ON WEYL–TITCHMARSH THEORY FOR SINGULAR FINITE DIFFERENCE HAMILTONIAN SYSTEMS

STEVE CLARK AND FRITZ GESZTESY

Dedicated with great pleasure to Norrie Everitt on the occasion of his 80th birthday.

Abstract. We develop the basic theory of matrix-valued Weyl–Titchmarsh M-functions and the associated Green's matrices for whole-line and half-line self-adjoint Hamiltonian finite difference systems with separated boundary conditions.

1. Introduction

This paper can be viewed as a natural continuation of our recent work on matrix-valued Schrödinger and Dirac-type operators (cf. [15] and [16]) to discrete Hamiltonian systems (i.e., Hamiltonian systems of difference equations). These investigations are part of a larger program which includes the following:
(*i*) A systematic asymptotic expansion of Weyl–Titchmarsh matrices and Green's matrices as the spectral parameter tends to infinity ([15], [16]).
(*ii*) The derivation of trace formulas for such systems ([16], [18], [39]).
(*iii*) The proof of certain uniqueness theorems (including Borg and Hochstadt-type theorems) for the operators in question ([8], [16], [18], [41], [46]).
(*iv*) The application of these results to related integrable systems (cf. [8], [44]).

Before we describe the content of this paper in more detail, it is appropriate to briefly comment on the literature devoted to general $2m \times 2m$ Hamiltonian systems ($m \geq 2$) and their (inverse) spectral theory as it relates to the topics of this paper and the next one in our series (see [19]). Due to the enormous amount of interest generated by continuous Hamiltonian systems over the past twenty years, we are forced to focus primarily on references in connection with discrete Hamiltonian systems, but we refer the reader to [8], [15]–[18], [41], and [44] which provide extensive documentation of pertinent material. The basic Weyl–Titchmarsh theory of regular Hamiltonian systems can be found in Atkinson's monograph [5]; Weyl–Titchmarsh theory of singular Hamiltonian systems and their basic spectral theory was developed by Hinton and Shaw and many others (see, e.g., [49, Sect. 10.7], [50]–[59], [64]–[67], [70], [74]–[77], [79, Ch. 9], [83,] and the references therein); the corresponding theory for Jacobi operators can be found in [9, Sect. VII.2], [37], [78, Ch. 10] and the literature therein. Deficiency indices of matrix-valued Jacobi operators are studied in [60]–[62]. Inverse spectral and scattering theory for matrix-valued finite difference systems and its intimate connection to matrix-valued orthogonal

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polynomials and the moment problem are treated in [1], [2], [9, Sect. VII.2], [20]–[22], [38], [68], [69], [72], [73], [78, Ch. 8], [80]. Finally, connections with nonabelian completely integrable systems are discussed in [10], [11], [71], [78, Chs. 9, 10].

In spite of these activities, the reader might perhaps be surprised to hear that Weyl–Titchmarsh theory for general discrete Hamiltonian systems appears to be underdeveloped. The only notable exceptions to this statement of course being the special case of matrix-valued Jacobi operators which are described in detail in [9, Sect. VII.2], [37], and a discussion of a class of canonical systems in [78, Ch. 8].) In fact, at the conclusion of a meeting held in honor of Professor Allan Krall at the University of Tennessee-Knoxville on October 10, 2002, Professor Krall noted in remarks, which he entitled "Linear Hamiltonian systems involving difference equations", that a Weyl–Titchmarsh theory for general Hamiltonian systems of difference equations has yet to be developed. By Hamiltonian system of difference equations he meant those systems that arise naturally as a discretization of linear Hamiltonian systems of differential equations (cf. (2.16)–(2.18)), and in analogy to the material developed in [52]–[57], the principal aim should be to construct the matrix-valued Weyl–Titchmarsh function and develop the related specral theory of such systems. In part, our paper is meant to follow up on the challenge extended by Professor Krall and develop Weyl–Titchmarsh theory for singular discrete Hamiltonian systems as a natural extension of the existing theory for scalar Jacobi equations (cf. [4], [9, Sect. VII.1], [42], [45], [47], [81, Ch. 2] and the references therein). The actual model we follow closely in this paper is our recent treatment of Dirac-type systems in [16].

In this paper we develop the basic theory of matrix-valued Weyl–Titchmarsh M-functions and the associated Green's matrices for whole-line and half-line self-adjoint Hamiltonian finite difference systems defined as follows. Let $m \in \mathbb{N}$ and

$$B = \{B(k)\}_{k\in\mathbb{Z}} \subset \mathbb{C}(\mathbb{Z})^{2m\times 2m}, \quad \rho = \{\rho(k)\}_{k\in\mathbb{Z}} \subset \mathbb{C}(\mathbb{Z})^{m\times m} \tag{1.1}$$

with $\mathbb{C}(\mathbb{Z})^{r\times s}$, the space of sequences of complex $r\times s$ matrices, $r,s\in\mathbb{N}$, where $B(k)$ and $\rho(k)$ are assumed to be self-adjoint and nonsingular matrices for all $k\in\mathbb{Z}$. We denote by $S^{\pm}$ the shift operators acting upon $\mathbb{C}(\mathbb{Z})^{m\times s}$, that is,

$$S^{\pm} f(\,\cdot\,) = f^{\pm}(\,\cdot\,) = f(\,\cdot\,\pm 1), \quad f\in\mathbb{C}(\mathbb{Z})^{m\times s}. \tag{1.2}$$

Moreover, let

$$A = \{A(k)\}_{k\in\mathbb{Z}} \subset \mathbb{C}(\mathbb{Z})^{2m\times 2m}, \tag{1.3}$$

such that

$$A(k) = \begin{pmatrix} A_{1,1}(k) & A_{1,2}(k) \\ A_{2,1}(k) & A_{2,2}(k) \end{pmatrix} \geqslant 0, \quad k\in\mathbb{Z}, \tag{1.4}$$

where $A_{u,v} = \{A_{u,v}(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}$, $u,v=1,2$. Introducing the following linear difference expression

$$\mathcal{S}_\rho - B, \quad \mathcal{S}_\rho = \begin{pmatrix} 0 & \rho S^+ \\ \rho^- S^- & 0 \end{pmatrix}, \tag{1.5}$$

the eigenvalue equation, or *discrete Hamiltonian system* on the whole-line considered in this paper, is then given by

$$\mathcal{S}_\rho \Psi(z,k) = [zA(k) + B(k)]\Psi(z,k), \quad z\in\mathbb{C},\ k\in\mathbb{Z}. \tag{1.6}$$

Here z plays the role of the spectral parameter and

$$\Psi(z,k) = \begin{pmatrix} \psi_1(z,k) \\ \psi_2(z,k) \end{pmatrix}, \quad \psi_j(z,\,\cdot\,) \in \mathbb{C}(\mathbb{Z})^{m\times r},\ j=1,2 \tag{1.7}$$

with $1 \leq r \leq 2m$, and $S^{\pm}\psi_j(z,\,\cdot\,) = \psi_j(z,\,\cdot\,\pm 1)$, $j=1,2$. Analogously, we will consider (1.6) on a half-line. Of course, at finite endpoints of the underlying interval (and possibly also at the point(s) at infinity), the formally self-adjoint Hamiltonian system (1.6) needs to be supplied with appropriate self-adjoint boundary conditions to render it self-adjoint. This will be discussed in Sections 2 and 3.

Forms such as (1.6) arise naturally when discretizing a Hamiltonian system of first-order ordinary differential equations,

$$J\Psi'(z,x) = [z\mathcal{A}(x) + \mathcal{B}(x)]\Psi(z,x), \quad x \in \mathbb{R}, \tag{1.8}$$

as discussed in the next section (cf. the discussion following (2.16)).

In Section 2 we set up the basic Weyl–Titchmarsh formalism associated with (1.6). We discuss possible normal forms of (1.6) and show that ρ can be assumed to be diagonal and positive definite without loss of generality. Subsequently, we introduce the necessary tools to discuss separated boundary conditions associated with (1.6) on a finite interval and then define the corresponding $m\times m$ matrix-valued Weyl–Titchmarsh function, the Weyl disk, and the Weyl circle. The latter is shown to correspond to regular boundary value problems associated with (1.6) on a finite interval with separated self-adjoint boundary conditions at the endpoints. Next, the Herglotz property of the Weyl–Titchmarsh function is established and different boundary conditions at one endpoint (keeping the boundary condition fixed at the other endpoint) are shown to be related by linear fractional transformations. The typical nesting property of Weyl disks associated with a finite interval then yield the existence of a limiting Weyl disk as the finite interval approaches a half-line. The limiting disk is nonempty, closed, and convex. The elements of the limit disk turn out to be $m \times m$ matrix-valued Herglotz functions of rank m. If the limiting Weyl disk consists of just a point, one then has the important limit point case.

In our final Section 3 we then consider boundary value problems and Green's functions associated with the discrete Hamiltonian system (1.6) and appropriate self-adjoint boundary conditions on the whole-line and on half-lines.

The results on Green's functions in Section 3 are fundamental for the concrete applications we have in mind in our subsequent paper [19]. There we will consider trace formulas and Borg-type uniqueness theorems associated with matrix-valued Jacobi operators and certain (supersymmetric) Dirac-type difference operators, which turn out to be interesting special cases of the discrete Hamiltonian system (1.6). These special cases have interesting applications to hierarchies of completely integrable nonabelian nonlinear evolution equations. In fact, the matrix-valued Jacobi difference expression (2.11) subject to (2.21) yields a Lax operator for the nonabelian Toda hierarchy (cf., e.g., [81, Sect. 12.2], [82, Sects. 3.1, 3.2],) and the Dirac-type difference expression (2.11) subject to (2.19) yields a Lax operator for the nonabelian Kac–van Moerbeke hierarchy (cf., e,g., [12], [40], [81, Sect. 14.1], [82, Sect. 3.8]).

Dedication. It is with great pleasure that we dedicate this paper to Norrie Everitt on the occasion of his 80th birthday. His enormous influence on the field of ordinary differential operators is universally admired. In the very special context of this paper, we refer, in particular, to his fundamental papers [14], [23]–[36], which paved the way for a systematic treatment of general Hamiltonian systems and inspired a whole generation of scientists to enter this field.

2. Weyl–Titchmarsh Matrices for Finite Difference Hamiltonian Systems

We now turn to the Weyl–Titchmarsh theory for Hamiltonian systems of finite difference operators. The model for this part of our discussion is the analogous development of the theory presented in [16] which in turn is based upon the theory developed by Hinton and Shaw in a series of papers devoted to the spectral theory of (singular) Hamiltonian systems of differential equations [52]–[57] (see also [64], [65]).

Throughout this paper, matrices will be considered over the field of complex numbers $\mathbb{C}$. With M in the space of $r\times s$ complex matrices, $\mathbb{C}^{r\times s}$, $r,s\in\mathbb{N}$, let $M^\top$ denote the transpose, and let M^* denote the adjoint or conjugate transpose of the matrix M. Let $M\geqslant 0$ and $M\leqslant 0$ indicate that M is nonnegative and nonpositive respectively. Similarly, $M>0$ (respectively, $M<0$) denotes a positive definite (respectively, negative definite) matrix. Moreover, let $\mathrm{Im}(M)=(M-M^*)/(2i)$ and $\mathrm{Re}(M)=(M+M^*)/2$ denote the imaginary and real parts of the matrix M.

Denote by $\mathbb{C}(I)^{r\times s}$ the space of sequences, defined on $I\subseteq\mathbb{Z}$, of complex $r\times s$ matrices where $1\leqslant s\leqslant 2m$, and where typically $r\in\{m,2m\}$. Denote by $\ell^\infty(I)^{r\times s}$ the sequence space of complex $r\times s$ matrices bounded on $I\subseteq\mathbb{Z}$ with respect to the norm $\|\cdot\|_{\ell^\infty(I)^{r\times s}}$, while $\ell^p(I)^{r\times s}$ denotes the space of sequences p-summable on $I\subseteq\mathbb{Z}$ with respect to the norm $\|\cdot\|_{\ell^p(I)^{r\times s}}$. Let $S^\pm$ denote the shift operators on $\mathbb{C}(\mathbb{Z})^{m\times r}$, that is,

$$S^\pm f(\,\cdot\,)=f(\,\cdot\,\pm 1),\quad f^\pm=S^\pm f,\quad f\in\mathbb{C}(\mathbb{Z})^{m\times r}. \tag{2.1}$$

Moreover, with $\Psi\in\mathbb{C}(\mathbb{Z})^{2m\times r}$, let

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \widehat\Psi=\begin{pmatrix}\psi_1\\ \psi_2^+\end{pmatrix},\quad \psi_j\in\mathbb{C}(\mathbb{Z})^{m\times r},\quad j=1,2. \tag{2.2}$$

Unless explicitly stated otherwise, $[c,d]\subset\mathbb{R}$ will mean the discrete interval $[c,d]\cap\mathbb{Z}$, with $c,d\in\mathbb{Z}$, possibly with $d<c$; the trivial interval occurring when $c=d$. If $c\neq d$, let ${}^+[c,d]$ denote the discrete interval

$${}^+[c,d]=[\min\{c,d\}+1,\max\{c,d\}]. \tag{2.3}$$

Evaluation may be expressed by

$$\psi\big|_c=\psi(c), \tag{2.4}$$

while differences may be expressed by

$$\psi\big|_{[c,d]}=\psi(\max\{c,d\})-\psi(\min\{c,d\}). \tag{2.5}$$

Sums over discrete intervals may be expressed by

$$\sum_{k\in[c,d]}\psi(k)=\sum_{n=\min\{c,d\}}^{\max\{c,d\}}\psi(k). \tag{2.6}$$

These conventions will turn out to be useful in connection with the functional $E_\ell(M)$ introduced in (2.50) in the sense that they permit us to avoid numerous case distinctions associated with $k_0>\ell$, $k_0=\ell$, $k_0<\ell$, etc.

Next, let $m \in \mathbb{N}$, and let $J \in \mathbb{C}^{2m\times 2m}$, $J_\rho(k), I_\rho(k) \in \mathbb{C}(\mathbb{Z})^{2m\times 2m}$ be defined for $k \in \mathbb{Z}$ by

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad J_\rho(k) = \begin{pmatrix} 0 & \rho(k) \\ -\rho(k) & 0 \end{pmatrix}, \quad I_\rho(k) = \begin{pmatrix} \rho(k) & 0 \\ 0 & \rho(k) \end{pmatrix}. \tag{2.7}$$

Here I_m denotes the $m \times m$ identity matrix in $\mathbb{C}^m$ and $\rho(k) \in \mathbb{C}^{m\times m}$ is self-adjoint and nonsingular for all $k \in \mathbb{Z}$. Let $A_{u,v}(k), B_{u,v}(k) \in \mathbb{C}(\mathbb{Z})^{m\times m}$ for $u, v = 1, 2$ and $k \in \mathbb{Z}$. Moreover, for $k \in \mathbb{Z}$, let

$$A(k) = \begin{pmatrix} A_{1,1}(k) & A_{1,2}(k) \\ A_{2,1}(k) & A_{2,2}(k) \end{pmatrix} \geqslant 0, \tag{2.8}$$

$$B(k) = \begin{pmatrix} B_{1,1}(k) & B_{1,2}(k) \\ B_{2,1}(k) & B_{2,2}(k) \end{pmatrix} = B(k)^*. \tag{2.9}$$

In terms of the operator ρS^+ and its formal adjoint $\rho^- S^-$, let $\mathcal{S}_\rho$ denote the formally self-adjoint matrix-valued difference expression given by

$$\mathcal{S}_\rho = \begin{pmatrix} 0 & \rho S^+ \\ \rho^- S^- & 0 \end{pmatrix}. \tag{2.10}$$

With $A(k)$, $B(k)$, $\mathcal{S}_\rho$ defined in (2.8) – (2.10), we consider the general difference expression given by

$$\mathcal{S}_\rho - B, \tag{2.11}$$

and its associated eigenvalue equation, or general *Hamiltonian system*, given by

$$\mathcal{S}_\rho \Psi(z,k) = [zA(k) + B(k)]\Psi(z,k), \quad z \in \mathbb{C},\ k \in \mathbb{Z}. \tag{2.12a}$$

Here z plays the role of the spectral parameter and

$$\Psi(z,k) = \begin{pmatrix} \psi_1(z,k) \\ \psi_2(z,k) \end{pmatrix}, \quad \psi_j(z,\,\cdot\,) \in \mathbb{C}(\mathbb{Z})^{m\times r},\ j = 1,2 \tag{2.12b}$$

with $1 \le r \le 2m$, and $S^\pm \psi_j(z,\,\cdot\,) = \psi_j(z,\,\cdot\, \pm 1)$, $j = 1,2$. Such a Hamiltonian system is said to be *well-posed* when it possesses unique solutions defined for all $k \in \mathbb{Z}$ associated with prescribed initial values of the type

$$\widehat{\Psi}(z,k_0) \in \mathbb{C}^{2m}. \tag{2.12c}$$

A necessary and sufficient condition for well-posedness is given in (2.14) below.

For our discussion concerning the Weyl–Titchmarsh theory of the Hamiltonian system (2.12a), we also adopt a definiteness condition like that of Atkinson [5]. We briefly sum up all hypotheses on the coefficients in (2.12a) as follows:

Hypothesis 2.1. We assume that our Hamiltonian system satisfies

$$A(k) \ge 0, \quad B(k) = B(k)^*, \quad \rho(k) > 0 \text{ for all } k \in \mathbb{Z}, \tag{2.13}$$

$$zA_{1,2}(k) + B_{1,2}(k) \text{ is invertible for all } k \in \mathbb{Z} \text{ and } z \in \mathbb{C}, \tag{2.14}$$

and for all nontrivial solutions $\Psi \in \mathbb{C}^{2m}$ of (2.12a), we suppose that

$$\sum_{k\in[c,d]} \Psi(z,k)^* A(k) \Psi(z,k) > 0, \tag{2.15}$$

for every nontrivial discrete interval $[c,d] \subset \mathbb{Z}$ in the case of the whole-line (resp., $[c,d] \subset [k_0,\infty)$ or $[c,d] \subset (-\infty,k_0]$ for some $k_0 \in \mathbb{Z}$ in the case of half-lines).

Remark 2.2. *Of course, condition* (2.14) *requires invertibility of* $B_{1,2}$ (*and hence that of* $B_{2,1}$). *Moreover, it is equivalent to invertibility of* $zA_{2,1}(k)+B_{2,1}(k)$ *for all* $k\in\mathbb{Z}$ *and* $z\in\mathbb{C}$. *In addition, condition* (2.14) *guarantees existence and uniqueness of the initial value problem* (2.12a), (2.12c) *by an explicit step by step construction of the solution* $\Psi(z,k)$, $k\in\mathbb{Z}$: *Given* $\widehat{\Psi}(z,k_0)$, *one needs to invert* $[zA_{2,1}(z,\ell)+B_{2,1}(z,\ell)]$ *for all* $\ell\geq k_0+1$ *to construct* $\Psi(z,k)$ *for all* $k\geq k_0+1$ *and one needs to invert* $[zA_{1,2}(z,\ell)+B_{1,2,}(z,\ell)]$ *for all* $\ell\leq k_0-1$ *to construct* $\Psi(z,k)$ *for all* $k\leq k_0$.

To avoid numerous case distinctions we will suppose the whole-line part of Hypothesis 2.1 throughout this section. We will make an explicit distinction between the whole-line and half-line cases in Section 3.

Forms such as (2.12a) arise naturally when discretizing a Hamiltonian system of first-order ordinary differential equations,

$$J\Psi'(z,x)=[z\mathcal{A}(x)+\mathcal{B}(x)]\Psi(z,x),\quad x\in\mathbb{R}, \tag{2.16}$$

by replacing $\Psi'(z,x)$ with the difference expression given by

$$\begin{pmatrix} -\partial^*\psi_1(z,k)\\ \partial\psi_2(z,k)\end{pmatrix}, \tag{2.17}$$

where the formally adjoint operators ∂ and ∂^* are defined by

$$\partial=S^+-I_m,\quad \partial^*=S^--I_m, \tag{2.18}$$

and where I_m represents the identity matrix in $\mathbb{C}(\mathbb{Z})^{m\times m}$. These forms also arise when considering matrix-valued Jacobi operators (cf. [41]), or when considering the matrix-valued generalizations of the super-symmetric Dirac-type operators considered in [12], [40], and [81, Sect. 14.1]. In particular (2.11) represents a super-symmetric Dirac-type operator in (2.9) when $B_{11}(k)=B_{22}(k)=0$, that is,

$$A(k)=I_{2m},\quad B(k)=\begin{pmatrix}0 & b(k)\\ b(k)^* & 0\end{pmatrix},\quad k\in\mathbb{Z}, \tag{2.19}$$

and is relevant to the Kac–van Moerbeke system. Alternatively, (2.12a) represents

$$\partial\, p\,\partial^* y+qy=zy, \tag{2.20}$$

the matrix-valued Sturm–Liouville difference equation, when

$$\rho(k)=I_m\ ,\quad A(k)=\begin{pmatrix}I_m & 0\\ 0 & 0\end{pmatrix},\quad B(k)=\begin{pmatrix}-q(k) & I_m\\ I_m & p(k)^{-1}\end{pmatrix},\quad k\in\mathbb{Z}, \tag{2.21}$$

and $B(k)^*=B(k)$, $k\in\mathbb{Z}$. Equation (2.20) is intimately related to the Jacobi operator H. More precisely, introducing the matrix-valued Jacobi difference expression L by

$$L=aS^++a^-S^-+b, \tag{2.22}$$

where

$$a=-p^+,\quad b=p^++p+q, \tag{2.23}$$

$$p=-a^-,\quad q=a+a^-+b \tag{2.24}$$

are $m\times m$ matrices, the matrix-valued Sturm–Liouville difference equation (2.20) is equivalent to the equation

$$Ly=zy. \tag{2.25}$$

Note that the examples cited in (2.19) and in (2.21) satisfy the requirements (2.13) and (2.15) in Hypothesis 2.1. Moreover, (2.14) is automatically satisfied for the example described in (2.21).

When considering the spectral or inverse spectral theory of (2.11) (especially in the context of [19]), we may choose, without loss of generality, a more restrictive normal form of ρ in which ρ represents a diagonal and positive definite matrix.

Lemma 2.3. *The difference expression in* (2.11) *is unitarily equivalent to another such expression in which ρ is diagonal and positive definite.*

Proof. Let $Q(k) \in \mathbb{C}^{m\times m}$ define a unitary matrix such that $Q(k)\rho(k)Q(k)^{-1} = \widetilde{d}(k)$, where $\widetilde{d}(k) \in \mathbb{R}^{m\times m}$ is diagonal and self-adjoint for all $k \in \mathbb{Z}$. Then,

$$U_\rho(\mathcal{S}_\rho - B)U_\rho^{-1} = \mathcal{S}_{\widetilde{d}} - \widetilde{B}, \quad \widetilde{B} = U_\rho B U_\rho^{-1}, \quad U_\rho = \begin{pmatrix} Q & 0 \\ 0 & Q^- \end{pmatrix}. \tag{2.26}$$

Next, let $\widetilde{\epsilon}(k) \in \mathbb{R}^{m\times m}$ be a diagonal matrix for which $(\widetilde{\epsilon}(k))_{\ell,\ell} \in \{+1,-1\}$, $\ell = 1,\dots,m$. Define $\epsilon(k) \in \mathbb{R}^{m\times m}$ by $\epsilon(k) = \widetilde{\epsilon}(k)\widetilde{\epsilon}(k+1)$ and choose $\widetilde{\epsilon}(k)$ so that $d = \epsilon\widetilde{d} > 0$. Then,

$$U_\epsilon(\mathcal{S}_d - \widetilde{B})U_\epsilon^{-1} = \mathcal{S}_d - \mathcal{B}, \quad \mathcal{B} = U_\epsilon \widetilde{B} U_\epsilon^{-1}, \quad U_\epsilon = \begin{pmatrix} \widetilde{\epsilon} & 0 \\ 0 & \widetilde{\epsilon} \end{pmatrix}, \tag{2.27}$$

thus showing that $\mathcal{S}_\rho - B$ is unitarily equivalent to a difference expression of type (2.11) for which ρ is diagonal and positive definite. $\square$

Definition 2.4. By a general difference expression and its associated Hamiltonian system of first-order difference equations, we mean (2.11) and (2.12), respectively, subject to Hypothesis 2.1 as well as the additional assumption that the matrix ρ is positive definite.

Thus, we assume the following set of assumptions for the remainder of this paper:

Hypothesis 2.5. In addition to Hypothesis 2.1 assume that ρ is positive definite.

Next, we introduce a set of matrices which will serve to describe boundary data for separated boundary conditions to be associated with the Hamiltonian system given in (2.12a):

Definition 2.6. Let $\mathcal{B}_d$ denote the set of matrices $\gamma = (\gamma_1\ \gamma_2)$ with $\gamma_j \in \mathbb{C}^{m\times m}$, $j = 1,2$, which satisfy the following conditions,

$$\operatorname{rank}(\gamma) = m, \tag{2.28a}$$

and that either

$$\operatorname{Im}(\gamma_1\gamma_2^*) \leqslant 0, \quad \text{or} \quad \operatorname{Im}(\gamma_1\gamma_2^*) \geqslant 0, \tag{2.28b}$$

where $(2i)^{-1}\gamma J\gamma^* = \operatorname{Im}(\gamma_1\gamma_2^*)$. Given the rank condition in (2.28a), we assume, without loss of generality in what follows, the normalization

$$\gamma\gamma^* = I_m. \tag{2.28c}$$

With $\rho(k) \in \mathbb{C}^{m\times m}$ positive definite and diagonal, and I_ρ as given in (2.7), $\widetilde{\gamma}$ is given by

$$\widetilde{\gamma}(k) = \gamma I_\rho(k)^{1/2}. \tag{2.28d}$$

In (2.28d) (and in the remainder of this paper) $\rho^{1/2}$ will always denote the unique positive definite square root of $\rho > 0$.

Remark 2.7. *With $\gamma \in \mathbb{C}^{m\times 2m}$, the conditions*

$$\gamma\gamma^* = I_m, \quad \gamma J\gamma^* = 0 \tag{2.29}$$

imply that $\gamma \in \mathcal{B}_d$, and explicitly read

$$\gamma_1\gamma_1^* + \gamma_2\gamma_2^* = I_m, \quad \gamma_1\gamma_2^* - \gamma_2\gamma_1^* = 0. \tag{2.30}$$

In fact, from (2.29) one also obtains

$$\gamma_1^*\gamma_1 + \gamma_2^*\gamma_2 = I_m, \quad \gamma_1^*\gamma_2 - \gamma_2^*\gamma_1 = 0, \tag{2.31}$$

as is clear from

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_1^* & -\gamma_2^* \\ \gamma_2^* & \gamma_1^* \end{pmatrix} = I_{2m} = \begin{pmatrix} \gamma_1^* & -\gamma_2^* \\ \gamma_2^* & \gamma_1^* \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix}, \tag{2.32}$$

since any left inverse matrix is also a right inverse, and vice versa. Moreover, from (2.31) or (2.32), we obtain

$$\gamma^*\gamma J + J\gamma^*\gamma = J. \tag{2.33}$$

With $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.29) and with $\widetilde\alpha = \widetilde\alpha(k_0)$ defined according to (2.28d), let $\Psi(z,k,k_0,\widetilde\alpha)$ denote a normalized fundamental system of solutions for the Hamiltonian system (2.12a) described in Definition 2.4 which for some $k_0 \in \mathbb{Z}$ satisfies

$$\widehat\Psi(z,k_0,k_0,\widetilde\alpha) = I_\rho(k_0)^{-1}(\widetilde\alpha^* \ J\widetilde\alpha^*) = I_\rho(k_0)^{-1/2}(\alpha^* \ J\alpha^*). \tag{2.34a}$$

We partition $\Psi(z,k,k_0,\widetilde\alpha)$ as follows,

$$\begin{aligned} \Psi(z,k,k_0,\widetilde\alpha) &= (\Theta(z,k,k_0,\widetilde\alpha) \quad \Phi(z,k,k_0,\widetilde\alpha)) & (2.34\text{b}) \\ &= \begin{pmatrix} \theta_1(z,k,k_0,\widetilde\alpha) & \phi_1(z,k,k_0,\widetilde\alpha) \\ \theta_2(z,k,k_0,\widetilde\alpha) & \phi_2(z,k,k_0,\widetilde\alpha) \end{pmatrix}, & (2.34\text{c}) \end{aligned}$$

where $\theta_j(z,k,k_0,\widetilde\alpha)$ and $\phi_j(z,k,k_0,\widetilde\alpha)$ for $j=1,2$ are $m\times m$ matrices, entire with respect to $z\in\mathbb{C}$, and normalized according to (2.34a). One can now prove the following result.

Lemma 2.8. *Let $\Theta(z,k) = \Theta(z,k,k_0,\widetilde\alpha)$ and $\Phi(z,k) = \Phi(z,k,k_0,\widetilde\alpha)$ be defined in* (2.34) *with $\alpha,\beta \in \mathcal{B}_d$, and with $\operatorname{Im}(\alpha_1\alpha_2^*) = 0$. Let $\widetilde\alpha = \widetilde\alpha(k_0)$ and $\widetilde\beta = \widetilde\beta(\ell)$ be defined according to* (2.28d). *Then, for $\ell \neq k_0$, $\widetilde\beta\widehat\Phi(z,\ell)$ is singular if and only if z is an eigenvalue for the regular boundary value problem given by* (2.12a) *together with the separated boundary conditions*

$$\widetilde\alpha\widehat\Psi(z,k_0) = 0, \quad \widetilde\beta\widehat\Psi(z,\ell) = 0. \tag{2.35}$$

One observes that both regular boundary conditions described in (2.35) are self-adjoint when $\operatorname{Im}(\beta_1\beta_2^*) = 0$.

In light of Lemma 2.8, it is possible to introduce, under appropriate conditions, the $m\times m$ matrix-valued meromorphic function, $M(z,\ell,k_0,\widetilde\alpha,\widetilde\beta)$, as follows.

Definition 2.9. Let (2.34) define $\Theta(z,k,k_0,\widetilde\alpha)$, and $\Phi(z,k,k_0,\widetilde\alpha)$ with $\alpha,\beta\in\mathcal{B}_d$, and $\operatorname{Im}(\alpha_2\alpha_1^*) = 0$. For $\ell\neq k_0$, and $\widetilde\beta\widehat\Phi(z,\ell,k_0,\widetilde\alpha)$ nonsingular, define

$$M(z,\ell,k_0,\widetilde\alpha,\widetilde\beta) = -[\widetilde\beta\widehat\Phi(z,\ell,k_0,\widetilde\alpha)]^{-1}[\widetilde\beta\widehat\Theta(z,\ell,k_0,\widetilde\alpha)]. \tag{2.36}$$

$M(z,\ell,k_0,\widetilde\alpha,\widetilde\beta)$ is said to be the *Weyl–Titchmarsh M-function* for the regular boundary value problem described in Lemma 2.8.

By means of the equations

$$v_1(z,k) = \rho(k)^{1/2}\psi_1(z,k), \quad v_2(z,k) = \rho^-(k)^{1/2}\psi_2(z,k), \tag{2.37}$$

and

$$\widetilde{\alpha} = \widetilde{\alpha}(k_0) = \alpha I_\rho(k_0)^{1/2}, \quad \widetilde{\beta} = \widetilde{\beta}(\ell) = \beta I_\rho(\ell)^{1/2}, \tag{2.38}$$

the boundary value problem described in Lemma 2.8 is transformed into one described by

$$\mathcal{S}_{I_m} V(z,k) = [z\widetilde{A}(k) + \widetilde{B}(k)]V(z,k), \quad V(z,k) = \begin{pmatrix} v_1(z,k) \\ v_2(z,k) \end{pmatrix}, \tag{2.39}$$

$$\alpha\widehat{V}(z,k_0) = 0, \quad \beta\widehat{V}(z,\ell) = 0, \tag{2.40}$$

where

$$\widetilde{A} = \mathcal{D}^{-1/2}A\mathcal{D}^{-1/2}, \quad \widetilde{B} = \mathcal{D}^{-1/2}B\mathcal{D}^{-1/2}, \quad \mathcal{D} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^- \end{pmatrix}. \tag{2.41}$$

The following remark will play a role in connection with inverse spectral theory considerations in [19].

Remark 2.10. (i) *In general,* (2.37) *and* (2.38) *do not define a unitary transformation. However, if* $\Psi_{I_m}(z,k,k_0,\alpha)$ *represents the fundamental solution matrix of* (2.39) *such that* $\Psi_{I_m}(z,k_0,k_0,\alpha) = (\alpha^* \; J\alpha^*)$ *and if*

$$M_{I_m}(z,\ell,k_0,\alpha,\beta) = -[\beta\widehat{\Phi}_{I_m}(z,\ell,k_0,\alpha)]^{-1}[\beta\widehat{\Theta}_{I_m}(z,\ell,k_0,\alpha)] \tag{2.42}$$

represents the associated M-function for (2.39)*, while* $M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)$ *is defined in* (2.36)*, then*

$$M_{I_m}(z,\ell,k_0,\alpha,\beta) = M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big). \tag{2.43}$$

So one can replace ρ *by* I_m *and* $\widetilde{\alpha}$, $\widetilde{\beta}$ *by* α, β *but possibly at the expense of more complex expressions for* A *and* B.
(ii) *Assume that* $M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)$ *is the* M*-function defined in* (2.36) (*with* $\rho(k) > 0$, $k \in \mathbb{Z}$) *and* $M_{\widetilde{d}}(z,\ell,k_0,\gamma,\delta)$ *is the* M*-function corresponding to*

$$\mathcal{S}_{\widetilde{d}}\mathit{\Phi}(z,k) = [z\widetilde{A}(k) + \widetilde{B}(k)]\mathit{\Phi}(z,k), \tag{2.44}$$

$$\widetilde{A}(k) = U_\rho(k)A(k)U_\rho(k)^{-1}, \quad \widetilde{B}(k) = U_\rho(k)B(k)U_\rho(k)^{-1}, \tag{2.45}$$

$$U_\rho(k) = \begin{pmatrix} Q(k) & 0 \\ 0 & Q(k)^- \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{2.46}$$

with $\widetilde{d}(k) > 0$ *a diagonal matrix as discussed in the proof of Lemma* 2.3 (*and* $Q(k)$ *a unitary* $m \times m$ *matrix as in the proof of Lemma* 2.3*, except that* $\rho(k)$ *was not assumed to be positive definite in Lemma* 2.3)*. Then,*

$$M_{\widetilde{d}}(z,\ell,k_0,\gamma,\delta) = M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big), \tag{2.47}$$

where

$$\gamma = \widetilde{\alpha}\begin{pmatrix} Q(k_0)^{-1} & 0 \\ 0 & Q(k_0)^{-1} \end{pmatrix}, \quad \delta = \widetilde{\beta}\begin{pmatrix} Q(\ell)^{-1} & 0 \\ 0 & Q(\ell)^{-1} \end{pmatrix}. \tag{2.48}$$

The Weyl–Titchmarsh M-function in (2.36) is an $m \times m$ matrix-valued function with meromorphic entries whose poles correspond to eigenvalues for the regular boundary value problem given by the difference equation (2.12a) and the boundary conditions (2.35). Moreover, given the normalized fundamental matrix, $\Psi(z,k,k_0,\widetilde{\alpha})$, defined in (2.34) and given $M \in \mathbb{C}^{m\times m}$, one defines

$$U(z,k,k_0,\widetilde{\alpha}) = \begin{pmatrix} u_1(z,k,k_0,\widetilde{\alpha}) \\ u_2(z,k,k_0,\widetilde{\alpha}) \end{pmatrix} = \Psi(z,k,k_0,\widetilde{\alpha}) \begin{pmatrix} I_m \\ M \end{pmatrix}, \tag{2.49}$$

with $u_j(z,k,k_0,\widetilde{\alpha}) \in \mathbb{C}^{m\times m}$, $j=1,2$. Then $U(z,k,k_0,\widetilde{\alpha})$ will satisfy the boundary condition at $k=\ell$ in (2.35) when $M = M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)$. Intimately connected with the matrices introduced in Definition 2.9 is the set of $m\times m$ complex matrices known as the Weyl disk. Several characterizations of this set have appeared in the literature for the Hamiltonian system of differential equations given in (2.16) (see, e.g., [5]–[7], [50], [52], [64], [70]). By analogy, such definitions also exist for the Hamiltonian difference equation (2.12a).

To describe this set, we first introduce the matrix-valued function $E_\ell(M)$: With $\ell \neq k_0$, $z \in \mathbb{C}\backslash\mathbb{R}$, and with $U(z,\ell,k_0,\widetilde{\alpha})$ defined by (2.49) in terms of a matrix $M \in \mathbb{C}^{m\times m}$, let

$$E_\ell(M) = \sigma(\ell,k_0,z)\widehat{U}(z,\ell,k_0,\widetilde{\alpha})^*(iJ_\rho(\ell))\widehat{U}(z,\ell,k_0,\widetilde{\alpha}), \tag{2.50}$$

where

$$\sigma(\ell,k,z) = \frac{(\ell-k)\mathrm{Im}(z)}{|(\ell-k)\mathrm{Im}(z)|}, \quad \sigma(\ell,k) = \sigma(\ell,k,i) \tag{2.51}$$

with $\ell \neq k$, and $\ell,k \in \mathbb{Z}$.

Definition 2.11. Let the following be fixed: Integers k_0 and $\ell \neq k_0$, $\alpha \in \mathcal{B}_d$, and $z \in \mathbb{C}\backslash\mathbb{R}$. $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$ will denote the collection of all $M \in \mathbb{C}^{m\times m}$ for which $E_\ell(M) \leqslant 0$, where $E_\ell(M)$ is defined in (2.50). $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$ is said to be a *Weyl disk*. The set of $M \in \mathbb{C}^{m\times m}$ for which $E_\ell(M) = 0$ is said to be a *Weyl circle* (even when $m>1$). The interior of the Weyl disk is denoted by $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})^\circ$.

This definition leads to a representation that is a generalization of the description first given by Weyl [84] in the context of Sturm-Liouville differential expressions: a representation in which $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$ is homeomorphic to the set of contractive matrices, that is, those matrices $V \in \mathbb{C}^{m\times m}$ for which $VV^* \leqslant I_m$. This provides the justification for the geometric terms of circle and disk (cf., e.g., [50], [52], [64], [70]). From this representation it is also seen that the *interior* of the Weyl disk is nonempty and corresponds to the collection of all $M \in \mathbb{C}^{m\times m}$ for which $E_\ell(M) < 0$.

We next discuss some basic properties associated with elements of the disk $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$. To this end, we introduce the assumptions contained in the next hypothesis for the parameters k_0 and ℓ:

Hypothesis 2.12. If for the Hamiltonian system satisfying Hypothesis 2.5 it is given that $A(k) > 0$ for $k \in \mathbb{Z}$, then $k_0 \neq \ell$; otherwise, we assume that ${}^+[k_0,\ell]$ is nontrivial.

In the next lemma, we note that the Weyl circle corresponds to the regular boundary value problems with separated, self-adjoint boundary conditions described in in Lemma 2.8. This lemma is the analog in our discrete setting of Lemma 2.8 in [16]. For convenience of the reader, and to achieve a reasonable level of completeness, we produce the corresponding short proof below.

Lemma 2.13. *Given Hypothesis* 2.12*, let $M \in \mathbb{C}^{m\times m}$, and let $z \in \mathbb{C}\backslash\mathbb{R}$. Then, $E_\ell(M) = 0$ if and only if there is a $\beta \in \mathbb{C}^{m\times 2m}$ satisfying* (2.29) *such that*

$$0 = \widetilde{\beta}\widehat{U}(z,\ell,k_0,\widetilde{\alpha}), \tag{2.52}$$

where $U(z,\ell,k_0,\widetilde{\alpha})$ is defined in (2.49) *in terms of M, and where $\widetilde{\beta}=\widetilde{\beta}(\ell)$. With β so defined,*

$$M=-[\widetilde{\beta}\widehat{\Phi}(z,\ell,k_0,\widetilde{\alpha})]^{-1}[\widetilde{\beta}\widehat{\Theta}(z,\ell,k_0,\widetilde{\alpha})], \tag{2.53}$$

that is, $M=M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)$. Moreover, $\beta\in\mathcal{B}_d$ and may be chosen to satisfy (2.28c).

Proof. Let $z\in\mathbb{C}\backslash\mathbb{R}$, and suppose for a given $M\in\mathbb{C}^{m\times m}$ that there is a $\beta\in\mathbb{C}^{m\times 2m}$ which satisfies (2.29) and such that (2.52) is satisfied. Given that $\beta J\beta^*=0$ and that $\operatorname{rank}(\widetilde{\beta})=\operatorname{rank}(I_\rho(\ell)^{-1}J\widetilde{\beta}^*)=m$, then by (2.52) there is a nonsingular $C\in\mathbb{C}^{m\times m}$ such that $\widehat{U}(z,\ell,k_0,\widetilde{\alpha})=I_\rho(\ell)^{-1}J\widetilde{\beta}^*C$. Hence, $E_\ell(M)=-i\sigma(\ell,k_0,z)C^*\beta J\beta^*C=0$.

Upon showing that $\widetilde{\beta}\widehat{\Phi}(z,\ell,k_0,\widetilde{\alpha})$ is nonsingular, (2.53) will then follow from (2.52). If $\widehat{\Phi}(z,\ell)=\widehat{\Phi}(z,\ell,k_0,\widetilde{\alpha})$ and $\widetilde{\beta}\widehat{\Phi}(z,\ell)$ is singular, then there are nonzero vectors $v,w\in\mathbb{C}^m$ such that $\widetilde{\beta}\widehat{\Phi}(z,\ell)v=0$, and such that $\widehat{\Phi}(z,\ell)v=I_\rho(\ell)^{-1}J\widetilde{\beta}^*w$. Let $\Psi_j=\Psi_j(z,k)$, $j=1,2$, denote solutions of (2.12a) with $z=z_j$, $j=1,2$. Noting that

$$\partial(\widehat{\Psi}_1^*J_\rho\widehat{\Psi}_2)^-=\Psi_1^*\mathcal{S}_\rho\Psi_2-(\mathcal{S}_\rho\Psi_1)^*\Psi_2=(z_2-\bar{z}_1)\Psi_1^*A\Psi_2. \tag{2.54}$$

and recalling that $\Phi(z,\,\cdot\,)=\Phi(z,\,\cdot\,,k_0,\widetilde{\alpha})$ is defined in (2.34), we obtain

$$\widehat{\Phi}^*J_\rho\widehat{\Phi}\big|_{[k_0,\ell]}=(z-\bar{z})\sum_{+[k_0,\ell]}\Phi^*A\Phi. \tag{2.55}$$

by (2.54). Since $\widehat{\Phi}(z,k_0)^*J_\rho(k_0)\widehat{\Phi}(z,k_0)=0$, we then see that

$$2i\operatorname{Im}(z)\sum_{k\in+[k_0,\ell]}v^*\Phi^*(z,k)A(k)\Phi(z,k)v=\sigma(\ell,k_0)v^*\widehat{\Phi}(z,\ell)^*J_\rho(\ell)\widehat{\Phi}(z,\ell)v \tag{2.56}$$

$$=w^*\beta J\beta^*w=0. \tag{2.57}$$

By Hypothesis 2.12, $\operatorname{Im}(z)=0$. This contradicts the assumption that $z\in\mathbb{C}\backslash\mathbb{R}$.

Conversely, if $E_\ell(M)=0$ for a given $M\in\mathbb{C}^{m\times m}$, then for $z\in\mathbb{C}\backslash\mathbb{R}$ let $\widetilde{\beta}=[I_m\ \ M^*]\widehat{\Psi}(z,\ell,k_0,\widetilde{\alpha})^*J_\rho(\ell)=\widehat{U}(z,\ell,k_0,\widetilde{\alpha})^*J_\rho(\ell)$, and let $\beta=\widetilde{\beta}I_\rho(\ell)^{-1/2}$. Thus (2.52) is satisfied and $\operatorname{rank}(\beta)=\operatorname{rank}(\widetilde{\beta})=m$. Moreover, $0=E_\ell(M)=2\sigma(\ell,k_0,z)\operatorname{Im}(\beta_1\beta_2^*)$. If for this choice of β, (2.28c) is not yet satisfied Let $\delta=(\beta\beta^*)^{-1/2}\beta$, and $\widetilde{\delta}=\delta I_\rho(\ell)^{1/2}$. Note that $0=\widetilde{\delta}U(z,\ell,k_0,\widetilde{\alpha})$, that $\operatorname{Im}(\delta_1\delta_2^*)=(\beta\beta^*)^{-1/2}\operatorname{Im}(\beta_1\beta_2^*)(\beta\beta^*)^{-1/2}$, and hence that $\delta\in\mathcal{B}_d$. □

Next, we observe that a fundamental property holds for matrices in $\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$.

Lemma 2.14. *Given Hypothesis* 2.12*, let $M\in\mathbb{C}^{m\times m}$, and let $z\in\mathbb{C}\backslash\mathbb{R}$. Then,*

$$\sigma(\ell,k_0,z)\operatorname{Im}(M)>0, \tag{2.58}$$

whenever $M\in\mathcal{D}(z,\ell,k_0,\widetilde{\alpha})$. Moreover, whenever $\beta\in\mathbb{C}^{m\times 2m}$ satisfies (2.29)*,*

$$M\big(\bar{z},\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)=M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big)^*. \tag{2.59}$$

Proof. By (2.54),

$$2i\operatorname{Im}(z)\sum_{+[k_0,\ell]}U^*AU=\widehat{U}^*J_\rho\widehat{U}\big|_{[k_0,\ell]} \tag{2.60a}$$

$$=2i\sigma(\ell,k_0)\operatorname{Im}(M)+\sigma(\ell,k_0)\widehat{U}^*J_\rho\widehat{U}\big|_\ell \tag{2.60b}$$

with $U = U(z, \cdot, k_0, \widetilde{\alpha})$ defined in (2.49). Moreover, by the definition of $E_\ell(M)$ in (2.50), one obtains

$$2\sigma(\ell, k_0, z)\mathrm{Im}(M) = -E_\ell(M) + 2|\mathrm{Im}(z)| \sum_{+[k_0,\ell]} U^* A U. \tag{2.61}$$

By Hypothesis 2.12 and Definition 2.11, one infers that $\sigma(\ell, k_0, z)\mathrm{Im}(M) > 0$.

To prove (2.59), we first let $\Psi(z) = \Psi(z, \cdot, k_0, \widetilde{\alpha})$, where Ψ is defined in (2.34). By (2.54) we note that $\widehat{\Psi}(\bar{z})^* J_\rho \widehat{\Psi}(z) = -J$. As a consequence,

$$I_\rho^{1/2} J \widehat{\Psi}(z)(\widehat{\Psi}(\bar{z})J)^* I_\rho^{1/2} = -I_{2m}, \tag{2.62}$$

and hence $\widehat{\Psi}(z) J \widehat{\Psi}(\bar{z})^* = -J I_\rho^{-1}$. We obtain

$$\begin{aligned}(\widetilde{\beta}\widehat{\Phi}(z))(\widetilde{\beta}\widehat{\Theta}(\bar{z}))^*\big|_\ell - (\widetilde{\beta}\widehat{\Theta}(z))(\widetilde{\beta}\widehat{\Phi}(\bar{z}))^*\big|_\ell &= -\widetilde{\beta} J I_\rho^{-1} \widetilde{\beta}^*\big|_\ell && \text{(2.63a)}\\ &= -\beta J \beta^* = 0. && \text{(2.63b)}\end{aligned}$$

Equation (2.59) then follows immediately. □

For $\ell > k_0$, the function $M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$, defined by (2.36), whose values satisfy (2.58), thus represents a matrix-valued *Herglotz* function of rank m. Hence, for $\mathrm{Im}(\beta_2\beta_1^*) = 0$, poles of $M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$, $\ell > k_0$, are at most of first-order, are real, and have nonpositive residues. For $\ell < k_0$, $-M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ is a Herglotz matrix. Thus, one obtains a representation of $\sigma(\ell, k_0)M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ of the form (cf. [13], [48], [52], [53], [57])

$$\begin{aligned}\sigma(\ell, k_0)M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big) &= C_1\big(\ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big) + zC_2\big(\ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)\\ &\quad + \int_{-\infty}^{\infty} d\Omega\big(\lambda, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right),\end{aligned} \tag{2.64}$$

where $C_2\big(\ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big) \geqslant 0$ and $C_1\big(\ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ are self-adjoint $m \times m$ matrices, and where $\Omega\big(\lambda, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ is a nondecreasing $m \times m$ matrix-valued function such that

$$\int_{-\infty}^{\infty} \|d\Omega\big(\lambda, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)\|_{\mathbb{C}^{m\times m}}\, (1+\lambda^2)^{-1} < \infty, \tag{2.65a}$$

$$\Omega\big((\lambda,\mu], \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big) = \lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}\big[\sigma(\ell, k_0)M\big(\nu + i\epsilon, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)\big]. \tag{2.65b}$$

In general, for self-adjoint boundary value problems, $\Omega\big(\lambda, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ is piecewise constant with jump discontinuities precisely at the eigenvalues of the boundary value problem, and that in the matrix-valued Schrödinger and Dirac-type cases $C_2 = 0$ in (2.64). Analogous statements apply to $-M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ if $\ell < k_0$. For such problems, we note in the subsequent lemma that for fixed β, varying the boundary data α produces Weyl–Titchmarsh matrices $M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$ which are related to each other by a linear fractional transformations (see also [43], [48] for a general approach to such linear fractional transformations).

Lemma 2.15. *Assume Hyothesis* 2.5. *If* $\alpha, \beta, \gamma \in \mathbb{C}^{m\times 2m}$ *satisfy* (2.29) *and if* $\widetilde{\alpha} = \widetilde{\alpha}(k_0)$, $\widetilde{\gamma} = \widetilde{\gamma}(k_0)$, $\widetilde{\beta} = \widetilde{\beta}(\ell)$, *then*

$$M_{\widetilde{\alpha}} = [-\alpha J \gamma^* + \alpha\gamma^* M_{\widetilde{\gamma}}][\alpha\gamma^* + \alpha J \gamma^* M_{\widetilde{\gamma}}]^{-1}, \tag{2.66}$$

where $M_{\widetilde{\alpha}} = M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$*, and* $M_{\widetilde{\gamma}} = M\big(z, \ell, k_0, \widetilde{\gamma}, \widetilde{\beta}\big)$.

Remark 2.16. *From the proof of the previous lemma one infers, in general, that*

$$U_{\widetilde{\gamma}}(z,k) = U_{\widetilde{\alpha}}(z,k)(\alpha\gamma^* + \alpha J\gamma^* M_{\widetilde{\gamma}}). \tag{2.67}$$

Moreover, if $\alpha_0 = (I_m\ 0)$ *and* $\gamma_0 = (0\ I_m)$ *one observes, in particular, that*

$$M\big(z, \ell, k_0, \widetilde{\alpha}_0, \widetilde{\beta}\big) = -M\big(z, \ell, k_0, \widetilde{\gamma}_0, \widetilde{\beta}\big)^{-1}. \tag{2.68}$$

We further note that the sets $\mathcal{D}(z, \ell, k_0, \widetilde{\alpha})$ are closed, and convex, (cf., e.g., [50], [56], [64], [70]). Moreover, by (2.61) and Hypothesis 2.5, one concludes that $E_\ell(M)$ is increasing. This fact implies that, as a function of ℓ, the sets $\mathcal{D}(z, \ell, k_0, \widetilde{\alpha})$ are nesting in the sense that

$$\mathcal{D}(z, \ell_2, k_0, \widetilde{\alpha}) \subset \mathcal{D}(z, \ell_1, k_0, \widetilde{\alpha}) \quad \text{for} \quad k_0 < \ell_1 < \ell_2 \quad \text{or} \quad \ell_2 < \ell_1 < k_0. \tag{2.69}$$

Hence, the intersection of this nested sequence, as $\ell \to \pm\infty$, is nonempty, closed and convex. We say that this intersection is a limiting set for the nested sequence.

Definition 2.17. Let $\mathcal{D}_\pm(z, k_0, \widetilde{\alpha})$ denote the closed, convex set in the space of $m \times m$ matrices which is the limit, as $\ell \to \pm\infty$, of the nested collection of sets $\mathcal{D}(z, \ell, k_0, \widetilde{\alpha})$ given in Definition 2.11. $\mathcal{D}_\pm(z, k_0, \widetilde{\alpha})$ is said to be a limiting *disk*. Elements of $\mathcal{D}_\pm(z, k_0, \widetilde{\alpha})$ are denoted by $M_\pm(z, k_0, \widetilde{\alpha}) \in \mathbb{C}^{m\times m}$.

In light of the containment described in (2.69) and Hypothesis 2.5, for $\ell \neq k_0$ and $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\mathcal{D}_\pm(z, k_0, \widetilde{\alpha}) \subset \mathcal{D}(z, \ell, k_0, \widetilde{\alpha}), \tag{2.70}$$

with emphasis on strict containment of the disks in (2.70). Moreover, by (2.61),

$$M \in \mathcal{D}_\pm(z, k_0, \widetilde{\alpha}) \text{ precisely when } E_\ell(M) < 0 \text{ for all } \ell \in (k_0, \pm\infty). \tag{2.71}$$

In the next lemma, the interior points of the Weyl disk are characterized in terms of certain elements of $\mathcal{B}_d$ (cf. (2.28d)). This lemma is the analog in our discrete setting both in its statement and proof of Lemma 2.13 of [16].

Lemma 2.18. *Given Hypothesis* 2.12*, let* $M \in \mathbb{C}^{m\times m}$*, and let* $z \in \mathbb{C}\backslash\mathbb{R}$*. Then,* $E_\ell(M) < 0$ *if and only if there is a* $\beta \in \mathbb{C}^{m\times 2m}$ *satisfying the condition*

$$\sigma(\ell, k_0, z)\mathrm{Im}(\beta_1\beta_2^*) > 0, \tag{2.72}$$

and such that (2.52) *holds with* $u_j(z,\ell) = u_j(z, \ell, k_0, \widetilde{\alpha})$*,* $j = 1, 2$*, defined in* (2.49) *in terms of* M*. With* β *so defined,* (2.53) *holds, that is,* $M = M\big(z, \ell, k_0, \widetilde{\alpha}, \widetilde{\beta}\big)$*. Moreover,* $\beta \in \mathcal{B}_d$ *and* β *may be chosen to satisfy* (2.28c)*.*

Proof. Let $z \in \mathbb{C}\backslash\mathbb{R}$, and for a given $M \in \mathbb{C}^{m\times m}$ suppose that there is a $\beta \in \mathbb{C}^{m\times 2m}$ satisfying (2.72) such that (2.52) holds. The matrices β_j, $j = 1, 2$, are invertible by (2.72), and by (2.52) it follows that

$$\widehat{U}(z,\ell) = \begin{pmatrix} -\widetilde{\beta}_1^{-1}\widetilde{\beta}_2 \\ I_m \end{pmatrix} u_2^+(z, \ell, k_0, \widetilde{\alpha}). \tag{2.73}$$

By (2.50) and (2.73), we see that

$$E_\ell(M) = -2\sigma(\ell, k_0, z) u_2^+(z,\ell)^* \rho^{1/2} \beta_1^{-1} \mathrm{Im}(\beta_1\beta_2^*)(\beta_1^*)^{-1} \rho^{1/2} u_2^+(z,\ell), \tag{2.74}$$

and hence that $E_\ell(M) < 0$ whenever (2.72) holds.

Upon showing that $\widetilde{\beta}\widehat{\Phi}(z,\ell)$ is nonsingular, (2.53) will follow from (2.52). If $\widetilde{\beta}\widehat{\Phi}(z,\ell)$ is singular, then there is a nonzero vector $v \in \mathbb{C}^m$ such that $\widetilde{\beta}\widehat{\Phi}(z,\ell)v = 0$.

By the nonsingularity of β_j, $j=1,2$, $\phi_1(z,\ell)v = -\widetilde{\beta}_1^{-1}\widetilde{\beta}_2\phi_2^+(z,\ell)v$, and as a result, (2.56) yields

$$\begin{aligned} &2|\mathrm{Im}(z)| \sum_{k\in^+[k_0,\ell]} v^*\Phi^*(z,k)A(k)\Phi(z,k)v \\ &\quad = -2\sigma(\ell,k_0,z)v^*u_2^+(z,\ell)^*\rho^{1/2}\beta_1^{-1}\mathrm{Im}(\beta_1\beta_2^*)(\beta_1^*)^{-1}\rho^{1/2}u_2^+(z,\ell)v, \end{aligned} \tag{2.75}$$

and hence a contradiction given (2.72) (cf. (2.15)).

Conversely, if $E_\ell(M) = 2\sigma(\ell,k_0,z)\mathrm{Im}(u_1^*\rho u_2^+)\big|_\ell < 0$ for a given $M\in\mathbb{C}^{m\times m}$, then for $z\in\mathbb{C}\backslash\mathbb{R}$, $u_2^+(z,\ell,k_0,\widetilde{\alpha})$ is nonsingular. Let $\beta_1 = I_\rho(\ell)^{-1/2}$ and let $\beta_2 = -u_1(u_2{}^+)^{-1}I_\rho{}^{-1/2}\big|_\ell$. Then, for this choice of β_j, $j=1,2$, (2.52) holds and (2.74) now implies that $\sigma(\ell,k_0,z)\mathrm{Im}(\beta_1\beta_2^*)>0$ for k_0 and ℓ satisfying Hypothesis 2.12, and for $z\in\mathbb{C}\backslash\mathbb{R}$. For this choice, β does not satisfy (2.28c). However, we may normalize the boundary data as described in the proof of Lemma 2.13. □

Note that if $M\in\mathcal{D}_\pm(z,k_0,\widetilde{\alpha})$, then as a result of Lemma 2.18 and (2.70) there is a $\beta\in\mathbb{C}^{m\times 2m}$ satisfying (2.72) such that

$$M_\pm(z,k_0,\widetilde{\alpha}) = M\big(z,\ell,k_0,\widetilde{\alpha},\widetilde{\beta}\big). \tag{2.76}$$

Definition 2.19. When $\mathcal{D}_+(z,k_0,\widetilde{\alpha})$ (resp., $\mathcal{D}_-(z,k_0,\widetilde{\alpha})$) is a singleton matrix, the system (2.12a) is said to be in the *limit point* (l.p.) case at ∞ (resp., $-\infty$). If $\mathcal{D}_+(z,k_0,\widetilde{\alpha})$ (resp., $\mathcal{D}_-(z,k_0,\widetilde{\alpha})$) has nonempty interior, then (2.12a) is said to be in the *limit circle* (l.c.) case at ∞ (resp., $-\infty$).

Indeed, for the case $m=1$, the limit point case corresponds to a point in $\mathbb{C}$, whereas the limit circle case corresponds to $\mathcal{D}_\pm(z,k_0,\widetilde{\alpha})$ being a closed disk in $\mathbb{C}$.

By analogy with the continuous case, these apparent geometric properties for the disk correspond to analytic properties for the solutions of the Hamiltonian system (2.12a). To describe this correspondence, we introduce the following spaces in which we assume that $I\subseteq\mathbb{Z}$:

$$\ell_A^2(I) = \bigg\{\phi : I\to\mathbb{C}^{2m} \,\bigg|\, \sum_{k\in I}(\phi(k),A\phi(k))_{\mathbb{C}^{2m}} < \infty\bigg\}, \tag{2.77a}$$

$$N(z,\infty) = \{\phi\in\ell_A^2([k_0,\infty)) \,|\, \mathcal{S}_\rho\phi = (zA+B)\phi\}, \tag{2.77b}$$

$$N(z,-\infty) = \{\phi\in\ell_A^2((-\infty,k_0]) \,|\, \mathcal{S}_\rho\phi = (zA+B)\phi\}, \tag{2.77c}$$

for $z\in\mathbb{C}$ and some $k_0\in\mathbb{Z}$. (Here $(\phi,\psi)_{\mathbb{C}^n} = \sum_{j=1}^n \overline{\phi}_j\psi_j$ denotes the standard scalar product in $\mathbb{C}^n$, abbreviating $\chi\in\mathbb{C}^n$ by $\chi=(\chi_1,\dots,\chi_n)^t$.) Both dimensions of the spaces in (2.77b) and (2.77c), $\dim_{\mathbb{C}}(N(z,\infty))$ and $\dim_{\mathbb{C}}(N(z,-\infty))$, are constant for $z\in\mathbb{C}_\pm$ (see, e.g., [5], [59]), where

$$\mathbb{C}_\pm = \{\zeta\in\mathbb{C} \,|\, \pm\mathrm{Im}(\zeta)>0\}. \tag{2.78}$$

One then observes that the Hamiltonian system (2.12a) is in the limit point case at ∞ (resp., $-\infty$) whenever

$$\dim_{\mathbb{C}}(N(z,\infty)) = m \text{ (resp., } \dim_{\mathbb{C}}(N(z,-\infty)) = m \text{ for all } z\in\mathbb{C}\backslash\mathbb{R} \tag{2.79}$$

and in the limit circle case at ∞ (resp., $-\infty$) whenever

$$\dim_{\mathbb{C}}(N(z,\infty)) = 2m \text{ (resp., } \dim_{\mathbb{C}}(N(z,-\infty)) = 2m) \text{ for all } z\in\mathbb{C}. \tag{2.80}$$

For the boundary condition given by

$$\widetilde{\alpha}\widehat{\Psi}(z,k_0)=0 \tag{2.81}$$

with $\alpha\in\mathbb{C}^{m\times 2m}$ satisfying (2.29), there is an associated boundary set $\partial\mathcal{D}_\pm(z,k_0,\widetilde{\alpha})$ for the limiting disk. In either the limit point or limit circle cases, $M_\pm(z,k_0,\widetilde{\alpha})\in\partial\mathcal{D}_\pm(a,k_0,\widetilde{\alpha})$ is said to be a *half-line Weyl–Titchmarsh matrix*. Each such matrix is associated with the construction of a Green's matrix for certain boundary value problems involving separated boundary conditions which are posed on the whole-line and on half-lines as will be discussed in Section 3.

Given the definition of $M_\pm(z,k_0,\widetilde{\alpha})$ as the limit of a sequence $M\big(z,\ell_n,k_0,\widetilde{\alpha},\widetilde{\beta}\big)$, $n\in\mathbb{N}$, and given the geometry of Weyl disks, as for the case of Hamiltonian systems of differential equations discussed in [15] and [16], we see that $M_\pm(z,k_0,\widetilde{\alpha})$ possesses certain properties as a complex, matrix-valued function of z. For the convenience of the reader, we now summarize some of the principal properties of half-line Weyl–Titchmarsh matrices:

Theorem 2.20 ([3], [13], [48], [52], [53], [57], [63])**.** *Assume Hypotheses* 2.5 *and let* $z\in\mathbb{C}\backslash\mathbb{R}$*,* $k_0\in\mathbb{R}$*, and denote by* $\alpha,\gamma\in\mathbb{C}^{m\times 2m}$ *matrices satisfying* (2.29)*. Then,*
(i) $\pm M_\pm(z,k_0,\alpha)$ *is an* $m\times m$ *matrix-valued Herglotz function of maximal rank. In particular,*

$$\operatorname{Im}(\pm M_\pm(z,k_0,\alpha))>0,\quad z\in\mathbb{C}_+, \tag{2.82}$$

$$M_\pm(\overline{z},k_0,\alpha)=M_\pm(z,k_0,\alpha)^*, \tag{2.83}$$

$$\operatorname{rank}(M_\pm(z,k_0,\alpha))=m, \tag{2.84}$$

$$\lim_{\varepsilon\downarrow 0}M_\pm(\lambda+i\varepsilon,k_0,\alpha)\ \textit{exists for a.e. } \lambda\in\mathbb{R}, \tag{2.85}$$

$$\begin{aligned}M_\pm(z,k_0,\alpha)=[-\alpha J\gamma^*+\alpha\gamma^*M_\pm(z,k_0,\gamma)]\times\\ \times[\alpha\gamma^*+\alpha J\gamma^*M_\pm(z,k_0,\gamma)]^{-1}.\end{aligned} \tag{2.86}$$

Local singularities of $\pm M_\pm(z,k_0,\alpha)$ *and* $\mp M_\pm(z,k_0,\alpha)^{-1}$ *are necessarily real and at most of first order in the sense that*

$$\mp\lim_{\epsilon\downarrow 0}(i\epsilon\, M_\pm(\lambda+i\epsilon,k_0,\alpha))\geq 0,\quad \lambda\in\mathbb{R}, \tag{2.87}$$

$$\pm\lim_{\epsilon\downarrow 0}\left(\frac{i\epsilon}{M_\pm(\lambda+i\epsilon,k_0,\alpha)}\right)\geq 0,\quad \lambda\in\mathbb{R}. \tag{2.88}$$

(ii) $\pm M_\pm(z,k_0,\alpha)$ *admit the representations*

$$\pm M_\pm(z,k_0,\alpha)=F_\pm(k_0,\alpha)+\int_{\mathbb{R}}d\Omega_\pm(\lambda,k_0,\alpha)\left((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\right) \tag{2.89}$$

$$=\exp\left(C_\pm(k_0,\alpha)+\int_{\mathbb{R}}d\lambda\,\Xi_\pm(\lambda,k_0,\alpha)\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big)\right), \tag{2.90}$$

where

$$F_\pm(k_0,\alpha)=F_\pm(k_0,\alpha)^*,\quad \int_{\mathbb{R}}\|d\Omega_\pm(\lambda,k_0,\alpha)\|_{\mathbb{C}^{m\times m}}(1+\lambda^2)^{-1}<\infty, \tag{2.91}$$

$$C_\pm(k_0,\alpha)=C_\pm(k_0,\alpha)^*,\quad 0\leqslant\Xi_\pm(\,\cdot\,,k_0,\alpha)\leqslant I_m\ \textit{ a.e.} \tag{2.92}$$

Moreover,

$$\Omega_\pm((\lambda,\mu],k_0,\alpha)=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta}d\nu\,\operatorname{Im}(\pm M_\pm(\nu+i\varepsilon,k_0,\alpha)), \tag{2.93}$$

$$\Xi_\pm(\lambda,k_0,\alpha)=\lim_{\varepsilon\downarrow 0}\pi^{-1}\mathrm{Im}(\ln(\pm M_\pm(\lambda+i\varepsilon,k_0,\alpha)))\ \textit{for a.e. } \lambda\in\mathbb{R}. \tag{2.94}$$

(iii) Define the $2m\times m$ matrices

$$\begin{aligned}U_\pm(z,\ell,k_0,\alpha)&=\begin{pmatrix}u_{\pm,1}(z,\ell,k_0,\alpha)\\ u_{\pm,2}(z,\ell,k_0,\alpha)\end{pmatrix}=\Psi(z,\ell,k_0,\alpha)\begin{pmatrix}I_m\\ M_\pm(z,k_0,\alpha)\end{pmatrix}\\ &=\begin{pmatrix}\theta_1(z,\ell,k_0,\alpha) & \phi_1(z,\ell,k_0,\alpha)\\ \theta_2(z,\ell,k_0,\alpha) & \phi_2(z,\ell,k_0,\alpha)\end{pmatrix}\begin{pmatrix}I_m\\ M_\pm(z,k_0,\alpha)\end{pmatrix},\end{aligned} \tag{2.95}$$

with $\theta_j(z,\ell,k_0,\alpha)$, and $\phi_j(z,\ell,k_0,\alpha)$, $j=1,2$, defined by (2.34c)*. Then, for every $\zeta\in\mathbb{C}^{2m}$, $U_\pm(z,\ell,k_0,\alpha)\zeta\in\ell^2_A([k,\pm\infty))$. Moreover,*

$$\mathrm{Im}(M_+(z,k_0,\alpha))=\mathrm{Im}(z)\sum_{+[k_0,\infty)}U_+(z,s,k_0,\alpha)^*A(s)U_+(z,s,k_0,\alpha), \tag{2.96}$$

$$\mathrm{Im}(M_-(z,k_0,\alpha))=\mathrm{Im}(z)\sum_{(-\infty,k_0]}U_-(z,s,k_0,\alpha)^*A(s)U_-(z,s,k_0,\alpha). \tag{2.97}$$

We conclude this section by first giving another characterization of the elements of the limiting disks $\mathcal{D}_\pm(z,k_0,\widetilde\alpha)$ and then noting a connection between these elements and certain solutions of a related Riccati equation.

Lemma 2.21. *Assume Hypothesis* 2.12 *and let $z\in\mathbb{C}\backslash\mathbb{R}$. Moreover, suppose that $U(z,k,k_0,\widetilde\alpha)$ is defined by* (2.49) *in terms of an $M\in\mathbb{C}^{m\times m}$ so that the columns of $U(z,k,k_0,\widetilde\alpha)$ are in $\ell^2_A([k_0,\pm\infty))$. Then, $M\in\mathcal{D}_\pm(z,k_0,\widetilde\alpha)$ if and only if for $k\in\mathbb{Z}$,*

$$-\sigma(k,k_0,z)\mathrm{Im}(u_1(z,k)^*\rho(k)u_2^+(z,k))>0, \tag{2.98}$$

or equivalently,

$$-\sigma(k,k_0,z)\mathrm{Im}(\rho(k)u_2^+(z,k)u_1(z,k)^{-1})>0. \tag{2.99}$$

Proof. By (2.54) and (2.60a),

$$\begin{aligned}-2\sigma(k,k_0,z)\mathrm{Im}(u_1(z,k)^*\rho(k)u_2^+(z,k))&=\sigma(k,k_0,z)\widehat U^*(iJ_\rho)\widehat U\big|_k\\ &=-E_\ell(M)+2|\mathrm{Im}(z)|\sum_{+[k,\ell]}U^*AU.\end{aligned} \tag{2.100}$$

The summation expression in (2.100) is positive by Hypothesis 2.12, and is decreasing as a function of k with fixed ℓ while increasing as a function of ℓ with fixed k. As a consequence, should a column of $U(z,k,k_0,\widetilde\alpha)$ not be in $\ell^2_A([k_0,\pm\infty))$, then there is a vector $\zeta\in\mathbb{C}^{2m}$ such that $\zeta^*E_\ell(M)\zeta>0$ for large $|\ell|$ and hence $M\notin\mathcal{D}_\pm(z,k_0,\widetilde\alpha)$. Thus, we assume that the columns of $U(z,k,k_0,\widetilde\alpha)$ are in $\ell^2_A([k_0,\pm\infty))$.

If $M\in\mathcal{D}(z,k_0,\widetilde\alpha)$ then (2.98) follows because $-E_\ell(M)\geqslant 0$ and because the sum in (2.100) is positive for large $|\ell|$. If $M\notin\mathcal{D}(z,k_0,\widetilde\alpha)$ then $\zeta^*E_{\ell_0}(M)\zeta>0$ for some $\zeta\in\mathbb{C}^{2m}$ and some ℓ_0 and hence $\zeta^*E_\ell(M)\zeta\geqslant\zeta^*E_{\ell_0}(M)\zeta$ for $|\ell|>|\ell_0|$. Then, for sufficiently large $|\ell|$ and $|k|$, the left-hand side of (2.98) is negative.

The equivalence of (2.98) and (2.99) follows because

$$u_1^*\mathrm{Im}(\rho u_2^+u_1^{-1})u_1=\mathrm{Im}(u_1^*\rho u_2^+)>0. \tag{2.101}$$

□

The Hamiltonian system (2.12), described in Definition 2.4, can be written as

$$\rho\psi_2^+ = (zA_{1,1} + B_{1,1})\psi_1 + (zA_{1,2} + B_{1,2})\psi_2, \tag{2.102a}$$

$$\rho^-\psi_1^- = (zA_{2,1} + B_{2,1})\psi_1 + (zA_{2,2} + B_{2,2})\psi_2. \tag{2.102b}$$

When $\Psi(z,k) \in \mathbb{C}^{2m\times m}$ represents a solution of (2.12a) for which $\psi_j(z,k) \in \mathbb{C}^{m\times m}$, $j = 1,2$, are nonsingular for $k \in [k_0,\ell]$, then (2.102a) and (2.102b) respectively yield,

$$\rho\psi_2^+\psi_1^{-1} = (zA_{1,1} + B_{1,1}) + (zA_{1,2} + B_{1,2})\psi_2\psi_1^{-1}, \tag{2.103a}$$

$$\psi_1^{-1} = [\rho^-\psi_1^- - (zA_{2,2} + B_{2,2})\psi_2]^{-1}(zA_{2,1} + B_{2,1}), \tag{2.103b}$$

from which it follows that

$$\begin{aligned} \rho\psi_2^+\psi_1^{-1} = (zA_{1,1} + B_{1,1}) &+ (zA_{1,2} + B_{1,2})[\rho^-\psi_1^-\psi_2^{-1} - (zA_{2,2} + B_{2,2})]^{-1} \\ &\times (zA_{2,1} + B_{2,1}). \end{aligned} \tag{2.104}$$

Thus, $V(z,k) = \rho(k)\psi_2^+(z,k)\psi_1(z,k)^{-1}$, $k \in [k_0,\ell]$, yields a solution of the Riccati equation given by

$$\begin{aligned} V = (zA_{1,1} + B_{1,1}) &+ (zA_{1,2} + B_{1,2})[\rho^-(V^-)^{-1}\rho^- - (zA_{2,2} + B_{2,2})]^{-1} \\ &\times (zA_{2,1} + B_{2,1}). \end{aligned} \tag{2.105}$$

As an immediate consequence of these observations, we obtain the following result.

Lemma 2.22. *Assume Hypothesis* 2.12 *and let* $z \in \mathbb{C}\backslash\mathbb{R}$. *Moreover, suppose that* $U(z,k,k_0,\widetilde{\alpha})$ *is defined by* (2.49) *in terms of an* $M \in \mathbb{C}^{m\times m}$ *so that the columns of* $U(z,k,k_0,\widetilde{\alpha})$ *are in* $\ell_A^2([k_0,\pm\infty))$. *Then,* $M \in \mathcal{D}_\pm(z,k_0,\widetilde{\alpha})$ *if and only if for* $k \in \mathbb{Z}$, $V(z,k) = \rho(k)u_2^+(z,k)u_1(z,k)^{-1}$ *represents a solution of the Riccati equation* (2.105) *for* $z \in \mathbb{C}\backslash\mathbb{R}$.

3. Boundary Value Problems and Green Matrices on the Whole Line and on Half-Lines

In this section, we consider the nonhomogeneous equation given by

$$\mathcal{S}_\rho\psi = (zA + B)\psi + Af \tag{3.1}$$

associated with the Hamiltonian system (2.12a) on the whole-line or on half-lines.

First, we discuss the whole-line case and assume Hypothesis 2.5 on $\mathbb{Z}$. Hence, we consider (3.1) with $f \in \ell_A^2(\mathbb{Z})$. To keep matters reasonably short, the endpoints ∞ and $-\infty$ will separately be assumed to be either of *limit point* or *limit circle* type for (2.12a). (These cases typically receive most attention and the sequel [19] to this paper will, in particular, focus on the limit point case at ∞ and $-\infty$.) We describe for $k,\ell \in \mathbb{Z}$, and $z \in \mathbb{C}\backslash\mathbb{R}$, a matrix $K(z,k,\ell) \in \mathbb{C}^{2m\times 2m}$ for which the following properties hold:

$$\sum_{\ell\in\mathbb{Z}} K(z,k,\ell)A(\ell)K(z,k,\ell)^* < \infty, \quad k \in \mathbb{Z}. \tag{3.2}$$

If $f \in \ell_A^2(\mathbb{Z})$ and if

$$y(z,k) = \sum_{\ell\in\mathbb{Z}} K(z,k,\ell)A(\ell)f(\ell), \quad k \in \mathbb{Z}, \tag{3.3}$$

then $y(z,\cdot) \in \ell^2_A(\mathbb{Z})$ and $y(z,\cdot)$ satisfies (3.1) on $\mathbb{Z}$. In addition, it will be seen that $y(z,\cdot)$ satisfies certain boundary conditions at ∞ (resp., $-\infty$) if (2.12a) is in the limit circle case at ∞ (resp., $-\infty$).

As a matter of convenience, we state the next theorem assuming that $z \in \mathbb{C}_+$ (cf. (2.78)) and note that the theorem can be restated for $z \in \mathbb{C}_-$ with the details of the proof essentially unchanged.

Theorem 3.1. *Assume Hypothesis* 2.5 *on* $\mathbb{Z}$ *and suppose that* $z \in \mathbb{C}_+$ *and* $k, \ell \in \mathbb{Z}$. *Let* $K(z,k,\ell)$ *be defined by*

$$K(z,k,\ell) = \begin{cases} U_+(z,k)\omega(z)U_-(\bar z,\ell)^*, & k > \ell, \\ \begin{pmatrix} u_{+,1}(z,k)\omega(z)u_{-,1}(\bar z,k)^* & u_{+,1}(z,k)\omega(z)u_{-,2}(\bar z,k)^* \\ u_{-,2}(z,k)\omega(z)u_{+,1}(\bar z,k)^* & u_{-,2}(z,k)\omega(z)u_{+,2}(\bar z,k)^* \end{pmatrix}, & k = \ell, \\ U_-(z,k)\omega(z)U_+(\bar z,\ell)^*, & k < \ell. \end{cases} \tag{3.4}$$

Here $U_\pm(z,k)$ *is defined in* (2.49) *with* $M = M_\pm(z) \in \partial\mathcal{D}_\pm(z,k_0,\widetilde\alpha)$, $U_\pm(\bar z,k)$ *is defined in* (2.49) *with* $M = M_\pm(\bar z) = M_\pm(z)^* \in \partial\mathcal{D}_\pm(\bar z,k_0,\widetilde\alpha)$, *and*

$$U_\pm(z,k) = \begin{pmatrix} u_{\pm,1}(z,k) \\ u_{\pm,2}(z,k) \end{pmatrix} \tag{3.5}$$

with $u_{\pm,j}(z,k) \in \mathbb{C}^{m\times m}$, $j = 1,2$. *Moreover,*

$$\omega(z) = \pm\big(\widehat U_\mp(\bar z,k_0)^* J_\rho(k_0)\widehat U_\pm(z,k_0)\big)^{-1} = [M_-(z) - M_+(z)]^{-1}. \tag{3.6}$$

With $K(z,k,\ell)$ *so defined,* (3.2) *is satisfied. Moreover, as defined in* (3.3), $y(z,\cdot)$ *satisfies* (3.1) *on* $\mathbb{Z}$ *and is in* $\ell^2_A(\mathbb{Z})$.

Proof. We begin by defining notation to be used for the remainder of this section. We adopt the following convention:

$$F^{\circledast}(z,k) = F(\bar z,k)^* \text{ for } F \in \mathbb{C}^{m\times m} \text{ (or } F \in \mathbb{C}^{2m\times m}). \tag{3.7}$$

Let the matrices $a(z,k)$, $b(z,k)$, $c(z,k) \in \mathbb{C}^{m\times m}$ be defined by

$$zA(k) + B(k) = \begin{pmatrix} a(z,k) & b(z,k) \\ b(\bar z,k)^* & c(z,k) \end{pmatrix} = \begin{pmatrix} a(z,k) & b(z,k) \\ b^{\circledast}(z,k) & c(z,k) \end{pmatrix}, \tag{3.8}$$

where $A(k)$, $B(k)$ are given in (2.8) and (2.9) respectively, and are subject to Hypothesis 2.5 on $\mathbb{Z}$. We also note that $a^{\circledast}(z,k) = a(z,k)$, and that $c^{\circledast}(z,k) = c(z,k)$. Lastly, let $\varphi_\pm(z,k)$ and $\vartheta_\pm(z,k)$ be defined by

$$\varphi_\pm(z,k) = u_{\pm,1}(z,k), \qquad \vartheta_\pm(z,k) = u_{\pm,2}(z,k). \tag{3.9}$$

With this convention, $\widehat U^{\circledast}_\mp(z,k)J_\rho(k)\widehat U_\pm(z,k) = \widehat U_\mp(\bar z,k)^* J_\rho(k)\widehat U_\pm(z,k)$. Then, by (2.54) we note that

$$\widehat U^{\circledast}_\mp(z,k)J_\rho(k)\widehat U_\pm(z,k) = \widehat U^{\circledast}_\mp(z,k_0)J_\rho(k_0)\widehat U_\pm(z,k_0) \tag{3.10a}$$

$$= M_\mp(z) - M_\pm(z). \tag{3.10b}$$

Given that $\operatorname{Im}(M_\pm(z)) \gtrless 0$, we see that $\widehat U^{\circledast}_\mp(z,k)J_\rho(k)\widehat U_\pm(z,k)$ is invertible for $k \in \mathbb{Z}$.

Next, we note that for $k \neq \ell$, $0 = \big((\mathcal{S}_\rho - zA - B)K(z,\cdot,\ell)\big)(k)$. Thus, to verify that y defined in (3.3) solves (3.1), it is necessary to show that

$$I_{2m} = \big((\mathcal{S}_\rho - zA - B)K(z,\cdot,k)\big)(k) \tag{3.11a}$$

$$= \big(\mathcal{S}_\rho K(z,\cdot,k)\big)(k) - \begin{pmatrix} a(z,k) & b(z,k) \\ b^{\circledast}(z,k) & c(z,k) \end{pmatrix} K(z,k,k). \tag{3.11b}$$

Then, by (3.4) for $k=\ell$, and (3.9),

$$\begin{pmatrix} a(z,k) & b(z,k) \\ b^{\circledast}(z,k) & c(z,k) \end{pmatrix} K(z,k,k) = \begin{pmatrix} a & b \\ b^{\circledast} & c \end{pmatrix} \begin{pmatrix} \varphi_+\omega\varphi_-^{\circledast} & \varphi_+\omega\vartheta_-^{\circledast} \\ \vartheta_-\omega\varphi_+^{\circledast} & \vartheta_-\omega\vartheta_+^{\circledast} \end{pmatrix}, \tag{3.12}$$

and by (1.5) and (3.4) for $k \neq \ell$,

$$\big(\mathcal{S}_\rho K(z,\cdot,k)\big)(k) = \begin{pmatrix} \rho\vartheta_+^+\omega\varphi_-^{\circledast} & \rho\vartheta_+^+\omega\vartheta_-^{\circledast} \\ \rho^-\varphi_-^-\omega\varphi_+^{\circledast} & \rho^-\varphi_-^-\omega\vartheta_+^{\circledast} \end{pmatrix}. \tag{3.13}$$

Thus, (3.11) is equivalent to the following system:

$$I_m = \rho\vartheta_+^+\omega\varphi_-^{\circledast} - a\varphi_+\omega\varphi_-^{\circledast} - b\vartheta_-\omega\varphi_+^{\circledast}, \tag{3.14a}$$
$$0_m = \rho\vartheta_+^+\omega\vartheta_-^{\circledast} - a\varphi_+\omega\vartheta_-^{\circledast} - b\vartheta_-\omega\vartheta_+^{\circledast}, \tag{3.14b}$$
$$0_m = \rho^-\varphi_-^-\omega\varphi_+^{\circledast} - b^{\circledast}\varphi_+\omega\varphi_-^{\circledast} - c\vartheta_-\omega\varphi_+^{\circledast}, \tag{3.14c}$$
$$I_m = \rho^-\varphi_-^-\omega\vartheta_+^{\circledast} - b^{\circledast}\varphi_+\omega\vartheta_-^{\circledast} - c\vartheta_-\omega\vartheta_+^{\circledast}. \tag{3.14d}$$

Given that $0 = \big((\mathcal{S}_\rho - zA - B)U_\pm(z,\cdot)\big)(k)$, we obtain additionally that

$$\rho\vartheta_\pm^+ = a\varphi_\pm + b\vartheta_\pm, \tag{3.15a}$$
$$\rho^-\varphi_\pm^- = b^{\circledast}\varphi_\pm + c\vartheta_\pm. \tag{3.15b}$$

From (3.15), we obtain

$$(\vartheta_\pm^{\circledast})^+\rho = \varphi_\pm^{\circledast}a + \vartheta_\pm^{\circledast}b^{\circledast}, \tag{3.16a}$$
$$(\varphi_\pm^{\circledast})^-\rho^- = \varphi_\pm^{\circledast}b + \vartheta_\pm^{\circledast}c. \tag{3.16b}$$

Substituting into (3.14) the expressions for $\rho\vartheta_+^+$ and for $\rho_-\varphi_-^-$ in (3.15), we obtain the equivalent system

$$I_m = b\vartheta_+\omega\varphi_-^{\circledast} - b\vartheta_-\omega\varphi_+^{\circledast}, \tag{3.17a}$$
$$0_m = b\vartheta_+\omega\vartheta_-^{\circledast} - b\vartheta_-\omega\vartheta_+^{\circledast}, \tag{3.17b}$$
$$0_m = b^{\circledast}\varphi_-\omega\varphi_+^{\circledast} - b^{\circledast}\varphi_+\omega\varphi_-^{\circledast}, \tag{3.17c}$$
$$I_m = b^{\circledast}\varphi_-\omega\vartheta_+^{\circledast} - b^{\circledast}\varphi_+\omega\vartheta_-^{\circledast}. \tag{3.17d}$$

Verification of (3.11) comes with first showing the consistency of the equations in (3.17) and hence the consistency of those in (3.14). Following this, we identify ω by (3.6) as terms by which the equations in (3.17) hold.

Given that $b(z,k)$ and $b(z,k)^{\circledast}$ are invertible by Hypothesis 2.1, and given the invertibility of $\varphi_\pm(z,k)$ and $\vartheta_\pm(z,k)$, $z \in \mathbb{C}_+$, $k \in \mathbb{Z}$ which follows from Lemma 2.22, we obtain from (3.17b)

$$\omega = \vartheta_+^{-1}\vartheta_-\omega\vartheta_+^{\circledast}(\vartheta_-^{\circledast})^{-1}, \tag{3.18}$$

and from (3.17c)

$$\omega = \varphi_+^{-1}\varphi_-\omega\varphi_+^{\circledast}(\varphi_-^{\circledast})^{-1}. \tag{3.19}$$

Replacing in (3.17a) the expression for ω in (3.19) one obtains

$$\omega = \varphi_-^{-1}(\vartheta_+\varphi_+^{-1} - \vartheta_-\varphi_-^{-1})^{-1}b^{-1}(\varphi_+^{\circledast})^{-1}, \tag{3.20}$$

and replacing in (3.17d) the expression for ω in (3.19) one obtains

$$\omega = \varphi_-^{-1}(b^{\circledast})^{-1}\big((\varphi_+^{\circledast})^{-1}\vartheta_+^{\circledast} - (\varphi_-^{\circledast})^{-1}\vartheta_-^{\circledast}\big)^{-1}(\varphi_+^{\circledast})^{-1}. \tag{3.21}$$

Equating the right-hand sides of (3.20) and (3.21) then yields

$$b(\vartheta_+\varphi_+^{-1} - \vartheta_-\varphi_-^{-1}) = \big((\varphi_+^{\circledast})^{-1}\vartheta_+^{\circledast} - (\varphi_-^{\circledast})^{-1}\vartheta_-^{\circledast}\big)b^{\circledast}. \tag{3.22}$$

Substituting on the left-hand side of (3.22) using the expression for $b\vartheta_\pm$ given in (3.15a), and substituting on the right-hand side of (3.22) using the expression for $\vartheta_\pm^{\circledast}b^{\circledast}$ given in (3.16a) one obtains

$$\rho\vartheta_+^+\varphi_+^{-1} - \rho\vartheta_-^+\varphi_-^{-1} = (\varphi_+^{\circledast})^{-1}(\vartheta_+^{\circledast})^+\rho - (\varphi_-^{\circledast})^{-1}(\vartheta_-^{\circledast})^+\rho. \tag{3.23}$$

That equation (3.23) holds follows from (2.54), from the fact that

$$\widehat{U}_\pm^{\circledast}(z,k)J_\rho(k)\widehat{U}_\pm(z,k) = \widehat{U}_\pm^{\circledast}(z,k_0)J_\rho(k_0)\widehat{U}_\pm(z,k_0) = 0, \tag{3.24}$$

and hence that

$$(\vartheta_\pm^{\circledast})^+\rho\varphi_\pm = \varphi_\pm^{\circledast}\rho\vartheta_\pm^+. \tag{3.25}$$

As a consequence, we see that (3.17a), (3.17c), (3.17d) are consistent.

Replacing ω in (3.19) with the expression for ω in (3.20) yields

$$\omega = \varphi_+^{-1}(\vartheta_+\varphi_+^{-1} - \vartheta_-\varphi_-^{-1})^{-1}b^{-1}(\varphi_-^{\circledast})^{-1}, \tag{3.26}$$

and replacing ω in (3.18) with the expression for ω in (3.20) yields

$$\omega = \big(\vartheta_-^{\circledast}(\vartheta_+^{\circledast})^{-1}\varphi_+^{\circledast}b(\vartheta_+\varphi_+^{-1}\varphi_-\vartheta_-^{-1}\vartheta_+ - \vartheta_+)\big)^{-1}. \tag{3.27}$$

Showing the equivalence of (3.26) and (3.27) will yield the equivalence of (3.17b) and (3.17c), and hence the consistency of all equations in (3.17).

Replacing $\varphi_+^{\circledast}b$ in (3.27) with the expression obtained from (3.16b) yields

$$\omega = \big[\vartheta_-^{\circledast}((\vartheta_+^{\circledast})^{-1}(\varphi_+^{\circledast})^-\rho^- - c)(\vartheta_+\varphi_+^{-1}\varphi_-\vartheta_-^{-1}\vartheta_+ - \vartheta_+)\big]^{-1}. \tag{3.28}$$

By (3.25), $(\vartheta_+^{\circledast})^{-1}(\varphi_+^{\circledast})^-\rho^- = \rho^-\varphi_+^-\vartheta_+^{-1}$, and thus,

$$\omega = \big[\vartheta_-^{\circledast}(\rho^-\varphi_+^-\vartheta_+^{-1} - c)(\vartheta_+\varphi_+^{-1}\varphi_-\vartheta_-^{-1}\vartheta_+ - \vartheta_+)\big]^{-1}. \tag{3.29}$$

By (3.15b), $\rho^-\varphi_+^-\vartheta_+^{-1} - c = b^{\circledast}\varphi_+\vartheta_+^{-1}$, and hence,

$$\omega = \big[\vartheta_-^{\circledast}b^{\circledast}\varphi_+\vartheta_+^{-1}(\vartheta_+\varphi_+^{-1}\varphi_-\vartheta_-^{-1} - I_m)\vartheta_+\big]^{-1}, \tag{3.30}$$

$$= \big[\vartheta_-^{\circledast}b^{\circledast}(\varphi_-\vartheta_-^{-1} - \varphi_+\vartheta_+^{-1})\vartheta_+\big]^{-1}, \tag{3.31}$$

$$= \varphi_+^{-1}\big[(\varphi_-^{\circledast})^{-1}\vartheta_-^{\circledast}b^{\circledast}(\varphi_-\vartheta_-^{-1} - \varphi_+\vartheta_+^{-1})\vartheta_+\varphi_+^{-1}\big]^{-1}(\varphi_-^{\circledast})^{-1}. \tag{3.32}$$

By (3.16a), $\vartheta_-^{\circledast}b^{\circledast} = (\vartheta_-^{\circledast})^+\rho - \varphi_-^{\circledast}a$, and thus,

$$\omega = \varphi_+^{-1}\big[((\varphi_-^{\circledast})^{-1}(\vartheta_-^{\circledast})^+\rho - a)(\varphi_-\vartheta_-^{-1}\vartheta_+\varphi_+^{-1} - I_m)\big]^{-1}(\varphi_-^{\circledast})^{-1}. \tag{3.33}$$

By (3.25), $(\varphi_-^{\circledast})^{-1}(\vartheta_-^{\circledast})^+\rho = \rho\vartheta_-^+\varphi_-^{-1}$, and hence,

$$\omega = \varphi_+^{-1}\big[(\rho\vartheta_-^+\varphi_-^{-1} - a)(\varphi_-\vartheta_-^{-1}\vartheta_+\varphi_+^{-1} - I_m)\big]^{-1}(\varphi_-^{\circledast})^{-1} \tag{3.34}$$

$$= \varphi_+^{-1}\big[(\rho\vartheta_-^+ - a\varphi_-)(\vartheta_-^{-1}\vartheta_+\varphi_+^{-1} - \varphi_-^{-1})\big]^{-1}(\varphi_-^{\circledast})^{-1}. \tag{3.35}$$

By (3.15a), $\rho\vartheta_-^+ - a\varphi_- = b\vartheta_-$. Using this equivalence in (3.35) yields the right-hand side of (3.26) and thus establishes the equivalence of (3.26) and (3.27). Thus the equations in (3.14) and (3.17) are consistent.

To obtain (3.6), we first note that (3.20) and (3.26) can be written as

$$\omega = \varphi_\pm^{-1}(\vartheta_+\varphi_+^{-1} - \vartheta_-\varphi_-^{-1})^{-1}b^{-1}(\varphi_\mp^{\circledast})^{-1} \tag{3.36}$$

$$= \big[\varphi_\mp^{\circledast}(b\vartheta_+\varphi_+^{-1} - b\vartheta_-\varphi_-^{-1})\varphi_\pm\big]^{-1}. \tag{3.37}$$

By (3.15a), $b\vartheta_\pm = \rho\vartheta_\pm^+ - a\varphi_\pm$, and hence,

$$\omega = \big[\varphi_\mp^{\circledast}(\rho\vartheta_+^+\varphi_+^{-1} - \rho\vartheta_-^+\varphi_-^{-1})\varphi_\pm\big]^{-1}. \tag{3.38}$$

$$= \pm\big[\varphi_\mp^{\circledast}(\rho\vartheta_\pm^+\varphi_\pm^{-1} - \rho\vartheta_\mp^+\varphi_\mp^{-1})\varphi_\pm\big]^{-1}. \tag{3.39}$$

By (3.25), $\rho\vartheta_\mp^+\varphi_\mp^{-1} = (\varphi_\mp^{\circledast})^{-1}(\vartheta_\mp^{\circledast})^+\rho$, and thus,

$$\omega = \pm\big[\varphi_\mp^{\circledast}\rho\vartheta_\pm^+ - (\vartheta_\mp^{\circledast})^+\rho\varphi_\pm\big]^{-1} = \pm\big(\widehat{U}_\mp^{\circledast}J_\rho\widehat{U}_\pm\big)^{-1} \tag{3.40}$$

which by (3.10a) yields (3.6), and thus completes the demonstration that y, as defined by (3.3), satisfies (3.1).

For the remaining properties of $K(z,k,\ell)$, we first note that (2.96) and (2.97) imply that $K(z,k,\ell)$ satisfies (3.2). And finally, by an argument in direct analogy with that given in [54, Lemma 4.2] (see also [56, Lemma 4.1]), we see that y given by (3.3) satisfies

$$\sum_{k\in\mathbb{Z}} y(z,k)^*A(k)y(z,k) \leqslant (\operatorname{Im}(z))^{-2}\sum_{k\in\mathbb{Z}} f(k)^*A(k)f(k). \tag{3.41}$$

As a result, we note that $y(z,\cdot) \in \ell_A^2(\mathbb{Z})$ whenever $f \in \ell_A^2(\mathbb{Z})$. $\square$

For $k \neq \ell$, there exists an alternative expression for $K(z,k,\ell)$ in terms of the half-line M-matrices and the fundamental solution $\Psi(z,k,k_0,\widetilde{\alpha})$ which is defined in (2.34). In direct analogy with equation (4.5) of [54], we note that

$$\begin{aligned} K(z,k,\ell) &= U_+(z,k)\omega(z)U_-(\bar{z},\ell)^* = U_+(z,k)[M_-(z)-M_+(z)]^{-1}U_-(\bar{z},\ell)^* \\ &= \Psi(z,k,k_0,\widetilde{\alpha}) \\ &\quad\times\begin{pmatrix} [M_-(z)-M_+(z)]^{-1} & [M_-(z)-M_+(z)]^{-1}M_-(z) \\ M_+(z)[M_-(z)-M_+(z)]^{-1} & M_+(z)[M_-(z)-M_+(z)]^{-1}M_-(z)\end{pmatrix} \\ &\quad\times\Psi(\bar{z},\ell,k_0,\widetilde{\alpha})^*, \quad k>\ell \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} K(z,k,\ell) &= U_-(z,k)\omega(z)U_+(\bar{z},\ell)^* = U_-(z,k)[M_-(z)-M_+(z)]^{-1}U_+(\bar{z},\ell)^* \\ &= \Psi(z,k,k_0,\widetilde{\alpha}) \\ &\quad\times\begin{pmatrix} [M_-(z)-M_+(z)]^{-1} & [M_-(z)-M_+(z)]^{-1}M_+(z) \\ M_-(z)[M_-(z)-M_+(z)]^{-1} & M_-(z)[M_-(z)-M_+(z)]^{-1}M_+(z)\end{pmatrix} \\ &\quad\times\Psi(\bar{z},\ell,k_0,\widetilde{\alpha})^*, \quad k<\ell, \end{aligned} \tag{3.43}$$

noting that $M_+(M_- - M_+)^{-1}M_- = M_-(M_- - M_+)^{-1}M_+$.

Given the notation introduced in Theorem 3.1, specifically in (3.9), let

$$V_\pm(z,k) = \rho(k)u_{\pm2}^+(z,k)u_{\pm1}(z,k)^{-1} = \rho(k)\vartheta_\pm^+(z,k)\varphi_\pm(z,k)^{-1}. \tag{3.44}$$

We note that $V_\pm(z,k)$ is a solution of the Riccati equation given in (2.105). We also note that by (3.38),

$$\omega = (M_- - M_+)^{-1} = \varphi_\pm^{-1}(\rho\vartheta_+^+\varphi_+^{-1} - \rho\vartheta_-^+\varphi_-^{-1})^{-1}(\varphi_\mp^\circledast)^{-1} \tag{3.45}$$
$$= \varphi_\pm^{-1}(V_+ - V_-)^{-1}(\varphi_\mp^\circledast)^{-1}. \tag{3.46}$$

Then, by (3.4) for $k=\ell$,

$$K(z,k,k) = \begin{pmatrix} \varphi_+(z,k)\omega(z)\varphi_-^\circledast(z,k) & \varphi_+(z.k)\omega(z)\vartheta_-^\circledast(z,k) \\ \vartheta_-(z,k)\omega(z)\varphi_+^\circledast(z,k) & \vartheta_-(z,k)\omega(z)\vartheta_+^\circledast(z,k) \end{pmatrix}. \tag{3.47}$$

An alternative representation for the entries of the matrix $K(z,k,k)$ also exists:

$$\varphi_+\omega\varphi_-^\circledast = (V_+ - V_-)^{-1}, \tag{3.48}$$
$$\begin{aligned}\varphi_+\omega\vartheta_-^\circledast &= (V_+ - V_-)^{-1}(\varphi_-^\circledast)^{-1}\vartheta_-^\circledast, \\ &= (V_+ - V_-)^{-1}(u_{-,1}^\circledast)^{-1}u_{-,2}^\circledast,\end{aligned} \tag{3.49}$$
$$\begin{aligned}\vartheta_-\omega\varphi_+^\circledast &= \vartheta_-\varphi_-^{-1}(V_+ - V_-)^{-1}, \\ &= u_{-,2}(u_{-,1})^{-1}(V_+ - V_-)^{-1}\end{aligned} \tag{3.50}$$
$$\begin{aligned}\vartheta_-\omega\vartheta_+^\circledast &= \vartheta_-\varphi_-^{-1}(V_+ - V_-)^{-1}(\varphi_+^\circledast)^{-1}\vartheta_+^\circledast \\ &= u_{-,2}(u_{-,1})^{-1}(V_+ - V_-)^{-1}(u_{+,1}^\circledast)^{-1}u_{+,2}^\circledast.\end{aligned} \tag{3.51}$$

The proof of the next lemma relies upon an argument involving the geometry of the Weyl disks described in Definition 2.11. We refer the reader to [56, Theorem 2.1] for details while noting that the discussion in [56] occurs in the context of Hamiltonian systems of ordinary differential equations but that the argument remains the same for the current setting.

Lemma 3.2. *Assume that the Hamiltonian system* (2.12a) *which satisfies Hypothesis* 2.5 *is in the limit point or the limit circle case at* ∞. *Let* $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$. *Then,*

$$\lim_{k\to\infty} \begin{bmatrix} I & M_+(z_2)^* \end{bmatrix} \widehat{\Psi}(z,k,k_0,\widetilde{\alpha})^* J_\rho \widehat{\Psi}(z,k,k_0,\widetilde{\alpha}) \begin{bmatrix} I \\ M_+(z_1) \end{bmatrix} = 0, \tag{3.52}$$

where $M_+(z_j) \in \partial\mathcal{D}(z_j,k_0,\widetilde{\alpha})$, $j=1,2$, *and where* $\Psi(z,k,k_0,\widetilde{\alpha})$ *is the fundamental matrix defined in* (2.34).

Of course, an analogous result can be stated when the Hamiltonian system (2.12a) is in the limit point or the limit circle case at $-\infty$. Moreover, an immediate consequence of this result, like that of its continuous counterpart, is the following corollary. Again see [56, Corollary 2.3] for details of the proof that also remains the same for the current setting.

Corollary 3.3. *The Hamiltonian system* (2.12a) *which satisfies Hypothesis* 2.5 *is in the limit point case at* ∞ *if and only if*

$$\lim_{k\to\infty} \widehat{y}(z_1,k)^* J_\rho \widehat{y}(z_2,k) = 0, \quad z_1, z_2 \in \mathbb{C}\backslash\mathbb{R} \tag{3.53}$$

for all $\ell_A^2([k_0,\infty))$*-solutions* $y(z_j,\cdot)$, $j=1,2$, *of* (2.12a).

As before, we note that an analogous result for Corollary 3.3 can be stated when the Hamiltonian system (2.12a) is in the limit point case at $-\infty$.

As a consequence of the preceding results of this section, we have the following theorem which effectively characterizes solutions of the nonhomogeneous system (3.1) described in Theorem 3.1.

Theorem 3.4. *Assume that the Hamiltonian system* (2.12a) *which satisfies Hypothesis* 2.5 *is either in the limit circle case or in the limit point case at* $-\infty$ *and is either in the limit circle or in the limit point case at* ∞*. Let* $f \in \ell^2_A(\mathbb{Z})$ *and let* $y(z,\cdot)$ *represent the* $\ell^2_A(\mathbb{Z})$*-solution of the nonhomogeneous system* (3.1) *which is defined by* (3.3)*. Then,* $y(z,\cdot)$ *represents the unique* $\ell^2_A(\mathbb{Z})$*-solution of* (3.1) *which satisfies the boundary conditions given by*

$$\lim_{k\to\infty} \widehat{U}^{\circledast}_{+}(z,k)J_\rho \widehat{y}(z,k) = 0, \tag{3.54}$$

$$\lim_{k\to-\infty} \widehat{U}^{\circledast}_{-}(z,k)J_\rho \widehat{y}(z,k) = 0, \tag{3.55}$$

where $U_\pm(z,k) = U_\pm(z,k,k_0,\widetilde{\alpha})$ *and* $U^{\circledast}_{\pm}(z,k)$ *are defined in Theorem* 3.1*. Moreover, when* (2.12a) *is in the limit point case at* ∞ *(resp.,* $-\infty$*), the corresponding boundary condition* (3.54) *(resp.,* (3.55)*) is superfluous and can be dropped.*

Proof. There are four cases to be considered. However, by symmetry this may be reduced to three. The first of these cases we consider assumes that the limit circle case holds at $-\infty$ while the limit point case holds at ∞.

Let $u(z,k)$ and $v(z,k)$ be $\ell^2_A(\mathbb{Z})$-solutions of (3.1) which satisfy (3.54) and (3.55). Then, $w(z,k) = u(z,k) - v(z,k)$ satisfies (3.54) and is an $\ell^2_A(\mathbb{Z})$-solution of the Hamiltonian system (2.12a), and as a result, $w(z,k) = U_+(z,k)\zeta$ for some $\zeta \in \mathbb{C}^{2m}$. Thus, by Lemma 3.2, (3.54), and (3.10), we see that

$$0 = \lim_{k\to-\infty} \widehat{U}^{\circledast}_{-}(z,k)J_\rho \widehat{w}(z,k) \tag{3.56}$$

$$= \lim_{k\to-\infty} \widehat{U}^{\circledast}_{-}(z,k)J_\rho \widehat{U}_{+}(z,k)\zeta \tag{3.57}$$

$$= [M_-(z) - M_+(z)]\zeta. \tag{3.58}$$

Given the invertibility of $(M_- - M_+)$, we see that $\zeta = 0$. Note that the boundary condition at ∞ given in (3.55) was not used in this argument. However, note that it is automatically satisfied by $y(z,k)$ due to Corollary 3.3.

Suppose that the limit point case holds at both ∞ and at $-\infty$. Then, with u, v, and w as previously defined, we again see that $w(z,k) = U_+(z,k)\zeta$ for some $\zeta \in \mathbb{C}^{2m}$, but that $\zeta = 0$ by the reasoning in the previous case. However, now note that (3.54) is automatically satisfied by $y(z,k)$ by Corollary 3.3 as it can be restated when the limit point case holds at $-\infty$.

Lastly, we suppose that the limit circle case holds at both ∞ and at $-\infty$. Once again, let $u(z,k)$ and $v(z,k)$ be $\ell^2_A(\mathbb{Z})$-solutions of (3.1) which satisfy (3.54) and (3.55) and let $w(z,k) = u(z,k) - v(z,k)$. Now note that the columns of $U_-(z,k)$ and of $U_+(z,k)$ together form a basis for all solutions of the Hamiltonian system (2.12a). Then, $w(z,k) = U_-(z,k)\zeta + U_+(z,k)\eta$ for some $\zeta,\eta \in \mathbb{C}^{2m}$. Then, by Lemma 3.2, (3.54), (3.10), and (3.24),

$$0 = \lim_{k\to-\infty} \widehat{U}^{\circledast}_{-}(z,k)J_\rho \widehat{w}(z,k) \tag{3.59}$$

$$= \lim_{k\to-\infty} \widehat{U}^{\circledast}_{-}(z,k)J_\rho \begin{bmatrix} \widehat{U}_-(z,k) & \widehat{U}_+(z,k) \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \tag{3.60}$$

$$= \begin{bmatrix} 0 & (M_-(z) - M_+(z)) \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix}. \tag{3.61}$$

Given the invertibility of $(M_- - M_+)$, we see that $\eta = 0$. By similar reasoning using (3.55), (3.10), and (3.24), we see that $\zeta = 0$. □

In analogy to the treatment in [54], [56] in the continuous context, we will call the kernel $K(z,\cdot,\cdot)$ defined in (3.4) the $2m\times 2m$ Green's matrix of the Hamiltonian system (2.12a) on $\mathbb{Z}$ associated with the boundary conditions (3.54) (resp., (3.55)) if (2.12a) is in the limit circle case at ∞ (resp., $-\infty$). If (2.12a) is in the limit point case at ∞ and $-\infty$, $K(z,\cdot,\cdot)$ represents the unique Green's matrix corresponding to (2.12a) on $\mathbb{Z}$.

Next, we turn to the analogous considerations for half-lines and start with the righ half-line $[k_0,\infty)$. We assume Hypothesis 2.5 on $[k_0+1,\infty)$ and again consider the nonhomogeneous system (3.1) associated with the Hamiltonian system (2.12a) which is in the limit point or the limit circle case at ∞. We assume that $f(k)$ is defined for $k\in[k_0+1,\infty)$ and that $f\in\ell^2_A([k_0+1,\infty))$.

We describe for $k,\ell\in[k_0,\infty)$, and $z\in\mathbb{C}\backslash\mathbb{R}$, a matrix $K_+(z,k,\ell)\in\mathbb{C}^{2m\times 2m}$ for which the following properties hold:

$$\sum_{\ell\in[k_0,\infty)} K_+(z,k,\ell)A(\ell)K_+(z,k,\ell)^*<\infty,\quad k\in[k_0,\infty). \tag{3.62}$$

If $f\in\ell^2_A([k_0+1,\infty))$ and if

$$y(z,k)=\sum_{\ell\in[k_0+1,\infty)} K_+(z,k,\ell)A(\ell)f(\ell),\quad k\in[k_0,\infty), \tag{3.63}$$

then $y(z,\cdot)\in\ell^2_A([k_0,\infty))$ and $y(z,\cdot)$ satisfies (3.1) on $[k_0+1,\infty)$. In addition, it will be seen that $y(z,\cdot)$ satisfies certain boundary conditions at $k=k_0$ and at ∞ (if (2.12a) is in the l.c. case at ∞).

As in Theorem 3.1, we assume for convenience that $z\in\mathbb{C}_+$.

Theorem 3.5. *Assume Hypothesis* 2.5 *on* $[k_0,\infty)$ *and suppose that* $z\in\mathbb{C}_+$ *and* $k,\ell\in[k_0,\infty)$. *Let* $K_+(z,k,\ell)$ *be defined by*

$$K_+(z,k,\ell)=\begin{cases} U_+(z,k)\Phi(\bar z,\ell)^*, & k>\ell,\\ \begin{pmatrix} u_{+,1}(z,k)\phi_1(\bar z,k)^* & u_{+,1}(z,k)\phi_2(\bar z,k)^*\\ \phi_2(z,k)u_{+,1}(\bar z,k)^* & \phi_2(z,k)u_{+,2}(\bar z,k)^*\end{pmatrix}, & k=\ell,\\ \Phi(z,k)U_+(\bar z,\ell)^*, & k<\ell.\end{cases} \tag{3.64}$$

Here $U_+(z,k)$ *is defined in* (2.49) *with* $M=M_+(z)\in\partial\mathcal{D}_+(z,k_0,\widetilde\alpha)$, $U_+(\bar z,k)$ *is defined in* (2.49) *with* $M=M_+(\bar z)=M_+(z)^*\in\partial\mathcal{D}_+(\bar z,k_0,\widetilde\alpha)$, *and*

$$U_+(z,k)=\begin{pmatrix} u_{+,1}(z,k)\\ u_{+,2}(z,k)\end{pmatrix} \tag{3.65}$$

with $u_{+,j}(z,k)\in\mathbb{C}^{m\times m}$, $j=1,2$, $k\in[k_0,\infty)$. *In addition,* $\Phi(z,k)$ *is defined in* (2.34). *With* $K_+(z,k,\ell)$ *so defined,* (3.62) *is satisfied. Moreover, as defined in* (3.63), $y(z,\cdot)$ *satisfies* (3.1) *on* $[k_0+1,\infty)$ *and is in* $\ell^2_A([k_0,\infty))$.

Proof. This result follows using the same steps already given for the proof of Theorem 3.1 with the following identifications replacing those of (3.9):

$$\varphi_+(z,k)=u_{+,1}(z,k),\qquad \varphi_-(z,k)=\phi_1(z,k), \tag{3.66}$$

$$\vartheta_+(z,k)=u_{+,2}(z,k),\qquad \vartheta_-(z,k)=\phi_2(z,k). \tag{3.67}$$

This assigns the same meaning to $U_+(z,k)$ as in Theorem 3.1, but unlike Theorem 3.1 it makes the further assignment given by $U_-(z,k)=\Phi(z,k)$.

The principal effect of this set of assignments in modifying the proof of Theorem 3.1 comes with the realization that (3.10) is now replaced by

$$\widehat{U}_-(\bar{z},k)^* J_\rho(k)\widehat{U}_+(z,k) = \widehat{U}_-(\bar{z},k_0)^* J_\rho(k_0)\widehat{U}_+(z,k_0) \tag{3.68}$$

$$= \widehat{\Phi}(\bar{z},k_0)^* J_\rho(k_0)\widehat{U}_+(z,k_0) \tag{3.69}$$

$$= I_m, \tag{3.70}$$

together with

$$\widehat{U}_+(\bar{z},k)^* J_\rho(k)\widehat{U}_-(z,k) = \widehat{U}_+(\bar{z},k_0)^* J_\rho(k_0)\widehat{U}_-(z,k_0) \tag{3.71}$$

$$= \widehat{U}_+(\bar{z},k_0)^* J_\rho(k_0)\widehat{\Phi}(z,k_0) \tag{3.72}$$

$$= -I_m. \tag{3.73}$$

As a consequence of the identifications now given, we make the further assignment and modification to the proof given in the previous theorem: $\omega = I_m$.

As in Theorem3.1, we note that (2.96) and (2.97) imply that $K_+(z,k,\ell)$ satisfies (3.62). And finally, by an argument in direct analogy with that given in [54, Lemma 4.2], or in [59, Lemma 2.1] for the one singular endpoint case, we see that y given by (3.63) satisfies

$$\sum_{k\in[k_0+1,\infty)} y(z,k)^* A(k) y(z,k) \leqslant (\mathrm{Im}(z))^{-2} \sum_{k\in[k_0+1,\infty)} f(k)^* A(k) f(k). \tag{3.74}$$

As a result, we note that $y(z,\cdot) \in \ell^2_A([k_0,\infty))$ whenever $f \in \ell^2_A([k_0+1,\infty))$. □

We now state a result whose proof is analogous to that of Theorem 3.4.

Theorem 3.6. *Assume that the Hamiltonian system* (2.12a) *which satisfies Hypothesis* 2.5 *on* $[k_0,\infty)$ *is in the limit point or limit circle case at* ∞. *Let* $f(k) \in \mathbb{C}^{2m}$ *be defined for* $k \in [k_0+1,\infty)$ *with* $f \in \ell^2_A([k_0+1,\infty))$ *and let* $y(z,k)$ *be described by* (3.63). *Then,* $y(z,\cdot) \in \ell^2_A([k_0,\infty))$ *and* $y(z,\cdot)$ *satisfies* (3.1) *on* $[k_0+1,\infty)$. *Moreover,* $y(z,\cdot)$ *is uniquely defined by the boundary conditions*

$$\widetilde{\alpha}\widehat{y}(z,k_0) = 0, \tag{3.75}$$

$$\lim_{k\to\infty} \widehat{U}_+^{\circledast}(z,k) J_\rho \widehat{y}(z,k) = 0, \tag{3.76}$$

where $U_+(z,k) = U_+(z,k,k_0,\widetilde{\alpha})$ *and* $U_+^{\circledast}(z,k)$ *are defined in Theorem* 3.1. *When* (2.12a) *is in the limit point case at* ∞, *the corresponding boundary condition in* (3.76) *is superfluous and can be dropped.*

Again, in analogy to the treatment in [52] in the continuous context, we will call the kernel $K_+(z,\cdot,\cdot)$ defined in (3.64) the $2m \times 2m$ half-line Green's matrix of the Hamiltonian system (2.12a) on $[k_0,\infty)$ associated with the boundary conditions (3.75) and (3.76) (if (2.12a) is in the limit circle case at ∞).

Finally, we briefly turn to the left half-line case $(-\infty,k_0]$. We assume Hypothesis 2.5 on $(-\infty,k_0]$ and again consider the nonhomogeneous system (3.1) associated with the Hamiltonian system (2.12a) which is in the limit point or the limit circle case at $-\infty$. We assume that $f(k)$ is defined for $k \in (-\infty,k_0-1]$ and that $f \in \ell^2_A((-\infty,k_0-1])$.

We describe for $k,\ell \in (-\infty,k_0]$, and $z\in\mathbb{C}\backslash\mathbb{R}$, a matrix $K_-(z,k,\ell)\in\mathbb{C}^{2m\times 2m}$ for which the following properties hold:

$$\sum_{\ell\in(-\infty,k_0]} K_-(z,k,\ell)A(\ell)K_-(z,k,\ell)^* < \infty, \quad k\in(-\infty,k_0]. \tag{3.77}$$

If $f\in\ell^2_A((-\infty,k_0-1])$ and if

$$y(z,k) = \sum_{\ell\in(-\infty,k_0-1]} K_-(z,k,\ell)A(\ell)f(\ell), \quad k\in(-\infty,k_0], \tag{3.78}$$

then $y(z,\cdot)\in\ell^2_A((-\infty,k_0])$ and $y(z,\cdot)$ satisfies (3.1) on $(-\infty,k_0-1]$. In addition, it will be seen that $y(z,\cdot)$ satisfies certain boundary conditions at $k=k_0$ and at $-\infty$ (if (2.12a) is in the l.c. case at $-\infty$).

As in Theorems 3.1 and 3.5, we assume for convenience that $z\in\mathbb{C}_+$.

Theorem 3.7. *Assume Hypothesis* 2.5 *on* $(-\infty,k_0]$ *and suppose that* $z\in\mathbb{C}_+$ *and* $k,\ell\in(-\infty,k_0]$. *Let* $K_-(z,k,\ell)$ *be defined by*

$$K_-(z,k,\ell) = \begin{cases} -\Phi(z,k)U_-(\bar z,\ell)^*, & k>\ell, \\ \begin{pmatrix} -\phi_1(z,k)u_{-,1}(\bar z,k)^* & -\phi_1(z,k)u_{-,2}(\bar z,k)^* \\ -u_{-,2}(z,k)\phi_1(\bar z,k)^* & -u_{-,2}(z,k)\phi_2(\bar z,k)^* \end{pmatrix}, & k=\ell, \\ -U_-(z,k)\Phi(\bar z,\ell)^*, & k<\ell. \end{cases} \tag{3.79}$$

Here $U_-(z,k)$ *is defined in* (2.49) *with* $M=M_-(z)\in\partial\mathcal{D}_-(z,k_0,\widetilde\alpha)$, $U_-(\bar z,k)$ *is defined in* (2.49) *with* $M=M_-(\bar z)=M_-(z)^*\in\partial\mathcal{D}_-(\bar z,k_0,\widetilde\alpha)$, *and*

$$U_-(z,k) = \begin{pmatrix} u_{-,1}(z,k) \\ u_{-,2}(z,k)\end{pmatrix} \tag{3.80}$$

with $u_{-,j}(z,k)\in\mathbb{C}^{m\times m}$, $j=1,2$, $k\in(-\infty,k_0]$, *and* $z\in\mathbb{C}_+$. *In addition,* $\Phi(z,k)$ *is defined in* (2.34). *With* $K_-(z,k,\ell)$ *so defined,* (3.77) *is satisfied. Moreover, as defined in* (3.78), $y(z,\cdot)$ *satisfies* (3.1) *on* $(-\infty,k_0-1]$ *and is in* $\ell^2_A((-\infty,k_0])$.

Proof. As seen in the proof of Theorem 3.5, this result follows using the same steps already given for the proof of Theorem 3.1 with the following identifications replacing those of (3.9):

$$\varphi_+(z,k)=\phi_1(z,k), \qquad \varphi_-(z,k)=u_{-,1}(z,k), \tag{3.81}$$

$$\vartheta_+(z,k)=\phi_2(z,k), \qquad \vartheta_-(z,k)=u_{-,2}(z,k). \tag{3.82}$$

This assigns the same meaning to $U_-(z,k)$ as in Theorem 3.1, but unlike Theorem 3.1 it makes the further assignment given by $U_+(z,k)=\Phi(z,k)$. As in Theorem 3.5, we again find that

$$\widehat U_\mp(\bar z,k)^*J_\rho(k)\widehat U_\pm(z,k) = \widehat U_\mp(\bar z,k_0)^*J_\rho(k_0)\widehat U_\pm(z,k_0) \tag{3.83}$$

$$= \mp I_m, \tag{3.84}$$

and hence that $\omega=-I_m$.

As in Theorem 3.5, we note that (2.96) and (2.97) imply that $K_-(z,k,\ell)$ satisfies (3.77). And finally, by an argument in direct analogy with that given in [59, Lemma 2.1] for the one singular endpoint case, we see that y given by (3.78) satisfies

$$\sum_{k\in(-\infty,k_0-1]} y(z,k)^*A(k)y(z,k) \leqslant (\mathrm{Im}(z))^{-2}\sum_{k\in(-\infty,k_0-1]} f(k)^*A(k)f(k). \tag{3.85}$$

As a result, we note that $y(z,\cdot)\in\ell^2_A((-\infty,k_0])$ whenever $f\in\ell^2_A((-\infty,k_0-1])$. $\square$

Lastly, we state a result whose proof is again analogous to that of Theorem 3.4.

Theorem 3.8. *Assume that the Hamiltonian system* (2.12a) *which satisfies Hypothesis* 2.5 *on* $(-\infty, k_0]$ *is in the limit point or limit circle case at* $-\infty$. *Let* $f(k) \in \mathbb{C}^{2m}$ *be defined for* $k \in (-\infty, k_0 - 1]$ *with* $f \in \ell^2_A((-\infty, k_0 - 1])$ *and let* $y(z,k)$ *be described by* (3.78). *Then,* $y(z,\cdot) \in \ell^2_A((-\infty, k_0])$ *and* $y(z,\cdot)$ *satisfies* (3.1) *on* $(-\infty, k_0 - 1]$. *Moreover,* $y(z,\cdot)$ *is uniquely defined by the boundary conditions*

$$\widetilde{\alpha}\widehat{y}(z,k_0) = 0, \tag{3.86}$$

$$\lim_{k\to-\infty} \widehat{U}_-^{\circledast}(z,k) J_\rho \widehat{y}(z,k) = 0, \tag{3.87}$$

where $U_-(z,k) = U_-(z,k,k_0,\widetilde{\alpha})$ *and* $U_-^{\circledast}(z,k)$ *are defined in Theorem* 3.1. *When* (2.12a) *is in the limit point case at* $-\infty$, *the corresponding boundary condition* (3.87) *is superfluous and can be dropped.*

As in the previous half-line case, we will call the kernel $K_-(z,\cdot,\cdot)$ defined in (3.79) the $2m \times 2m$ half-line Green's matrix of the Hamiltonian system (2.12a) on $(-\infty, k_0]$ associated with the boundary conditions (3.86) and (3.87) (if (2.12a) is in the limit circle case at $-\infty$).

In our subsequent paper [19], the explicit formulas (3.4) for the Green's function on $\mathbb{Z}$ together with their asymptotic expnsions as $|z| \to \infty$ will be used to prove trace formulas of the matrix-valued Jacobi operator (2.11), (2.21) and the Dirac-type difference expression (2.11), (2.19). This in turn then yields Borg-type uniqueness theorems for these Jacobi and Dirac-type difference operators in analogy to our treatment of Schrödinger and Dirac-type differential operators in [16] and [18]. As indicated at the end of the introduction, these results are relevant in connection with the nonabelian Toda and Kac–van Moerbeke hierarchies of completely integrable evolution equations.

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Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409, USA

E-mail address: `sclark@umr.edu`

URL: `http://www.umr.edu/~clark`

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: `fritz@math.missouri.edu`

URL: `http://www.math.missouri.edu/people/fgesztesy.html`