boundary value problems,

$$-(\Delta + \lambda)v^\pm = g\,, \quad x \in \Pi^\pm\,, \qquad v^\pm = 0\,, \quad x \in \partial\Pi^\pm\,, \tag{4.2}$$

where g is an arbitrary function from $L^2(\Pi)$ with the support contained in Π_A for some $A \geq \max\{a, b-1\}$. This choice is given by the requirement that Π_A

contains both the window and the support of f in such a way which will make the smooth interpolation (4.6) used below possible. The problems (4.2) can be easily solved by separation of the variables; using the explicit form of Green's function of Laplace-Dirichlet problem on a halfline we get

$$v^{\pm}(x) = \int_{\Pi^{\pm}} G^{\pm}(x,t,\lambda)\, g(t)\, d^2t\,, \tag{4.3}$$

$$G^{\pm}(x,t,\lambda) = \sum_{j=1}^{\infty} \frac{1}{\pi\kappa_j(\lambda)} \left(\mathrm{e}^{-\kappa_j(\lambda)|x_1-t_1|} - \mathrm{e}^{\mp\kappa_j(\lambda)(x_1+t_1)}\right) \sin jx_2 \sin jt_2\,, \tag{4.4}$$

where $\kappa_j(\lambda) := \sqrt{j^2-\lambda}$. In the following we will also employ the "glued" function v equal to v^+ if $x_1 \geq 0$ and to v^- if $x_1 < 0$. The functions $v^{\pm}$ can be naturally regarded as results of action of the bounded linear operators $T_1^{\pm}(\lambda)$, i.e. we have $v^{\pm} = T_1^{\pm}(\lambda)g$, where $T_1^{\pm} : L^2(\Pi_A^{\pm}) \to W_2^2(\Pi^{\pm})$ with the "halved" rectangles $\Pi_A^{\pm} = \Pi \cap \{x : 0 < \pm x_1 < A\}$. It is easy to check that the operator families $T_1^{\pm}$ are holomorphic in $\lambda \in \mathcal{D}_\delta$. In the next step we consider the problem

$$\Delta w = \Delta v\,,\;\; x \in \Pi_A\,, \quad \frac{\partial w}{\partial x_2} = 0\,,\;\; x \in \gamma(a)\,, \quad w = v\,,\;\; x \in \partial\Pi_A \setminus \gamma(a)\,. \tag{4.5}$$

The function v may have according to its definition given above a weak discontinuity, i.e. a jump of the first derivatives. Thus we have to say what we mean by Δv in (4.5): it is the function from $L^2(\Pi)$ which coincides with Δv^+ if $x_1 > 0$ and with Δv^- if $x_1 < 0$. With the problem (4.2) in mind we can also write $\Delta v = -(\lambda v + g)$. The problem (4.5) is posed in a bounded domain, hence the standard theory of elliptic boundary value problems is applicable. In particular, we can infer using [La] that the function w exists, it is unique and belongs to $W_2^1(\Pi_A)$. We will also consider its restriction avoiding the points where the boundary condition changes, regarded as an element of $W_2^1(\Pi_A) \cap W_2^2(\Pi_A \setminus S_r)$ for any $r > 0$, where $S_r = \{x : (x_1 \pm a)^2 + x_2^2 < r^2\}$. In this way we introduce a linear bounded operator $T_2 : L^2(\Pi_A) \to W_2^1(\Pi_A) \cap W_2^2(\Pi_A \setminus S_r)$ (for any r) such that $w = T_2 g$.

Next we employ a smooth interpolation. Let χ be an infinitely differentiable mollifier function such that $\chi(\tau) = 1$ if $|\tau| < A-1$ while for $|\tau| > A$ it vanishes. We will construct a solution to the problem (4.1) interpolating between the functions v and w, specifically

$$u(x) = \chi(x_1)w(x) + (1-\chi(x_1))v(x). \tag{4.6}$$

Since $w = T_2 g$ and $v^{\pm} = T_1^{\pm}(\lambda)g$, we can also regard u as the result of an action of some linear operator $T_3(\lambda)$ which maps $L^2(\Pi_A)$ into $W_2^1(\Pi) \cap W_2^2(\Pi \setminus S_r)$ for a fixed $r > 0$. Such an operator T_3 is linear and bounded, and as an operator family with respect to λ it is again holomorphic.

Owing to the definition of w and v the function u satisfies all the boundary conditions involved in (4.1), and consequently, it represents a solution to (4.1) if and only if it satisfies the differential equation in question. Substituting (4.6) into the latter and taking into account (4.2), (4.5), we arrive at the equation

$$g + T_4(\lambda)g = f\,, \tag{4.7}$$

where $T_4 : L^2(\Pi_A) \to L^2(\Pi_A)$ is a linear bounded operator defined by

$$T_4(\lambda)g := -2\nabla_x\chi \cdot \nabla_x(w-v) - (w-v)(\Delta+\lambda)\chi\,, \tag{4.8}$$

where the dot in the first term denotes the inner product in $\mathbb{R}^2$. The relation (4.7) is the sought Fredholm equation, considered in the space $L^2(\Pi_A)$. Naturally the first thing to do here is to check the compactness of the operator T_4. It can be done as follows. The function $w-v$ belongs to $W_2^1(\Pi_A)$, thus the operator mapping g into $w-v$ is bounded as an operator from $L^2(\Pi_A)$ into $W_2^1(\Pi_A)$, and consequently, it is compact as an operator in the space $L^2(\Pi_A)$; this solves the question for the second term at the right-hand side of (4.8). Furthermore, due to the definition of the mollifier χ the support of $\nabla_x\chi$ lies within $\overline{\Pi}_A \setminus \Pi_{A-1}$. This domain does not contain the endpoints of the segment γ. Hence $w-v \in W_2^2(\operatorname{supp}\nabla_x\chi)$, and therefore $\nabla_x(w-v)$ considered as an element of $L^2(\operatorname{supp}\nabla_x\chi)$ results from action of a compact operator mapping $L^2(\Pi_A)$ into $L^2(\operatorname{supp}\nabla_x\chi)$; this concludes the proof of compactness $T_4(\lambda)$ considered as an operator in the space $L^2(\Pi_A)$. In a similar way one can check that $T_4(\lambda)$ is a holomorphic operator family w.r.t. λ.

This conclusion allows us to apply to (4.7) the standard Fredholm technique; we will see that solution to (4.7) exists and is unique for almost all λ except for points where a nontrivial solution for (4.7) with zero right-hand side exists. This will yield a solution to our original problem because the two are equivalent; this is the contents of the following lemma the proof of which we skip because it is completely analogous to that of Proposition 3.2 in [BEG].

Lemma 4.1. *To any solution g of (4.7) there is a unique solution $u = T_3(\lambda)g$ of (4.1), and vice versa, for each solution of (4.1) there exists a unique g solving (4.7) such that $u = T_3(\lambda)g$. The equivalence holds for any $\lambda \in \mathcal{D}_\delta$.*

Thus the equation (4.7) says how to find a bounded solution to (4.1): one should solve the equation (4.8) and then to construct the solution of (4.1) by the procedure described above, i.e. by putting $u = T_3(\lambda)g$.

Since the operator family $T_4(\lambda)$ is holomorphic, the corresponding resolvent family $(I+T_4(\lambda))^{-1}$ is meromorphic and its only poles are exactly the eigenvalues of $H(a)$ – cf. [SP, Chap. 16, Th. 7.1]. In order to prove Theorem 2.1, we need to know more about the behavior of $(I+T_4(\lambda))^{-1}$ in the vicinity of these poles.

Lemma 4.2. *Let $\lambda_0 < 1$ be an eigenvalue of $H(a)$. Then for any λ close enough to λ_0 the following representation is valid,*

$$(I+T_4(\lambda))^{-1} = \frac{\phi}{\lambda-\lambda_0}T_5 + T_6(\lambda)\,, \tag{4.9}$$

where $T_5 f := -(f,\psi)_{L^2(\Pi)}$ and $T_6 : L^2(\Pi_A) \to L^2(\Pi_A)$ is a bounded linear operator which is holomorphic in λ. Furthermore, ϕ is such that $\psi = T_3(\lambda_0)\phi$, where ψ is an eigenfunction of $H(a)$ associated with λ_0 and normalized in $L^2(\Pi)$.

Proof. We assume throughout that $\lambda \in \mathcal{D}_\delta$ lies in a small neighborhood of λ_0 containing no other eigenvalues of $H(a)$. As we have already mentioned, the operator family $(I+T_4(\lambda))^{-1}$ has a pole at λ_0. It means that the vector-valued function $g : \lambda \mapsto (I+T_4(\lambda))^{-1} f$ satisfies

$$g(\lambda) = \frac{g_{-q}}{(\lambda-\lambda_0)^q} + \frac{\widetilde{g}(\lambda)}{(\lambda-\lambda_0)^{q-1}}\,, \tag{4.10}$$

where q is a positive integer and $\widetilde{g}$ is holomorphic in λ. Substituting this representation into (4.7) and calculating the coefficients of $(\lambda-\lambda_0)^{-q}$ we see that g_{-q} must satisfy the equation $g_{-q} + T(\lambda_0) g_{-q} = 0$, in other words $g_{-q} = \phi T_5 f$, where $T_5 f$ is a number depending on f. Together with (4.10) this means that the solution to (4.1) associated with g, i.e. $u = T_3(\lambda) g$, can be written as

$$u(x,\lambda) = \frac{T_5 f}{(\lambda-\lambda_0)^q}\psi(x) + \frac{\widetilde{u}(x,\lambda)}{(\lambda-\lambda_0)^{q-1}}\,, \tag{4.11}$$

where is $\widetilde{u}$ is holomorphic in λ. Due to the definition of $T_3(\lambda)$ this formula is valid in the sense of $W_2^1(\Pi)$-norm as well as in $W_2^2(\Pi\setminus\Pi_A)$. Taking the inner product of (4.1) with ψ, using the fact that the latter is an eigenfunction of $H(a)$, and performing an integration by parts in Π_R with R large enough we find

$$-\int\limits_{\partial\Pi_R} \left(\psi\frac{\partial u}{\partial\nu} - u\frac{\partial\psi}{\partial\nu}\right) + \lambda_0 (u,\psi)_{L^2(\Pi_R)} = (f,\psi)_{L^2(\Pi_R)} + \lambda(u,\psi)_{L^2(\Pi_R)}\,. \tag{4.12}$$

The functions u and ψ behave at infinity as $\mathcal{O}(\mathrm{e}^{-|x_1|\sqrt{1-\lambda}})$ and $\mathcal{O}(\mathrm{e}^{-|x_1|\sqrt{1-\lambda_0}})$, respectively. With this fact in mind we can pass to the limit $R\to\infty$ in (4.12) for each fixed value of λ; this implies the identity

$$\lambda_0(u,\psi)_{L^2(\Pi)} = (f,\psi)_{L^2(\Pi)} + \lambda(u,\psi)_{L^2(\Pi)}\,.$$

Substituting to it from (4.11) and computing the coefficients at the same powers of $\lambda-\lambda_0$, we see first that $q=1$, and furthermore, that $T_5 f = -(f,\psi)_{L^2(\Pi)}$. This completes the proof. ∎

We will also need to know the behavior of the inverse $(I+T_4(\lambda))^{-1}$ as $\lambda\to 1$. For the right hand side f in (4.1) with a definite parity w.r.t. $x_1 = 0$ it was done in [BEG], here we have just to show how to extend this result to our case. We will assume that λ lies in a small neighborhood of one and that this neighborhood contains no eigenvalues of $H(a)$. First of all, however, we should characterize the class of functions in which we will seek the solution of (4.1) in this case. Instead of λ we introduce another parameter by setting $\lambda = 1-\kappa^2$, where κ lies in a small neighborhood of zero. The only restriction to the size of this neighborhood correction is that the associated values of λ should not coincide with eigenvalues of the operator $H(a)$. If κ is real and a solution to the problem (4.1) exists, it is unique and holomorphic in κ. This fact follows from the arguments given above, because for such a $\lambda = 1 - \kappa^2$ the equation (4.7) is uniquely solvable. The said solution can be extended to all values of κ in the vicinity of zero so that this extension will be an analytic function of κ. The existence of such an extension is guaranteed by the definition of the functions $v^\pm$ in (4.3) where $\kappa_1(\lambda)$ is nothing else than κ introduced above. We see that the formulae (4.3) are valid not only for real κ but also in a complex neighborhood including $\kappa = 0$, because the kernels (4.4) have finite limits as $\kappa = 0$, namely

$$\begin{aligned} G^\pm(x,t,1) = &-\frac{1}{\pi}(|x_1 - t_1| \mp (x_1 + t_1)) \sin x_2 \sin t_2 \\ &+ \sum_{j=2}^{\infty} \frac{1}{\pi \kappa_j(1)} \left(\mathrm{e}^{-\kappa_j(1)|x_1 - t_1|} - \mathrm{e}^{\mp \kappa_j(1)(x_1+t_1)} \right) \sin jx_2 \sin jt_2 \,. \end{aligned} \tag{4.13}$$

This is why we are able to extend the solution of the problem (4.1) analytically to all values of κ in the vicinity of zero. We should also stress that the function u given by (4.6) decays exponentially at infinity if $\mathrm{Re}\,\kappa > 0$, it is bounded for $\mathrm{Re}\,\kappa = 0$ and increases exponentially provided $\mathrm{Re}\,\kappa < 0$.

In this approach all the operators introduced above preserve their properties when we vary the range of the variables passing from unbounded domains the cut-off ones treating, for instance, $T_1^\pm$ as operators mapping $L^2(\Pi_A^\pm)$ into $W_2^2(\Pi_R^\pm)$ for any R. As another notational simplification we will not introduce an extra symbol for the composed mapping $\kappa \mapsto T_i(1-\kappa^2)$ and write instead just $T_i(\kappa)$.

Mimicking the argument used in the proof of [BEG, Thm 3.4], one can check the following claim:

Lemma 4.3. *If the Neumann segment of the boundary does not have a critical size, the operator $(I+T_4(\kappa))^{-1}$ exists and is uniformly bounded in κ in the vicinity of zero. In the opposite case, i.e. $a = a_n$, we have in a punctured neighborhood of zero the following representation,*

$$(I + T_4(\kappa))^{-1} = \frac{\phi^n}{\kappa} T_7 + T_8(\kappa) \,, \tag{4.14}$$

where $T_7 f := \frac{1}{2}(f, \psi^n)_{L^2(\Pi)}$ and $T_8 : L^2(\Pi_A) \to L^2(\Pi_A)$ is a bounded linear operator which is holomorphic in κ. Furthermore, ϕ^n is such that $\psi^n = T_3(\kappa = 0)\phi^n$, where ψ^n solves the equation $(H(a_n) + 1)\psi^n = 0$ and behaves at infinity in accordance with (2.1).

5 Analysis of perturbed operator

The main purpose of this section is to reduce the problem

$$
\begin{gathered}
-(\Delta+\lambda)u = f\,, \quad x\in\Pi^l\,, \\
\psi=0\,,\ \ x\in\Gamma(a)\,, \qquad \frac{\partial\psi}{\partial x_1}=0\,,\ \ x\in\gamma(a)\,, \qquad hu=0\,,\ \ x_1=0\,,
\end{gathered}
\tag{5.1}
$$

to an operator equation similar to (4.7). We will show that the problem (5.1) can be reduced to solution of a Fredholm equation which is a regular perturbation of the equation (4.7). We will start from the case of Dirichlet condition at the cut $x_1=0$, i.e., $hu=u$. We are going to employ the same scheme as in previous section and use the same notations unless stated otherwise.

First we will treat the case $\lambda\in\mathcal{D}_\delta$. In analogy with (4.2) we consider two problems,

$$
-(\Delta+\lambda)v_l^+ = g\,,\ \ x\in\Pi^+\,, \qquad v_l^+=0\,,\ \ x\in\partial\Pi^+\,, \tag{5.2}
$$

$$
-(\Delta+\lambda)v_l^- = g\,,\ \ x\in\Pi_l^-\,, \qquad v_l^-=0\,,\ \ x\in\partial\Pi_l^-\,. \tag{5.3}
$$

The first one coincides with (4.2) for v_l^+, while in (5.3) we take into account the perturbation. Consequently, we have $v_l^+ := v^+$, where v^+ is the function from (4.3). The problem (5.3) differs from (4.2) but it can be solved again by separation of variables. It is convenient to write its solution v_l^- in the following form,

$$
v_l^-(x) = v^-(x) + \int\limits_{\Pi^-} G_l^-(x,t,\lambda)\,g(t)\,d^2t\,, \tag{5.4}
$$

$$
G_l^-(x,t,\lambda) = -\sum_{j=1}^{\infty}\frac{2\,\mathrm{e}^{-\kappa_j(\lambda)l}}{\pi\kappa_j(\lambda)\sinh\kappa_j(\lambda)l}\,\sinh\kappa_j(\lambda)x_1\,\sinh\kappa_j(\lambda)t_1\sin jx_2\,\sin jt_2\,, \tag{5.5}
$$

where v^- is given by (4.3); we keep in mind here that g is finite, and therefore its support lies inside Π_l for all l large enough. As in previous section we can introduce a linear bounded operator $T_9(\lambda) : L^2(\Pi_A^-)\to W_2^2(\Pi_l^-)$ such that $v_l^- = T_9(\lambda,l)g$. This operator can be represented as the sum $T_9(\lambda,l) = T_1^-(\lambda)+T_{10}(\lambda,l)$, where $T_{10}(\lambda,l) : L^2(\Pi_A^-)\to W_2^2(\Pi_l^-)$ is holomorphic in λ, jointly continuous with respect to (λ,l) provided $\lambda\in\mathcal{D}_\delta$, $l\in[l_0,+\infty]$, and l_0 is a fixed number large enough. The norm of the operator T_{10} is of order $\mathcal{O}(\mathrm{e}^{-l\sqrt{1-\lambda}})$ as $l\to+\infty$, hence for $\lambda\in\mathcal{D}_\delta$ we may consider this operator as an exponentially small perturbation.

The analogue of the function w (denoted here by w_l) is defined as above without any changes, i.e. as a solution of the problem (4.5) with v replaced by

$$
v_l := \begin{cases} v_l^+\,, & x_1>0\,,\\ v_l^-\,, & x_1<0\,. \end{cases}
$$

The solution of (5.1) is then constructed as an interpolation (4.6) with v and w replaced by v_l and w_l; this leads us to the desired operator equation,

$$g + T_4(\lambda)g + T_{11}(\lambda, l) = f\,. \tag{5.6}$$

Here T_4 is the operator appearing in (4.7) and $T_{11}(\lambda, l) : L^2(\Pi_A) \to L^2(\Pi_A)$ is a compact linear operator which is holomorphic in λ and jointly continuous w.r.t. (λ, l) provided $\lambda \in \mathcal{D}_\delta$, and $l \in [l_0, +\infty]$. The norm of the last named operator is exponentially small as $l \to +\infty$ uniformly in $\lambda \in \mathcal{D}_\delta$:

$$\|T_{11}\| = \mathcal{O}(\mathrm{e}^{-2l\sqrt{1-\lambda}})\,. \tag{5.7}$$

The solution to the problem (5.1) can be reconstructed from the function g by $u = T_3(\lambda)g + T_{12}(\lambda, l)g$, where $T_{12} : L^2(\Pi_A) \to W_2^1(\Pi^l)$ is a linear bounded operator the norm of which satisfies

$$\|T_{12}\| = \mathcal{O}(\mathrm{e}^{-l\sqrt{1-\lambda}})\,. \tag{5.8}$$

This operator is also holomorphic in λ and jointly continuous with respect to $(\lambda, l) \in \mathcal{D}_\delta \times [\lambda_0, +\infty]$. The equation (5.6) is a second-kind Fredholm operator equation and it is equivalent to the problem (5.1); this claim can be checked in the same way as we did it for (4.7) in the previous section.

The case of $hu = \frac{\partial u}{\partial x_1}$ is treated in full analogy. The only difference due to another boundary condition at $x_1 = 0$ is the definition of the operator T_{10} which is now described by the kernel

$$G_l^-(x, t, \lambda) = \sum_{j=1}^{\infty} \frac{2\mathrm{e}^{-\kappa_j(\lambda) l}}{\pi \kappa_j(\lambda) \cosh \kappa_j(\lambda) l} \sinh \kappa_j(\lambda)x_1 \, \sinh \kappa_j(\lambda)t_1 \, \sin jx_2 \, \sin jt_2\,. \tag{5.9}$$

All the arguments used above remain valid.

On the other hand, for λ in the vicinity of one almost all the above arguments remain valid provided we replace λ by $(1 - \kappa^2)$. In analogy with the previous section the operators introduced here may be considered on cut-off strips, i.e. as $T_9(\kappa, l) : L^2(\Pi_A^-) \to W_2^2(\Pi_R^-)$, $T_{10}(\kappa, l) : L^2(\Pi_A^-) \to W_2^2(\Pi_R^-)$, $T_{11}(\kappa, l) : L^2(\Pi_A) \to L^2(\Pi_A)$, $T_{12}(\kappa, l) : L^2(\Pi_A) \to W_2^1(\Pi_R)$ for any fixed R. However, we are not longer allowed to say that these operators are holomorphic in κ because of the terms

$$\frac{2\mathrm{e}^{-\kappa l}}{\sinh \kappa l}\,, \quad \frac{2\mathrm{e}^{-\kappa l}}{\cosh \kappa l}$$

in (5.5), (5.9), since these terms have poles at $\kappa = \frac{\pi \mathrm{i}}{l} j$ and $\kappa = \frac{\pi \mathrm{i}}{l}(j + \frac{1}{2})$. Moreover, these terms are also responsible for the fact that the operators have no proper limit as $\kappa \to 0$ and $l \to +\infty$. At the same time, restricting the range of κ we will be able to show that the operators T_{10}, T_{11}, and T_{12} are small for small κ and large l, thus we will be allowed to consider them as small perturbations again. This claim leans on the following lemma.

Lemma 5.1. *Let $\varkappa \in (0, \frac{\pi}{2})$ be fixed and $\mathcal{Q}_\varkappa := \{\kappa : |\arg\kappa \pm \frac{\pi}{2}| \geq \varkappa\}$. Then there is $C > 0$ such that for small $\kappa \in \mathcal{Q}_\varkappa$ and large l the following estimate is valid,*

$$\max\left\{\left|\frac{\kappa\,\mathrm{e}^{-\kappa l}}{\sinh\kappa l}\right|, \left|\frac{\kappa\,\mathrm{e}^{-\kappa l}}{\cosh\kappa l}\right|\right\} \leq C\left(|\kappa| + l^{-1}\right) .$$

Proof. We will show how to derive the first estimate, the proof of the second one is similar. We start by introducing the function

$$P(z) := \frac{z}{\mathrm{e}^z - 1} .$$

Suppose that $z \in \mathcal{Q}_\varkappa$. If we have in addition $|z| \leq 1$, one can check that

$$|P(z)| \leq C \tag{5.10}$$

with some C independent on z. On the other hand, if $|z| > 1$, $z \in \mathcal{Q}_\varkappa$, and $\operatorname{Re} z > 0$, then the exponent in the function P increases as $|z| \to \infty$ and we arrive at (5.10) again (in general with another C). Finally, if $|z| > 1$, $z \in \mathcal{Q}_\varkappa$, and $\operatorname{Re} z < 0$ then the exponent in the function P decreases and we have a uniform estimate,

$$|P(z)| \leq C|z|.$$

Combining it with (5.10) we get the inequality

$$|P(z)| \leq C_1|z| + C_2$$

valid for $z \in \mathcal{Q}$ and suitable C_1, C_2. The obvious identity

$$\frac{\kappa\,\mathrm{e}^{-\kappa l}}{\sinh\kappa l} = \frac{1}{l}\,P(2\kappa l)$$

then completes the proof of the lemma. ∎

Using this result one can check that the operators T_{10}, T_{11}, and T_{12} are small for small $\kappa \in \mathcal{Q}_\varkappa$ and large l, holomorphic in κ, and jointly continuous in (κ, l).

6 Proof of Theorem 2.1

In this section we are going to derive the asymptotic expansions for the eigenvalues of $H_l(a)$ separated from the continuum. We will also find the asymptotic behavior of the associated eigenfunctions.

The main idea behind the calculation of the asymptotics is borrowed from [Ga1, Ga2, BEG]. Instead of dealing with eigenvectors of $H_l(a)$ directly we consider here those of the problems (2.9). In order to find eigenvalues of the latter we should look in accordance with the results of the previous sections for λ such that the operator equation

$$\Phi + T_4(\lambda)\Phi + T_{11}(\lambda, l)\Phi = 0 \tag{6.1}$$

has a nontrivial solution. We will deal with eigenvalues which are close to a fixed eigenvalue $\lambda_j(a)$ of the limiting operator $H(a)$; for simplicity we will denote the latter as λ_0 in the following. Also the parameter λ will be assumed to be close to λ_0, more specifically, it will be supposed to lie in a neighborhood of λ_0 containing neither any other limiting eigenvalue nor the point $\lambda = 1$.

By the definition of T_{11} the term $T_{11}(\lambda, l)\Phi$ in (6.1) is supported inside Π_A. Hence considering it as the right hand side, we arrive at the equation (4.7) with $f = -T_{11}(\lambda, l)\Phi$. Choosing $\lambda \neq \lambda_0$, we can invert the operator $I + T_4(\lambda)$ obtaining

$$\Phi + (I + T_4(\lambda))^{-1} T_{11}(\lambda, l)\Phi = 0\,.$$

Using the Lemma 4.2, we can rewrite the last equation in the form

$$\Phi - \frac{\phi}{\lambda - \lambda_0}(\psi, T_{11}(\lambda, l)\Phi)_{L^2(\Pi)} + T_6(\lambda) T_{11}(\lambda, l)\Phi = 0\,; \tag{6.2}$$

recall that $\psi \in L^2(\Pi)$ here is the normalized eigenfunction associated with λ_0 and $\phi \in L^2(\Pi_A)$ is a function such that $\psi = T_3(\lambda_0)\phi$.

The operator $T_{11}(\lambda, l)$ is small in the asymptotic region, $l \to +\infty$, while $T_6(\lambda)$ is holomorphic in λ. Thus we may invert the operator $I + T_6(\lambda)T_{11}(\lambda, l)$ and apply the result to the equation (6.2), which then acquires the form

$$\Phi - \frac{1}{\lambda - \lambda_0}(\psi, T_{11}(\lambda, l)\Phi)_{L^2(\Pi)}(I + T_6(\lambda)T_{11}(\lambda, l))^{-1}\phi = 0\,. \tag{6.3}$$

The inner product $(\psi, T_{11}(\lambda, l)\Phi)_{L^2(\Pi)}$ does not vanish. Indeed, otherwise the function Φ would be zero too, however, we seek a nontrivial solution of the equation (6.1). With this fact in mind, we express the function Φ from the equation (6.3) and then calculate the inner product $(\psi, T_{11}(\lambda, l)\Phi)_{L^2(\Pi)}$. This procedure leads us to the equation

$$1 - \frac{1}{\lambda - \lambda_0}\left(\psi, T_{11}(\lambda, l)(I + T_6(\lambda)T_{11}(\lambda, l))^{-1}\phi\right)_{L^2(\Pi)} = 0\,,$$

or in a more convenient form

$$\lambda - \lambda_0 - \left(\psi, T_{11}(\lambda, l)(I + T_6(\lambda)T_{11}(\lambda, l))^{-1}\phi\right)_{L^2(\Pi)} = 0\,. \tag{6.4}$$

This is the sought equation determining the perturbed eigenvalues of the problem (2.9), and, thus, of the operator $H_l(a)$. The associated solution of the equation (6.1), as it follows from (6.3), can be written as

$$\Phi = (I + T_6(\lambda)T_{11}(\lambda, l))^{-1}\phi\,; \tag{6.5}$$

we naturally keep in mind the fact that the eigenfunctions are defined up to a multiplicative constant.

The equation (6.4) determine all eigenvalues of $H_l(a)$; due to the equivalence between (6.1) and (2.9) only the eigenvalues of $H_l(a)$ satisfy this equation. Thus, by Proposition 3.1, for every T_{11} there exists an unique solution of the equation (6.4) converging to λ_0 as $l \to +\infty$.

The desired asymptotic expansions for the perturbed eigenvalues can be calculated directly from the equation (6.4). First of all we recall the assertion (5.7) which implies that for λ close to λ_0 the norm T_{11} can be estimated by $\mathcal{O}(\mathrm{e}^{-(2\sqrt{1-\lambda_0}-\sigma)l})$. It allows us first to establish the estimate

$$\lambda - \lambda_0 = \mathcal{O}\left(\mathrm{e}^{-(2\sqrt{1-\lambda_0}-\sigma)l}\right), \tag{6.6}$$

and secondly to expand the second term in the equation (6.4) obtaining

$$\lambda - \lambda_0 - (\psi, T_{11}(\lambda, l)\phi)_{L^2(\Pi)} + \mathcal{O}(\mathrm{e}^{-2(2\sqrt{1-\lambda_0}-\sigma)l}) = 0\,. \tag{6.7}$$

We can also extract the leading term from the operator $T_{11}(\lambda, l)$, which obviously comes from the lowest-mode contribution to the sum at the right hand side of (5.5). We will do that for $hu = u$, in the other case one proceeds analogously.

First we introduce additional notations setting

$$V(x) := \begin{cases} -\dfrac{4\mathrm{e}^{-2\kappa_1(\lambda_0)l}}{\pi\kappa_1(\lambda_0)} \sinh\kappa_1(\lambda_0)x_1 \sin x_2 \displaystyle\int\limits_{\Pi^-} \sinh\kappa_1(\lambda_0)t_1 \sin t_2 \phi\, d^2t\,, & x_1 < 0\,, \\ 0\,, & x_1 > 0\,. \end{cases} \tag{6.8}$$

Suppose that a function W solves the problem (4.5) with $v = V$, then

$$T_{11}(\lambda, l)\phi = -(\Delta + \lambda_0)\,(V + \chi(W - V)) + \mathcal{O}(l\mathrm{e}^{-2(2\sqrt{1-\lambda_0}-\sigma)l}) \quad \text{in } L^2(\Pi_A)\,.$$

Using this identity together with the fact that the function $T_{11}(\lambda, l)\phi$ is finite, we can calculate the leading term of the second summand in (6.7),

$$\begin{aligned} (\psi, T_{11}(\lambda, l)\phi)_{L^2(\Pi)} &= -\int\limits_{\Pi} (\Delta + \lambda_0)\,(V + \chi(W - V))\; d^2x + \mathcal{O}(l\mathrm{e}^{-2(2\sqrt{1-\lambda_0}-\sigma)l})\,, \\ \int\limits_{\Pi} (\Delta + \lambda_0)(V + \chi(W - V))\, d^2x &= \lim_{R\to+\infty} \int\limits_{\{x:\,|x_1|=R,\,0<x_2<\pi\}} \left(\psi\frac{\partial V}{\partial\nu} - V\frac{\partial\psi}{\partial\nu}\right) ds \\ &= -\lim_{R\to+\infty} \int\limits_{\{x:\,x_1=-R,\,0<x_2<\pi\}} \left(\psi\frac{\partial V}{\partial x_1} - V\frac{\partial\psi}{\partial x_1}\right) ds\,. \end{aligned} \tag{6.9}$$

In order to calculate the last integral we use the fact that in view of the relation $\psi = T_3(\lambda_0)\phi$ and the definition of T_3 the constant $\alpha = \alpha_j$ in (2.2) is given by

$$\alpha = -\frac{2\rho}{\pi\kappa_1(\lambda_0)} \int\limits_{\Pi^-} \sinh(\kappa_1(\lambda_0)t_1)\sin t_2\, \phi(t)\, d^2t\,,$$

where ρ is 1 if ψ even and -1 if it is odd. Using this relation together with (2.2) and (6.8), we can finish our calculations in (6.9) arriving at

$$\lim_{R\to+\infty} \int\limits_{\{x:\, x_1=-R,\, 0<x_2<\pi\}} \left(\psi\frac{\partial V}{\partial x_1} - V\frac{\partial \psi}{\partial x_1}\right) ds = \pi\alpha^2\kappa_1(\lambda_0)\, \mathrm{e}^{-2\kappa_1(\lambda_0)l}\,.$$

Combining this with (6.9) and (6.7) we get the asymptotic (2.3), (2.4) for $\lambda_j^-(a)$. In the case $hu = \frac{\partial u}{\partial x_1}$ a similar reasoning leads to asymptotics (2.3), (2.4) for $\lambda_j^+(a)$. In order to prove relation (2.5) it is sufficient to express α in terms of suitable integrals. Keeping the parity of ψ in mind we compute

$$\begin{aligned}
0 &= \lim_{R\to+\infty} \int\limits_{\Pi_R} \mathrm{e}^{\kappa_1(\lambda_0)x_1} \sin x_2\, (\Delta+\lambda_0)\psi(x)\, d^2x \\
&= \lim_{R\to+\infty} \int\limits_{\partial\Pi_R} \left(\mathrm{e}^{\kappa_1(\lambda_0)x_1} \sin x_2 \frac{\partial}{\partial\nu}\psi(x) - \psi(x)\frac{\partial}{\partial\nu}\, \mathrm{e}^{\kappa_1(\lambda_0)x_1} \sin x_2\right) ds \\
&= \int\limits_{\gamma(a)} \psi(x)\, \mathrm{e}^{\sqrt{1-\lambda_0}x_1}\, dx_1 - \alpha\pi\sqrt{1-\lambda_0}\,.
\end{aligned}$$

This result leads us to formulae (2.5).

The asymptotics of the eigenfunctions can be derived easily. The definite parity of those associated with $\lambda_j^{\pm}(l,a)$ is obvious. The relation (6.5) tells us that

$$\Phi^{\pm} = \phi + \mathcal{O}(\mathrm{e}^{-(l\sqrt{1-\lambda_0}-\sigma)})\,. \tag{6.10}$$

The symbol "$\pm$" indicate here two variants of definition of the operator T_{11}. Now in order to prove the expansions for the eigenfunctions one has just to use this expression and to employ the arguments of the previous two section. More precisely, we have $\psi = T_3(\lambda_0)\phi$ and $\Psi^{\pm} = (T_3(\lambda^{\pm}) + T_{12}(\lambda^{\pm}, l))\Phi^{\pm}$, where $\Psi^{\pm}$ is the eigenfunction of the problem (2.9) associated with the chosen eigenvalue and chosen variant of boundary operator h. Using (6.10) and the holomorphy of T_3, T_{12}, the estimates (6.6) and (5.8), we arrive at the asymptotical formula

$$\Psi^{\pm} = \phi + \mathcal{O}\left(\mathrm{e}^{-l(\sqrt{1-\lambda_0}-\sigma)l}\right) \tag{6.11}$$

in $W_2^1(\Pi^l)$. Recovering now the eigenfunctions of $H_l(a)$ we obtain all their properties stated in Theorem 2.1.

Let us finally prove that there are no other eigenvalues of $H_l(a)$ in $\mathcal{D}_1$. Consider the equation (6.1) where λ is close to one and does not lie in real semi-axis $[1,+\infty)$, more specifically, suppose that $\kappa \in \mathcal{Q}_{\varkappa}$. Then we can invert the operator $(I + T_4(\kappa))^{-1}$, and arrive at the equation

$$\Phi + (I + T_4(\kappa))^{-1} T_{11}(\kappa, l)\Phi = 0\,,$$

where the operator $(I + T_4(\kappa))^{-1}$ is uniformly bounded in κ, because the Neumann segment does not have by assumption a critical size – see Lemma 4.3 – while $T_{11}(\kappa, l)$ is small for all possible values κ and l. Hence the operator $(I + T_4(\kappa))^{-1}T_{11}(\kappa, l)$ is also small, and therefore we can invert in turn the operator $(I + (I + T_4(\kappa))^{-1}T_{11}(\kappa, l))$ which immediately leads us to the unique solution $\Phi = 0$. Moreover, the operator $H_l(a)$ cannot have eigenvalues corresponding to κ satisfying $|\arg\kappa \pm \frac{\pi}{2}| < \varkappa$, $\operatorname{Re}\kappa \neq 0$, simply because it is self-adjoint and all its eigenvalues are real, thus there is no other eigenvalues to $H_l(a)$ in $\mathcal{D}_1$. By this the proof of Theorem 2.1 is complete. ∎

7 Proof of Theorem 2.2

It is sufficient to consider in detail only the eigenvalue λ_{n+1} emerging from the continuum because all the statements related to the other eigenvalues verify in a way completely analogous to the previous section.

We know from Proposition 3.1 that the eigenfunction associated with the indicated eigenvalue is even with respect to x_1, thus we have to consider here only the case $hu = \frac{\partial u}{\partial x_1}$. Assuming $\kappa \in \mathcal{Q}_{\varkappa}$, we start with the equation

$$\Phi + T_4(\kappa)\Phi + T_{13}(\kappa, l)\Phi = 0\,, \tag{7.1}$$

which is how (6.1) looks like in the present case, with $T_{13}(\kappa, l)$ being the perturbation operator associated with (5.9). The operator $T_{13}(\kappa, l)$ is small by Lemma 5.1, and an argument analogous to that which lead us to (6.3) yields the equation

$$\Phi + \frac{1}{2\kappa}(\psi, T_{13}(\kappa, l)\Phi)_{L^2(\Pi)}(I + T_8(\lambda)T_{13}(\lambda, l))^{-1}\phi = 0\,. \tag{7.2}$$

Recall that $\psi = \psi^n$ is a solution to the equation $(H(a_n)+1)\psi^n = 0$ which behaves at infinity in accordance with (2.1) and $\phi \in L^2(\Pi_A)$ such that $\psi = T_3(\kappa = 0)\phi$. From this equation one can deduce an analogue of the equation (6.4), namely

$$2\kappa + \big(\psi, T_{13}(\kappa, l)(I + T_8(\kappa)T_{13}(\kappa, l))^{-1}\phi\big)_{L^2(\Pi)} = 0\,. \tag{7.3}$$

The value of κ associated with the eigenvalue emerging from the continuum solves this equation and by Proposition 3.1 it tends to zero. Using these two facts we will deduce the asymptotic formula stated in Theorem 2.2. First of all, in the following we will consider the equation (7.3) for real positive κ only. This restriction can be justified easily, since for negative κ the associated function u given by (4.6) increases at infinity and thus it does not belong to $L^2(\Pi)$. In order to calculate the asymptotics, we extract the leading part of the second term in the equation (7.3); for small positive κ we have

$$\big(\psi, T_{13}(\kappa, l)(I+T_8(\kappa)T_{13}(\kappa, l))^{-1}\phi\big)_{L^2(\Pi)} = (\psi, T_{13}(\kappa, l)\phi)_{L^2(\Pi)} + T_{14}(\kappa, l)\,, \tag{7.4}$$

where $T_{13}(\kappa,l):\mathbb{R}^2\to\mathbb{R}$ is a function defined for $(\kappa,l)\in\mathcal{Q}_\varkappa\times[\lambda_0,+\infty)$ which satisfies the relation

$$T_{14}(\kappa,l)=\mathcal{O}\left(\frac{\kappa^2\mathrm{e}^{-2\kappa l}}{\cosh^2\kappa l}+\mathrm{e}^{-4\sqrt{3}l}\right) \tag{7.5}$$

as $(\kappa,l)\to(0,+\infty)$. To get this estimate one has to employ the relation $T_{14}(\kappa,l)=\mathcal{O}(\|T_{13}(\kappa,l)\|^2)$ and the fact that

$$\|T_{13}(\kappa,l)\|=\mathcal{O}\left(\frac{\kappa\,\mathrm{e}^{-\kappa l}}{\cosh\kappa l}+\mathrm{e}^{-2\sqrt{3}l}\right) \tag{7.6}$$

implied by the definition of T_{13} – see (5.9). Our next step is to extract the leading term from the first summand at the right hand side of (7.4). We will do it in the same way as in last section, the only difference is that now we have to take into account also the second transverse-mode contribution to (5.9).

We introduce the function V_1 that is an analogue of (6.8) by

$$V_1(x):=\begin{cases}\dfrac{2\kappa\,\mathrm{e}^{-\kappa l}}{\pi\cosh\kappa l}x_1\sin x_2\displaystyle\int\limits_{\Pi^-}t_1\sin t_2\,\phi(t)\,d^2t\,, & x_1<0\,,\\ 0\,, & x_1>0\,.\end{cases} \tag{7.7}$$

Let W_1 be a solution to the problem (4.5) with $v=V_1$. We also introduce the function V_2 in the following way

$$V_2(x):=\begin{cases}\dfrac{2\mathrm{e}^{-\sqrt{3}l}}{\pi\sqrt{3}\cosh\sqrt{3}l}\sinh\sqrt{3}x_1\sin 2x_2\displaystyle\int\limits_{\Pi^-}\sinh\sqrt{3}t_1\sin 2t_2\,\phi(t)\,d^2t\,, & x_1<0\,,\\ 0\,, & x_1>0\end{cases} \tag{7.8}$$

and suppose that W_2 is a solution of (4.5) with $v=V_2$. One can check that

$$(\psi,T_{13}(\kappa,l)\phi)_{L^2(\Pi)}=-\left(\psi,(\Delta+1)\left(\widetilde{V}+\chi(\widetilde{W}-\widetilde{V})\right)\right)_{L^2(\Pi)}+T_{15}(\kappa,l)\,,$$

where $\widetilde{V}=V_1+V_2$, $\widetilde{W}=W_1+W_2$, and the function $T_{15}(\kappa,l)$ satisfies the estimate

$$T_{15}(\kappa,l)=\mathcal{O}\left(\frac{\kappa^3\mathrm{e}^{-\kappa l}}{\cosh\kappa l}+\kappa^2\mathrm{e}^{-2\sqrt{3}l}+\mathrm{e}^{-2\sqrt{8}l}\right) \tag{7.9}$$

as $(\kappa,l)\to(0,+\infty)$. Calculating the inner product $(\psi,T_{13}(\kappa,l)\phi)_{L^2(\Pi)}$ in the same way how we deduced (2.3) and bearing in mind the asymptotics (2.1) for ψ together with (7.4), (7.5), and (7.7)–(7.9) we obtain the equation

$$\begin{aligned}&2\kappa+\rho\frac{\sqrt{2}\kappa\mathrm{e}^{-\kappa l}}{\sqrt{\pi}\cosh\kappa l}\int\limits_{\Pi^-}t_1\sin t_2\,\phi\,d^2t\\ &\quad+\rho\frac{\beta\,\mathrm{e}^{-\sqrt{3}l}}{\cosh\sqrt{3}l}\int\limits_{\Pi^-}\sinh\sqrt{3}t_1\sin 2t_2\,\phi\,d^2t+T_{16}(\kappa,l)=0\,,\end{aligned} \tag{7.10}$$

where ρ is again the parity of ψ. The function $T_{16}(\kappa, l)$ satisfies

$$T_{16}(\kappa, l) = O\left(\frac{\kappa^2 \mathrm{e}^{-2\kappa l}}{\cosh^2 \kappa l} + \frac{\kappa^3 \mathrm{e}^{-\kappa l}}{\cosh \kappa l} + \kappa^2 \mathrm{e}^{-2\sqrt{3}l} + \mathrm{e}^{-2\sqrt{8}l}\right) \tag{7.11}$$

as $(\kappa, l) \to (0, +\infty)$. Since the function ϕ obeys $\psi = T_3(\kappa = 0)\phi$, we can take into account the definition of the last operator (see (4.13)) and the asymptotics (2.1) to conclude that

$$\rho\sqrt{\frac{2}{\pi}} = -\frac{2}{\pi}\int\limits_{\Pi^-} t_1 \sin t_2\, \phi(t)\, d^2t\,,$$

$$\rho\beta = -\frac{2}{\pi\sqrt{3}}\int\limits_{\Pi^-} \sinh\sqrt{3}t_1 \sin 2t_2\, \phi(t)\, d^2t\,,$$

which together with (7.10) leads us to ($\beta = \beta_n$)

$$2\kappa - \frac{\kappa\, \mathrm{e}^{-\kappa l}}{\cosh \kappa l} - \frac{\beta^2\pi\sqrt{3}}{2}\frac{\mathrm{e}^{-\sqrt{3}l}}{\cosh\sqrt{3}l} + T_{16}(\kappa, l) = 0\,,$$

or equivalently,

$$\frac{\kappa\, \mathrm{e}^{\kappa l}}{\cosh \kappa l} - \frac{\beta^2\pi\sqrt{3}}{2}\frac{\mathrm{e}^{-\sqrt{3}l}}{\cosh\sqrt{3}l} + T_{16}(\kappa, l) = 0\,. \tag{7.12}$$

We know that this equation has a positive solution tending to zero as $\kappa \to 0$. In view of (7.11), (7.12), and the trivial inequality

$$1 \le \frac{\mathrm{e}^{\tau}}{\cosh\tau} \le 2$$

we have for this solution the following estimate,

$$C_1 \mathrm{e}^{-2\sqrt{3}l} \le \kappa \le C_2 \mathrm{e}^{-2\sqrt{3}l}$$

with constants C_1, C_2 independent on l. Using it we can expand the first term in the equation (7.12) with respect to κl, which is small, and to estimate T_{15} – see (7.11). In this way we arrive at the relation,

$$\kappa - \beta^2\pi\sqrt{3}\,\mathrm{e}^{-2\sqrt{3}l} + \mathcal{O}(\mathrm{e}^{-2\sqrt{8}l}) = 0\,,$$

which implies the sought asymptotical expansion (2.6), (2.7). The second formula for μ stated in Theorem 2.2 can be proven completely by analogy with the proof of (2.5). One just should multiply the equation $(\Delta + 1)\psi^n$ by $\mathrm{e}^{\sqrt{3}x_1} \sin 2x_2$ and integrate then by parts over Π_R passing then to the limit as $R \to +\infty$.

The argument concerning the asymptotics for the associated eigenfunction is completely analogous to that of the previous section. The solution to the equation (7.1) is given by

$$\Phi = (I + T_8(\kappa)T_{13}(\kappa, l))^{-1}\phi\,.$$

Now one has just to perform the expansion using the fact that the operator $T_8(\kappa)T_{13}(\kappa, l)$ is small, then using the obtained asymptotics for κ, to apply to the remainder the estimate (7.6), to construct the corresponding eigenfunction of the problem (2.9) by the scheme described in the Section 5, and finally, to recover the eigenfunctions of $H_l(a_n)$. This completes the proof of the second theorem. $\blacksquare$

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