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SPECTRAL PROPERTIES OF THE PERIODIC MAGNETIC SCHRÖDINGER OPERATOR IN THE HIGH-ENERGY REGION. TWO-DIMENSIONAL CASE.

YULIA KARPESHINA

ABSTRACT The goal is to investigate spectral properties of the operator $H=(-i\nabla+\vec{a}(x))^2+a_0(x)$ in the two-dimensional situation, $(\vec{a}(x), a_0(x))$ being periodic. We construct asymptotic formulae for Bloch eigenvalues and eigenfunctions in the high-energy region, describe properties of isoenergetic curves in the space of quasimomenta and give a new proof of the Bethe-Sommerfeld conjecture.

1. Introduction.

We consider the operator

$$H=\left(\frac{1}{i}\frac{\partial}{\partial x_1}+a_1(x)\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial x_2}+a_2(x)\right)^2+\alpha a_0(x) \tag{1}$$

in $L_2(R^2)$, where $a_0(x)$, $a_1(x)$, $a_2(x)$ are real periodic functions, α is a real parameter. The operator (1) describes motion of an electron in the plane (x_1,x_2) when the periodic magnetic field $B(x)=a_1(x)_{x_2}-a_2(x)_{x_1}$ acts in the direction x_3 orthogonal to the plane and the periodic electric field $\alpha\nabla a_0(x)$ acts in a direction in the plane. Note that not every periodic magnetic field can be described in terms of a periodic magnetic potential $\vec{a}(x)=(a_1(x),a_2(x))$. In particular, $\vec{a}(x)$ is a linear function for a constant magnetic field. The case of a periodic $B(x)$ with non-periodic $\vec{a}(x)$ is significantly more difficult than ours. Most of the results in this situation are obtained for the case of a constant magnetic field (e.g [LL, DN, Sj] and literature there).

We assume that $a_0(x)$, $a_1(x)$, $a_2(x)$ are trigonometric polynomials and have the same periods $\vec{b}_1,\vec{b}_2$. For the sake of simplicity, we consider the periods to be orthogonal, however all the results are also valid for nonorthogonal periods. Without the loss of generality we assume that the periods are directed along coordinate axis $\vec{b}_1=(b_1,0),\vec{b}_2=(0,b_2)$, $b_1\geq b_2$, and

$$\int_Q a_s(x)dx=0,\quad s=0,1,2,\quad Q=[0,b_1]\times[0,b_2]. \tag{2}$$

$$\operatorname{div}\vec{a}(x)=0,\quad \vec{a}(x)=(a_1(x),a_2(x))\,. \tag{3}$$

We use the notation

$$\vec{A}(x)=(a_0(x),a_1(x),a_2(x)) \tag{4}$$

and the Fourier representation of the trigonometric polynomials $a_0(x)$, $a_1(x)$, $a_2(x)$:

$$a_s(x)=\sum_{j\in Z^2,\ 0<|j|\leq R_0} a_{sj}\exp i(\vec{p}_j(0),x),\quad s=0,1,2,\quad R_0<\infty, \tag{5}$$

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here $(\cdot,\cdot)$ is the standard inner product in R^2 and $\vec{p}_j(0)$ is a vector of the dual lattice:

$$\vec{p}_j(0) = 2\pi(j_1b_1^{-1}, j_2b_2^{-1}). \tag{6}$$

Spectral study of the operator (1) is essentially more complicated than study of the operator with just an electric potential ($\vec{a}(x) = 0$). The difficulties are caused by the fact that function $\vec{a}(x)$ is a coefficient at $\nabla u(x)$, while $a_0(x)$ is a coefficient at $u(x)$. As it is well-known, perturbations of coefficients at higher order derivatives produce stronger effects on spectral properties of a differential operator. Let us shortly describe existing results on (1).

It was proved by I.M.Gel'fand [G] that spectral analysis of H can be reduced to analysis of a family of operators $H(t)$, $t \in K$, where K is the elementary cell of the dual lattice,

$$K = [0, 2\pi b_1^{-1}) \times [0, 2\pi b_2^{-1}).$$

The vector t is called the *quasimomentum*. The operator $H(t)$, $t \in K$, acts in $L_2(Q)$. It is described by formula (1) and the quasiperiodic conditions:

$$u(b_1, x_2) = \exp(it_1b_1)u(0, x_2), \quad u(x_1, b_2) = \exp(it_2b_2)u(x_1, 0). \tag{7}$$

$$u_{x_1}(b_1, x_2) = \exp(it_1b_1)u_{x_1}(0, x_2), \quad u_{x_2}(x_1, b_2) = \exp(it_2b_2)u_{x_2}(x_1, 0).$$

Each operator $H(t)$, $t \in K$, has a discrete bounded below spectrum $\Lambda(t)$:

$$\Lambda(t) = \cup_{n=1}^{\infty}\lambda_n(t), \ \lambda_n(t) \to_{n\to\infty} \infty,$$

here eigenvalues $\lambda_n(t)$ being labelled in the increasing order. The spectrum Λ of the operator H is the union of the spectra of the operators $H(t)$: $\Lambda = \cup_{t\in K}\Lambda(t) = \cup_{n\in N, t\in K}\lambda_n(t)$. The functions $\lambda_n(t)$ are continuous, so Λ has a band structure:

$$\Lambda = \cup_{n=1}^{\infty}[q_n, Q_n], \quad q_n = \min_{t\in K}\lambda_n(t), \quad Q_n = \max_{t\in K}\lambda_n(t). \tag{8}$$

The eigenfunctions of $H(t)$ and H are simply related. Extending all the eigenfunctions of the operators $H(t)$ quasiperiodically (see (7)) to R^2, we obtain a complete system of generalized eigenfunctions of H.

Let H_0 be the operator (1) corresponding to $\vec{A} = 0$, $H_0 = -\Delta$. It is well-known that the spectrum of H_0 is absolutely continuous and fills the semiaxis $[0, \infty)$. The generalized eigenfunctions of H_0 are plane waves $\exp i(\vec{k}, x)$, $|\vec{k}|^2$ being the corresponding eigenvalues. The eigenfunctions of an operator $H_0(t)$, $t \in K$, are those plane waves which satisfy (7). They are naturally indexed by the points of Z^2:

$$\Psi_j^0(t, x) = |Q|^{-1/2}\exp i(\vec{p}_j(t), x), \quad j \in Z^2, \tag{9}$$

here and below

$$\vec{p}_j(t) = \vec{p}_j(0) + t \tag{10}$$

$\vec{p}_j(0)$ being given by (6), $|Q| = b_1b_2$. The eigenvalue corresponding to $\Psi_j^0(t, x)$ is equal to $p_j^2(t)$, $p_j^2(t) = |\vec{p}_j(t)|^2$.

An important property of (1) is absolute continuity of its spectrum. For the operator (1), absolute continuity is equivalent to the fact that no band $[q_n, Q_n]$ in the spectrum (8) of (1) degenerates into a point [Ku]. For a small magnetic potential $\vec{a}$ and any dimension d absolute continuity of the spectrum was proved by R.Hempel and I.Herbst [HH1]. They used L.Thomas' approach [Th] originally developed for the Schrödinger operator with just an electric potential ($\vec{a} = 0$). The approach is based on the fact that the resolvent of an operator $H(t)$ is compact for any $t \in K$ and admits an analytic extension to complex values of each component t_j, $j = 1, ...d$, of t, provided that others are real and fixed. Using perturbation considerations, L.Thomas [Th] proved that the resolvent goes to zero in the class of bounded operators when $\vec{a}=0$, Im $t_j \to \infty$ for some j, while Im $t_r = 0$ for $r \neq j$. It is easy to see that this can not happen if H has a band degenerated into a point, i.e., it has an eigenvalue constant with respect to t and, hence, proves absolute continuity of the spectrum. M.Birman and T.Suslina [BiSu1, BiSu2] generalized L.Thomas' approach for a general class of magnetic potentials $\vec{a}$ in dimension two. Later A.Sobolev [So] extended L.Thomas' approach to any dimension d and smooth magnetic potentials of arbitrary magnitude. P.Kuchment and S.Levendorskii [KuLe] simplified Sobolev's proof and obtained stronger results (weaker assumptions on $\vec{a}$, a validity for non-selfadjoint case). The latest paper on the subject belongs to L.Friedlander [F]. He obtained a completely new proof of absolute continuity of the spectrum for a second order self-adjoint elliptic differential operator with periodic coefficients satisfying additional symmetry conditions. In the case of operator (1) these conditions mean that functions $a_0(x)$ and $\vec{a}(x)$ have to be even with respect to x_1 and x_2.

The next important question is whether the number of gaps in the spectrum (8) of H is finite. Let us consider the analog of (1) in the one-dimensional situation:

$$H_1 = \left(\frac{1}{i}\frac{\partial}{\partial x_1} + a_1(x)\right)^2 + \alpha a_0(x), \quad x \in R. \tag{11}$$

The spectrum of this operator has a band structure (8), where $Q_n \leq q_{n+1}$ for all n-s. This means that zones can overlap by no more than one point. However, the situation $Q_n = q_{n+1}$ is exceptional. For a generic potential, $Q_n < q_{n+1}$ for all n. This means that bands $[q_n, Q_n]$ of the spectrum do not overlap and the number of gaps (Q_n, q_{n+1}) in the spectrum is infinite. It was noticed long time ago that the multidimensional situation $d>1$ is different from the case $d=1$. The conjecture that the Schrödinger operator with just an electric periodic potential in R^2, R^3 has only a finite number of gaps was formulated by physicists Bethe and Sommerfeld [BeSo]. Rigorous mathematical proofs of finiteness of the number of gaps for an electric potential in dimensions 2,3 and 4 were obtained later ([Sk1, Sk2, Sk3, V1, K1, K2, K3, HM, PaSo1, PaSo2]). A recent preprint [V2] based on ideas of [V3] claims the proof for an arbitrary dimension. Note that finiteness of the number of gaps in the multidimensional situation is closely related to "thickness" of the spectrum in the multidimensional case: it has infinite multiplicity unlike the spectrum in the one-dimensional case, which has the multiplicity two. The structure of the spectrum is also much more complicated in higher dimensions. Even in the simplest case $\vec{A}=0$ zones do overlap: $q_{n+1} > Q_n$ for all n, the spectrum filling the semiaxis $[0,\infty)$. This thickness of the spectrum can not be destroyed by an electric potential when energy is large enough, however it can be destroyed by other bounded perturbations [Sk2]. Thus, validity of the Bethe-Sommerfeld conjecture for a magnetic operator (1) is far from being obvious, even though it is proven for Scrödinger operators with just electric potentials. Obviously, the perturbation by a magnetic potential is much stronger than that by an electric potential. The question is whether it can destroy even a thick spectrum and produce gaps. So far, the Bethe-Sommerfeld conjecture is proven for electromagnetic potentials only in the case $d=2$, i.e., for (1). The proof belongs to A.Morame (A.Mohamed), who also obtained an asymptotic formula for an integrated density of states at high energies. The integrated density of states is closely related to A.Morame's approach to the proof of Bethe-Sommerfeld conjecture. The integrated density of states is defined by the formula:

$$D(H,\lambda)=\int_K N(H,\lambda,t)dt, \tag{12}$$

here $N(H,\lambda,t)$ is a counting function for Bloch eigenvalues:

$$N(H,\lambda,t)=\sum_{i:\lambda_i(t)<\lambda}1.$$

The notion of integrated density of states is important in physics: $D(H,\lambda+\delta\lambda)-D(H,\lambda)$ characterizes the number of possible electron states in the energy interval $[\lambda,\lambda+\delta\lambda]$. It is easy to see that $N(H_0,\lambda,t)$, corresponding to $H_0=-\Delta$, $d=2$, is just the number of the lattice points $\{\vec{p}_j(t)\}_{j\in Z^2}$ (see (10)) in the disk of the radius $\sqrt{\lambda}$ centered at the origin, and $D(H_0,\lambda)$ is the area of the disk:

$$D(H_0,\lambda)=\pi\lambda. \tag{13}$$

Using methods of microlocal analysis A.Morame proved:

$$D(H,\lambda)_{\lambda\to\infty}=D(H_0,\lambda)+O(\lambda^{\varepsilon}) \tag{14}$$

for any positive ε. Moreover, he generalized formulae (12) and (14) considering

$$D_q(H,\lambda)=\int_K e^{i(q,t)}N(H,\lambda,t)dt,\quad q=(q_1b_1,q_2b_2),\ q_1,q_2\in Z, \tag{15}$$

and having proved that

$$D_q(H,\lambda)_{\lambda\to\infty}=D_q(H_0,\lambda)+O(\lambda^{\varepsilon}). \tag{16}$$

A.Morame deduced a proof of the Bethe-Sommerfeld conjecture from (16), following the scheme of B.E.J.Dahlberg and E.Trubowitz [DaTr], which was developed earlier to prove the conjecture in the case of just an electric periodic potential in dimension two.

The next natural question is how Bloch eigenvalues and eigenfunctions of (1) look like. Bloch eigenvalues and eigenfunctions of the unperturbed operator $H_0(t)$ are given by formulae (9). What happens with $\Psi_j^0(t)$, $\lambda_j^0(t)$ under the perturbation? It is easier to answer this question in the high energy region, since one can use energy as a large parameter to construct asymptotic expansions. Until now the results were obtained only in the case of an electric potential ($\vec{a}=0$). It was proven in [K1] -[K3] that most of perturbed eigenvalues and eigenfunctions are close to unperturbed ones when $\vec{a}=0$, the result being valid for dimensions two and three. From physical point of view, closeness of perturbed eigenfunctions to unperturbed ones means weak diffraction inside the crystal. It is well known that there are also cases of strong diffraction (see e.g. [Ma]) and they are extremely important for applications. However, strong diffraction happens rarely: most plane waves are only weakly diffracted in the crystal. Let us precisely state the results obtained in [K1] for the case $d=2$. We introduce an unperturbed isoenergetic set, i.e., the set of all such t that $H_0(t)$ has an eigenvalue equal to a given λ:

$$\mathcal{S}_0(\lambda)=\{t,t\in K:\exists j\in Z^2:\lambda_j^0(t)=\lambda\}.$$

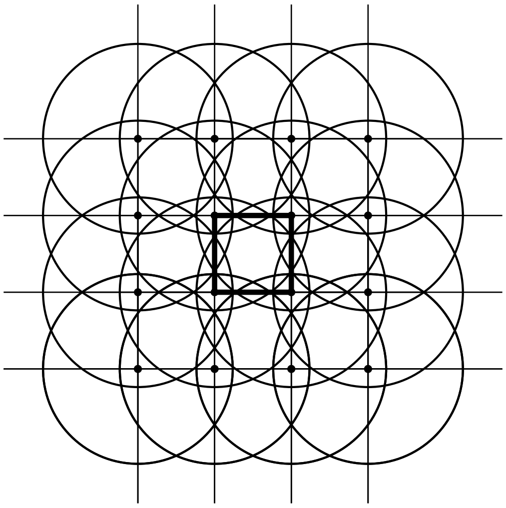

FIGURE 1. Isoenergetic Set of the Free Operator.

Considering that $\lambda_j^0(t) = p_j^2(t)$, we easily figure out that $\mathcal{S}_0(\lambda)$ is a circle of the radius $\sqrt{\lambda}$ "packed" into the box K (Fig.1): to obtain $\mathcal{S}_0(\lambda)$ we take the intersection of the cell K with the union of all the circles of the radius $\sqrt{\lambda}$ centered at the points $\{\vec{p}_m(0)\}_{m\in Z^2}$ or, in other words, the sphere of radius k centered at the origin of R^n is divided into pieces by the dual lattice $\{\vec{p}_m(t)\}_{m\in Z^2}$, and then all these pieces are shifted into the cell K.

It is proved [K1] that for any λ large enough there is a rich subset χ_0 of $\mathcal{S}_0(\lambda)$ such that if $t \in \chi_0$, then the perturbations of $\lambda_j^0(t)$ and $\Psi_j^0(t)$ by an electric potential are small. The rigorous statement of the result is presented by the next theorem.

Theorem 1. *Suppose $\vec{a}(x) = 0$. Then, for any positive δ and any sufficiently large λ, $\lambda > \lambda_0\,(a_0(x), \delta)$, there is a subset $\chi_0(\lambda, \delta)$ of $\mathcal{S}_0(\lambda)$ with the following properties:*

1) For every $t \in \chi_0(\lambda, \delta)$ there is a single $j \in Z^2$ such that

$$p_j^2(t) = \lambda,$$

the following inequality being satisfied:

$$\min_{m\in Z^2, m\neq j} |p_j^2(t) - p_m^2(t)| > 2\lambda^{-\delta}. \tag{17}$$

For this t the operator $H(t)$ has a simple eigenvalue $\lambda_j(t)$ close to λ, $\lambda = p_j^2(t)$,

$$\lambda_j(t) = p_j^2(t) + O(\lambda^{-1+8\delta}). \tag{18}$$

This eigenvalue is unique: there are no other eigenvalues of $H(t)$ in the interval $(\lambda - \lambda^{-\delta}, \lambda + \lambda^{-\delta})$.

2) An eigenfunction $\Psi_j(t, x)$ corresponding to $\lambda_j(t)$ is close to the plane wave $\Psi_j^0(t, x)$:

$$\left\|\Psi_j(t, x) - \Psi_j^0(t, x)\right\|_{L_\infty} = O(\lambda^{-1/2+8\delta}). \tag{19}$$

3) The measure of $\chi_0(\lambda,\delta)$ *is asymptotically close to that of* $\mathcal{S}_0(\lambda)$ *in the following sense:*

$$\frac{l\left(\chi_0(\lambda,\delta)\right)}{l(\mathcal{S}_0(\lambda))} > 1 - c\lambda^{-\delta/8},$$

here c *is a positive constant and* $l(\cdot)$ *is a curve length.* [1]

Note that inequality (17) means that $\lambda = p_j^2(t)$ is a simple eigenvalue of $H_0(t)$ and the distance from $p_j^2(t)$ to all other eigenvalues of $H_0(t)$ is greater than $2\lambda^{-\delta}$. The set $\chi_0(\lambda,\delta)$ is defined by a system of inequalities, (17) being just one of them.

It is proved also in [K1] that the asymptotic formula (18) is differentiable with respect to t in the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_0(\lambda,\delta)$. The isoenergetic set of H is defined as the set of t for which $H(t)$ has an eigenvalue equal to a given λ. Using (18) and its differentiability with respect to t one can easily show that there is a single isoenergetic curve corresponding to H in the $(\lambda^{-3/2+8\delta})$-neighborhood of each component of $\chi_0(\lambda,\delta)$. In other words, the perturbation of H_0 by an electric potential causes only a small distortion of $\chi_0(\lambda,\delta)$. The validity of the Bethe-Sommerfeld conjecture obviously follows from the last result, since existence of an isoenergetic curve for a given λ implies that λ belongs to the spectrum. Thus, every λ large enough belongs to the spectrum and, therefore, there is only a finite number of gaps in the spectrum, because only a finite number of gaps can happen below certain energy [RS].

The main goal of this paper is to construct asymptotic formulae in the high energy region for Bloch eigenvalues and eigenfunctions of operator (1). A natural question is whether Theorem 1 can be generalized to the case of a nonzero magnetic potential. The answer is no. It turns out that perturbed eigenvalues and eigenfunctions for $\vec{a} \neq 0$ are not close to unperturbed ones – the perturbation is too big. To understand what is really happening when $\vec{a} \neq 0$, we construct *model functions*. The model functions solve the equations for eigenfunctions approximately. They will be described explicitly. The model functions are not close to plane waves (9). They have the form: $e^{i(\vec{p}_j(t),x)+S_j(t,x)}$, $S_j(t,x)$ being a periodic function in x. We obtain $S_j(t,x)$ as an approximate solution of an eikonal equation. The imaginary part of $S_j(t,x)$ is proved to be small. This means that the amplitude of the wave is almost constant with respect to x. The main theorem proved in this paper states that most of Bloch eigenfunctions of H are close to model functions. The statement of the theorem is analogous to that of Theorem 1, $\Psi_j^0(t,x)$ being replaced by a model function, $\lambda_j^0(t)$ – by a number $\hat{\lambda}_j(t)$ corresponding to the model function. Further, using the asymptotic formulae for Bloch eigenvalues of H, we describe behavior of the isoenergetic curves of H in the high energy region. We prove that the length of isoenergetic curves in the high energy region is at least $2\pi\lambda^{1/2}(1+o(1))$ as λ goes to infinity. This implies that every sufficiently large λ is in the spectrum of H, i.e., the Bethe-Sommerfeld conjecture holds.

Finally, let us introduce some notations. Let $|Q| = b_1b_2$, $b_0 = \max\{2\pi/b_2, 1\}$, here and below we consider $b_2 < b_1$. We denote by c an absolute constant independent of the parameters of H, by c_b a constant defined only by b_1, b_2, by $c_{b,\vec{A}}$ a constant defined only by $b_1, b_2, \vec{A}$, etc. The next section describes the results obtained in the paper.

[1] The length of $\mathcal{S}_0(\lambda)$ is obviously equal to $2\pi\sqrt{\lambda}$.

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2. The Main Results.

2.1. Construction of model functions. We consider the equation $H\Psi = \lambda\Psi$, H being the differential expression (1), and look for its approximate solutions in the form of modified plane waves:

$$\hat{\Psi}(\vec{k}, x) = e^{i(\vec{k},x)+S(\vec{k},x)}, \quad \vec{k} \in R^2, \tag{20}$$

a function $S(\vec{k}, x)$ being periodic in x. Without the loss of generality we can assume $\int_Q S(\vec{k}, x)dx = 0$, since an additional constant in $S(\vec{k}, x)$ corresponds to a constant multiplier in front of the exponent. We look for $S(\vec{k}, x)$ which is, generally speaking, not small, however, has a small real part. This means that $S(\vec{k}, x)$ changes the phase of the plane wave $e^{i(\vec{k},x)}$ without a significant change of its amplitude. This situation we classify as the case of a weak diffraction. It is well known from physics [Ma] that there is a strong diffraction inside a crystal if momentum $\vec{k}$ of the plane wave satisfies the von Laue diffraction condition:

$$|\vec{k}|^2 = |\vec{k} + \vec{p}_r(0)|^2, \quad \text{for some } r \in Z^2 \setminus \{0\}. \tag{21}$$

Under this condition, the plane wave $e^{i(\vec{k},x)}$ incident upon a crystal, is essentially refracted inside the crystal becoming close to a linear combination of $e^{i(\vec{k},x)}$ and $e^{i(\vec{k}+\vec{p}_r(0),x)}$. Hence, we can not expect to find an approximate solution of the form (20), when $\vec{k}$ belongs to one of the lines (21) or close to it. For our purpose of construction of an approximate solution it is enough to assume that $|\vec{k}|^2 \neq |\vec{k} + \vec{p}_r(0)|^2$ for relatively small r, namely for $0 < |r| < k^{\varepsilon}$, ε being a parameter, $0 < \varepsilon < 1/4$. Moreover, we assume that $\vec{k}$ is relatively far from diffraction lines (21):

$$\Big||\vec{k}|^2 - |\vec{k} + \vec{p}_r(0)|^2\Big| > k^{1/2+\beta}, \quad \text{for } 0 < |r| < k^{\varepsilon}, \quad k = |\vec{k}|, \tag{22}$$

here β is a parameter $0 < \beta < 1/2 - 2\varepsilon$. We denote the set of $\vec{k}$ satisfying (22) by $\Pi(\beta)$ omitting ε for shortness.

Let $\vec{k} \in \Pi(\beta)$. We are looking for a solution of the equation

$$H\hat{\Psi}(\vec{k}, x) = \hat{\lambda}(\vec{k})\hat{\psi}(\vec{k}, x) + \eta(\vec{k}, x), \tag{23}$$

$\psi(\vec{k}, x)$ being of the form (20), $\hat{\lambda}(\vec{k})$ is a large number, $\eta(\vec{k}, x)$ is small. We fix a number N, $N \geq -1$ and require $|\eta(\vec{k}, x)| < k^{1-2(N+2)\beta}$. Clearly, a bigger N corresponds to a better approximation. Equation (23) leads to that for $S(\vec{k}, x)$:

$$2i\left(\vec{k}, \nabla S(\vec{k}, x)\right) + \Delta S(\vec{k}, x) + \left(\nabla S(\vec{k}, x), \nabla S(\vec{k}, x)\right) + 2i\left(\nabla S(\vec{k}, x), \vec{a}(x)\right) =$$

$$2\left(\vec{k}, \vec{a}(x)\right) + (\vec{a}(x), \vec{a}(x)) + \alpha a_0(x) + \hat{\lambda}(\vec{k}) - k^2 + \epsilon_N(\vec{k}, x). \tag{24}$$

where ∇ and Δ are taken with respect to x, a function $\epsilon_N(x)$ admits the estimate:

$$|\epsilon_N(x)| < c_{\vec{A}} k^{1-2(N+2)\beta}. \tag{25}$$

Let us find a solution of (24). First, we project both parts of equation (24) on the subspace **1** of constant functions in $L_2(Q)$. Taking into account that

$$\int_Q \left(\vec{k}, \nabla S(\vec{k},x)\right) dx = \int_Q \Delta S(\vec{k},x)dx = 0,$$

since $S(\vec{k},x)$ is periodic, and

$$\int_Q \left(\vec{k}, \vec{a}(x)\right) dx = \int_Q \alpha a_0(x)dx = 0,$$

because of conditions (2), we obtain:

$$\xi_0(S) = \hat{\lambda}(\vec{k}) - k^2 + O(k^{1-2(N+2)\beta}), \tag{26}$$

here and below

$$\xi_0(S) = \int_Q \left[\left(\nabla S(\vec{k},x), \nabla S(\vec{k},x)\right) + 2i\left(\nabla S(\vec{k},x), \vec{a}(x)\right) - (\vec{a}(x), \vec{a}(x))\right] dx. \tag{27}$$

Second, we project (24) on the subspace $L_2(Q) \ominus \mathbf{1}$. We get

$$2i\left(\vec{k}, \nabla S(\vec{k},x)\right) + \Delta S(\vec{k},x) + \left(\nabla S(\vec{k},x), \nabla S(\vec{k},x)\right) + 2i\left(\nabla S(\vec{k},x), \vec{a}(x)\right) - \xi_0(S) =$$

$$2\left(\vec{k}, \vec{a}(x)\right) + (\vec{a}(x), \vec{a}(x)) + \alpha a_0(x) + \epsilon_N(x), \tag{28}$$

here $\epsilon_N(x)$ is in $L_2(Q) \ominus \mathbf{1}$ and satisfies (25). We solve equation (28) by the method of successive approximations considering k as a large parameter, and, then substitute its solution $S(\vec{k},x)$ in (26) to find $\hat{\lambda}(\vec{k})$.

The formal procedure for solving (28) is the following. To obtain the first term of approximation we assume $|\nabla S(\vec{k},x)|^2 = o(k)$, and, hence, $\xi_0(S) = o(k)$. We keep in the equation (24) only the terms containing $\vec{k}$ and $\alpha a_0(x)$, since we consider that $|\alpha|$ can be comparable with k. This gives us the equation:

$$2i\left(\vec{k}, \nabla S(\vec{k},x)\right) + \Delta S(\vec{k},x) = 2\left(\vec{k}, \vec{a}(x)\right) + \alpha a_0(x)o(k). \tag{29}$$

This equation admits an explicit solution $S^{(0)}(\vec{k},x)$:

$$S^{(0)}(\vec{k},x) = -\sum_{r \in Z^2 \setminus \{0\}} \frac{2(\vec{k}, \vec{a}_r) + \alpha a_{0r}}{|\vec{k} + \vec{p}_r(0)|^2 - |\vec{k}|^2} e^{i(\vec{p}_r(0),x)}, \tag{30}$$

$\vec{a}_r$, a_{0r} being Fourier coefficients of $\vec{a}(x)$ and $a_0(x)$ correspondingly. Note that the solution makes sense only if $|\vec{k}|^2 \neq \left|\vec{k} + \vec{p}_r(0)\right|^2$ for all $r : 0 < |r| \leq R_0$, R_0 being the length of trigonometric polynomial $\vec{A}(x)$ (see (5)). Further, we look for $S(\vec{k},x)$ in the form of a sum:

$$S(\vec{k},x) = \sum_{n=0}^{N+1} S^{(n)}(\vec{k},x), \tag{31}$$

where the first term $S^{(0)}(\vec{k},x)$ is given by (30) and

$$S^{(1)}(\vec{k},x) = \sum_{r \in Z^2 \setminus \{0\}} \frac{\tilde{a}_r^{(1)}(\vec{k})}{|\vec{k} + \vec{p}_r(0)|^2 - |\vec{k}|^2} e^{i(\vec{p}_r(0),x)}, \tag{32}$$

$\tilde{a}_r^{(1)}(\vec{k})$ being Fourier coefficients of the function

$$\tilde{a}^{(1)}(\vec{k},x) = 2i\left(\nabla S^{(0)}(\vec{k},x), \vec{a}(x)\right) + \left(\nabla S^{(0)}(\vec{k},x), \nabla S^{(0)}(\vec{k},x)\right) - (\vec{a}(x), \vec{a}(x)). \quad (33)$$

The functions $S^{(n)}(\vec{k},x)$, $n \geq 2$ are defined by a recurrent formula:

$$S^{(n)}(\vec{k},x) = \sum_{r \in Z^2 \setminus \{0\}} \frac{\tilde{a}_r^{(n)}(\vec{k})}{|\vec{k}+\vec{p}_r(0)|^2 - |\vec{k}|^2} e^{i(\vec{p}_r(0),x)}, \quad (34)$$

where $a_r^{(n)}(\vec{k})$ are Fourier coefficients of the function

$$\tilde{a}^{(n)}(\vec{k},x) = 2i\left(\nabla S^{(n-1)}(\vec{k},x), \vec{a}(x)\right) + 2\left(\nabla S^{(n-1)}(\vec{k},x), \sum_{s=0}^{n-2} \nabla S^{(s)}(\vec{k},x)\right)$$
$$+ \left(\nabla S^{(n-1)}(\vec{k},x), \nabla S^{(n-1)}(\vec{k},x)\right). \quad (35)$$

The following lemma shows that expansion (31) has an asymptotic character for large k. We denote by $\|f\|_\nu$ the Wiener norm of a function $f(x)$:

$$\|f\|_\nu = |f_0| + \sum_{r \in Z^2 \setminus \{0\}} |f_r||r|^\nu,$$

f_r being Fourier coefficients of $f(x)$. If $\vec{f}(x)$ is a vector function: $f: Q \to \mathbf{C}^M$, $\vec{f}(x) = (f_1(x), ..., f_M(x))$, then

$$\|\vec{f}\|_\nu = \sum_{s=1}^{M} \|f_s\|_\nu.$$

Obviously, $\|f\|_{\nu_1} \leq \|f\|_{\nu_1}$, when $\nu_1 < \nu_2$ and $\|D^{\vec{\nu}} f\|_0 \leq \left(\frac{2\pi}{b_2}\right)^\nu \|f\|_\nu$, here $\vec{\nu} = (\nu_1, \nu_2)$, $\nu = \nu_1 + \nu_2$.

Lemma 1. *Suppose $\vec{A}$ is a trigonometric polynomial of the length R_0, $0 < \varepsilon < 1/4$, $0 < \beta < 1/2 - 2\varepsilon$, a vector $\vec{k}$ satisfies (22), and $-1 \leq N < k^\varepsilon R_0^{-1}$. Then, the following relations hold for the functions $S^{(n)}(\vec{k},x)$, n=0,1,...,N+1, and $\tilde{a}^{(n)}(\vec{k},x)$, $n = 1, ..., N+1$:*

$$S_r^{(n)} = \tilde{a}_r^{(n)} = 0, \quad \text{when } |r| > 2^n R_0, \quad (36)$$

$S_r^{(n)}$, $\tilde{a}_r^{(n)}$, $r \in Z^2$ being Fourier coefficients of $S^{(n)}(\vec{k},x)$ and $\tilde{a}^{(n)}(\vec{k},x)$, respectively;

$$\left\|S^{(n)}(\vec{k},x)\right\|_\nu \leq c_{n,\nu,b} \left\|\vec{A}\right\|_{\nu+n}^{n+1} k^{1/2-(1+2n)\beta}, \quad (37)$$

$$\left\|\Re S^{(n)}(\vec{k},x)\right\|_\nu \leq c_{n,\nu,b} \left\|\vec{A}\right\|_{\nu+n+2}^{n+1} k^{-2\beta(n+1)}, \quad (38)$$

$$\left\|D_{\vec{k}}^\mu S^{(n)}(\vec{k},x)\right\|_\nu \leq c_{n,\nu,b,\mu} \left\|\vec{A}\right\|_{\nu+n}^{n+1} k^{1/2-(1+2n)\beta-(1/2+\beta)|\mu|}, \quad (39)$$

$D_{\vec{k}}^\mu$ being the derivative with respect to $\vec{k}$ of order $\mu = (\mu_1, \mu_2)$, $|\mu| = \mu_1 + \mu_2$;

$$\left\|\left(S^{(n)}(\vec{k},x) - S^{(n)}(\vec{k}',x)\right)\right\|_\nu \leq c_{n,\nu,b} \left\|\vec{A}\right\|_{\nu+n}^{n+1} |\vec{k}-\vec{k}'| k^{-2(n+1)\beta}, \quad (40)$$

if $|\vec{k}-\vec{k}'| < (4b_0)^{-1} k^{1/2+\beta-\varepsilon}$, $b_0 = \max\{2\pi/b_2, 1\}$;

$$\left\|\tilde{a}^{(n)}(\vec{k},x)\right\|_\nu \leq c_{n,\nu,b} \left\|\vec{A}\right\|_{\nu+n}^{n+1} k^{1-2n\beta}, \quad (41)$$

$$\left\| Im\ \tilde{a}^{(n)}(\vec{k},x)\right\|_{\nu} \leq c_{n,\nu,b}\left\|\vec{A}\right\|_{\nu+n+2}^{n+1} k^{1/2-(1+2n)\beta}, \tag{42}$$

$$\left\| D^{\mu}_{\vec{k}}\tilde{a}^{(n)}(\vec{k},x)\right\|_{\nu} \leq c_{n,\nu,b,\mu}\left\|\vec{A}\right\|_{\nu+n}^{n+1} k^{1-2n\beta-(1/2+\beta)|\mu|}, \tag{43}$$

$$\left\|\tilde{a}^{(n)}(\vec{k},x)-\tilde{a}^{(n)}\left(\vec{k}',x\right)\right\|_{\nu} \leq c_{n,\nu,b}\left\|\vec{A}\right\|_{\nu+n}^{n+1}|\vec{k}-\vec{k}'|k^{1/2-(2n+1)\beta}, \tag{44}$$

if $|\vec{k}-\vec{k}'|<(4b_0)^{-1}k^{1/2+\beta-\varepsilon}$. *The estimates (37) – (42) hold for any* $\nu=0,1,...$ *and* k *large enough,* $k>k_0(\vec{A},\alpha,N,\beta,\varepsilon,\nu)$, *namely if* $k:\ k>\alpha,\ k^{\varepsilon}>2^{N+2}R_0$, $k^{2\beta}>(5b_0)^{N+2+\nu}\|\vec{A}\|_{\nu+N+2}+b_0^2$.

Corollary 1.

$$\left\|S(\vec{k},x)\right\|_{\nu} \leq c_{N,\nu,b}\left\|\vec{A}\right\|_{\nu} k^{1/2-\beta}, \tag{45}$$

$$\left\|\Re S(\vec{k},x)\right\|_{\nu} \leq c_{N,\nu,b}\left\|\vec{A}\right\|_{\nu+2} k^{-2\beta}, \tag{46}$$

$$\left\|\left|e^{S(\vec{k},x)}\right|-1\right\|_{0} < c_{N,b}\left\|\vec{A}\right\|_{2} k^{-2\beta}, \tag{47}$$

$$\left\|S(\vec{k},x)-S\left(\vec{k}',x\right)\right\|_{\nu} \leq c_{N,\nu,b}\left\|\vec{A}\right\|_{\nu}|\vec{k}-\vec{k}'|k^{-2\beta}, \tag{48}$$

if $|\vec{k}-\vec{k}'|<(4b_0)^{-1}k^{1/2+\beta-\varepsilon}$.

Corollary 2.

$$\|\hat{\Psi}(\vec{k},x)\|_{L_2(Q)} =_{k\to\infty} 1+O(k^{-2\beta}). \tag{49}$$

Corollary 2 follows from formula (20) for $\hat{\Psi}(\vec{k},x)$ and (47).

Lemma 2. *Under the conditions of Lemma 1 the function* $S(\vec{k},x)$ *defined by (31) – (35) satisfies the relation (24), where the error term* $\varepsilon_N(x)$ *obeys the estimate*

$$\|\epsilon_N(x)\|_0 < c_{\vec{A},N}k^{1-2(N+2)\beta}; \tag{50}$$

the model number $\hat{\lambda}(\vec{k})$ *is given by the formula*

$$\hat{\lambda}(\vec{k})=k^2+\delta\hat{\lambda}(\vec{k}),\quad \delta\hat{\lambda}(\vec{k})=\sum_{n=1}^{N+1}\tilde{a}_0^{(n)}(\vec{k}), \tag{51}$$

and satisfy the following estimates:

$$\left|\delta\hat{\lambda}(\vec{k}))\right| < c_{N,b}\|\vec{A}\|_1^2 k^{1-2\beta}, \tag{52}$$

$$\left|\nabla_{\vec{k}}\delta\hat{\lambda}(\vec{k}))\right| < c_{N,b}\|\vec{A}\|_1^2 k^{1/2-3\beta}, \tag{53}$$

$$\left|\delta\hat{\lambda}(\vec{k})-\delta\hat{\lambda}(\vec{k}')\right| < c_{N,b}\left\|\vec{A}\right\|_1^2|\vec{k}-\vec{k}'||k^{1/2-3\beta}. \tag{54}$$

if $|\vec{k}-\vec{k}'|<(4b_0)^{-1}k^{1/2+\beta-\varepsilon}$.

From the previous lemma we obtain that $\hat{\Psi}_j(\vec{k},x)$ satisfies an equation for eigenfunctions with a high accuracy.

Lemma 3. *Under the conditions of Lemma 1 the function $\hat{\Psi}(\vec{k},x)$ satisfies the equation:*

$$H\hat{\Psi}(\vec{k},x) = \hat{\lambda}(\vec{k})\hat{\Psi}(\vec{k},x) + \delta\hat{\Psi}(\vec{k},x), \tag{55}$$

where the model value $\hat{\lambda}(\vec{k})$ is defined by (51), $\delta\hat{\Psi}(\vec{k},x)$ obeys the estimate

$$\|\delta\hat{\Psi}(\vec{k},x)\|_0 \le c_{\vec{A},N} k^{1-2(N+2)\beta}. \tag{56}$$

Corollary 3.

$$\left| Im\ \hat{\lambda}(\vec{k}) \right| < c_{\vec{A},N} k^{1-2(N+2)\beta}. \tag{57}$$

Proofs of Lemmas 1 – 3 are in Section 3.1.

Now we define $D(\lambda)$ as the level curve for the function $\Re\hat{\lambda}(\vec{k})$:

$$D(\lambda) = \left\{ \vec{k} : \vec{k} \in \Pi(\beta), \Re\hat{\lambda}(\vec{k}) = \lambda \right\}, \quad \lambda > 0. \tag{58}$$

The next lemma asserts that when λ is large, $D(\lambda)$ is a slightly distorted circle with holes. The deviation of $D(\lambda)$ from the exact circle of radius $\sqrt{\lambda}$ centered at the origin is no larger than $c_{\vec{A}}\lambda^{-\beta}$. The holes come from the restriction $\vec{k} \in \Pi(\beta)$. However, the holes are relatively small; the length of $D(\lambda)$ is relatively close to that of the whole circle. Before formulating the lemma we introduce some notations. Let S_ρ be the circle of the radius $\rho = \sqrt{\lambda}$ centered at the origin and S'_ρ be the circle with holes:

$$S'_\rho = S_\rho \cap \Pi(\beta) =$$
$$\left\{ \vec{k} : |\vec{k}| = \rho,\ \left| |\vec{k} + \vec{p}_r(0)|^2 - |\vec{k}|^2 \right| > k^{1/2+\beta},\ \text{for any } r : 0 < |r| < \rho^\varepsilon \right\}. \tag{59}$$

To understand location of holes in S'_ρ let us rewrite inequality (22) defining $\Pi(\beta)$ in the form:

$$\left| 2\left(\vec{k}, \vec{p}_r(0)\right) + p_r^2(0) \right| > k^{1/2+\beta}. \tag{60}$$

Considering that the equation $2\left(\vec{k}, \vec{p}_r(0)\right) + p_r^2(0) = 0$ describes a line, we conclude that any $\vec{k} \in \Pi(\beta)$ lies outside the $\left(k^{1/2+\beta}/2p_r(0)\right)$-neighborhood of the lines $2\left(\vec{k}, \vec{p}_r(0)\right) + p_r^2(0) = 0$, $0 < |r| < k^\varepsilon$. The situation looks even simpler if we neglect $p_r^2(0)$ in (60). We can do this since $p_r^2(0) < c_b k^{2\varepsilon} = o(k^{1/2+\beta})$. We rewrite (60) in the form:

$$\left| 2\left(\vec{k}, \vec{p}_r(0)\right) \right| > k^{1/2+\beta} \quad \text{or} \quad \left| 2\left(\vec{\nu}, \vec{p}_r(0)\right) \right| > k^{-1/2+\beta}, \quad \nu = \frac{\vec{k}}{k}. \tag{61}$$

We see that if $\vec{k} \in \Pi(\beta)$, then the direction $\vec{\nu}$ of $\vec{k}$ cannot be orthogonal or close to being orthogonal to $\vec{p}_r(0)$. Thus, the set S'_ρ is a sphere where certain directions are excluded together with their neighborhoods. Let S_1 be the unit circle centered at the origin and $D_1(\lambda) \subset S_1$ be the set of directions corresponding to S'_ρ:

$$D_1(\lambda) = \left\{ \vec{\nu} \in S_1 : \rho\vec{\nu} \in S'_\rho \right\}, \quad \rho = \sqrt{\lambda}. \tag{62}$$

Obviously, $D_1(\lambda)$ depends not only on λ but also on β and ε, however, we omit them to keep the notation short. According to (61) $D_1(\lambda)$ approximately consists of vectors $\vec{\nu}$ satisfying the inequality $\left| 2\left(\vec{\nu}, \vec{p}_r(0)\right) \right| > k^{-1/2+\beta}$ for some $r : 0 < |r| < k^\varepsilon$. Considering that the number of r obeying the last inequality does not exceed $c_b k^{2\varepsilon}$, we easily see that $D_1(\lambda)$ has a length asymptotically close to that of a unit circle: $l\left(D_1\right) = 2\pi + O(\lambda^{-(1-2\beta-4\varepsilon)/4})$, $1 - 2\beta - 4\varepsilon > 0$.

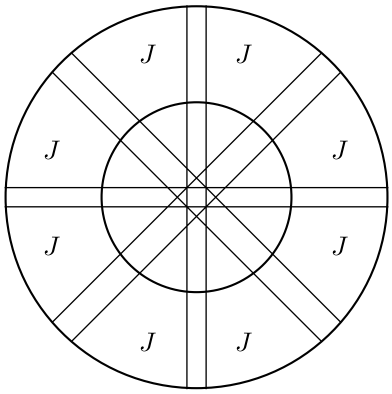

FIGURE 2. The set $J(\lambda_0)$.

Lemma 4. *If $0<\varepsilon<1/4$, $0<\beta<1/2-2\varepsilon$, than for sufficiently large λ, the set $D(\lambda)$ can be described by the formula:*

$$D(\lambda)=\left\{\vec{k}\in R^2:\vec{k}=k(\lambda,\vec{\nu})\vec{\nu},\quad \vec{\nu}\in D_1(\lambda),\quad k(\lambda,\vec{\nu})=\sqrt{\lambda}+h(\lambda,\vec{\nu})\right\}, \qquad (63)$$

function $h(\lambda,\vec{\nu})$ obeying the asymptotic formulae:

$$h(\lambda,\vec{\nu})=_{\lambda\to\infty}O(\lambda^{-\beta}), \qquad (64)$$

$$\frac{\partial h(\lambda,\vec{\nu})}{\partial\theta}=_{\lambda\to\infty}O(\lambda^{(1-6\beta)/4}), \qquad (65)$$

given $\vec{\nu}=(\cos\theta,\sin\theta)$. Furthermore, $D(\lambda)$ has a length which is asymptotically close to that of a circle of the radius $\sqrt{\lambda}$ in the following sense:

$$l\left(D(\lambda)\right)=2\pi\sqrt{\lambda}\left(1+O(\lambda^{-\gamma})\right),\quad \gamma=(1-2\beta-4\varepsilon)/4>0. \qquad (66)$$

The lemma is proved in Section 3.4.

2.2. **The Main Theorems.** Let us define the set of model functions $\hat{\Psi}_j(t,x)$ for an operator $H(t)$, $t\in K$, as follows:

$$\hat{\Psi}_j(t,x)=\hat{\Psi}\left(\vec{p}_j(t),x\right)\ \text{ if } j\in J(\lambda_0), \qquad (67)$$

$\hat{\Psi}\left(\vec{p}_j(t),x\right)$ being defined by (20) for $\vec{k}=\vec{p}_j(t)$, and a set $J(\lambda_0)$ (Fig.2) is defined by the formula:

$$J(\lambda_0)=\{j:\vec{p}_j(t)\in\Pi(\beta):\lambda_0/4<p_j^2(t)<4\lambda_0\}. \qquad (68)$$

We consider λ_0 to be large enough to satisfy the conditions of Lemma 1 for all $\vec{p}_j(t)$, $j\in J(\lambda_0)$. We use the notation:

$$\hat{\lambda}_j(t)=\hat{\lambda}\left(\vec{p}_j(t)\right), \qquad (69)$$

$\hat{\lambda}\left(\vec{p}_j(t)\right)$ being the model number (51) for $\vec{k}=\vec{p}_j(t)$. According to Lemma 3, the functions $\hat{\Psi}_j(t,x)$, $j\in J(\lambda_0)$, obey the equations

$$H(t)\hat{\Psi}_j(t,x)=\hat{\lambda}_j(t)\hat{\Psi}_j(t,x)+\delta\Psi_j(t,x), \qquad (70)$$

where $\|\delta\Psi_j(t,x)\|_0 = O(|j|^{1-2(N+2)\beta})$ for $j \in J(\lambda_0)$. Further, for simplicity, we assume that $1/3 < \beta < 1/2$, $N \geq 31$. Obviously for these ranges of β and N,

$$\|\delta\Psi_j(t,x)\|_0 = O(\lambda_0^{-21/2}). \tag{71}$$

It follows from (70) and (71) that

$$\Im\hat{\lambda}_j(t) = O(\lambda_0^{-21/2}). \tag{72}$$

Let us consider the level set $\chi(\lambda)$ of the functions $\Re\hat{\lambda}_j(t)$, $j \in J(\lambda_0)$:

$$\chi(\lambda) = \{t \in K : \exists j, j \in J(\lambda_0),\ \Re\hat{\lambda}_j(t) = \lambda\}, \quad \lambda_0/2 < \lambda < 2\lambda_0. \tag{73}$$

It is easy to see that

$$\chi(\lambda) = \mathcal{K}D(\lambda), \tag{74}$$

here $\mathcal{K} : R^2 \to K$ is the map:

$$\mathcal{K}\vec{p}_j(t) = t,$$

the map being correctly defined, since any $\vec{k} \in R^2$ can be represented in the form $\vec{k} = \vec{p}_j(t)$ in a unique way. Obviously, to obtain $\chi(\lambda)$ we break $D(\lambda)$ by the lattice with nodes in $\{\vec{p}_r(0)\}_{r\in Z^2}$ into pieces and then shift them in a parallel manner into K. Thus, $\chi(\lambda)$ is the set $D(\lambda)$ packed into the bag K.

We call $\chi(\lambda)$ a model isoenergetic set.

Theorem 2. *Let $0 < \varepsilon < 1/100$, $1/3 < \beta < 1/2 - 6\varepsilon$, $0 < \delta < \min\{1/2 - \beta - 6\varepsilon, \varepsilon\}$, $N \geq 31$.* [2]*. Then for any sufficiently large λ, $\lambda > \lambda_0(\vec{A}(x), \beta, \varepsilon, \delta)$, there exists a subset $\chi_*(\lambda, \delta)$ of $\chi(\lambda)$ with the following properties:*

1) For every $t \in \chi_*(\lambda, \delta)$ there is a single $j \in J(\lambda_0)$ such that $\Re\hat{\lambda}_j(t) = \lambda$, the following inequality being satisfied:

$$\min_{m\in J(\lambda_0), m\neq j} |\Re\hat{\lambda}_j\,(t) - \Re\hat{\lambda}_m\,(t)\,| > 2\lambda^{-\delta}. \tag{75}$$

The operator $H(t)$ has a simple eigenvalue $\lambda_j(t)$ close to $\Re\hat{\lambda}_j(t) = \lambda$,

$$\lambda_j(t) = \Re\hat{\lambda}_j\,(t) + O(\lambda^{-21/2}). \tag{76}$$

This eigenvalue is unique: there are no other eigenvalues of $H(t)$ in the interval $\epsilon(\lambda) = (\lambda - \lambda^{-\delta}, \lambda + \lambda^{-\delta})$.

2) Formula (76) is differentiable with respect to t:

$$\nabla_t\lambda_j(t) = \nabla_t\Re\hat{\lambda}_j(t) + O(\lambda^{-19/2+2\delta}). \tag{77}$$

3) An eigenfunction $\Psi_j(t,x)$ corresponding to $\lambda_j(t)$ is close to the model function $\hat{\Psi}_j(t,x)$:

$$\left\|\Psi_j(t,x) - \hat{\Psi}_j(t,x)\right\|_{L_2(Q)} = O(\lambda^{-10+2\delta}). \tag{78}$$

4) The results 1–3 are stable in the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_*(\lambda,\delta)$: if t belongs to the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_*(\lambda,\delta)$, then there is unique j: $|\hat{\lambda}_j(t) - \lambda| < 4\lambda^{-2\delta}$, the estimate (75) being satisfied. The operator $H(t)$ has a simple eigenvalue $\lambda_j(t)$ obeying (76). This eigenvalue is unique: there are no other eigenvalues of $H(t)$ in the interval $\epsilon(\lambda)$, the estimates (77) and (78) being true.

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[2] This is not the optimal set of parameters. For a example, an analogous theorem can be proved for smaller β. However, it would require longer considerations.

5) The set $\chi_*(\lambda,\delta)$ has a measure asymptotically close to that of a circle of the radius $\sqrt{\lambda}$ in the following sense:

$$\frac{l\left(\chi_*(\lambda,\delta)\right)}{2\pi\sqrt{\lambda}} > 1 - c_{\vec{A}}\lambda^{-\delta/4}, \quad c_{\vec{A}} > 0. \tag{79}$$

Corollary 4. *Under conditions of Theorem 2*

$$\nabla_t \lambda_j(t) = \vec{p}_j(t) + O(\lambda^{(1-6\beta)/4}). \tag{80}$$

Formula (80) follows from (77) and (53), where $k = p_j(t) \approx \sqrt{\lambda}$.

An isoenergetic set $I(\lambda)$ of H is defined as the set of t for which $H(t)$ has an eigenvalue equal to a given λ:

$$I(\lambda) = \left\{t \in K : \exists \lambda_n(t),\ n \in N, \lambda_n(t) = \lambda\right\}.$$

We denote by $I_*(\lambda)$ the part of $I(\lambda)$ which is in the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_*(\lambda,\delta)$.

Corollary 5. *For every $t \in I_*(\lambda)$ there is a single $j \in J(\lambda_0)$ such that $\lambda_j(t) = \lambda$.*

Proof of the corollary. If $t \in I_*(\lambda)$, then there is an eigenvalue $\lambda_n(t)$ of the operator $H(t)$ such that $\lambda_n(t) = \lambda$. Since t is in the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_*(\lambda,\delta)$, by the forth part of Theorem 2, there is a unique $j \in J(\lambda_0)$ such that $\lambda_j(t)$ is given by (76) and $|\Re\hat{\lambda}_j(t) - \lambda| < 4\lambda^{-2\delta}$. Since $\lambda_j(t)$ is a unique eigenvalue of $H(t)$ in the interval $\varepsilon(\lambda)$, we conclude that $\lambda_n(t) = \lambda_j(t)$. $\diamondsuit$

Since $\chi_*(\lambda,\delta) \subset \chi(\lambda) = \mathcal{K}D$ (see (74)), there is a subset $D_*(\lambda,\delta)$ of $D(\lambda)$ such that $\chi_* = \mathcal{K}D_*$. Obviously, $D_*(\lambda,\delta)$ is also a distorted sphere with holes, it just has more holes than $D(\lambda)$. Note that $D_*(\lambda,\delta)$ is defined uniquely, since, by (75), $\chi_*(\lambda,\delta)$ does not have selfintersections. Hence, $l(D_*) = l(\chi_*)$ and by Theorem 2:

$$\frac{l\left(D_*(\lambda,\delta)\right)}{2\pi\sqrt{\lambda}} > 1 - c_{\vec{A}}\lambda^{-\delta/4}, \quad c_{\vec{A}} > 0. \tag{81}$$

Let $D_{*1}(\lambda,\delta)$ be the set of directions corresponding to $D_*(\lambda,\delta)$:

$$D_{*1}(\lambda,\delta) = \{\vec{\nu} \in D_1(\lambda) : k(\lambda,\vec{\nu})\vec{\nu} \in D_*(\lambda,\delta)\}, \tag{82}$$

$D_1(\lambda)$ being defined by (62). In the next theorem we show that $I_*(\lambda)$ can be described as the shift $\mathcal{K}$ of a distorted sphere which is close to $D_*(\lambda,\delta)$.

Theorem 3. *Under conditions of Theorem 2, the set $I_*(\lambda,\delta)$ can be described by the formula $I_*(\lambda) = \mathcal{K}\tilde{D}_*(\lambda)$, where $\tilde{D}_*(\lambda)$ is a distorted sphere with holes:*

$$\tilde{D}_*(\lambda) = \{\vec{k} \in R^2 : \vec{k} = \tilde{\rho}(\lambda,\vec{\nu})\vec{\nu}, \vec{\nu} \in D_{*1}(\lambda,\delta), \tilde{\rho}(\lambda,\vec{\nu}) = \sqrt{\lambda} + h(\lambda,\vec{\nu}) + \tilde{h}(\lambda,\vec{\nu})\}, \tag{83}$$

function $\tilde{h}(\lambda,\vec{\nu})$ satisfying the estimates:

$$\tilde{h}(\lambda,\vec{\nu}) = O(\lambda^{-11}), \tag{84}$$

$$\frac{\partial \tilde{h}(\lambda,\vec{\nu})}{\partial\theta} = O(\lambda^{-19/2+2\delta}), \tag{85}$$

given $\vec{\nu} = (\cos\theta, \sin\theta)$. The set $\tilde{D}_*(\lambda)$ has a lengths which is asymptotically close to that of a circle of radius $\sqrt{\lambda}$ in the following sense:

$$\frac{l\left(\tilde{D}_*(\lambda)\right)}{2\pi\sqrt{\lambda}} > 1 - c_{\vec{A}}\lambda^{-\delta/4}, \quad c_{\vec{A}} > 0. \tag{86}$$

The parallel shift $\mathcal{K}$ establishes one-to-one correspondence between $\tilde{D}_*$ and I_* and between their $(\lambda^{-1/2-2\delta})$-neighborhoods in R^2 and K, respectively. The length of I_* is equal to the length of $\tilde{D}_*(\lambda)$.

Let $\vec{n}_{\tilde{D}_*}(\vec{\nu})$ and $\vec{n}_{D_*}(\vec{\nu})$ be normal vectors to $\tilde{D}_*$ and D_*, respectively, corresponding to the same $\vec{\nu}$. We consider their direction to be correlated, i.e., the angle between $\vec{n}_{\tilde{D}_*}(\vec{\nu})$ and $\vec{n}_{D_*}(\vec{\nu})$ is no large than $\pi/2$. Since $D_* \subset D$, D being the level set for the function $\hat{\lambda}(\vec{k})$, the normal vector to D_* can be defined by the formula

$$\vec{n}_{D_*}(\vec{\nu}) = \left.\frac{\nabla\hat{\lambda}(\vec{k})}{\left|\nabla\hat{\lambda}(\vec{k})\right|}\right|_{\vec{k}=k(\lambda,\nu)\vec{\nu}} = \vec{\nu} + O(\lambda^{-(1+6\beta)/4}),$$

the last estimate following from (53).

Corollary 6. *The set $\tilde{D}_*(\lambda,\delta)$ is situated in the $\left(c_{\vec{A}}\lambda^{-11}\right)$-neighborhood of $D_*(\lambda)$ and in the $(c_{\vec{A}}\lambda^{-\beta})$-neighborhood of $S'_{\sqrt{\lambda}}$. The normal vector $\vec{n}_{\tilde{D}_*}(\vec{\nu})$ to $\tilde{D}_*(\lambda,\delta)$ satisfies the estimate:*

$$\vec{n}_{\tilde{D}_*}(\vec{\nu}) = \vec{n}_{D_*}(\vec{\nu}) + O\left(\lambda^{-10+2\delta}\right) = \vec{\nu} + O(\lambda^{-(1+6\beta)/4}). \tag{87}$$

Since, there is one to one correspondence between $I_*(\lambda)$ and $\tilde{D}_*(\lambda)$, $I_*(\lambda)$ can be also parameterized by $\vec{\nu}$, $\vec{\nu} \in D_{*1}(\lambda,\delta)$, a normal vector $\vec{n}_{I_*}(\vec{\nu})$ being defined by the formula $\vec{n}_{I_*}(\vec{\nu}) = \vec{n}_{\tilde{D}_*}(\vec{\nu})$, since $I_*(\lambda)$ is obtained from $\tilde{D}_*$ by the parallel shift $\mathcal{K}$. Thus, the following corollary from Theorem 3 is valid.

Corollary 7. *The set $I_*(\lambda)$ is situated in the $(c_{\vec{A}}\lambda^{-11})$-neighborhood of $\chi_*(\lambda,\delta)$ and in the $(c_{\vec{A}}\lambda^{-\beta})$-neighborhood of $\mathcal{K}S'_{\sqrt{\lambda}}$. The normal vector $\vec{n}_{I_*}(\vec{\nu})$ satisfies the estimate:*

$$\vec{n}_{I_*}(\vec{\nu}) = \vec{n}_{\chi_*}(\vec{\nu}) + O\left(\lambda^{-10+2\delta}\right) = \vec{\nu} + O(\lambda^{-(1+6\beta)/4}). \tag{88}$$

The curve $I_*(\lambda)$ has the length which is asymptotically close to that of a circle of radius $\sqrt{\lambda}$ in the following sense:

$$\frac{l\left(I_*(\lambda)\right)}{2\pi\sqrt{\lambda}} > 1 - c_{\vec{A}}\lambda^{-\delta/4}, \quad c_{\vec{A}} > 0. \tag{89}$$

Theorem 4. (The validity of the Bethe-Sommerfeld conjecture) *The operator H has only a finite number of gaps in the spectrum.*

The validity of the Bethe-Sommerfeld conjecture is a simple consequence of Corollary 7. Existence of an isoenergetic set for a given λ implies that λ belongs to the spectrum. Thus, by Corollary 7, every large enough λ belongs to the spectrum, since the corresponding isoenergetic set exists and rather large (see (89)). This immediately implies finiteness of the number of gaps in the spectrum, since only a finite number of zones (and gaps) can be located below certain energy [RS].

Now we sketch a proof of Theorem 2. We consider the system of model functions and show (Lemma 6) that the system is almost orthogonal: its Gramm matrix $G_{jm} =< \hat{\Psi}_m, \hat{\Psi}_j >$ satisfies the estimate $\|G - I\| = o(1)$ as $\lambda \to \infty$. Thus, $\{\hat{\Psi}_j\}_{j\in J(\lambda_0)}$ is a basis-like system in its linear span. Another important property of model functions is that its span essentially contains all the exponents $\Psi_j^0(t,x)$, $j \in J'(\lambda_0)$, here $J'(\lambda_0) \subset J(\lambda_0)$, $J'(\lambda_0)$ being similar to $J(\lambda_0)$:

$$J'(\lambda_0) = \left\{j : p_j(0) \in \Pi(1/2 - 6\varepsilon),\ \lambda_0/2 < p_j^2(0) < 3\lambda_0/2\right\}. \tag{90}$$

By "essentially contains" we mean that every plane wave $\Psi_j^0(t,x)$, $j\in J'(\lambda_0)$, can be represented as a linear combination of model functions with a high accuracy:

$$\Psi_j^0(t,x)=\sum_{m\in J(\lambda_0)} c_{mj}\hat\Psi_m(t,x)+O(\lambda_0^{-10}). \tag{91}$$

This result formulated in terms of projections P and P_0 on the linear spans of model functions $\{\hat\Psi\}_{j\in J(\lambda_0)}$ and exponents $\{\Psi_j^0(t,x)\}_{j\in J'(\lambda_0)}$, respectively, constitutes Lemma 7. The property (91) looks natural if we recall that for $\vec A=0$, the set $\{\hat\Psi\}_{j\in J(\lambda_0)}$ consists just from the exponents $\{\Psi^0\}_{j\in J(\lambda_0)}$. The property (91) means that when $\vec A\neq 0$ the functions $\{\hat\Psi\}_{j\in J(\lambda_0)}$ to some extend inherit the above property of the unperturbed system. Two described properties of model functions are in the core of the proof of Theorem 2. Using model functions we introduce a family of model operators $\hat H(t)$ acting in $L_2(Q)$, $t\in K$. An operator $\hat H(t)$ is defined as follows. Let P be the projection on the linear span of model functions. The action of $\hat H(t)$ on $PL_2(Q)$ is defined by the formula:

$$\hat H(t)\hat\Psi_m(t,x)=\left(\Re\hat\lambda_m(t)\right)\hat\Psi_m(t,x),\quad m\in J(\lambda_0); \tag{92}$$

and the action of $\hat H(t)$ on $(I-P)L_2(Q)$ is defined as:

$$\hat H(t)(I-P)=(I-P)H(I-P). \tag{93}$$

Basing on (70) and (71), we show that

$$\|H(t)-\hat H(t)\|<c\lambda_0^{-10} \tag{94}$$

for any $t\in K$. Further we consider H as a small perturbation of $\hat H(t)$. We denote eigenvalues of $(I-P)H(I-P)$ by $\hat\lambda_m(t)$, $m\in Z^2\backslash J(\lambda_0)$. Obviously, $\hat\lambda_m(t)=\Re\hat\lambda_m(t)$ for $j\in Z^2\setminus J(\lambda_0)$.

We define the set $\chi_*(\lambda,\delta)$, $\delta>0$, as follows: for any $t\in\chi_*(\lambda,\delta)$ there is an eigenvalue $\Re\hat\lambda_j(t)$, $j\in J(\lambda_0)$ of $\hat H(t)$ equal to λ, and all other eigenvalues of $\hat H(t)$ are separated from $\Re\hat\lambda_j(t)$ by the distance greater than $4\lambda^{-\delta}$:

$$\chi_*=\left\{t\in K:\exists j\in J(\lambda_0):\Re\hat\lambda_j(t)=\lambda,\ \min_{m\in Z^2,m\neq j}\left|\Re\hat\lambda_j(t)-\Re\hat\lambda_m(t)\right|>4\lambda^{-\delta}\right\}. \tag{95}$$

Inequalities (76) – (78) follow by standard perturbation considerations from estimate (94) the definition of the set $\chi_*(\lambda,\delta)$. They are proved in Section 3.5.

The next important step is to prove that $\chi_*(\lambda,\delta)$ is not empty, moreover it has a measure of order of $2\pi\sqrt{\lambda}$. For this we need to describe properties of $\hat\lambda_m(t)$ as functions of t for all $m\in Z^2$. Properties of $\hat\lambda_m(t)$ when $m\in J(\lambda_0)$ are given in Lemma 2 where $\vec k=\vec p_m(t)$, the properties of eigenvalues $\hat\lambda_m(t)$, $m\in Z^2\setminus J(\lambda_0)$, of the operator $(I-P)H(I-P)$ are investigated in Section 3.3. Basing on this, we prove (Section 3.4) the following lemma:

Lemma 5. *The set $\chi_*(\lambda,\delta)$ has a length close to that of a circle of radius $\sqrt{\lambda}$ in the following sense:*

$$\frac{l\left(\chi_*(\lambda,\delta)\right)}{2\pi\sqrt{\lambda}}>1-c_{\vec A}\lambda^{-\delta/4},\quad c_{\vec A}>0. \tag{96}$$

Theorem 3 readily follows from Theorem 2 and Implicit Function Theorem. Theorem 4 easily follows from Theorem 3 (for explanations see Section 3.5.)

3. Proof of the Main Results.

3.1. **Construction of Model Functions.** In this section we prove Lemmas 1–3 and two main properties of eigenfunctions: first, that they form a basis-like system in its linear span (Lemma 6) and, second, is that the linear span essentially contains all the exponents $e^{i(\vec{p}_j(t),x)}$, $j \in J(\lambda_0)$ (Lemma 7).

Proof of Lemma 1. To shorten notations we denote in this lemma $S^{(n)}(\vec{k},x)$ and $\tilde{a}^{(n)}(\vec{k},x)$ just by $S^{(n)}(x)$ and $\tilde{a}^{(n)}(x)$. By (5), $\vec{A}_r = 0$ when $|r| > R_0$. From the definition (30) of $S^{(0)}(x)$ we get $S_r^{(0)} = 0$ when $|r| > R_0$. It follows now from (33) that $\tilde{a}_r^{(1)} = 0$ when $|r| > 2R_0$. Using (34) we immediately obtain $S_r^{(1)} = 0$ when $|r| > 2R_0$. Further, using (35) and (34) we obtain (36) for all n by mathematical induction.

Next we prove the pair of estimates (37) and (41) by mathematical induction. Using (22) and the inequality $2^{N+2}R_0 < k^{\varepsilon}$, given in the statement of the lemma, we get

$$\left| |\vec{k} + \vec{p}_r(0)|^2 - |\vec{k}|^2 \right| > k^{1/2+\beta} \text{ for all } r \in Z^2, 0 < |r| \le 2^{N+2}R_0. \tag{97}$$

Using the definition (30) of $S^{(0)}(x)$, inequality (97) and the estimate $|\alpha| \le k$, we easily obtain:

$$\|S^0(x)\|_\nu < c\|\vec{A}\|_\nu k^{1/2-\beta}, \tag{98}$$

i.e., (37) for $n = 0$. Next, we get (41) for $\tilde{a}^{(1)}(x)$. In fact, it is easy to see that for any $\vec{f}, \vec{g} : Q \to R^2$

$$\left\| \left(\vec{f}(x), \vec{g}(x) \right) \right\|_\nu \le 2^\nu \|\vec{f}\|_\nu \|\vec{g}\|_\nu. \tag{99}$$

Applying (99) in (33) we obtain:

$$\|\tilde{a}^1(x)\|_\nu \le 2^\nu \left(2\|\nabla S^0(x)\|_\nu \|\vec{a}(x)\|_\nu + \|\nabla S^{(0)}(x)\|_\nu^2 + \|\vec{a}(x)\|_\nu^2 \right).$$

Using (98) and considering that

$$\|\nabla S^{(0)}(x)\|_\nu \le 2b_0 \|S^{(0)}(x)\|_{\nu+1}, \tag{100}$$

we get (41) for $\tilde{a}^1(x)$. It follows from (34) and (97) that for all $\nu \ge 0$ and $n = 1, ..., N+1$,

$$\|S^{(n)}(x)\|_\nu \le \|\tilde{a}^{(n)}(x)\|_\nu k^{-1/2-\beta}. \tag{101}$$

Formulae (35) and (99) yield:

$$\|\tilde{a}^{(n)}(x)\|_\nu \le$$

$$2^{\nu+1} \left(\|\nabla S^{(n-1)}(x)\|_\nu \|\vec{a}(x)\|_\nu + \|\nabla S^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S^{(s)}(x)\|_\nu \right)$$

for any $n \ge 2$. Using (100) and (101), we obtain

$$\|\tilde{a}^{(n)}(x)\|_\nu \le 2^{\nu+1} b_0 k^{-1/2-\beta} \|\tilde{a}^{(n-1)}(x)\|_{\nu+1} \times$$

$$\left(\|\vec{a}(x)\|_\nu + 2b_0 \|S^{(0)}(x)\|_{\nu+1} + 2b_0 k^{-1/2-\beta} \sum_{s=1}^{n-1} \|\tilde{a}^{(s)}(x)\|_{\nu+1} \right).$$

Suppose (41) holds for $s = 1, ..., n-1$. Using (37) for $n = 0$ and (41) for $s = 1, ..., n-1$ on the right-hand side of the last inequality, we obtain

$$\|\vec{a}\|_\nu + 2b_0\|S^{(0)}(x)\|_{\nu+1} + 2b_0k^{-1/2-\beta}\sum_{s=1}^{n-1}\|\tilde{a}^{(s)}(x)\|_{\nu+1} \leq cb_0k^{1/2-\beta}\|\vec{A}\|_{\nu+n},$$

the term $2b_0\|S^{(0)}(x)\|_{\nu+1}$ being the biggest on the left-hand side when k is large enough, namely when $k^{2\beta} > (5b_0)^{N+\nu+2}\|\vec{A}\|_{N+\nu+2} + b_0^2$.

Thus, $\|\tilde{a}^{(n)}(x)\|_\nu < c\cdot 2^\nu b_0^2 k^{-2\beta}\|\vec{A}\|_{\nu+1}\|\tilde{a}^{(n-1)}(x)\|_{\nu+1}$. Using (41) for $\tilde{a}^{(n-1)}(x)$, we get (41) for $\tilde{a}^{(n)}(x)$. Estimate (37) follows from (41) and (101).

Let us prove the pair (38) and (42). By elementary calculation we obtain from (30):

$$\Re S^{(0)}(x) = -\sum_{r\in Z^2\setminus\{0\}}\frac{\left(2(\vec{k},\vec{a}_r) + \alpha a_{0r}\right)p_r^2(0)}{(|\vec{k}+\vec{p}_r(0)|^2 - |\vec{k}|^2)(|\vec{k}-\vec{p}_r(0)|^2 - |\vec{k}|^2)}e^{i(\vec{p}_r(0),x)}, \quad (102)$$

The estimate (38) for $n = 0$ easily follows from the last formula and (97). Using this estimate and the formula (33) for $\tilde{a}_j^{(1)}$, and taking into account (37) for $n = 0$, it is not difficult to show that $\|\Im\tilde{a}^{(1)}\|_\nu \leq c_{n,\nu,b}\|\vec{A}\|_{\nu+3}^2 k^{1/2-3\beta}$, i.e., (42) holds for $n = 1$. Formula (34) for $S(x)$ yields

$$\|\Re S^{(n)}\|_\nu < \|\tilde{a}^{(n)}\|_{\nu+2}k^{-1-2\beta} + \|\Im\tilde{a}^{(n)}\|_\nu k^{-1/2-\beta}.$$

Using estimate (41), we obtain:

$$\|\Re S^{(n)}\|_\nu < c_{n,\nu}\|A\|_{\nu+n+2}^{n+1}k^{-2\beta(n+1)} + \|\Im\tilde{a}^{(n)}\|_\nu k^{-1/2-\beta}. \quad (103)$$

Formulae (35) and (99) yield:

$$\|\Im\tilde{a}^{(n)}(x)\|_\nu \leq$$

$$2^{\nu+1}\left(\|\Re\nabla S^{(n-1)}(x)\|_\nu\|\vec{a}\|_\nu + \|\Re\nabla S^{(n-1)}(x)\|_\nu\sum_{s=0}^{n-1}\|\nabla S^{(s)}(x)\|_\nu + \right. \quad (104)$$

$$\left. \|\nabla S^{(n-1)}(x)\|_\nu\sum_{s=0}^{n-2}\|\Re\nabla S^{(s)}(x)\|_\nu\right).$$

Suppose (38) holds for $n \leq s-1$, and (42) for $n \leq s$, $s \geq 1$. Then, using (103), we get that (38) holds for n=s, and using (104), we get (42) for $n = s+1$.

Suppose $\vec{k}_0 \in \Pi(\beta)$. Then, obviously, $S^{(n)}(\vec{k}, x)$ is an analytic function of k_1 in the disc $|k_1 - k_{01}| < \frac{1}{4}k^{1/2+\beta}$, the estimate (37) holding in the whole disk. Therefore, $|D^\mu S^{(n)}(\vec{k}_0, x)|_\nu < c_{n,\nu,b,\mu}\|\vec{A}\|_{\nu+n}^{n+1}k_0^{1/2-\beta(1+2n)-(1/2+\beta)|\mu|}$, $\mu = (\mu_1, 0)$. It is easy to generalize the last estimate for $\mu = (\mu_1, \mu_2)$ using an analytic continuation with respect to k_2. Since $\vec{k}_0$ is any vector in $\Pi(\beta)$, we have (39). The estimate (43) can be proved in the analogous way.

To prove (40) we notice that every vector $\vec{k} + \tau(\vec{k}' - \vec{k})$, $0 \leq \tau \leq 1$, satisfies the estimate of the type (97), since $\vec{k}' - \vec{k}$ is sufficiently small. Using (39) for $|\mu| = 1$, we easily obtain $\|D_\tau S^{(n)}\left(\vec{k} + \tau(\vec{k}' - \vec{k})\right)\|_\nu < c_{n,\nu,b}\|\vec{A}\|_{\nu+n}^{n+1}k^{-2\beta(1+n)}|\vec{k}' - \vec{k}|$. Integrating this estimate with respect to τ, we get (40). Obviously, (44) can be proven in the analogous way. ♢

Proof of Lemma 2. Formula (30) and a direct calculation yield:

$$2i\left(\vec{k},\nabla S^{(0)}(x)\right)+\Delta S^{(0)}(x)-2(\vec{k},\vec{a}(x))-\alpha a_0(x)=0 \tag{105}$$

Using the definition of $S^{(1)}(x)$ (see (32) and (33)) it is easy to check that

$$2i\left(\vec{k},\nabla S^{(1)}(x)\right)+\Delta S^{(1)}(x)+2i(\nabla S^{(0)}(x),\vec{a}(x))+\left(\nabla S^{(0)}(x),\nabla S^{(0)}(x)\right)-$$
$$(\vec{a}(x),\vec{a}(x))=\tilde{a}_0^1, \tag{106}$$

$\tilde{a}_0^1$ being the zero-th Fourier coefficient of $\tilde{a}^1(x)$. We have got $\tilde{a}_0^1$ because the term corresponding to $r=0$ is omitted in formula (32). Formulae (34) and (35) easily lead to the equation for $S^{(n)}(x)$:

$$2i\left(\vec{k},\nabla S^{(n)}(x)\right)+\Delta S^{(n)}(x)+2i\left(\nabla S^{(n-1)}(x),\vec{a}(x)\right)$$
$$+\left(\sum_{s=0}^{n-1}\nabla S^{(s)}(x),\sum_{s=0}^{n-1}\nabla S^{(s)}(x)\right)-\left(\sum_{s=0}^{n-2}\nabla S^{(s)}(x),\sum_{s=0}^{n-2}\nabla S^{(s)}(x)\right)=\tilde{a}_0^{(n)}. \tag{107}$$

Adding formulae (105) – (107) for $n=0,1,2,...N+1$ and using (37) for $n=N+1$ and $\nu=1$, we get (24), (25) and (51).

Summarizing estimates (41) for $n=0,...,N+1$, we obtain (52). Similarly, we get (53) and (54) from (43) and (44), respectively. $\diamondsuit$

Proof of Lemma 3 We prove formula (55) substituting (20) into the equation and using (24). To get (56) from (25) we use (47). $\diamondsuit$

3.2. **Properties of the set of model functions.** Let G be the Gramm matrix of the system $\{\Psi_j(x)\}_{j\in J(\lambda_0)}$, and I_J be the identity matrix for $j\in J(\lambda_0)$:

$$G_{jm}=<\hat{\Psi}_m,\hat{\Psi}_j>,\ \ (I_J)_{jm}=\delta_{jm},\ j,m\in J(\lambda_0).$$

Lemma 6. *Let $0<\beta<1/2$, $0<2\varepsilon<\min\{1/2-\beta,4\beta\}$. Then, $G=I_J+\delta G$, where δG obeys the estimate:*

$$\|\delta G\|<C_{\|A\|_7}\lambda_0^{-\gamma_1},\quad \gamma_1=\beta-\varepsilon/2>0. \tag{108}$$

Proof. Obviously,

$$<\hat{\Psi}_m,\hat{\Psi}_j>=\frac{1}{|Q|}\int_Q\exp\left(i(\vec{p}_m(t)-\vec{p}_j(t),x)+S_m(x)+\bar{S}_j(x)\right)dx. \tag{109}$$

First we consider $m=j$. Clearly,

$$<\hat{\Psi}_j,\hat{\Psi}_j>=\frac{1}{|Q|}\int_Q\exp\left(2\Re S_j(x)\right)dx=1+O\left(\lambda_0^{-\beta}\right)=1+O\left(\lambda_0^{-\gamma_1}\right), \tag{110}$$
$$\left|O\left(\lambda_0^{-\gamma_1}\right)\right|<C_{\|A\|_2}\lambda_0^{-\gamma_1},$$

here (47) and the relation $2p_j(t)>\sqrt{\lambda_0}$ have been used. Suppose $m\neq j$. Let us consider $S_m(x)+\bar{S}_j(x)$. First, we check that

$$\left\|S_m(x)+\bar{S}_j(x)\right\|_\nu<C_{\|A\|_{\nu+2}}|\vec{p}_m(t)-\vec{p}_j(t)|\lambda_0^{-\gamma_1}. \tag{111}$$

Without the loss of generality we may assume $p_j(t)\geq p_m(t)$. If $|\vec{p}_m(t)-\vec{p}_j(t)|<(4b_0)^{-1}p_j(t)^{1/2+\beta-\varepsilon}$, then (111) follows from (46) and (48). If $|\vec{p}_m(t)-\vec{p}_j(t)|\geq(4b_0)^{-1}p_j(t)^{1/2+\beta-\varepsilon}$, then we estimate $\|S_m(x)\|_\nu$ and $\left\|\bar{S}_j(x)\right\|_\nu$ using (45). Applying the triangle inequality, we obtain:

$$\left\|S_m(x)+\bar{S}_j(x)\right\|_\nu<C_{\|A\|_{\nu+2}}p_j(t)^{1/2-\beta}<C_{\|A\|_{\nu+2}}|\vec{p}_m(t)-\vec{p}_j(t)|\lambda_0^{-\gamma_1}. \tag{112}$$

Thus, (111) is proved. Second, we represent integral (109) in the form $\int_Q vDudx$, where u is just the integrand itself, D is the derivative in the direction of $\vec{p}_m(t)-\vec{p}_j(t)$ and

$$v=\frac{1}{|\vec{p}_m(t)-\vec{p}_j(t)|+D\left(S_m+\bar{S}_j\right)(x)}.$$

Integrating (109) by parts and considering (111), we arrive at inequality:

$$<\hat{\Psi}_m,\hat{\Psi}_j> < \ C_{\|A\|_4}\lambda_0^{-\gamma_1}|\vec{p}_m(t)-\vec{p}_j(t)|^{-1}. \tag{113}$$

Repeating integration by parts four times yields:

$$<\hat{\Psi}_m,\hat{\Psi}_j> < \ C_{\|A\|_7}\lambda_0^{-\gamma_1}|\vec{p}_m(t)-\vec{p}_j(t)|^{-4}, \tag{114}$$

Therefore,

$$\sup_j\sum_{m\neq j}|<\hat{\Psi}_m,\hat{\Psi}_j>|<c_{\|A\|_7}\lambda_0^{-\gamma_1}. \tag{115}$$

Estimate (108) follows from (110) and (115). $\diamondsuit$

Let P_0 be the projection on the linear span of the exponents $\Psi_j^0(x)$, $j\in J'(\lambda_0)$, here $\Psi_j^0(x)$ are defined by (9). We recall that P is the projection on the linear span of model functions $\hat{\Psi}_j$, $j\in J(\lambda_0)$.

Lemma 7. *If $0<\varepsilon<1/100$, $0<\beta<1/2-6\varepsilon$, then*

$$PP_0=P_0+\delta P,\quad \|\delta P\|=O(\lambda_0^{-10}). \tag{116}$$

Corollary 8.

$$P_0P=P_0+\delta P^*,\quad \|\delta P^*\|=O(\lambda_0^{-10}). \tag{117}$$

Proof. We prove that each $\Psi_j^0(x)$, $j\in J'$, can be represented as a linear combination of $\hat{\Psi}_m$, $m\in J$ with a high accuracy:

$$\Psi_j^0(x)=\sum_{m\in J}c_{jm}\hat{\Psi}_m(x)+\epsilon_j(x), \tag{118}$$

$$\|\epsilon_j\|_0=O(\lambda_0^{-11}). \tag{119}$$

Considering that the set J contains no more than $c\lambda_0$ elements, we easily deduce (116) from (118) and (119). It is enough to prove these formulae for $\beta\geq 1/2-7\varepsilon$ since the set $J(\lambda_0)$ is getting larger when β is decreasing. Let us prove (118) and (119). We do this in three steps.

Step 1. First we prove that each $\Psi_j^0(x)$, $j\in J'$, can be represented as a linear combination of $\hat{\Psi}_m(x)$ with a good accuracy. Namely, for any $j\in J'$ and a $\mu>6\varepsilon$ there is a set of complex numbers $\{c_{jm}\}_{m:|j-m|<\lambda_0^{\mu}}$ such that

$$\Psi_j^0(x)=\sum_{m:|m-j|<\lambda_0^{\mu}}c_{jm}\hat{\Psi}_m(x)+\epsilon_j(x), \tag{120}$$

$$\|\epsilon_j\|_{L_2(Q)}=O(\lambda_0^{-\gamma_2}),\quad 2\gamma_2=\min\{1/2-6\varepsilon-2\mu,200\mu\}. \tag{121}$$

Obviously, the estimate is trivial for $2\mu\geq 1/2-6\varepsilon$. We assume further that $2\mu<1/2-6\varepsilon$. To prove (120), (121) it suffices to check that the projection of $\Psi_j^0(x)$ on the linear span of model functions $\hat{\Psi}_m$, $m:|m-j|<\lambda_0^{\mu}$, has the $L_2(Q)$-norm greater than $1-c\lambda_0^{-2\gamma_2}$. Since $\vec{p}_j(0)\in\Pi(1/2-6\varepsilon)$ and $\mu<1/2-6\varepsilon$, estimate (108) with $\beta=1/2-6\varepsilon$ can be applied to the Gram matrix of the system

$\{\hat{\Psi}_m\}_{m:|m-j|<\mu}$, i.e., $\|\delta G\| < c_A \lambda_0^{-1/2+13\varepsilon/2}$. Hence, (120) and (121) follow from the inequality

$$\sum_{m:|m-j|<\lambda_0^{\mu}} | < \hat{\Psi}_m, \Psi_j^0 > |^2 > 1 - c_{\vec{A},\mu} \lambda_0^{-2\gamma_2}, \tag{122}$$

because $2\gamma_2 < 1/2 - 13\varepsilon/2$. Let us prove (122). Obviously,

$$< \hat{\Psi}_m(x), \Psi_j^0(x) >= |Q|^{-1} \int_Q e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)+S_m(x)} dx.$$

We replace $S_m(x)$ by $-\bar{S}_j(x)$, with a small error following :

$$< \hat{\Psi}_m(x), \Psi_j^0(x) >= |Q|^{-1} \int_Q e^{-\bar{S}_j(x)+i(\vec{p}_m(t)-\vec{p}_j(t),x)} dx+ \tag{123}$$

$$O\left(|m-j||j|^{-1+12\varepsilon} + |j|^{-1+12\varepsilon}\right),$$

here we have used (46) and (48) for $S_m(x)$ and $S_j(x)$ with $\beta = 1/2 - 6\varepsilon$ (we can do this since $j \in J' \subset \Pi(1/2 - 6\varepsilon)$ and $|j - m| < \lambda_0^{\mu} = o\left(p_j(t)^{1-7\varepsilon}\right)$).

Note that the integral in (123) is the $(j-m)$-th Fourier coefficient of the function $e^{-\bar{S}_j(x)}$. We will denote it by $(e^{-\bar{S}_j(x)})_{j-m}$. Obviously,

$$O\left(|m-j||j|^{-1+12\varepsilon} + |j|^{-1+12\varepsilon}\right) = O(\lambda_0^{-1/2+6\varepsilon+\mu}),$$

since $|j| < c\sqrt{\lambda_0}$. Thus,

$$\sum_{m:|m-j|<\lambda_0^{\mu}} | < \hat{\Psi}_m(x), \Psi_j^0(x) > |^2 =$$

$$\sum_{m:|m-j|<\lambda_0^{\mu}} \left|(e^{-\bar{S}_j(x)})_{j-m} + O(\lambda_0^{-1/2+6\varepsilon+\mu})\right|^2.$$

Using the Cauchy-Schwarz inequality and considering that the sum contains no more than $c\lambda_0^{2\mu}$ terms, we obtain:

$$\sum_{m:|m-j|<\lambda_0^{\mu}} | < \hat{\Psi}_m(x), \Psi_j^0(x) > |^2 > \sum_{m:|m-j|<\lambda_0^{\mu}} |(e^{-\bar{S}_j(x)})_{j-m}|^2 +$$

$$O\left(\|e^{-\bar{S}_j}\|_{L_2(Q)} \lambda_0^{-1/2+6\varepsilon+2\mu}\right) + O\left(\lambda_0^{-1+12\varepsilon+4\mu}\right). \tag{124}$$

Now we estimate the incomplete sum of the Fourier coefficients on the right-hand side. We have a formula for the complete sum:

$$\sum_{m\in Z^2} |(e^{-\bar{S}_j(x)})_{j-m}|^2 = \|e^{-\bar{S}_j}\|^2_{L_2(Q)} = 1 + O(|j|^{-1+12\varepsilon}) = 1 + O(\lambda_0^{-1/2+6\varepsilon}), \tag{125}$$

which follows from (47) with $\beta = 1/2 - 6\varepsilon$. Now, to estimate the incomplete sum it suffices to estimate the tail of the series. Let us consider $(e^{-\bar{S}_j(x)})_{j-m}$ for $|j - m| \geq \lambda_0^{\mu}$. We represent the integral on the right-hand side of (123) in the form $\int UDV dx$, where

$$U = e^{-\bar{S}_j(x)}, \quad V = \frac{1}{|\vec{p}_m(t) - \vec{p}_j(t)|} e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)},$$

D is the derivative in the direction of $\vec{p}_m(t) - \vec{p}_j(t)$. Integrating M times by parts and considering estimate (45) with $\beta = 1/2 - 6\varepsilon$, we obtain:

$$|(e^{-\bar{S}_j(x)})_{j-m}| < \left(c_{\vec{A},\beta,\varepsilon,M} |j|^{6\varepsilon} |m-j|^{-1}\right)^M \tag{126}$$

for any fixed M. Squaring the last inequality, considering $M > 2$ and summarizing over m we arrive at the inequality:

$$\sum_{m:|j-m|\geq\lambda_0^{\mu}} |(e^{-\bar{S}_j(x)})_{j-m}|^2 = O(\lambda_0^{6\varepsilon M-2\mu(M-1)}). \tag{127}$$

Considering that $\mu > 6\varepsilon$ and choosing $M = 202$, we get:

$$\sum_{m:|j-m|\geq\lambda_0^{\mu}} |(e^{-\bar{S}_j(x)})_{j-m}|^2 = O(\lambda_0^{-200\mu}). \tag{128}$$

Combining (125) and (128) with (124), we obtain (122). Thus, (120) and (121) hold.

Step 2. In the first step we obtained $L_2(Q)$-estimate for ϵ_j. Now we estimate its Wiener norm $\|\epsilon_j\|_0$. We prove that

$$\|\epsilon_j\|_0 = O(\lambda_0^{-\gamma_2+\mu}). \tag{129}$$

Let us consider the set W_1: $W_1 : \{r : |r-j| < \mathbf{2}\lambda_0^{\mu}\}$ and let P_1 be the projection on the linear span of Ψ_r^0, $r \in W_1$. Estimate (129) follows from two estimates:

$$\|P_1\epsilon_j\|_0 = O(\lambda_0^{-\gamma_2+\mu}), \tag{130}$$

$$\|(I-P_1)\epsilon_j\|_0 < O(\lambda_0^{-200\mu}). \tag{131}$$

First we consider $P_1\epsilon_j$. Estimate (121) yields $\|P_1\epsilon_j\|_{L_2(Q)} = O(\lambda_0^{-\gamma_2})$. Considering that W_1 contains about $c\lambda_0^{2\mu}$ elements and using the Cauchy-Schwarz inequality, we get (130). To prove (131), we apply the projection $I - P_1$ to both sides of (120). Note that $(I-P_1)\Psi_j^0 = 0$ and, therefore,

$$\|(I-P_1)\epsilon_j\|_0 < \sum_{m:|m-j|<\lambda_0^{\mu}} |c_{jm}|\ \|(I-P_1)\hat{\Psi}_m(x)\|_0. \tag{132}$$

By the definitions of P_1 and $\|\cdot\|_0$, we have:

$$\|(I-P_1)\hat{\Psi}_m\|_0 \leq \sum_{r\notin W_1} |<\hat{\Psi}_m, \Psi_r^0>|.$$

Note that for any $r \notin W_1$, $|r-m| \geq |r-j| - |j-m| > \lambda_0^{\mu}$. Next, we estimate $<\hat{\Psi}_m, \Psi_r^0>$, when $|r-m| > \lambda_0^{\mu}$, $|m-j| < \lambda_0^{\mu}$. Obviously,

$$<\hat{\Psi}_m(x), \Psi_r^0(x)> = |Q|^{-1}\int_Q e^{i(\vec{p}_m(t)-\vec{p}_r(t),x)+S_m(x)}dx.$$

here $\|S_m(x)\|_{\nu} = O(|m|^{6\varepsilon})$ for all ν, since $|m-j| < \lambda_0^{\mu}$, $\mu < 1-7\varepsilon$ and $j \in J' \subset \Pi(1/2-6\varepsilon)$. Integrating by parts M times in the direction of $\vec{p}_m(t) - \vec{p}_r(t)$ the same way as we did to obtain (126), we arrive at the estimate:

$$|<\hat{\Psi}_m, \Psi_r^0>| < \left(c_{\|A\|_0,\varepsilon,\beta,\mu,M}|m|^{6\varepsilon}|m-r|^{-1}\right)^M, \tag{133}$$

for any $r \notin W_1$. Considering $M > 2$ and summarizing with respect to r, we obtain

$$\|(I-P_1)\Psi_m\|_0 \leq \lambda_0^{3\varepsilon M-\mu(M-2)}.$$

Choosing $M = 408$, we obtain $\|(I-P_1)\Psi_m\|_0 = O(\lambda_0^{-202\mu})$. Considering that the sum (132) contains no more than $c\lambda_0^{2\mu}$ terms and $\sum_m |c_{jm}|^2 \approx 1$ by Lemma 5, we obtain (131). Thus, (129) holds.

Step 3. We consider $\mu = 1/10$ It is easy to see that $-\gamma_2 + \mu < 0$. Let us consider sets W_n: $W_n : \{m : |m-j| < (n+1)\lambda_0^{\mu}\}$ and let P_n be the corresponding projections on the linear spans of Ψ_m^0, $m \in W_n$. Clearly, $\cup_{n=1}^{N_1} W_n$, $N_1 = [\lambda_0^{1/2-6\varepsilon-\mu}]$ belongs to the $(\lambda_0^{1/2-6\varepsilon})$-neighborhood of $J'(\lambda_0)$, which is up to technical multiplies is the same as $J'(\lambda_0)$. We can apply (120), (129) to each $m \in \cup_{n=1}^{N_1} W_n$. Applying formulae (120), (129) to each $m \in W_1$, we get

$$P_1 \varepsilon_j = \sum_{m \in W_1} (\epsilon_j)_m \Psi_m^0(x) = \sum_{r \in W_2} c_{jr} \hat{\Psi}_r(x) + \sum_{m \in W_1} (\epsilon_j)_m \epsilon_m(x), \tag{134}$$

here $c_{jr} = \sum_m (\epsilon_j)_m c_{mr}$. Clearly,

$$\| \sum_{m \in W_1} (\epsilon_j)_m \epsilon_m(x) \|_0 \leq \|\epsilon_j\|_0 \sup_m \|\epsilon_m\|_0 = O(\lambda_0^{-2(\gamma_2-\mu)}).$$

By (131), $\|(I-P_1)\varepsilon_j\| = O(\lambda_0^{-20})$. Using (120), (134) and the last two estimates, we obtain:

$$\Psi_j^0(x) = \sum_{m \in W_2} c_{jr} \hat{\Psi}_r(x) + \epsilon_j(x), \quad \|\epsilon_j\|_0 = O(\lambda_0^{-2(\gamma_2-\mu)}). \tag{135}$$

Repeating this procedure $12/(\gamma_2 - \mu)$ times, we get (118). $\diamondsuit$

3.3. **Model Operator.** At the end of Section 2.2 we introduced a family of model operators $\hat{H}(t)$, $t \in K$, acting in $L_2(Q)$. We recall that the action of $\hat{H}(t)$ on $PL_2(Q)$ is defined by the formula:

$$\hat{H}(t)\hat{\Psi}_j(t,x) = \left(\Re\hat{\lambda}_j(t)\right) \hat{\Psi}_j(t,x), \quad \text{when} \quad j \in J(\lambda_0); \tag{136}$$

and on $(I-P)L_2(Q)$ it is defined as follows:

$$\hat{H}(t)(I-P) = (I-P)H(I-P). \tag{137}$$

The operator $\hat{H}(t)$ is correctly defined on $L_2(Q)$, since the model functions have basis-like properties. [3]

Lemma 8. *Let $0 < \varepsilon < 1/4$, $1/3 < \beta < 1/2 - 2\varepsilon$ and $N \geq 31$. Then, for any $t \in K$,*

$$\|H(t) - \hat{H}(t)\| < C_{\vec{A}} \lambda_0^{-10}. \tag{138}$$

Proof. Considering formulae (136) and (70) – (72), for $\beta > 1/3$, $N \geq 31$, $k \approx \lambda_0$ and taking into account that $J(\lambda_0)$ contains no more than $c\lambda_0$ elements, we obtain

$$\left\| \left(H(t) - \hat{H}(t)\right) P \right\| < C_{\vec{A}} \lambda_0^{-10}, \tag{139}$$

and, hence, $\left\|(I-P)\left(H(t) - \hat{H}(t)\right) P\right\| < C_{\vec{A}} \lambda_0^{-10}$. From the definition of $\hat{H}(t)$ we have $(I-P)\hat{H}(t)P = 0$. Using the last two relations yields $\|(I-P)H(t)P\| < C_{\vec{A}} \lambda_0^{-10}$, and, therefore,

$$\|PH(t)(I-P)\| < C_{\vec{A}} \lambda_0^{-10}. \tag{140}$$

Considering that $P\hat{H}(t)(I-P) = 0$, we arrive at the inequality

$$\left\| P \left(H(t) - \hat{H}(t)\right) (I-P) \right\| < C_{\vec{A}} \lambda_0^{-10}.$$

[3] The operator $\hat{H}(t)$ is not necessary selfadjoint since the system of model functions is not exactly orthogonal.

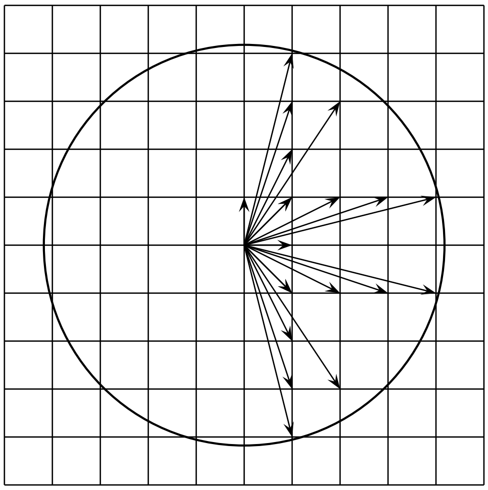

FIGURE 3. The set $\Gamma(\lambda_0^{\varepsilon})$

Taking into account (137), we obtain: $(I-P)\left(H(t)-\hat{H}(t)\right)(I-P)=0$ and, hence, $\left\|\left(H(t)-\hat{H}(t)\right)(I-P)\right\|<C_{\vec{A}}\lambda_0^{-10}$. The last inequality together with (139) gives (138). $\diamondsuit$

Now we consider the operator $(I-P)H(I-P)$ in $(I-P)L_2(Q)$. We denoted its eigenvalues by $\hat{\lambda}_j(t)$, $j\in Z^2\setminus J(\lambda_0)$ (there is no particular way of assigning j to eigenvalues). We are mostly interested in eigenvalues $\hat{\lambda}_j(t)$ which are close to λ_0, namely in those satisfying the inequality:

$$|\hat{\lambda}_j(t)-\lambda_0|<\lambda_0/8. \tag{141}$$

The goal is to construct asymptotic formulae for such eigenvalues and to prove that the inequality

$$\min_{j\in Z^2\setminus J(\lambda_0)}|\hat{\lambda}_j(t)-\lambda_0|>4\lambda_0^{-\delta}$$

holds for a rich set of t.

Let us consider the set of indices i such that $p_i^2(0)$ is sufficiently close to λ_0:

$$M(\lambda_0)=\{i:|p_i^2(0)-\lambda_0|<\lambda_0/2\},$$

i.e., $\vec{p}_i(0)$ is in the ring with radiuses $\lambda_0/\sqrt{2}$ and $\sqrt{3}\lambda_0/\sqrt{2}$. Looking at the definition (90) of $J'(\lambda_0)$, we see that $J'(\lambda_0)\subset M(\lambda_0)$ and $M(\lambda_0)\setminus J'(\lambda_0)$ consists indices i satisfying the inequality

$$\left|p_i^2(0)-|\vec{p}_i(0)+\vec{p}_q(0)|^2\right|<\lambda_0^{1/2-3\varepsilon}$$

for at least one q: $0<|q|<\lambda_0^{\varepsilon}$. Hence, $M(\lambda_0)\setminus J'(\lambda_0)=\cup_{0<|q|<\lambda_0^{\varepsilon}}\Omega_q$,

$$\Omega_q=\left\{i\in M_0(\lambda_0):\left|p_i^2(0)-|\vec{p}_i(0)+\vec{p}_q(0)|^2\right|<\lambda_0^{1/2-3\varepsilon}\right\}.$$

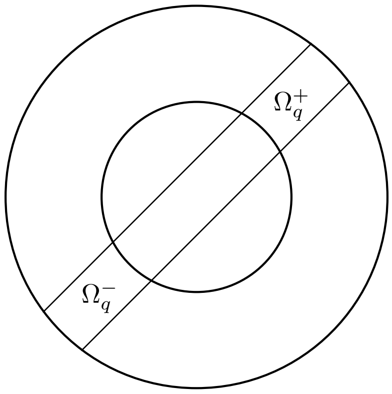

FIGURE 4. The set Ω_q.

Note that $\Omega_q \supset \Omega_{q'}$ if q and q' are colinear and $|q| < |q'|$. Therefore, in the formula for $M(\lambda_0) \setminus J'(\lambda_0)$ we can take the union over a smaller set of indices q:

$$M(\lambda_0) \setminus J'(\lambda_0) = \cup_{\Gamma_0(\lambda_0^\varepsilon)} \Omega_q, \tag{142}$$

where $\Gamma_0(\lambda_0^\varepsilon) \subset Z^2$ is defined as follows: we take all the vectors $q \in Z^2, 0 < |q| < \lambda_0^\varepsilon$. Some of these vectors are colinear with each other. From each family of colinear vector we choose its representative with the minimal length (Fig.3). The union of all minimal representatives we denote by $\Gamma(\lambda_0^\varepsilon)$. Clearly, for every $j \in Z^2, 0 < |j| < \lambda_0^\varepsilon$ there is a unique $q \in \Gamma(\lambda_0^\varepsilon)$ and a unique $n \in Z$, such that $j = nq$, $\vec{p}_j(0) = n\vec{p}_q(0)$.

It is easy to see that each Ω_q corresponds to the intersection of the ring with radiuses $\frac{1}{\sqrt{2}}\lambda_0$ and $\sqrt{\frac{3}{2}}\lambda_0$ and a strip of the width $\lambda_0^{1/2-3\varepsilon}/2p_q(0) = o(\sqrt{\lambda_0})$ going through the origin. Obviously, Ω_q consists of two disjoint parts Ω_q^+ and Ω_q^- on opposite sides of the ring, the distance between them being of order of $\sqrt{\lambda_0}$ (Fig.4).

It is easy to show that for λ_0 large enough

$$\Omega_q \cap \Omega_{q'} = \emptyset, \quad \text{when } q \neq q', \quad q, q' \in \Gamma(\lambda_0^\varepsilon). \tag{143}$$

since the angle between $\vec{p}_q(0)$ and $\vec{p}_{q'}(0)$ can be estimated from below by $c_b\lambda_0^{-2\varepsilon}$. Here is the point where we essentially use that the magnetic operator acts in the two-dimensional space. The relation (143) does not hold in higher dimensions.

Let P_q, $q \in \Gamma_0(\lambda_0^\varepsilon)$ be the projection in $L_2(Q)$ defined by a diagonal matrix $(P_q)_{im}$ in the basis of exponents $\{\Psi_i^0(x)\}_{i \in Z^2}$, the diagonal elements $(P_q)_{ii}$, $i \in Z^2$, being given by the formula:

$$(P_q)_{ii} = \begin{cases} 1, & \text{if } i \in \Omega_q; \\ 0, & \text{otherwise.} \end{cases} \tag{144}$$

Obviously, $P_q P_{q'} = 0$, if $q \neq q'$. We show later that eigenvalues of operators $P_q H(t) P_q$, $q \in \Gamma(\lambda_0^\varepsilon)$, have monotonous properties with respect to t. Now we prove that operators $P_q H(t) P_q$, $q \in \Gamma(\lambda_0^\varepsilon)$, play a special role: eigenvalues of $(I - P)H(t)(I - P)$ which are close to λ_0 can be well approximated by eigenvalues of the operators of $P_q H P_q$, $q \in \Gamma(\lambda_0^\varepsilon)$.

Lemma 9. *Let $\hat{\lambda}_j(t)$ be an eigenvalue of $(I-P)H(t)(I-P)$ satisfying the inequality:*

$$|\hat{\lambda}_j(t) - \lambda_0| < \lambda_0/8. \tag{145}$$

Then, there is an eigenvalue $\mu_n^q(t)$, $n \in N$, *of an operator* $P_qH(t)P_q$, $q \in \Gamma(\lambda_0^\varepsilon)$, *such that*

$$|\hat\lambda_j(t) - \mu_n^q(t)| < c_{\vec A}\lambda_0^{-9}. \tag{146}$$

Corollary 9. *If* $\min_{j\in Z^2\setminus J(\lambda_0)} |\hat\lambda_j(t) - \lambda_0| < \lambda_0^{-\mu}$, $\mu < 9$, *then* $\min_{n,q} |\mu_n^q(t) - \lambda_0| < 2\lambda_0^{-\mu}$.

Proof. The proof contains five steps.

1. Let $\hat\Psi_j(t)$ be an eigenfunction of $(I-P)H(t)(I-P)$ corresponding to an eigenvalue $\hat\lambda_j(t)$ which satisfies (145). It follows from (138) that

$$H(t)\hat\Psi_j = \hat\lambda_j(t)\hat\Psi_j + O(\lambda_0^{-10}), \tag{147}$$

here and below in this lemma we mean $O(\cdot)$ with respect to $L_2(Q)$-norm.

2. Let $\hat P$ be the projection in $L_2(Q)$ with a diagonal matrix $\hat P_{mr}$ in the basis of exponents $\{\Psi_r^0(t,x)\}_{r\in Z^2}$:

$$\hat P_{rr} = \begin{cases} 1, & \text{if } r \in M(\lambda_0); \\ 0, & \text{otherwise.} \end{cases} \tag{148}$$

We prove by standard perturbation methods that the function $\hat\Psi_j$ defined in the first step essentially belongs to $\hat P L_2(Q)$:

$$\hat\Psi_j = \hat P\hat\Psi_j + O(\lambda_0^{-10}), \tag{149}$$

$$H(t)\hat\Psi_j = H(t)\hat P\hat\Psi_j + O(\lambda_0^{-9}). \tag{150}$$

Indeed, let us consider $< \hat\Psi_j, \Psi_r^0 >$, $r : |p_r^2(0) - \lambda_0| > \lambda_0/4$. Considering that

$$< H(t)\hat\Psi_j, \Psi_r^0 >= p_r^2(t) < \hat\Psi_j, \Psi_r^0 > + O\left(\max_{s:|s-r|<R_0} p_s(t)| < \hat\Psi_j, \Psi_s^0 > |\right), \tag{151}$$

and using elementary perturbation considerations we obtain that:

$$\left|< \hat\Psi_j, \Psi_r^0 >\right| < c_{\vec A} p_r(0) \left|p_r(0)^2 - \hat\lambda_j(t)\right|^{-1} < \frac{c_{\vec A}}{p_r(0)}.$$

Taking into account that $\vec A(x)$ is a trigonometric polynomial and applying the last estimate, we obtain that for any $r : |p_r^2(0) - \lambda_0| > \lambda_0/4 + R_0$,

$$\left|< \hat\Psi_j, \Psi_r^0 >\right| < c_{\vec A}\left(\max_{s:|s-r|<R_0}\left|< \hat\Psi_j, \Psi_s^0 >\right|\right) p_r(0)\left|p_r(0)^2 - \hat\lambda_j(t)\right|^{-1} < \frac{c_{\vec A}}{p_r(0)^2},$$

and so on. If $r : |p_r^2(0) - \lambda_0| > \lambda_0/4 + nR_0$, then

$$\left|< \hat\Psi_j, \Psi_r^0 >\right| < c_{\vec A}\left(\max_{s:|s-r|<R_0}\left|< \hat\Psi_j, \Psi_s^0 >\right|\right) p_r(0)\left|p_r(0)^2 - \hat\lambda_j(t)\right|^{-1} < \frac{c_{\vec A}}{p_r(0)^{n+1}}.$$

Choosing $n = 20$, we obtain

$$\left|< \hat\Psi_j, \Psi_r^0 >\right| < \frac{c_{\vec A}}{p_r(0)^{21}}. \tag{152}$$

This estimate obviously holds for all $r : |p_r^2(0) - \lambda_0| > \lambda_0/2$. Squaring the last estimate and summarizing it over the r, we obtain (149). Considering that $\|H\hat P\| < 2\lambda_0$, we get (150).

3. We prove the relations:

$$\hat P\hat\Psi_j = P_Q\hat\Psi_j + O(\lambda_0^{-10}), \tag{153}$$

$$H(t)\hat P\hat\Psi_j = HP_Q\hat\Psi_j + O(\lambda_0^{-9}), \tag{154}$$

here

$$P_Q=\sum_{q\in\Gamma(\lambda_0^\varepsilon)}P_q.$$

In fact, considering (142), we represent $\hat P$ in the form: $\hat P=P_0+P_Q$, projection P_0 is defined before Lemma 7. By Corollary 8, $P_0(I-P)=O(\lambda_0^{-10})$. Considering also that $\hat\Psi_j=(I-P)\hat\Psi_j$, we obtain:

$$\hat P\hat\Psi_j=\hat P(I-P)\hat\Psi_j=P_Q(I-P)\hat\Psi_j+O(\lambda_0^{-10})=P_Q\hat\Psi_j+O(\lambda_0^{-10}).$$

Thus, (153) is proved. Taking into account that $\|H(t)\hat P\|<2\lambda_0$ and $\|H(t)P_Q\|<2\lambda_0$, we arrive at (154).

4. Combining (147), (149), (150), (153) and (154), we obtain: $H(t)P_Q\hat\Psi_j=\hat\lambda_jP_Q\hat\Psi_j+O(\lambda_0^{-9})$, and, therefore,

$$P_QH(t)P_Q\hat\Psi_j=\hat\lambda_jP_Q\hat\Psi_j+O(\lambda_0^{-9}). \tag{155}$$

The last relation means that

$$\text{there is an eigenvalue } \mu_n^Q \text{ of } P_QH(t)P_Q \text{ such that } |\hat\lambda_j(t)-\mu_n^Q(t)|<c\lambda_0^{-9}. \tag{156}$$

5. We prove that

$$P_QH(t)P_Q=\sum_{q\in\Gamma(\lambda_0^\varepsilon)}P_qH(t)P_q. \tag{157}$$

It is enough to show that

$$P_qH(t)P_{q'}=0,\quad \text{when}\quad q\neq q'. \tag{158}$$

Indeed, suppose (158) is not true. Then, there is $s\in\Omega_q$ and $r\in\Omega_{q'}$, such that $H_{sr}\neq 0$, here H_{rs} is a matrix element of $H(t)$ in the basis of exponents. Since $\vec A(x)$ is a trigonometric polynomial, H_{rs} can differ from zero only if $|r-s|<R_0$. In the same time from the definition of Ω_q and $\Omega_{q'}$ it follows that $|r-s|>c\lambda_0^{1/2-2\varepsilon}$, since the angle between $\vec p_q(0)$ and $\vec p_{q'}(0)$ is greater than $c\lambda_0^{-2\varepsilon}$. The contradiction proves (158), and, hence, (157). Recall that the projections P_q, $q\in\Gamma(\lambda_0^\varepsilon)$ are orthogonal to each other. Therefore, the spectrum of $P_QH(t)P_Q$ is the union of spectra of the operators $P_qH(t)P_q$, $q\in\Gamma(\lambda_0^\varepsilon)$. Combining the last statement with (156), we finish the proof. $\diamondsuit$

Next, we investigate eigenvalues of the operators $P_qH(t)P_q$, $q\in\Gamma(\lambda_0^\varepsilon)$. We enumerate the eigenvalues $\mu_n^q(t)$ of an operator $P_qH(t)P_q$ in the increasing order, $n=1,...,N_q$. It is easy to see from the definition of P_q and Ω_q that $N_q<c\lambda_0^{1-3\varepsilon}$.

Lemma 10. *Eigenvalues $\mu_n^q(t)$ of an operator $P_qH(t)P_q$ have the following properties as functions of t:*

1) Eigenvalues $\mu_n^q(t)$ are differentiable with respect to t for almost all $t\in K$ and

$$|\nabla_t\mu_n^q(t)|<c_b\lambda_0^{1/2} \tag{159}$$

at points of differentiability.

2) Eigenvalues $\mu_n^q(t)$ are monotonous functions of t in a direction $\vec u_q$ orthogonal to $\vec p_q(0)$ and their derivative $(\nabla_t\mu_n^q(t),\vec u_q)$ in this direction satisfy the following estimate:

$$|(\nabla_t\mu_n^q(t),\vec u_q)|>\lambda_0^{1/2}. \tag{160}$$

3) The number of eigenvalues $\mu_n^q(t)$ satisfying the inequality $|\mu_n^q(t)-\lambda|<c\sqrt{\lambda_0}$ for a given $t\in K$ and a positive constant c does not exceed $C_{\vec A}\lambda_0^{1-3\varepsilon}$.

Proof. 1) The first statement can be proved by standard considerations. Indeed, if $\mu_n^q(t)$ is a simple eigenvalue than it is differentiable with respect to t [Kato] and

$$\nabla_t \mu_n^q(t) =< \nabla_t \left(P_q H(t) P_q\right) \Psi_n, \Psi_n >, \tag{161}$$

$\Psi_n(x)$ being a normalized eigenfunction of $P_q H(t) P_q$ corresponding to $\mu_n^q(t)$. Note that P_q does not depend on t and $\nabla H(t) = \vec{D}(t) + 2\vec{a}$, where $\vec{D}(t) = \nabla H_0(t)$ is a diagonal vector-matrix: $\vec{D}(t)_{mj} = \delta_{mj} \nabla_t p_j^2(t)$ and $\vec{a}$ is the operator of multiplication by $\vec{a}(x)$. Hence,

$$\nabla_t \mu_n^q(t) =< \vec{D}(t)\Psi_n, \Psi_n > +O(1).$$

It is easy to show that $\|\vec{D}(t)\Psi_n\|^2 = c_b < H_0\Psi_n, \Psi_n >$. Elementary perturbation considerations yield $< H_0\Psi_n, \Psi_n > < 2\lambda_0$ and, therefore, $|\nabla_t \mu_n^q(t)| < c_b\sqrt{\lambda_0}$ for λ_0 large enough.

2) Note that a set Ω_q consists of two parts Ω_q^+ and Ω_q^- (Fig.4). Correspondingly, we can represent P_q as $P_q = P_q^+ + P_q^-$, here P_q^+ and P_q^- are defined similarly to P_q up to the replacement of Ω_q by Ω_q^+ and Ω_q^-, respectively. Obviously $P_q^+ P_q^- = 0$. Taking into account that the distance between Ω_q^+ and Ω_q^- is greater than $c_b \lambda_0^{1/2}$ and considering that $\vec{A}(x)$ is a trigonometric polynomial, we easily obtain $P_q^+ H P_q^- = 0$; and, therefore, $P_q H(t) P_q = P_q^+ H(t) P_q^+ + P_q^- H(t) P_q^-$. Since P_q^+ and P_q^- are projections onto orthogonal subspaces, the spectrum of $P_q H(t) P_q$ is the union of the spectra of $P_q^+ H(t) P_q^+$ and $P_q^- H(t) P_q^-$. Next, we show that both of these operators are monotonous with respect to t in the direction orthogonal to $\vec{p}_q(0)$. In fact, let $\vec{\nu}_q$ be the unit vector in the direction of $\vec{p}_q(0)$, and $\vec{u}_q \perp \vec{\nu}_q$, $|\vec{u}_q| = 1$. Any $\vec{p}_i(0)$, $i \in \Omega_q$ admits the representation $\vec{p}_i(0) = \xi_i \vec{\nu}_q + \eta_i \vec{u}_q$, here $\xi_i = (\vec{p}_i(0), \vec{\nu}_q)$, $|\xi_i| < c\lambda_0^{1/2-3\varepsilon}$, since $i \in \Omega_q$, and $|\eta_i|^2 > \lambda_0/2 + o(\lambda_0)$. Note that the sign of η_i is different in the sets Ω_q^+ and Ω_q^-, namely, $\eta_i > \frac{1}{\sqrt{2}}\lambda_0^{1/2} + o(\lambda_0^{1/2})$ in one of them (we denote it by Ω_q^+) and $\eta_i < -\frac{1}{\sqrt{2}}\lambda_0^{1/2} + o(\lambda_0^{1/2})$ in Ω_q^-. Similarly, we write a quasimomentum t in the form: $t = \tau\vec{\nu}_q + \zeta\vec{u}_q$, $\tau, \zeta \in R$. Let us show that the operator $P_q^+ H(t) P_q^+$ depends on ζ in a monotonous way. In fact, for any $f \in P_q^+ L_2(Q)$,

$$\frac{\partial}{\partial\zeta} < P_q^+ H(t) P_q^+ f, f >=< P_q^+ \left(\vec{D}(t) + \vec{a}, \vec{u}_q\right) P_q^+ f, f >=$$

$$< P_q^+ \left(\vec{D}(t), \vec{u}_q\right) P_q^+ f, f > +O(1). \tag{162}$$

Obviously, $P_q^+(\vec{D}(t), \vec{u}_q)P_q^+$ is a diagonal operator in the basis of exponents (9) and its diagonal elements are given by the formula:

$$\left(P_q^+ \left(\vec{D}(t), \vec{u}_q\right) P_q^+\right)_{ii} = 2(\eta_i + \zeta) > \sqrt{2}\lambda_0^{1/2} + o(\lambda_0^{1/2}), \quad \text{when } i \in \Omega_q^+.$$

Substituting the last estimate into (162), we obtain:

$$\frac{\partial}{\partial\zeta} < P_q^+ H(t) P_q^+ f, f > \; > \; \lambda_0^{1/2} \tag{163}$$

for any $f \in P_q^+ L_2(Q)$, $\|f\|_{L_2(Q)} = 1$. Similarly,

$$\frac{\partial}{\partial\zeta} < P_q^- H(t) P_q^- f, f > \; < \; -\lambda_0^{1/2} \tag{164}$$

for any $f \in P_q^- L_2(Q)$, $\|f\|_{L_2(Q)} = 1$. Let $\mu_n^{q+}(t)$ and $\mu_n^{q-}(t)$ be eigenvalues of $P_q^+ H(t) P_q^+$ and $P_q^- H(t) P_q^-$, respectively. It easily follows from inequalities (161), (163) and (164) that

$$\frac{\partial}{\partial\zeta}\mu_n^{q+}(t) = (\nabla_t \mu_n^{q+}(t), \vec{u}_q) > \lambda_0^{1/2}, \qquad \frac{\partial}{\partial\zeta}\mu_n^{q-}(t) = (\nabla_t \mu_n^{q-}(t), \vec{u}_q) < -\lambda_0^{1/2}.$$

Hence, (160) holds.

3) Using elementary perturbation considerations we show that the number of eigenvalues of $P_q H(t) P_q$ satisfying the inequality $|\mu_n^q(t) - \lambda| < c\sqrt{\lambda}$ does not exceed the number of eigenvalues $p_i^2(t)$, $i \in \Omega_q$ of the operator $P_q H_0(t) P_q$ satisfying the inequality $|p_i^2(t) - \lambda| < 2c\sqrt{\lambda}$. It is easy to estimate the number of lattice points $i \in \Omega_q$ satisfying the last inequality by computing the area of the region $\{\vec{k} : \left|k^2 - |\vec{k} + \vec{p}_q(0)|^2\right| < \lambda_0^{1/2-3\varepsilon}, \ \sqrt{\lambda} - c_{\vec{A}} < |x| < \sqrt{\lambda} + c_{\vec{A}}\}$. Obviously it does not exceed $c_{\vec{A}}\lambda^{1/2-3\varepsilon}$. $\diamondsuit$

3.4. **Geometric Considerations.** The purpose of this section is to construct the non-resonance set $\chi_*(\lambda, \delta)$, and to investigate its properties. In particular, we prove that it has a length asymptotically close to $2\pi\sqrt{\lambda}$. We start with proving Lemma 4.

The proof of Lemma 4. First, we show that for every $\vec{\nu} \in D_1(\lambda)$, there is a unique number $k(\vec{\nu})$ in the interval $[\sqrt{\lambda}/2, 2\sqrt{\lambda}]$ such that $\vec{k} = k(\lambda, \vec{\nu})\vec{\nu}$ satisfy the equation $\Re\hat{\lambda}(\vec{k}) = \lambda$ and

$$|k(\lambda, \vec{\nu}) - \sqrt{\lambda}| < c_{\vec{A}}\lambda^{-\beta}. \tag{165}$$

In fact, we consider the function $\hat{\lambda}(k\vec{\nu})$ as a function of k, $k \in [\sqrt{\lambda}/2, 2\sqrt{\lambda}]$, when $\vec{\nu}$ is fixed and $\vec{\nu} \in D_1(\lambda)$. Taking into account that $\hat{\lambda}(k\vec{\nu})$ satisfies (51) – (53) and (57), we obtain that the equation $\Re\hat{\lambda}(k\vec{\nu}) = \lambda$ has a unique solution $k(\lambda, \vec{\nu})$ in the interval $[\sqrt{\lambda}/2, 2\sqrt{\lambda}]$ and (165) holds. Using (165), we define $h(\lambda, \vec{\nu})$ by the formula $k(\lambda, \vec{\nu}) = \sqrt{\lambda} + h(\lambda, \vec{\nu})$, where $h(\lambda, \vec{\nu}) = O(\lambda^{-\beta})$. From the definition of $h(\lambda, \vec{\nu})$ it follows that $h_\theta(\lambda, \vec{\nu}) = k_\theta(\lambda, \vec{\nu})$. Differentiating the equation $\Re\hat{\lambda}(\vec{k}) = \lambda$ and using (53), we obtain:

$$h_\theta(\lambda, \vec{\nu}) = k_\theta(\lambda, \vec{\nu}) = -\frac{\left(\Re\delta\hat{\lambda}(\vec{k})\right)_\theta}{2k + \left(\Re\delta\hat{\lambda}(\vec{k})\right)_k} = O(k^{1/2-3\beta}) = O(\lambda^{(1-6\beta)/4}).$$

To prove (66) let us consider $D_1(\lambda)$ given by (62). It is the set of directions corresponding to $D(\lambda)$. According to (61) $D_1(\lambda)$ approximately consists of vectors $\vec{\nu}$ satisfying the inequality $|2\,(\vec{\nu}, \vec{p}_r(0))| > k^{-1/2+\beta}$ for all $r : 0 < |r| < k^{2\varepsilon}$. Considering that the number of r satisfying the last inequality does not exceed $c_b k^{2\varepsilon}$, we easily see that $D_1(\lambda)$ has a length asymptotically close to that of a unit circle: $l\,(D_1) = 2\pi + O(\lambda^{-(1-2\beta-4\varepsilon)/4})$. Taking into account that $D(\lambda)$ consists of slightly distorted arcs of radius $\sqrt{\lambda}$ (see (63) – (65)), defined by directions in $D_1(\lambda)$, we easily obtain (66). $\diamondsuit$

We already defined the set $\chi_*(\lambda, \beta)$ by the formula:

$$\chi_* = \left\{ t \in K : \exists j \in J(\lambda_0) : \Re\hat{\lambda}_j(t) = \lambda, \min_{m \in Z^2, m \neq j} \left|\Re\hat{\lambda}_j(t) - \Re\hat{\lambda}_m(t)\right| > 4\lambda^{-\delta} \right\}. \tag{166}$$

To understand properties of $\chi_*(\lambda,\delta)$ we represent it as a difference of two sets: $\chi_*(\lambda,\delta)=\chi_1(\lambda,\delta)\setminus\chi_2(\lambda,\delta)$. The definition of $\chi_1(\lambda,\delta)$ includes only model numbers $\hat\lambda_m(t)$ corresponding to $m\in J(\lambda_0)$:

$$\chi_1=\left\{t\in K:\exists j\in J(\lambda_0):\Re\hat\lambda_j(t)=\lambda,\min_{m\in J(\lambda_0),m\neq j}\left|\Re\hat\lambda_j(t)-\Re\hat\lambda_m(t)\right|>4\lambda^{-\delta}\right\},\tag{167}$$

while $\chi_2(\lambda,\delta)$ involves all $\hat\lambda_m(t)$, $m\in Z^2\setminus J(\lambda)$:

$$\chi_2=\left\{t\in K:\min_{m\in Z^2\setminus J(\lambda)}\left|\lambda-\Re\hat\lambda_m(t)\right|\leq 4\lambda^{-\delta}\right\}.\tag{168}$$

Note that in the last formula $\Re\hat\lambda_m(t)=\hat\lambda_m(t)$, since $\hat\lambda_m(t)$, $m\in Z^2\setminus J(\lambda_0)$, are eigenvalues of a self-adjoint operator. We start with investigating properties of $\chi_1(\lambda,\delta)$. Some properties of $\chi_*(\lambda,\delta)$ will follow immediately from those of $\chi_1(\lambda,\delta)$, since $\chi_*(\lambda,\delta)\subset\chi_1(\lambda,\delta)$.

First, we show that the inequalities defining $\chi_1(\lambda,\delta)$ are preserved in its small neighborhood (Lemma 11). Second, we describe geometric structure of $\chi_1(\lambda,\delta)$ (Lemmas 12 and 13). Third, we prove that $\chi_1(\lambda,\delta)$ has a length asymptotically close to $2\pi\sqrt{\lambda}$ (Lemma 14).

Lemma 11. *Suppose t belongs to the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1(\lambda,\delta)$. Then, there is a unique $j\in J(\lambda_0)$ such that*

$$|\Re\hat\lambda_j(t)-\lambda|<4\lambda^{-2\delta},\tag{169}$$

and, moreover,

$$|\Re\hat\lambda_m(t)-\Re\hat\lambda_j(t)|>2\lambda^{-\delta}\tag{170}$$

for all $m\in J(\lambda_0)$, $m\neq j$.

<u>Proof.</u> Let t belong to the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1(\lambda,\delta)$. Then, there is a $t_0\in\chi_1(\lambda,\delta)$ such that $|t-t_0|<\lambda^{-1/2-2\delta}$. By the definition of $\chi_1(\lambda,\delta)$, there is a single $j\in J(\lambda_0)$ such that $\Re\hat\lambda_j(t_0)=\lambda$ and, moreover,

$$|\Re\hat\lambda_m(t_0)-\Re\hat\lambda_j(t_0)|>4\lambda^{-\delta},\tag{171}$$

for all $m\in J(\lambda_0)$, $m\neq j$. Using (51) and (53) for $\vec k=\vec p_j(t)$ and considering the notation (69), we obtain $\left|\nabla\hat\lambda_j\right|<2\sqrt{\lambda}$ and, hence,

$$|\Re\hat\lambda_j(t)-\Re\hat\lambda_j(t_0)|<4\lambda^{-2\delta}.\tag{172}$$

Taking into account that $\Re\hat\lambda_j(t_0)=\lambda$, we get (169). Let us prove that (170) holds for all $m\neq j$. Suppose $m\in J(\lambda_0)$, $m\neq j$. Using again (51) and(53) yields:

$$|\Re\hat\lambda_m(t)-\Re\hat\lambda_m(t_0)|<4\lambda^{-2\delta}.\tag{173}$$

Combining the last inequality with (171) and (172), we get (170) for $m\in J(\lambda_0)$, $m\neq j$. $\diamondsuit$

Next, we investigate geometric properties of $\chi_1(\lambda,\delta)$. We start with considering a set $D_r(\lambda,\beta)$, which is the set $D(\lambda,\beta)$ (see (58)) shifted in R^2 by $\vec p_r(0)$, $r\neq 0$:

$$D_r(\lambda,\beta)=\left\{\vec k:\vec k-\vec p_r(0)\in D(\lambda,\beta)\right\}.$$

Obviously, the points of intersections of $D(\lambda,\beta)$ and $D_r(\lambda,\beta)$, $r \in Z^2 \setminus \{0\}$, are described by the equations: [4]

$$\Re\hat{\lambda}(\vec{k}) = \lambda, \quad \Re\hat{\lambda}\left(\vec{k}-\vec{p}_r(0)\right) = \lambda, \quad \vec{k},\ \vec{k}-\vec{p}_r(0) \in \Pi(\beta)\cap M_1(\lambda_0),\ r\neq 0,$$

$M_1(\lambda_0)$ being the centered at the origin ring with radiuses $\frac{1}{2}\sqrt{\lambda_0}$ and $2\sqrt{\lambda_0}$. If we replace the second equation in the above formula by the inequality

$$\left|\Re\hat{\lambda}\left(\vec{k}-\vec{p}_r(0)\right)-\lambda\right| < 4\lambda^{-\delta},$$

we obtain the intersection of $D(\lambda,\beta)$ with a small neighborhood of $D_r(\lambda,\beta)$. Since $D(\lambda,\beta)$ is a distorted circle with holes, this intersection is a small distorted arc. Let us exclude all such arcs, corresponding to different r, from $D(\lambda,\beta)$. The set obtained this way we denote by $\hat{D}(\lambda,\beta,\delta)$:

$$\hat{D}(\lambda,\beta,\delta) =$$
$$\left\{\vec{k}\in D(\lambda,\beta) : \min_{r:\vec{k}-\vec{p}_r(0)\in\Pi(\beta)\cap M_1(\lambda_0),\ r\neq 0}\left|\Re\hat{\lambda}\left(\vec{k}-\vec{p}_r(0)\right)-\lambda\right| \geq 4\lambda^{-\delta}\right\}. \tag{174}$$

Next, we prove that $\chi_1(\lambda,\delta)$ is just $\hat{D}(\lambda,\beta,\delta)$ packed into the bag K.

Lemma 12. *The following relation holds: $\mathcal{K}\hat{D} = \chi_1$. The parallel shift $\mathcal{K}$ establishes one-to-one correspondence between $\hat{D}$ and χ_1.*

Corollary 10. *The length of χ_1 is equal to that of $\hat{D}$.*

Proof. First we prove that $\chi_1 \subset \mathcal{K}\hat{D}$, i.e., for any $t\in\chi_1$, there is a j such that $\vec{p}_j(t)\in\hat{D}$. In fact, if $t\in\chi_1$, then there is a unique $j\in J(\lambda_0)$ such that $\Re\hat{\lambda}_j(t)=\lambda$. By the definition of $J(\lambda_0)$ (see (68)), the relation $j\in J(\lambda_0)$ is equivalent to the relation $\vec{p}_j(t)\in\Pi(\beta)\cap M_1(\lambda_0)$. By (69) the equation $\Re\hat{\lambda}_j(t)=\lambda$ can be rewritten in the form: $\Re\hat{\lambda}(\vec{p}_j(t))=\lambda$. Thus, $\vec{p}_j(t)\in D$, D being defined by (58). Let us prove that $\vec{p}_j(t)\in\hat{D}$. Suppose it is not so. Then there is $r\in Z^2\setminus\{0\}$, such that $\vec{p}_j(t)-\vec{p}_r(0)\in\Pi(\beta)\cap M_1(\lambda_0)$ and $\left|\Re\hat{\lambda}\left(\vec{p}_j(t)\right)-\Re\hat{\lambda}\left(\vec{p}_j(t)-\vec{p}_r(0)\right)\right| < 4k^{-\delta}$. Let $m=j-r$. Then, $\vec{p}_j(t)-\vec{p}_r(0)=\vec{p}_m(t)$ and $\left|\Re\hat{\lambda}\left(\vec{p}_j(t)\right)-\Re\hat{\lambda}\left(\vec{p}_m(t)\right)\right| < 4k^{-\delta}$. Since $\vec{p}_m(t)\in\Pi(\beta)\cap M_1(\lambda_0)$, we have $m\in J(\lambda_0)$. Recalling again notation (69), we get $|\Re\hat{\lambda}_j(t)-\Re\hat{\lambda}_m(t)|<4k^{-\delta}$. This contradicts to the assumption that $t\in\chi_1$. Hence, $\vec{p}_j(t)\in\hat{D}$. Thus, we have proved that $\chi_1\subset\mathcal{K}\hat{D}$.

Next, we prove that $\mathcal{K}\hat{D}\subset\chi_1$. Let $\vec{k}\in\hat{D}$. By definition of $\hat{D}$, $\Re\hat{\lambda}(\vec{k})=\lambda$. We represent it in the form $\vec{k}=\vec{p}_j(t)$, $t\in\mathcal{K}\hat{D}$, $j\in J(\lambda_0)$. Using (69), we obtain $\Re\hat{\lambda}_j(t)=\lambda$. It remains to prove that $|\Re\hat{\lambda}_j(t)-\Re\hat{\lambda}_m(t)|>4k^{-\delta}$ for all $m\in J(\lambda_0)$, $m\neq j$. Suppose the last inequality does not hold for some m. Since $m\in J(\lambda_0)$, $\hat{\lambda}_m(t)=\hat{\lambda}(\vec{p}_m(t))$, $\vec{p}_m(t)\in\Pi(\beta)\cap M_1(\lambda_0)$. Thus, $|\Re\hat{\lambda}(\vec{p}_m(t))-\Re\hat{\lambda}(\vec{p}_j(t))|\leq 4k^{-\delta}$. Defining r by the relation $m=j-r$, we obtain $|\Re\hat{\lambda}(\vec{k}-\vec{p}_r(0))-\Re\hat{\lambda}(\vec{k})|\leq 4k^{-\delta}$. This contradicts to the assumption that $\vec{k}\in\hat{D}$. Therefore, $\mathcal{K}\hat{D}\subset\chi_1$.

Combining two parts of the proof we obtain $\mathcal{K}\hat{D}=\chi_1$.

We establish one-to-one correspondence between $\hat{D}$ and χ_1 by contradiction. Suppose for some $t\in\chi_1$ there is a pair $j,m:\vec{p}_j(t),\vec{p}_m(t)\in\hat{D}$, $j\neq m$. This means

[4] Note that for $\vec{A}=0$, the equations have the form of von Laue diffraction conditions $|\vec{k}|^2 = |\vec{k}-\vec{p}_r(0)|^2=\lambda$.

that $\Re\hat{\lambda}_j(t) = \Re\hat{\lambda}_m(t) = \lambda$. This is in contradiction with the inequality in the definition of χ_1. $\diamondsuit$

Next, we establish that there is one-to-one correspondence not only between $\mathcal{K}\hat{D}$ and χ_1 but also between their $(\lambda^{-1/2-2\delta})-$neighborhoods.

Lemma 13. *The parallel shift $\mathcal{K}$ establishes one-to-one correspondence between the $(\lambda^{-1/2-2\delta})-$neighborhood of $\hat{D}$ in R^2 and the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_1 in K.*

Corollary 11. *The area of the $(\lambda^{-1/2-2\delta})-$neighborhood of $\hat{D}$ in R^2 is equal to the area of the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_1 in K.*

Proof. Suppose $\vec{k}$ is in the $(\lambda^{-1/2-2\delta})-$neighborhood of $\hat{D}$ in R^2. Then, there is a $\vec{k}_0 \in \hat{D}$, such that $|\vec{k}-\vec{k}_0| < \lambda^{-1/2-2\delta}$. This means that $t = \mathcal{K}\vec{k}$ is in the $(\lambda^{-1/2-2\delta})-$neighborhood of $t_0 = \mathcal{K}\vec{k}_0$ on the torus K. By the previous lemma $t_0 \in \chi_1$ and, therefore, t is in the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_1 in K. Thus, the parallel shift of the neighborhood of $\hat{D}$ belongs to the neighborhood of χ_1.

Suppose t is in the $(\lambda^{-1/2-2\delta})-$neighborhood of χ_1 in K. Then, there is $t_0 \in \chi_1$ such that $|t-t_0| < \lambda^{-1/2-2\delta}$. There is a single j such that $\lambda_j(t_0) = \lambda$. Let us consider $\vec{k}_0 = \vec{p}_j(t_0)$ and $\vec{k} = \vec{p}_j(t)$. Obviously, $|\vec{k}-\vec{k}_0| = |t-t_0| < \lambda^{-1/2-2\delta}$ and $\vec{k}_0 \in \hat{D}$. Therefore, $\vec{k}$ is in the $(\lambda^{-1/2-2\delta})$-neighborhood of $\hat{D}$ in R^2. Thus, the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_1 belongs to the parallel shift of the corresponding neighborhood of $\hat{D}$.

Let us establish that there is one-to-one correspondence between the neighborhoods. Suppose for a certain t in the neighborhood of χ_1 there is a pair of vectors $\vec{p}_j(t)$ and $\vec{p}_m(t)$, both being in the neighborhood of $\hat{D}$. Considering (53), we easily obtain that $|\Re\hat{\lambda}_j(t)-\lambda| < c\lambda^{-2\delta}$ and $|\Re\hat{\lambda}_m(t)-\lambda| < c\lambda^{-2\delta}$ By Lemma 11, $m = j$. $\diamondsuit$

Lemma 14. *Suppose $0 < \varepsilon < 1/8$, $1/4 < \beta < 1/2-2\varepsilon$, $0 < \delta < \min\{1-2\beta-4\varepsilon, 2\beta-1/2\}$. Then, the length of $\hat{D}(\lambda,\beta,\delta)$ obeys the estimate:*

$$l\left(\hat{D}(k,\beta,\delta)\right) = 2\pi\sqrt{\lambda}\left(1+O(\lambda^{-\delta/4})\right). \tag{175}$$

Corollary 12. *The length of $\chi_1(\lambda,\beta,\delta)$ obeys the estimate:*

$$l\left(\chi_1(k,\beta,\delta)\right) = 2\pi\sqrt{\lambda}\left(1+O(\lambda^{-\delta/4})\right). \tag{176}$$

Proof. We see from formula (174) that the set $\hat{D}(\lambda,\beta,\delta)$ is the set $D(\lambda,\beta)$ minus neighborhoods of intersections of $D(\lambda,\beta)$ with all $D_r(\lambda,\beta)$, $r \in Z^2 \setminus \{0\}$, i.e.,

$$\hat{D}(\lambda,\beta,\delta) = D(\lambda,\beta) \setminus L(\lambda,\beta,\delta),$$

here

$$L(\lambda,\beta,\delta) = \cup_{r\in Z^2\setminus\{0\}} L_r(\lambda,\beta,\delta),$$
$$L_r(\lambda,\beta,\delta) =$$
$$\left\{\vec{k} \in D(\lambda,\beta) : \vec{k}-\vec{p}_r(0) \in \Pi(\beta)\cap M_1(\lambda_0), \left|\hat{\lambda}(\vec{k})-\hat{\lambda}\left(\vec{k}-\vec{p}_r(0)\right)\right| < 4\lambda^{-\delta}\right\}.$$

To prove (175) it is enough to show that the length of $L(\lambda,\beta,\delta)$ satisfies the estimate:

$$l\left(L(\lambda,\beta,\xi)\right) < c\lambda^{1/2-\delta/4}, \tag{177}$$

since $D(\lambda,\beta)$ satisfies (66). Obviously,

$$l\left(L(\lambda,\beta,\delta)\right)\leq\sum_{r\in Z^2\setminus\{0\}}l\left(L_r(\lambda,\beta,\delta)\right).$$

We prove the following estimates:

$$l\left(L_r(\lambda,\beta,\delta)\right)<\frac{c\rho^{-2\delta+1}}{p_r(0)\sqrt{\rho^2-p_r^2(0)/4}},\quad \rho=\sqrt{\lambda} \tag{178}$$

if $p_r(0)/2\leq\rho-\rho^{\delta/2}$, and

$$l\left(L_r(\lambda,\beta,\xi)\right)<c\rho^{-\delta}, \tag{179}$$

if $\rho-\rho^{\delta/2}<p_r(0)/2<\rho+\rho^{\delta/2}$, and

$$l\left(L_r(\rho,\beta,\delta)\right)=0, \tag{180}$$

if $p_r(0)/2\geq\rho+\rho^{\delta}$. We estimate the sum over r of the right-hand sides of (178) as follows:

$$\sum_{m:p_r(0)/2\leq\rho-\rho^{\delta}}l\left(L_r(\rho,\beta,\delta)\right)\leq\sum_{r:p_r(0)/2\leq\rho-\rho^{\delta}}\frac{c\rho^{-2\delta+1}}{p_r(0)\sqrt{\rho^2-p_r^2(0)/4}}\leq$$

$$c\rho^{-2\delta+1}\int_{\tilde{B}}\frac{dx}{|x|\sqrt{\rho^2-|x|^2/4}}\leq c\rho^{-2\delta+1}=O(\lambda^{1/2-\delta}),$$

here $\tilde{B}$ is the disc of radius $\rho-\frac{1}{2}\rho^{\delta/2}$ in R^2. Considering that the sum over $r:\rho-\rho^{\delta/2}<p_r(0)/2<\rho+\rho^{\delta/2}$ contains no more than $c\rho^{1+\delta/2}$ terms, we obtain:

$$\sum_{r:\rho-\rho^{\delta}<p_r(0)/2<\rho+\rho^{\delta}}l\left(L_r(\lambda,\beta,\delta)\right)\leq c\rho^{1-\delta/2}=O(\lambda^{1/2-\delta/4}).$$

Adding the last two estimates and considering (180), we obtain (177).

Thus, it remains to prove (178) – (180). Note that $L_r(\lambda,\beta,\delta)$ is the intersection of a deformed sphere with a deformed plane layer. Assuming that k_1 is directed along $\vec{p}_r(0)$, we see that $L_r(\rho,\beta,\delta)$ is the set of $\vec{k}$ satisfying the equation for a deformed sphere and the inequality for a distorted plane layer:

$$k_1^2+k_2^2+f(k_1,k_2)=\rho^2, \tag{181}$$

$$\left|k_1-\frac{p_r(0)}{2}+\frac{g(k_1,k_2)}{2p_r(0)}\right|<\frac{2\rho^{-2\delta}}{p_r(0)}, \tag{182}$$

where $f(k_1,k_2)=\Re\delta\hat{\lambda}(\vec{k})$,

$$g(k_1,k_2)=\Re\delta\hat{\lambda}(\vec{k})-\Re\delta\hat{\lambda}\left(\vec{k}-\vec{p}_r(0)\right). \tag{183}$$

By (52) and (53),

$$\|f\|_{L_\infty}<c\rho^{1-2\beta}, \tag{184}$$

$$\|\nabla f\|_{L_\infty}<c\rho^{1/2-3\beta}, \tag{185}$$

$$\|g\|_{L_\infty}<c\rho^{1-2\beta}, \tag{186}$$

$$\|\nabla g\|_{L_\infty}<c\rho^{1/2-3\beta}. \tag{187}$$

Introducing polar coordinates $k_1=R\cos\varphi, k_2=R\sin\varphi$, we rewrite (181) and (182) in the form:

$$R^2+f(R,\varphi)=\rho^2, \tag{188}$$

$$\left|R\cos\varphi-\rho\cos\varphi_0+\frac{g(R,\varphi)}{2p_r(0)}\right|<\frac{2\rho^{-2\delta}}{p_r(0)}, \tag{189}$$

$$\cos\varphi_0 = \frac{p_r(0)}{2\rho}. \tag{190}$$

Solving (188) with respect to R, we obtain a function $R(\varphi)$,

$$R(\varphi) - \rho = O(\rho^{-2\beta}), \tag{191}$$

and

$$R'(\varphi) = O(\rho^{1/2-3\beta}). \tag{192}$$

We rewrite (188), (189) in the form:

$$\left| R(\varphi)\cos\varphi - \rho\cos\varphi_0 + \frac{g(R(\varphi),\varphi)}{2p_r(0)} \right| < \frac{2\rho^{-2\delta}}{p_r(0)}. \tag{193}$$

Let us introduce the notation:

$$A(\varphi) \equiv \left(R(\varphi)\cos\varphi - \rho\cos\varphi_0 + \frac{g(R(\varphi),\varphi)}{2p_r(0)} \right)',$$

here $'$ being the derivative with respect to φ. Suppose we have proved that

$$A(\varphi) =_{\rho\to\infty} \rho\sin\varphi_0\left(1+o(1)\right). \tag{194}$$

for all φ satisfying (193). Then considering that $\sin\varphi_0 \neq 0$ when $p_r(0)/2 \leq \rho - \rho^{\delta/2}$, we reduce (193) to the form:

$$|\varphi - \tilde{\varphi}_0| < \frac{c\rho^{-2\delta}}{p_r(0)\rho|\sin\varphi_0|} = \frac{c\rho^{-2\delta}}{p_r(0)\sqrt{\rho^2 - p_r^2(0)/4}}$$

for some $\tilde{\varphi}_0$ and, hence (178) follows. Thus, it remains to prove (194). Producing differentiation in the formula for $A(\varphi)$ and using the estimate (187) together with (191) and (192), we arrive at:

$$A(\varphi) = R(\varphi)\sin\varphi + O\left(\rho^{3/2-3\beta}/p_r(0)\right). \tag{195}$$

It remains to prove that the right-hand side of (195) can be rewriten as $\rho\sin\varphi_0(1+o(1))$. In fact, substituting (191) and (186) into (193), we rewrite it in the form:

$$\rho\cos\varphi - \rho\cos\varphi_0 = O(\rho^{-2\beta}) + O(\rho^{1-2\beta})p_r^{-1}(0) + O(\rho^{-2\delta})p_r^{-1}(0). \tag{196}$$

First, suppose $0 < p_r(0) < \rho$. Then, by (190), $\cos\varphi_0 < 1/2$ and $\varphi_0 > \pi/3$. From (196) it follows $\rho\cos\varphi - \rho\cos\varphi_0 = o(\rho)$, and, hence, $\varphi - \varphi_0 = o(1)$, i.e., $\varphi > \frac{\pi}{3}(1+o(1))$. Considering (195) and taking into account that $\beta > 1/4$, we obtain $A(\varphi) = \rho\sin\varphi + o(\rho) = \rho\sin\varphi_0(1+o(1))$. Second, suppose $\rho \leq p_r(0) < 2\rho - \rho^{\delta/2}$. Then, by (190) $\sin\varphi_0 > \rho^{-1/2+\delta/4}$, and, by (196), $\rho\cos\varphi - \rho\cos\varphi_0 = o(1)$. Hence, $\cos\varphi - \cos\varphi_0 = o(\rho^{-1})$, $\sin^2\frac{\varphi}{2} = \sin^2\frac{\varphi_0}{2} + o(\rho^{-1})$, $\sin^2\frac{\varphi}{2} = \sin^2\frac{\varphi_0}{2}(1+o(1))$, $\sin\varphi > \sin\varphi_0(1+o(1)) > \rho^{-1/2+\delta/4}$. Using (195), we obtain $A(\varphi) = \rho\sin\varphi + O(\rho^{1/2-3\beta}) = \rho\sin\varphi_0(1+o(1))$. Thus, we have proved (194) and, therefore, (178).

Suppose $2\rho - \rho^{\delta/2} < p_r(0) < 2\rho + \rho^{\delta/2}$. Using (186) and (191) in (193), we obtain: $\rho(\cos\varphi - \cos\varphi_0) = O(\rho^{-2\beta}) + O(\rho^{-2\delta-1}) = O(\rho^{-2\beta})$. Therefore, $|\cos\varphi - \cos\varphi_0| = O(\rho^{-2\beta-1})$ and, hence, $|\varphi - \varphi_0| < c\rho^{-\beta-1/2}$. Considering the last relation and (192), we obtain:

$$R(\varphi) - r(\varphi_0) = O(\rho^{-4\beta}).$$

Taking into account that $g(R(\varphi),\varphi)' = g_R R' + g_\varphi = O(\rho^{3/2-3\beta})$, we get:

$$g(R(\varphi),\varphi) - g(R(\varphi_0),\varphi_0) = O(\rho^{1-4\beta}).$$

Using the last two estimates, we can reduce (193) to the form:

$$|R(\varphi_0)\cos\varphi - a| = O(\rho^{-2\delta-1}) + (\rho^{-4\beta}),$$

here $a = \rho\cos\varphi_0 - g(R(\varphi_0),\varphi_0)/2p_r(0)$. Considering that $2\delta < 4\beta - 1$, we get $|R(\varphi_0)\cos\varphi - a| = O(\rho^{-2\delta-1})$. Hence, $L_r(\rho,\beta,\delta)$ can be described by the inequality $|\varphi - \tilde{\varphi}| < c\rho^{-\delta-1}$ for some $\tilde{\varphi}$. It easily follows that $l(L_r(\rho,\beta,\delta)) < c\rho^{-\delta}$. Thus, estimate (179) is checked. The relation (180) is easy to see considering that the deviation of D from the exact sphere of radius $\sqrt{\lambda}$ does not exceed $c\lambda^{-\beta}$ (see (191)). $\diamondsuit$.

The property of χ_1 established in the next lemma we will use later to estimate the area of $\chi_1 \setminus \chi_2$.

Lemma 15. *Let ε, β and δ satisfy the conditions in Lemma 14. Suppose $S \subseteq \chi_1$ and $l(S) > \lambda^{1/2-\mu}$, $\mu \geq 0$. Then the area of the $(\lambda^{-1/2-2\delta})$-neighborhood of S in K can be estimated from below by $c\lambda^{-\mu-2\delta}$.*

Proof. First, we prove the lemma for $S = \chi_1$, $\mu = 0$ By Lemmas 12 and 13, the parallel shifts establishes one-to-one correspondence between χ_1 and $\hat{D}$ and, moreover, between their $(\lambda^{-1/2-2\delta})$-neighborhoods. Since $\hat{D}$ is just a slightly distorted circle with holes with a length $2\pi\sqrt{\lambda}(1+O(\lambda^{-\delta/4}))$, its neighborhood can be estimated from below by $2\pi\lambda^{-2\delta}(1+O(\lambda^{-\delta/4}))$. By Corollary 11, the volume of the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_1 in K is equal to the volume of the $(\lambda^{-1/2-2\delta})$-neighborhood of $\hat{D}$ in R^2. Thus, the lemma is proved for $S = \chi_1$. Suppose $S \subset \chi_1$. Then, obviously, there is $\hat{S} \subset \hat{D}$ such that $\mathcal{K}\hat{S} = S$ and there is one-to one neighborhoods $\hat{S}$ and S and, moreover, between their $(\lambda^{-1/2-2\delta})$-neighborhoods. Obviously, the area of the neighborhood of $\hat{S}$ exceeds $c\lambda^{-\mu-2\delta}$ and the same is true for the neighborhood of S. $\diamondsuit$

Let us consider the set $\chi_2(\lambda,\delta)$. In the following lemma we obtain the estimate for the area of its $(\lambda^{-1/2-2\delta})$-neighborhood.

Lemma 16. *The area of the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_2(\lambda,\delta)$ does not exceed $c_{\vec{A},b}\lambda^{-\delta-\varepsilon}$.*

Proof. It follows from Corollary 9 that $\chi_2 \subseteq \chi_2'$, where

$$\chi_2' = \left\{t \in K : \exists q \in \Gamma(\lambda^{\varepsilon}),\ \ n \in \{1,...,N_q\},\ \ |\mu_n^{(q)}(t) - \lambda| < 5\lambda^{-\delta}\right\}, \tag{197}$$

$\left\{\mu_n^{(q)}(t)\right\}_1^{N_q}$ being eigenvalues of the operator $P_qH(t)P_q$ numerated in the increasing order. Therefore, the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_2 belongs to the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_2'. We denote the latter neighborhood by χ_2''. Since the area of the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_2 does not exceed the area of χ_2'', it suffices to show that the area of χ_2'' is no larger than $c_{\vec{A},b}\lambda^{-\delta-\varepsilon}$. Taking into account that $|\nabla_t\mu_n^{(q)}| < c_b\sqrt{\lambda}$ (see Lemma 10), we obtain that χ_2'' can be described by a formula similar to (197):

$$\chi_2'' \subset \left\{t \in K : \exists q \in \Gamma(\lambda^{\varepsilon}),\ \ n \in \{1,...,N_q\},\ \ |\mu_n^{(q)}(t) - \lambda| < c_b\lambda^{-\delta}\right\}.$$

With some abuse of notations since now we denote by χ_2'' the set on the right-hand side of the last formula. We represent χ_2'' in the form:

$$\chi_2'' = \cup_{q,n}\ \chi_2''(q,n),$$

where $\chi_2''(q,n)$ is the set corresponding to just one inequality:

$$\chi_2''(q,n) = \left\{ t \in K : |\mu_n^{(q)}(t) - \lambda| < c_b \lambda^{-\delta} \right\}.$$

First, we estimate the number of non-empty sets $\chi_2''(n,q)$. In fact, if $\chi_2''(q,n)$ is not empty, than $|\mu_n^{(q)}(t) - \lambda| < c_b \lambda^{-\delta}$, for at least one t in K. Considering that $|\nabla \mu_n^{(q)}(t)| < c_b \sqrt{\lambda}$ (Lemma 10), we obtain: $|\mu_n^{(q)}(t) - \lambda| < \tilde{c}_b \sqrt{\lambda}$ for all $t \in K$. By Lemma 10 the number of eigenvalues of $P_q H(t) P_q$ satisfying the last inequality does not exceed $c_{\vec{A},b} \lambda^{1-3\varepsilon}$. Considering that $\Gamma(\lambda^\varepsilon)$ contains no more than $c_b \lambda^{2\varepsilon}$ elements, we obtain that the number of eigenvalues $\mu_n^{(q)}(t)$, $q \in \Gamma(\lambda^\varepsilon)$, $n \in \{1, ..., N_q\}$, satisfying the inequality $|\mu_n^{(q)}(t) - \lambda| < c_b \sqrt{\lambda}$ for all t does not exceed $c_{\vec{A},b} \lambda^{1/2-\varepsilon}$. Therefore, there is no more than $c_{\vec{A},b} \lambda^{1/2-\varepsilon}$ non-empty sets $\chi_2''(q,n)$.

Second, we estimate the area of a non-empty $\chi_2''(q,n)$. By Lemma 10, each $\mu_n^{(q)}(t)$ is a monotonous continuous function of t in a direction $\vec{u}_q$ orthogonal to $\vec{p}_q(0)$. Since functions $\mu_n^{(q)}(t)$ can be expended periodically from K to the whole plane R^2, we can define $\mu_n^{(q)}(t)$ as a function of (ζ, τ) in R^2, here $\zeta = (t, \vec{u}_q)$ and $\tau = (t, \vec{\nu}_q)$, $\vec{\nu}_q$ being a unit vector in the direction of $\vec{p}_q(0)$. We consider ζ and τ changing in intervals big enough to cover the cell K. Because of monotonicity, an equation $\mu_n^{(q)}(t) = \lambda$ can be solved with respect to $\zeta = (t, \vec{u}_q)$, i.e., the curve $\mu_n^{(q)}(t) = \lambda$ can be described by the equation $\zeta = f_n(\tau)$. It is not difficult to show that $f_n(\tau)$ is a piece-wise continuous function. [5] Again, because of monotonicity and estimate (160), the set $|\mu_n^{(q)}(t) - \lambda| < c_b \lambda^{-\delta}$ is included in a neighborhood of the curve $\zeta = f_n(\tau)$, namely in the set $\{t = (\zeta, \tau) : |\zeta - f_n(\tau)| < c_b \lambda^{-1/2-\delta}\}$. Obviously, the area of this set is less than $c_b \lambda^{-1/2-\delta}$. Thus, the area of $\chi_2''(q,n)$ does not exceed $c_b \lambda^{-1/2-\delta}$. Considering that the number of non-empty sets $\chi_2''(q,n)$ is no larger than $c_{\vec{A},b} \lambda^{1/2-\varepsilon}$, we obtain that the area of χ_2'' and, therefore, of the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_2 does not exceed $c_{\vec{A},b} \lambda^{-\delta-\varepsilon}$. $\diamondsuit$

Lemma 17. *Let $0 < \varepsilon < 1/8$, $1/4 < \beta < 1/2 - 2\varepsilon$, $0 < 2\delta < \min\{1/2 - \beta - 2\varepsilon, 2\beta - 1/2, \varepsilon/2\}$. Then the length of $(\chi_1 \cap \chi_2)(\lambda, \delta)$ admits the estimate:*

$$l\,(\chi_1 \cap \chi_2) < c_{\vec{A}} \lambda^{1/2-\delta}. \tag{198}$$

Proof. We prove (198) by contradiction. Suppose (198) does not hold. Let us consider the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1 \cap \chi_2$. On one hand, by Lemma 15 ($S = \chi_1 \cap \chi_2$) the area of this neighborhood is greater than $c_b \lambda^{-3\delta}$. On the other hand, $\chi_1 \cap \chi_2 \subset \chi_2$ and, therefore, the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1 \cap \chi_2$ belongs to the $(\lambda^{-1/2-2\delta})$-neighborhood of χ_2. By Lemma 16, the area of the latter neighborhood is less than $c_b \lambda^{-\varepsilon-\delta}$, and, hence the area of the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1 \cap \chi_2$ is less than $c_b \lambda^{-\varepsilon-\delta}$. Considering that $2\delta < \varepsilon$, we see that two estimates for the area of the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1 \cap \chi_2$ are in contradiction for λ large enough. $\diamondsuit$

The proof of Lemma 5 Lemma 5 immediately follows from Corollary 12 and Lemma 17.

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[5] To show this we notice that the curves $\mu_n^{(q)}(t) = \lambda$, $n = 1, ..., N_q$, form a family of algebraic curves, since the determinant of $P_q H P_q$ is a polynomial with respect to t_1, t_2. These curves can have only a finite number of intersections, except the case when they are identical.

3.5. **Proof of the Main Theorems.** Proof of Theorem 2. If $t \in \chi_*(\lambda,\delta)$, then there is a single $j \in J(\lambda_0)$ such that $\Re\hat{\lambda}_j(t) = \lambda$. From the definition (92) of the operator $\hat{H}(t)$ we see that $\Re\hat{\lambda}_j(t)$ is its eigenvalue. The inequality in the definition (95) of $\chi_*(\lambda,\delta)$ means that $\Re\hat{\lambda}_j(t)$ is a simple eigenvalue of $\hat{H}(t)$, moreover all other eigenvalues are located at the distance greater than $4\lambda^{-\delta}$ from $\Re\hat{\lambda}_j(t)$, in particular, (75) holds.

From (70) – (72) it follows that the operator $H(t)$ has an eigenvalue $\lambda_j(t)$ satisfying the inequality $|\lambda_j(t) - \Re\hat{\lambda}_j(t)| < C_{\vec{A}}\lambda^{-21/2}$, here $\Re\hat{\lambda}_j(t) = \lambda$ when $t \in \chi_*(\lambda,\delta)$. By Lemma 8, $\|H(t) - \hat{H}(t)\| < C_{\vec{A}}\lambda^{-10}$. Using standard perturbation considerations [Kato] we obtain that $\lambda_j(t)$ is a unique eigenvalue of $H(t)$ in the interval $\epsilon(\lambda)$ and estimate (78) holds. The proof of (77) is somewhat more complicated. In fact, let us note that the relations (70) and (71) hold not only for real t in K but also for all $t : \Re t \in K, |\Im t| < p_j(0)^{1/2+\beta-2\varepsilon}$, the model values $\hat{\lambda}_j(t)$ and coefficients of the function $\hat{\Psi}_j(t,x)$ in the basis (9) being analytic with respect to each component of t. We consider $\hat{\Psi}_j(t,x)$ as a vector in the basis of free waves (9). Correspondingly, by $\nabla_t\hat{\Psi}_j(t,x)$ we denote a quasiperiodic vector-functions whose coefficients in the basis (9) are gradients of the coefficients of $\hat{\Psi}_j(t,x)$ with respect to t. Taking into account that $\|\Psi_j(t,x)\|_0 < 2$ in the mentioned above neighborhood and using a Cauchy integral we obtain:

$$\|\nabla_t\hat{\Psi}_j(t,x)\|_0 < 2p_j(0)^{-(1/2+\beta-2\varepsilon)} < 1. \tag{199}$$

Considering that the matrix of $H(t)$ is analytic, we obtain that $\delta\hat{\Psi}_j(t,x)$ is analytic with respect to each component of t. Using (71) and a Cauchy integral we obtain:

$$\left\|\nabla_t\delta\hat{\Psi}_j(t,x)\right\|_0 < \lambda^{-21/2}p_j(t)^{-(1/2+\beta-2\varepsilon)} < \lambda^{-21/2}. \tag{200}$$

Differentiating (70) with respect to t and multiplying both parts by an eigenfunction $\Psi_j(t,x)$ of $H(t)$, we obtain:

$$< D(t)\hat{\Psi}_j(t), \Psi_j(t) >= \nabla_t\hat{\lambda}_j(t) < \hat{\Psi}_j(t), \Psi_j(t) > +O(\lambda^{-21/2}), \tag{201}$$

here $D(t) = \nabla_t H(t) = \nabla_t H_0(t)$ and estimates (199) and (200) have been used. Obviously,

$$\left|< D(t)\hat{\Psi}_j(t), \Psi_j(t) > - < D(t)\Psi_j(t), \Psi_j(t) >\right| \leq$$
$$\|\hat{\Psi}_j(t) - \Psi_j(t)\|_{L_2(Q)}\|D(t)\Psi_j(t)\|_{L_2(Q)}.$$

By elementary perturbation considerations we show that $\|D(t)\Psi_j(t)\|_{L_2(Q)} < c_{\vec{A}}\sqrt{\lambda}$. Therefore, by (78),

$$< D(t)\hat{\Psi}_j(t), \Psi_j(t) > - < D(t)\Psi_j(t), \Psi_j(t) >= O(\lambda^{-19/2+2\delta}). \tag{202}$$

It is easy to show that

$$< D(t)\Psi_j(t), \Psi_j(t) >= \nabla_t\lambda_j(t) < \Psi_j(t), \Psi_j(t) > . \tag{203}$$

Subtracting (203) from (201) and using (202), we obtain

$$\nabla_t\lambda_j(t) < \Psi_j(t), \Psi_j(t) > -\nabla_t\hat{\lambda}_j(t) < \hat{\Psi}_j(t), \Psi_j(t) >= O(\lambda^{-19/2+2\delta}).$$

Using (78) and (49), we arrive at (77). The forth part of the theorem follows from Lemma 11 and considerations as above. Formula (79) is proved by Lemma 5. $\diamondsuit$

Proof of Theorem 3. By Lemma 13, the parallel shift $\mathcal{K} : \mathcal{K}\vec{p}_j(t) = t$ establishes one-to-one correspondence between the $(\lambda^{-1/2-2\delta})$-neighborhood of $\chi_1(\lambda,\delta)$ and the $(\lambda^{-1/2-2\delta})$-neighborhood of $\hat{D}(\lambda,\delta)$. Since $\chi_*(\lambda,\delta)$ is a subset of $\chi_1(\lambda,\delta)$, there is a unique subset of $D_*(\lambda,\delta)$ of $\hat{D}(\lambda,\delta)$ such that $\mathcal{K}D_*(\lambda,\delta)=\chi_*(\lambda,\delta)$ and there is one-to-one correspondence between their $(\lambda^{-1/2-2\delta})$-neighborhoods (the opposite sides of K are identified). By analogy with (69), we define the function $\lambda(\vec{k})$ in the neighborhood of $D_*(\lambda,\delta)$ by the formula: $\lambda(\vec{k})=\lambda_j(t)$, here $\vec{k}=\vec{p}_j(t)$. It is differentiable in the $(\lambda^{-1/2-2\delta})$-neighborhood of $D_*(\lambda,\delta)$ and by Corollary 4, $\nabla_{\vec{k}}\lambda(\vec{k})=\vec{k}+O(\lambda^{(1-6\beta)/4})$. The last estimate yields: $\lambda_k(\vec{k})=k+O(\lambda^{(1-6\beta)/4})$, the notation $k=|\vec{k}|$ has been used. Formula (76) gives: $\lambda(\vec{k})-\Re\hat{\lambda}(\vec{k})=O(\lambda^{-21/2})$. Using Implicit Function Theorem and considering (63), we obtain that there is a surface $\tilde{D}_*$ in the $(c\lambda^{-11})$-neighborhood of D_* such that $\lambda(\vec{k})=\lambda$ for any $\vec{k}\in\tilde{D}_*$ and formulas (83) – (85) hold. If $t\in I_*=\mathcal{K}\tilde{D}_*$, then $\lambda_j(t)=\lambda$. Thus, I_* is a piece of the isoenergetic surface of the operator H.

The set $D_{*1}(\lambda,\delta)$ is defined as the set of directions corresponding to $D_*(\lambda,\delta)$ (see (82)). It follows immediately from (81) that $D_{*1}(\lambda,\delta)$ has an asymptotically full measure on the unit circle:

$$\frac{l\left(D_{*1}(\lambda,\delta)\right)}{2\pi}>1-c_{\vec{A}}\lambda^{-\delta/4}.\ \ c_{\vec{A}}>0.$$

Using the last estimate and formula (83) for $\tilde{D}_*$ we readily prove (86). $\diamondsuit$

Proof of Theorem 4. Existence of an isoenergetic set for a given λ implies that λ belongs to the spectrum. Thus, by Corollary 7, every large enough λ belongs to the spectrum, since the corresponding isoenergetic set exists and rather large (see (89)). This immediately implies finiteness of the number of gaps in the spectrum, since only a finite number of zones (and gaps) can be located below certain energy [RS]. $\diamondsuit$

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