Notes on Rank One Perturbed Resolvent. Perturbation of Isolated Eigenvalue.

S.A. Chorošavin

Abstract

This paper is a didactic commentary (a transcription with variations) to the paper of S.R. Foguel *Finite Dimensional Perturbations in Banach Spaces.*

Addressed, mainly: postgraduates and related readers.

Subject: Suppose we have two linear operators, A, B, so that

$$B - A \text{ is rank one.}$$

Let λ_o be an *isolated* point of the spectrum of A:

$$\lambda_o \in \sigma(A).$$

In addition, let λ_o be an *eigenvalue* of A:

$$\lambda_o \in \sigma_{pp}(A).$$

The question is: Is λ_o in $\sigma_{pp}(B)$? – i.e., is λ_o an eigenvalue of B ?

And, if so, is the multiplicity of λ_o in $\sigma_{pp}(B)$ equal to the multiplicity of λ_o in $\sigma_{pp}(A)$? – or less? – or greater?

Keywords: M.G.Krein's Formula, Finite Rank Perturbation

Introduction

We continue to discuss the paper of S.R. Foguel, *Finite Dimensional Perturbations in Banach Spaces*, and we assume that the reader is familiar with our previous paper arXiv:math-ph/0312016.

The situatian we will discuss is:

Let A and B, so that $A - B$ is rank one [1], i.e.,

$$B - A = -f_a < l_a|$$

for an element f_a and a linear functional l_a. Next, let λ_o be an *isolated* point of the spectrum of A:

$$\lambda_o \in \sigma(A).$$

In addition, let λ_o be an *eigenvalue* of A:

$$\lambda_o \in \sigma_{pp}(A).$$

The question is:

Is λ_o in $\sigma_{pp}(B)$? – i.e., is λ_o an eigenvalue of B ?

And, if so, is the multiplicity of λ_o in $\sigma_{pp}(B)$ equal to the multiplicity of λ_o in $\sigma_{pp}(A)$? – or less? – or greater?

Foguel gave an answer in a very general situation. We will not discuss all his constructions. Instead, for technical reasons, we assume the underlying space $\mathcal{H}$ to be *Hilbert*, and A , B to be bounded and symmetric, hence *self-adjoint*, with respect to

$$(,) \quad = \quad \text{Hilbert inner product on } \mathcal{H}.$$

Thus we restrict ourselves by discussing the situation where there are fewer complications.

[1] more accurately expressed, rank one or less

Recall some facts.

If $\mathcal{A}$ is a self-adjoint operator, and if λ_o is a **non-real** number, $Im\,\lambda_o \neq 0$, then

$$\lambda_o - \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

is a bijection, and, in addition

$$(\lambda_o - \mathcal{A})^{-1} : \mathcal{H} \to D(\mathcal{A}) \subset \mathcal{H}$$

is bounded.

If $\mathcal{A}$ is a self-adjoint operator, and if λ_o is a **real** number, $Im\,\lambda_o = 0$, then

$$P^{\mathcal{A}}_{\lambda_o} \quad := \quad strong - \lim_{\epsilon \downarrow 0} (i\epsilon(\lambda_o + i\epsilon - \mathcal{A})^{-1})$$

exists and the **range** of $P^{\mathcal{A}}_{\lambda_o}$ is the set of all $f_{\lambda_o} \in D(\mathcal{A})$ such that

$$(\lambda_o - \mathcal{A}) f_{\lambda_o} = 0\,.$$

In addition, $P^{\mathcal{A}}_{\lambda_o}$ is a *projection* operator and a *self-adjoint* operator.

If λ_o is a **real** number, $Im\,\lambda_o = 0$, so that for all λ near λ_o, and $\lambda \neq \lambda_o$, it has occurred that

$$\lambda \in \rho(\mathcal{A}) \quad = \quad \text{the resolvent set of } \mathcal{A}\,,$$

in other words, if $\lambda_o \in \rho(\mathcal{A})$ or λ_o is an *isolated* point of $\sigma(\mathcal{A}) =$ spectrum of $\mathcal{A}$, then

$$\begin{aligned}(\lambda - \mathcal{A})^{-1} \quad = \quad & \frac{P^{\mathcal{A}}_{\lambda_o}}{\lambda - \lambda_o} \\ & + \mathcal{A}_{\lambda_o} - (\lambda - \lambda_o)\mathcal{A}^2_{\lambda_o} + (\lambda - \lambda_o)^2 \mathcal{A}^3_{\lambda_o} - \cdots \\ & \text{for all } \lambda \text{ near } \lambda_o\,, \lambda \neq \lambda_o\,, \\ & \text{and for a } \textbf{bounded self-adjoint } \mathcal{A}_{\lambda_o}\,.\end{aligned}$$

In particular, if λ_o is an *isolated* point of $\sigma(\mathcal{A})$, then $\lambda_o \in \sigma_{pp}(\mathcal{A})$.

Recall in addition, that

if A^{-1} exists and $1- < l_a | A^{-1} f_a > \neq 0$, then B^{-1} exists and, in addition,

$$B^{-1} - A^{-1} = \frac{A^{-1} f_a < l_a | A^{-1}}{1- < l_a | A^{-1} f_a >}$$

On the other hand,

$$\text{if } 1- < l_a | A^{-1} f_a >= 0\,, \text{ then } BA^{-1} f_a = 0$$

if

$$Bv_0 = 0 \text{ and } v_0 \neq 0\,,$$

then

$$< l_a | v_0 > \neq 0\,,\ 1- < l_a | A^{-1} f_a >= 0\,,\ BA^{-1} f_a = 0\,, \text{ and } v_0 = A^{-1} f_a < l_a | v_0 > \ .$$

Notice that

$$(\lambda - B) - (\lambda - A) = -(B - A) = f_a < l_a |\,.$$

Thus, we conclude:

if $(\lambda - A)^{-1}$ exists and $1+ < l_a | (\lambda - A)^{-1} f_a > \neq 0$, then $(\lambda - B)^{-1}$ exists and, in addition,

$$(\lambda - B)^{-1} - (\lambda - A)^{-1} = -\frac{(\lambda - A)^{-1} f_a < l_a | (\lambda - A)^{-1}}{1+ < l_a | (\lambda - A)^{-1} f_a >}$$

On the other hand,

$$\text{if } 1+ < l_a | (\lambda - A)^{-1} f_a >= 0\,, \text{ then } (\lambda - B)(\lambda - A)^{-1} f_a = 0$$

if

$$(\lambda - B) v_0 = 0 \text{ and } v_0 \neq 0\,,$$

then

$$< l_a | v_0 > \neq 0\,,\ 1+ < l_a | (\lambda - A)^{-1} f_a >= 0\,,\ (\lambda - B)(\lambda - A)^{-1} f_a = 0\,,$$

and

$$v_0 = (\lambda - A)^{-1} f_a < l_a | v_0 > \ .$$

Before starting, we shall recall that we prefer Dirac's "bra-ket" style of expressing, in the following form:

Notation 1. If f is an element of a linear space, X, over a field, K, then $|f>$ stands for the mapping $K \to X$, defined by

$$|f > \lambda := \lambda f \quad .$$

Notation 2. If l is a functional and we wish to emphasise this factor, then we write $< l|$ instead of l. We also write $< l|f >$ instead of $< l||f > 1$, and write the terms $|f >< l|$ and $f < l|$ interchangeably:

$$< l|f > \equiv < l||f > 1 \equiv l(f) \quad , \quad f < l| \equiv |f >< l| \quad .$$

Finally, we will restrict ourselves to the case where $< l_a|$ is of the form:

$$< l_a|u > \quad := \quad \alpha(f_a|u) \quad (u \in \mathcal{H})\,, \text{ for a real number } \alpha\,.$$

In this case, it is naturally to use the notations:

$$(f_a| := < l_a| \text{ and } |f_a > := |f_a)\,.$$

1 Perturbation of Isolated Eigenvalue.

Now, we turn to the relations, which links $(\lambda - A)^{-1}$ and $(\lambda - B)^{-1}$, and which we now write as follows:

if $(\lambda - A)^{-1}$ exists and $1 + \alpha(f_a|(\lambda - A)^{-1} f_a) \neq 0$, then $(\lambda - B)^{-1}$ exists and, in addition,

$$(\lambda-\lambda_o)(\lambda-B)^{-1}-(\lambda-\lambda_o)(\lambda-A)^{-1} = -\alpha\frac{(\lambda-\lambda_o)(\lambda-A)^{-1}f_a(f_a|(\lambda-\lambda_o)(\lambda-A)^{-1}}{(\lambda-\lambda_o)\Big(1+\alpha(f_a|(\lambda-A)^{-1}f_a)\Big)}$$

Note that the denominator is equal to

$$\begin{array}{rcl} (\lambda-\lambda_o) & + & \alpha(f_a|P^A_{\lambda_o} f_a) \\ & + & (\lambda-\lambda_o)\alpha(f_a|A_{\lambda_o} f_a) \\ & - & (\lambda-\lambda_o)^2\alpha(f_a|A^2_{\lambda_o} f_a) + (\lambda-\lambda_o)^3\alpha(f_a|A^3_{\lambda_o} f_a) - \cdots \end{array}$$

We distinguish three cases:

(a)
$$\alpha(f_a|P^A_{\lambda_o} f_a) \neq 0$$

(b)
$$\alpha(f_a|P^A_{\lambda_o} f_a) = 0\,, 1 + \alpha(f_a|A_{\lambda_o} f_a) \neq 0$$

(c)
$$(f_a|P^A_{\lambda_o} f_a) = 0\,,\ 1 + \alpha(f_a|A_{\lambda_o} f_a) = 0\,,\ \alpha \neq 0\,.$$

It is worthy to note that if

$$1 + \alpha(f_a|A_{\lambda_o} f_a) = 0$$

then

$$1 = |\alpha(f_a|A_{\lambda_o} f_a)|^2 \leq |\alpha|^2(f_a|f_a)(A_{\lambda_o} f_a|A_{\lambda_o} f_a) = |\alpha|^2(f_a|f_a)(f_a|A^2_{\lambda_o} f_a)$$

As a result,

$$\text{if } 1 + \alpha(f_a|A_{\lambda_o} f_a) = 0\,,\ \text{then } (f_a|A^2_{\lambda_o} f_a) \neq 0$$

Now let

$$\lambda := \lambda_o + i\epsilon \text{ and } \epsilon \downarrow 0\,.$$

Then we infer:

(a)

$$\alpha(f_a|P^A_{\lambda_o}f_a) \neq 0$$

In this case,

$$\begin{aligned} P^B_{\lambda_o} - P^A_{\lambda_o} &= -\alpha\frac{P^A_{\lambda_o}f_a(f_a|P^A_{\lambda_o}}{\alpha(f_a|P^A_{\lambda_o}f_a)} \\ &= -\frac{P^A_{\lambda_o}f_a(f_a|P^A_{\lambda_o}}{(f_a|P^A_{\lambda_o}f_a)} \quad \text{(note that } \alpha \neq 0\text{)} \end{aligned}$$

In particular,

$$dim P^B_{\lambda_o} = dim P^A_{\lambda_o} - 1\,.$$

(b)

$$\alpha(f_a|P^A_{\lambda_o}f_a) = 0\,,\, 1 + \alpha(f_a|A_{\lambda_o}f_a) \neq 0\,.$$

In this case, if $\alpha = 0$, then $B = A$, $P^B_{\lambda_o} = P^A_{\lambda_o}$, $dim P^B_{\lambda_o} = dim P^A_{\lambda_o}$. Otherwise, $(f_a|P^A_{\lambda_o}f_a) = 0$ and

$$(P^A_{\lambda_o}f_a|P^A_{\lambda_o}f_a) = (f_a|P^A_{\lambda_o}f_a) = 0\,.$$

Hence

$$P^A_{\lambda_o}f_a = 0\,,$$

and

$$\begin{aligned} (\lambda - A)^{-1}f_a &= +A_{\lambda_o}f_a - (\lambda-\lambda_o)A^2_{\lambda_o}f_a + (\lambda-\lambda_o)^2A^3_{\lambda_o}f_a - \cdots \\ &\quad \text{for all } \lambda \text{ near } \lambda_o\,, \lambda \neq \lambda_o\,, \\ &\quad \text{and for a } \textbf{bounded self-adjoint } A_{\lambda_o}\,. \end{aligned}$$

$$P^B_{\lambda_o} - P^A_{\lambda_o} = 0\,.$$

In particular,

$$dim P^B_{\lambda_o} = dim P^A_{\lambda_o}\,,$$

as well.

(c)

$$(f_a|P^A_{\lambda_o}f_a) = 0\,,\, 1 + \alpha(f_a|A_{\lambda_o}f_a) = 0\,,\, \alpha \neq 0\,.$$

In this case, as well as in the case (b),

$$P^A_{\lambda_o}f_a = 0\,,$$

and

$$\begin{aligned} (\lambda - A)^{-1}f_a &= +A_{\lambda_o}f_a - (\lambda-\lambda_o)A^2_{\lambda_o}f_a + (\lambda-\lambda_o)^2A^3_{\lambda_o}f_a - \cdots \\ &\quad \text{for all } \lambda \text{ near } \lambda_o\,, \lambda \neq \lambda_o\,, \end{aligned}$$

However

$$\begin{aligned} P^B_{\lambda_o} - P^A_{\lambda_o} &= -\alpha\frac{A_{\lambda_o}f_a(f_a|A_{\lambda_o}}{-\alpha(f_a|A^2_{\lambda_o}f_a)} \\ &= \frac{A_{\lambda_o}f_a(f_a|A_{\lambda_o}}{(f_a|A^2_{\lambda_o}f_a)} \quad (\alpha \neq 0) \end{aligned}$$

In particular,

$$dim P^B_{\lambda_o} = dim P^A_{\lambda_o} + 1\,.$$

2 Example

Let $\mathcal{T}$ stands for the functions transformation defined by

$$(\mathcal{T}u)(x) := -\frac{\partial^2 u(x)}{\partial x^2}\,;$$

$\mathcal{T}_{DD}$ be the restriction of $\mathcal{T}$ so that $\mathcal{T}_{DD}$ acts on that functions, u, for which $(\mathcal{T}u)(x)$ is defined at $0 \leq x \leq 1$ and, in addition:

$$\begin{aligned} u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

We take $L_2(0,1)$, as the underlying space $\mathcal{H}$. In this space, $\mathcal{T}_{DD}$ is closable and symmetric. Moreover, it is essentially self-adjoint. That means that its closure, T_{DD}, is self-adjoint.

One can check that T_{DD}^{-1} exists and is an integral operator; its integral kernel is

$$G_{DD}(x,\xi) \quad = \quad -\left\{ \begin{array}{lcl} x\cdot(\xi-1) & , & \text{if } x \leq \xi \\ (x-1)\cdot\xi & , & \text{if } \xi \leq x \end{array} \right\}$$

One can also check that

$$\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots$$

are eigenfunctions of T_{DD} and, of course, of T_{DD}^{-1}. The corresponding eigenvalues of T_{DD} are

$$\pi^2, (2\pi)^2, (3\pi)^2, \ldots$$

and that of T_{DD}^{-1} are

$$\frac{1}{\pi^2}, \frac{1}{(2\pi)^2}, \frac{1}{(3\pi)^2}, \ldots$$

All eigenvalues are multiplicity-free. It is not very difficult to describe

$$(z - T_{DD})^{-1}\,.$$

This is an integral operator. Its integral kernel is

$$G_{DD}(x,\xi,z) \quad = \quad -\frac{1}{k\sin(k)} \left\{ \begin{array}{lcl} \sin(kx)\sin(k(1-\xi)) & , & \text{if } x \leq \xi \\ \sin(k\xi)\sin(k(1-x)) & , & \text{if } \xi \leq x \end{array} \right\}$$
$$\text{where } k \text{ is defined by } k^2 = z\,,$$

and where, of course, z is to be so, that

$$\sin(k) \neq 0\,.$$

As for

$$(\lambda - T_{DD}^{-1})^{-1}\,,$$

it is not very difficult to describe it as well: A general (and quite standard) argumentation is:

$$\begin{aligned} (\lambda - T_{DD}^{-1})^{-1} \quad &= \quad T_{DD}(\lambda T_{DD} - I)^{-1} \\ &= \frac{1}{\lambda}\Big(\lambda T_{DD}(\lambda T_{DD} - I)^{-1}\Big) \\ &= \frac{1}{\lambda}\Big((\lambda T_{DD} - I + I)(\lambda T_{DD} - I)^{-1}\Big) \\ &= \frac{1}{\lambda}\Big(I + (\lambda T_{DD} - I)^{-1}\Big) \\ &= \frac{1}{\lambda}\Big(I - \frac{1}{\lambda}(\frac{1}{\lambda} - T_{DD})^{-1}\Big) \quad (\text{ naturally, here } \lambda \neq 0)\,. \end{aligned}$$

Now, we let

$$\begin{aligned} A &:= T_{DD}^{-1} \\ B &:= A + \alpha f_a (f_a| \end{aligned}$$

where f_a is defined by

$$f_a(x) := (x - \frac{1}{2})\,,$$

and let us apply the theory described in the previous section. So, let

$$f_z := z(z - T_{DD})^{-1} f_a\,,$$

i.e.,

$$\begin{aligned} z f_z(x) + \frac{\partial^2 f_z(x)}{\partial x^2} &= z(x - \frac{1}{2})\,, \\ f_z(0) &= 0\,, \\ f_z(1) &= 0\,. \end{aligned}$$

Hence

$$\begin{aligned} f_z(x) &= (x - \frac{1}{2}) - \frac{1}{2}\frac{\sin(k(x - \frac{1}{2}))}{\sin(k(\frac{1}{2}))} \\ &\quad \text{where } k \text{ is defined by } k^2 = z\,, \end{aligned}$$

and where, recall, z is such that

$$\sin(k) \neq 0\,.$$

Thus we deduce:

$$\begin{aligned} \Big((-I + z(z - T_{DD})^{-1}) f_a\Big)(x) &= -f(x) + f_z(x) \\ &= -(x - \frac{1}{2}) + (x - \frac{1}{2}) - \frac{1}{2}\frac{\sin(k(x - \frac{1}{2}))}{\sin(k(\frac{1}{2}))} \\ &= -\frac{1}{2}\frac{\sin(k(x - \frac{1}{2}))}{\sin(k(\frac{1}{2}))} \\ &\quad \text{where } k \text{ is defined by } k^2 = z\,, \end{aligned}$$

$$\begin{aligned} (f_a|(-I + z(z - T_{DD})^{-1}) f_a) &= -\frac{1}{2}\int_0^1 (\xi - \frac{1}{2})\frac{\sin(k(\xi - \frac{1}{2}))}{\sin(k(\frac{1}{2}))} d\xi \\ &= \frac{1}{2}\int_{\xi=0}^1 (\xi - \frac{1}{2})\frac{d\cos(k(\xi - \frac{1}{2}))}{k\sin(k(\frac{1}{2}))} \\ &= \frac{1}{2}\frac{\cos(k(\frac{1}{2}))}{k\sin(k(\frac{1}{2}))} - \frac{1}{2}\int_{\xi=0}^1 \frac{\cos(k(\xi - \frac{1}{2}))}{k\sin(k(\frac{1}{2}))} d(\xi - \frac{1}{2}) \\ &= \frac{1}{2}\frac{\frac{1}{2}\cos(k(\frac{1}{2}))}{k\sin(k(\frac{1}{2}))} - \frac{1}{k^2} \end{aligned}$$

$$\begin{aligned} 1 + z\alpha(f_a|(-I + z(z - T_{DD})^{-1}) f_a) &= 1 + k^2\alpha\Big(\frac{\frac{1}{2}\cos(k(\frac{1}{2}))}{k\sin(k(\frac{1}{2}))} - \frac{1}{k^2}\Big) \\ &= 1 + \alpha\Big(\frac{k}{2}\frac{\cos(\frac{k}{2})}{\sin(\frac{k}{2})} - 1\Big) \end{aligned}$$

We conclude :

The new eigenvalues, λ_n, are defined by

$$1 + z_n \alpha (f_a | (-I + z_n (z_n - T_{DD})^{-1}) f_a) = 0 \,,$$

i.e., by

$$1 + \alpha \Big(\frac{k_n}{2} \frac{\cos(\frac{k_n}{2})}{\sin(\frac{k_n}{2})} - 1 \Big) = 0 \,,$$

and the associated eigenfunctions are

$$\Big((-I + z_n (z_n - T_{DD})^{-1}) f_a \Big)(x) \quad = \quad -\frac{1}{2} \frac{\sin(k_n (x - \frac{1}{2}))}{\sin(k_n (\frac{1}{2}))}$$

Now, let

$$\alpha \quad := \quad 1 \,.$$

In this case, the solutions to

$$1 + \alpha \Big(\frac{k_n}{2} \frac{\cos(\frac{k_n}{2})}{\sin(\frac{k_n}{2})} - 1 \Big) = 0$$

are

$$k_n \quad = \quad \pi, 3\pi, 5\pi, \dots \,,$$

which conflicts with

$$\lambda_n \quad \in \quad \rho(T_{DD}^{-1}) \,.$$

Hence, there are **no new eigenvalues**, if $\alpha = 1$ (!!)

Now, let us analyse

$$\lambda_o - B$$

in the case where

$$\lambda_o \quad \in \quad \sigma(T_{DD}^{-1}) \,, \lambda_o \neq 0 \,,$$

i.e., where

$$k_o \quad \in \quad \{\pi, 2\pi, 3\pi, 4\pi \dots\} \,.$$

Firstly we notice that λ_o is multiplicity-free (in $\sigma(A)$) and that a corresponding eigenfunction, f_{λ_o}, is such that

$$f_{\lambda_o}(x) \quad = \quad \sin(k_o x) \quad (x \in [0,1]) \,.$$

We have: $(f_{\lambda_o} | f_{\lambda_o}) = 1/2$, hence,

$$P^{\mathcal{A}}_{\lambda_o} \quad = \quad 2 f_{\lambda_o} (f_{\lambda_o} |$$

In accordance with the scheme which had been formed in the previous section, we analyse $(f_a|P^A_{\lambda_o}f_a)$. We have,

$$\begin{aligned}(f_a|P^A_{\lambda_o}f_a) &= (f_a|2f_{\lambda_o})(f_{\lambda_o}|f_a)\\ &= 2\Big|\int_0^1 (\xi-\frac{1}{2})\sin(k_o\xi)d\xi\Big|^2\\ &= 2\Big|\Big(-\frac{\xi}{k_o}\cos(k_o\xi)+\frac{1}{k_o^2}\sin(k_o\xi)+\frac{1}{2k_o}\cos(k_o\xi)\Big)\Big|_{\xi=0}^{\xi=1}\Big|^2\\ &= 2\Big|\frac{1}{2k_o}\big(1+\cos(k_o)\big)\Big|^2 .\end{aligned}$$

We divide the analysis into two parts.

If

$$k_o \in \{2\pi, 4\pi, 6\pi\ldots\},$$

then

$$(f_a|P^A_{\lambda_o}f_a) = \frac{2}{k_o^2} .$$

In this case,

$$(f_a|P^A_{\lambda_o}f_a) \neq 0 .$$

Hence, if

$$k_o \in \{2\pi, 4\pi, 6\pi\ldots\},$$

then

$$dimP^B_{\lambda_o} = dimP^A_{\lambda_o} - 1 = 1 - 1 = 0$$

and

$$\lambda_o \text{ is no eigenvalue of } B .$$

On the contrary, if

$$k_o \in \{\pi, 3\pi, 5\pi\ldots\},$$

then

$$(f_a|P^A_{\lambda_o}f_a) = 0 .$$

Hence, if

$$k_o \in \{\pi, 3\pi, 5\pi\ldots\},$$

then

$$dimP^B_{\lambda_o} \geq dimP^A_{\lambda_o} = 1$$

and

$$\lambda_o \text{ is an eigenvalue of } B .$$

And what is equal then the multiplicity of λ_o to?

To answer this question, let us analyse

$$\begin{aligned} 1+\alpha(f_a|A_{\lambda_o}f_a) &= 1+\lim_{\epsilon\downarrow 0}(f_a|\Big((\lambda_o+i\epsilon-A)^{-1}-\frac{P^{\mathcal{A}}_{\lambda_o}}{i\epsilon}\Big)|f_a) \\ &= 1+\lim_{\epsilon\downarrow 0}(f_a|\Big((\lambda_o+i\epsilon-A)^{-1}\Big)|f_a) \end{aligned}$$

Notice,

$$\begin{aligned} 1+(f_a|\Big((\lambda-A)^{-1}\Big)|f_a) &= 1+z\alpha(f_a|(-I+z(z-T_{DD})^{-1})f_a) \\ &= 1+k^2\alpha\Big(\frac{\frac{1}{2}\cos(k(\frac{1}{2}))}{k\sin(k(\frac{1}{2}))}-\frac{1}{k^2}\Big) \\ &= 1+\alpha\Big(\frac{k}{2}\frac{\cos(\frac{k}{2})}{\sin(\frac{k}{2})}-1\Big) \\ &= \frac{k}{2}\frac{\cos(\frac{k}{2})}{\sin(\frac{k}{2})} \quad (\text{ because } \alpha=1)\,. \end{aligned}$$

Hence,

$$1+\alpha(f_a|A_{\lambda_o}f_a) \quad = \quad 0\,.$$

Thus, we conclude:

If

$$k_o \quad \in \quad \{\pi, 3\pi, 5\pi\ldots\}\,,$$

then

$$\text{the multiplicity of } \lambda_o \quad = \quad 2\,.$$

Exercise. *Compare B with the Green's function generated by $-\frac{\partial^2}{\partial x^2}$ on $[0,1]$ and relations*

$$u(0)=-u(1) \quad , \quad \frac{\partial u}{\partial x}\Big|_{x=0}=-\frac{\partial u}{\partial x}\Big|_{x=1}\,.$$

References

[Fog] S.R. Foguel, *Finite Dimensional Perturbations in Banach Spaces*, American Journal of Mathematics, Volume 82, Issue 2 (Apr., 1960), 260-270