MONOTONICITY AND THERMODYNAMIC LIMIT FOR SHORT RANGE DISORDERED MODELS[1]

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To Giovanni Jona-Lasinio on his 70-th birthday

Abstract

If the variance of a short range Gaussian random potential grows like the volume its quenched thermodynamic limit is reached monotonically.

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Introduction

The question of the existence of thermodynamic limit for all standard (i.e, short range, see Remark 2 below) models of spin glasses with two-body interactions has been settled long ago by Khanin and Sinai [KS] and later generalized to more general interactions by Zegarlinski [Ze] (previous references on the subject include [Le], [Vu], [PF]). The sharper property of the monotonicity of the free energy in the volume has been proved by van Enter and van Hemmen[EH]. In the long-range case (the most important examples being the Sherrington-Kirckpatrick model [MPV], the REM and the GREM [DG]) the existence and the monotonicity has been instead proved only very recently [GT],[GT2], [CDGG]. The proof relies on an interpolation argument introduced in [GT] which has the advantage of yielding the subadditivity of the free energy (equivalently, superadditivity of the pressure). Exactly as in the ferromagnetic case [Ru], and in [EH], the subadditivity entails the important property of the *monotonicity* of the free energy (pressure) as the volume increases.

In this paper we show that the above interpolation argument can be applied (actually in a slightly different form) to the short range case. For Gaussian couplings, in the summable case and in the non-summable one as well, we generalize the Khanin-Sinai and van Enter-van Hemmen results. Namely, for general, mixed, short-range $n-$body interactions (n arbitrary: for the physical relevance of $n > 2$ see e.g.[Li]), and assuming free boundary conditions, the free energy, internal energy, ground state energy are not only bounded but also decreasing in the volume. Hence their thermodynamic limit is reached monotonically. We remark that in the disordered case the monotonicity is even more relevant than in the ferromagnetic one because the ground state energy is tacitly assumed monotonic in all numerical simulations; for a discussion of this point, see e.g.[Ri], [BCDG].

The conclusion which may be drawn by this paper, together with references [GT] and [CDGG], is that as far as the thermodynamic limit is concerned Gaussian spin glasses in free boundary conditions do not differ from ordinary ferromagnets: in both cases pressure, internal energy and ground state energy are bounded and monotonic in the volume.

Definitions and Examples

Let M be a countable set and consider a finite subset $\Lambda \subset M$ of cardinality $|\Lambda| = N$. To each element $i \in \Lambda$ we associate a dynamical variable $\sigma_i \in \mathcal{S} \subset \mathbb{R}^k$ (for some fixed integer k) equipped with an *a priori* probability measure ν_i. For each $X \subset \Lambda$ we consider $\sigma_X = \{\sigma_i\}_{i\in X}$ and a function $\Phi_X : \sigma_X \to \Phi_X(\sigma_X) \in \mathbb{R}$.

In analogy to [Ru] (Sect.2.4, formula 4.3) and [Ze] we define the *random potential* as

$$U_\Lambda(J,\sigma) = \sum_{X\subset\Lambda} J_X \Phi_X(\sigma_X) \ , \tag{1}$$

(with $\Phi_\emptyset = 0$) under the following assumption: *the coefficients J_X are independent Gaussian variables with zero mean and variance depending only on X (and not on* Λ)

$$\mathrm{Av}(J_X) = 0 \ , \quad \mathrm{Av}(J_X^2) = \Delta_X^2 \ . \tag{2}$$

Examples:

Here $M = \mathbb{Z}^d$, and Λ is a cube.

1. The Edwards-Anderson model. $\mathcal{S} = \{+1, -1\}$, $\nu(\sigma_i) = \frac{1}{2}[\delta_1 + \delta_{-1}]$. The nearest neighbor case is defined by $\Phi_{n,n'}(\sigma_n, \sigma_{n'}) = \sigma_n \sigma'_n$ for $|n - n'| = 1$, $\Phi_X = 0$ otherwise, and $\Delta_X^2 = c^2$. More generally one may consider a short range interaction with with $\Delta_X^2 = |n - n'|^{-2d\alpha}$, $\alpha > 1/2$, or a many-body interaction with a suitable decay law.

2. Multicomponent spin models (Potts models): $\mathcal{S} = \{1, 2, ..., q\}$, $\nu(\sigma_i) = \frac{1}{q}\sum_{l=1}^{q} \delta_l$, $\Phi_X(\sigma_X) = \delta_{\sigma_X}$ where $\delta_{\sigma_X} = 1$ if all components of σ_X are equal and zero otherwise.

3. Continuous spin models: $\mathcal{S} = \mathbb{R}^k$, $\nu(\sigma_i) = d\mu(x) \geq 0$, $\int_{R^k} d\mu(x) = 1$ (unbounded case) or $\mathcal{S} = \mathbb{T}^k$, $\nu(\sigma_i) = d\phi$ (bounded case);

4. Lattice gases: here $\mathcal{S} = \{0, 1\}$, $\nu(\sigma_i) = \frac{1}{2}[\delta_0 + \delta_1]$.

Remarks:

1. Of course the examples may be considered on every finite dimensional lattice like $\mathbb{Z}^d$ or the triangular lattice etc.

2. The property that Δ_X^2 is independent of the volume Λ characterizes the short range case, such as the Edwards-Anderson one. In mean field (long range) models, such as the Sherrington-Kirckpatrick one, the variance has to decrease with N in order to have finite energy density.

Denoting $P_\Lambda(d\sigma) = \prod_{i\in\Lambda} d\nu_i(\sigma_i)$ we define:

1. The random partition function

$$Z_\Lambda(J) := \int_{\mathcal{S}^N} P_\Lambda(d\sigma) e^{U_\Lambda(J,\sigma)} , \tag{3}$$

2. The random Gibbs-Boltzmann state

$$\omega(-) := \frac{\int_{\mathcal{S}^N} P_\Lambda(d\sigma) - e^{U_\Lambda(J,\sigma)}}{Z_\Lambda(J)} , \tag{4}$$

3. The quenched state

$$< - > := \mathrm{Av}(\omega(-)) , \tag{5}$$

4. The quenched pressure

$$P_\Lambda := \mathrm{Av}\left(\ln Z_\Lambda(J)\right) . \tag{6}$$

5. The quenched potential

$$U_\Lambda := < U_\Lambda(J,\sigma) > \tag{7}$$

We remind that the free energy F_Λ is $-\beta^{-1}P_\Lambda$, and the internal energy E_Λ is $\beta^{-1}U_\Lambda$.

Superadditivity

Lemma 1

$$< J_X \Phi_X > \geq 0 . \tag{8}$$

Proof.

We remind the integration by parts for Gaussian variables

$$\mathrm{Av}\,(J_X f(J)) \;=\; \Delta_X^2 \mathrm{Av}\left(\frac{df(J)}{dJ_X}\right) \;, \tag{9}$$

and the correlation derivative formula

$$\frac{d\omega(\Phi_X)}{dJ_X} \;=\; \omega(\Phi_X^2) - \omega(\Phi_X)^2 \;\geq 0. \tag{10}$$

By applying successively (9) and (10) we obtain

$$\begin{aligned} < J_X \Phi_X > \;&=\; \mathrm{Av}\,(J_X \omega(\Phi_X)) \;= \\ &=\; \Delta_X^2 \mathrm{Av}\left(\omega(\Phi_X^2) - \omega(\Phi_X)^2\right) \geq 0 \;. \end{aligned} \tag{11}$$

As a corollary of lemma 1 we have

$$< U_\Lambda(J,\sigma) >= \sum_{X \subset \Lambda} \Delta_X^2 \mathrm{Av}\left(\omega(\Phi_X^2) - \omega(\Phi_X)^2\right) \geq 0 \tag{12}$$

Definition 1 *Consider a partition of Λ into n non empty disjoint sets Λ_s:*

$$\Lambda = \bigcup_{s=1}^{n} \Lambda_s \;, \tag{13}$$

$$\Lambda_s \cap \Lambda_{s'} = \emptyset \;. \tag{14}$$

For each partition the potential generated by all interactions among different subsets is defined as

$$\tilde{U}_\Lambda = U_\Lambda - \sum_{s=1}^{n} U_{\Lambda_s} \;; \tag{15}$$

from (1) we have that

$$\tilde{U}_\Lambda = \sum_{X \in \mathcal{C}_\Lambda} J_X \Phi_X \tag{16}$$

where $\mathcal{C}_\Lambda$ is the set of all $X \subset \Lambda$ which are not subsets of any Λ_s.

Theorem 1 *The quenched potential is superadditive:*

$$< U_\Lambda > \geq \sum_{s=1}^{n} < U_{\Lambda_s} > \tag{17}$$

Proof. Direct consequence of (8). In fact:

$$< \tilde{U}_\Lambda > = \sum_{X \in \mathcal{C}_\Lambda} < J_X \Phi_X > = \sum_{X \in \mathcal{C}_\Lambda} \Delta_X^2 \mathrm{Av} \left(\omega(\Phi_X^2) - \omega(\Phi_X)^2 \right) \geq 0\,. \tag{18}$$

Theorem 2 *The quenched pressure is superadditive:*

$$P_\Lambda \; \geq \sum_{s=1}^{n} P_{\Lambda_s} \; . \tag{19}$$

Proof.

To each partition of Λ we associate the interpolating potential for $0 \leq t \leq 1$

$$U_\Lambda(t) \; = \sum_{s=0}^{n} \sqrt{t_s} U_{\Lambda_s}^{(s)} \; , \tag{20}$$

with $t_0 = t$, $t_s = (1-t)$ for $1 \leq s \leq n$, $U_{\Lambda_0}^{(0)} = U_\Lambda$ and

$$U_{\Lambda_s}^{(s)} \; = \sum_{X \subset \Lambda_s} J_X^{(s)} \Phi_X \; , \tag{21}$$

where any $J_X^{(s)}$ is a centered independent Gaussian

$$\mathrm{Av} \left(J_X^{(s)} J_Y^{(q)} \right) = \delta_{s,q} \delta_{X,Y} \Delta_X^2 \tag{22}$$

(the symbol Av is here the average with respect to all the J's). We define the interpolating partition function

$$Z_\Lambda(t) \; = \int_{\mathcal{S}^N} P_\Lambda(d\sigma) e^{U_\Lambda(t)} \; , \tag{23}$$

and we observe that

$$Z_\Lambda(0) \; = \prod_{s=1}^{n} Z_{\Lambda_s}(J^{(s)}) \;, \quad Z_\Lambda(1) = Z_\Lambda(J) \; . \tag{24}$$

Consider the interpolating pressure

$$P_\Lambda(t) := \; \mathrm{Av}\left(\ln Z_\Lambda(t)\right) \;, \tag{25}$$

and the corresponding states $\omega_t(-)$ and $< - >_t$. Thanks to (24) we get

$$P_\Lambda(0) \; = \sum_{s=1}^{n} P_{\Lambda_s} \;, \quad P_\Lambda(1) = P_\Lambda \; . \tag{26}$$

We observe now that

$$\frac{d}{dt}P_\Lambda(t) \; = \sum_{s=0}^{n} \frac{\epsilon_s}{\sqrt{t_s}} < U^{(s)}_{\Lambda_s} >_t \;, \tag{27}$$

with $\epsilon_0 = 1$ and $\epsilon_s = -1$ for $1 \leq s \leq n$. For each s we have

$$< U^{(s)}_{\Lambda_s} >_t = \sum_{X \subset \Lambda_s} < J^{(s)}_X \Phi_X >_t \; ; \tag{28}$$

Integrating by parts each addend we obtain

$$< J^{(s)}_X \Phi_X >_t = \; \sqrt{t_s}\Delta^2_X \mathrm{Av}\left(\omega_t(\Phi^2_X) - \omega_t(\Phi_X)^2\right) \tag{29}$$

and summing up all the contributions in (27):

$$\begin{aligned} \frac{d}{dt}P_\Lambda(t) \; &= \; \sum_{X \subset \Lambda} \Delta^2_X \mathrm{Av}\left(\omega_t(\Phi^2_X) - \omega_t(\Phi_X)^2\right) - \\ &- \; \sum_{s=1}^{n} \sum_{X \subset \Lambda_s} \Delta^2_X \mathrm{Av}\left(\omega_t(\Phi^2_X) - \omega_t(\Phi_X)^2\right) \\ &= \; \sum_{X \in \mathcal{C}_\Lambda} \Delta^2_X \mathrm{Av}\left(\omega_t(\Phi^2_X) - \omega_t(\Phi_X)^2\right) \geq 0 \; . \end{aligned} \tag{30}$$

¿From (26) and (30) we immediately get formula (19).

Boundedness

For any random potential we define the quantity

$$||U|| \;=\; \sup_{\Lambda} \frac{1}{N} \mathrm{Av}\left(U_\Lambda(J,\sigma)^2\right) \;=\; \sup_{\Lambda} \frac{1}{N} \sum_{X \subset \Lambda} \Delta_X^2 \Phi_X^2 \,. \tag{31}$$

Potentials with a finite $||U||$ are called stable.

Theorem 3 *A stable random potential admits an internal energy and a quenched pressure bounded by the volume.*

Proof. By (12):

$$< U_\Lambda(J,\sigma) > = \sum_{X \subset \Lambda} \Delta_X^2 \mathrm{Av}\left(\omega(\Phi_X^2) - \omega(\Phi_X)^2\right) \le 2||U||N \,. \tag{32}$$

Using the Jensen inequality

$$\begin{aligned} P_\Lambda \;=\; \mathrm{Av}\left(\ln Z_\Lambda(J)\right) &\;\le\; \ln \mathrm{Av}\left(Z_\Lambda(J)\right) \;= \\ = \ln \int_{\mathcal{S}^N} P_\Lambda(d\sigma) \mathrm{Av}\left(e^{U_\Lambda(J,\sigma)}\right) &\;=\; \ln \int_{\mathcal{S}^N} P_\Lambda(d\sigma) e^{\frac{1}{2}\sum_{X\subset\Lambda} \Delta_X^2 \Phi_X^2} \;= \\ &\;\le\; \frac{1}{2}||U||N \end{aligned} \tag{33}$$

As a consequence for finite $||U||$ one has

$$\sup_{\Lambda} \frac{1}{N} U_\Lambda \le \infty \;, \tag{34}$$

and

$$\sup_{\Lambda} \frac{1}{N} P_\Lambda \le \infty \;. \tag{35}$$

Thermodynamic Limit

Let us verify the stability condition in the above examples.

1. Edwards-Anderson.

   For the nearest neighbor case

$$\sum_{(n,n')} \Delta_X^2 = 2dNC^2 \tag{36}$$

2. More generally for the short range case with $\alpha > 1/2$

$$\sum_{n,n'} \Delta_X^2 = \sum_{n,n'} \frac{1}{|n-n'|^{2d\alpha}} \leq \text{const}\, N \tag{37}$$

By Theorems 1 and 2 the previous models have an internal energy per particle and a free energy per particle which exist in the thermodynamic limit.

Remarks:

1. We point out that for short range models we only need to impose the boundedness condition (31) while the superadditivity always holds thanks to the condition of independence of the variance Δ_X^2 from the volume (see also the remark 2). In the mean field case the variance of the interactions depends on the volume and subadditivity is based on an inequality among the covariances [CDGG]

$$N_1 c_{N_1}(\sigma,\tau) + N_2 c_{N_2}(\sigma,\tau) - N c_N(\sigma,\tau) \geq 0\,. \tag{38}$$

   One may check that such an inequality reduces, in the short range case, to the positivity of the right hand side of (30).

2. Our result may be extended in two directions by exactly the same procedure of [GT]. First one can prove by standard probability arguments

that the above statement entails the almost sure convergence of pressure and ground state energy per particle. Second our result may be extended to non Gaussian J (see section 4.2 of [GT] and [T]): if J_X is for all X an even random variable with a finite 4th moment the integration by parts (9) is replaced by the more general formula

$$\mathrm{Av}(J_X F(J)) = \mathrm{Av}(J_X^2 F'(J)) - \frac{1}{4}\mathrm{Av}(|J_X| \int_{-|J_X|}^{|J_X|} (J_X^2 - x^2) F'''(x) dx) \,. \tag{39}$$

When used within Theorem 2 it generates in formula (30) a correction of order $O(\sqrt{N})$. Once the density is taken the correction vanishes in the thermodynamic limit. In general, however, we cannot establish its sign and the monotonicity is lost.

3. The present result holds for free boundary conditions. In general it is proved[Ze] by standard surface over volume arguments that the quenched quantities are independent of the boundary conditions, but the monotonicity property is lost.

4. It is interesting to observe that the interpolating strategy does apply also to standard ferromagnetic systems. Consider for instance the d-dimensional ferromagnetic Ising model with nearest neighbor Hamiltonian $H_\Lambda(\sigma) = -\sum_{(n,n')} \sigma_n \sigma_{n'}$. An interpolating functional would be

$$\alpha(t) = \log \sum_\sigma e^{-\beta[tH_\Lambda(\sigma)+(1-t)\sum_{s=1}^n H_{\Lambda_s}(\sigma)]} \,. \tag{40}$$

An easy calculation which goes parallel to Theorem 2 yields

$$\frac{d\alpha}{dt}(t) = \sum_{(n,n')\in\mathcal{C}} \omega_t(\sigma_n \sigma_{n'}) \,. \tag{41}$$

Since $0 \leq t \leq 1$ the t-interaction on (40) is still ferromagnetic and the Griffiths inequality (see for instance [Ru]) $\omega_t(\sigma_n \sigma_{n'}) > 0$ gives the positive sign of the former expression ensuring the monotonicity of the limit.

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