On the quantum Boltzmann equation

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Abstract

We give a nonrigorous derivation of the nonlinear Boltzmann equation from the Schrödinger evolution of interacting fermions. The argument is based mainly on the assumption that a quasifree initial state satisfies a property called *restricted quasifreeness* in the weak coupling limit at any later time. By definition, a state is called restricted quasifree if the four-point and the eight-point functions of the state factorize in the same manner as in a quasifree state.

1 Introduction

The fundamental equation governing many-body quantum dynamics, the Schrödinger equation, is a time reversible hyperbolic equation. The quantum

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dynamics of many-body systems, however, are often modelled by a nonlinear, time irreversible (quantum) Boltzmann equation, which exhibits a particle-like behavior. This apparent contradiction has attracted a lot of attention over the years and, in particular, Hugenholtz [7] gave a derivation based on a perturbation expansion involving multiple commutators. He selected a class of terms from this expansion and argued that it gives the Boltzmann equation. To the second order in the coupling constant, Hugenholtz's claim was proved by Ho and Landau [6]. Beyond that it is not even clear that these terms satisfy the Boltzmann equation order by order, partly due to the complicated selection rules.

In this paper, we present a derivation of the quantum Boltzmann equation under the main assumption that in the weak coupling limit (8) the four-point and the eight-point functions of the state factorize at any time in the same manner as in a quasifree state, see (22) and (23). A state with such a factorization property is called a *restricted quasifree state*. To rigorously verify this assumption, one has to analyze the connected m–point functions, a very difficult problem in our view. So it might appear that we have not improved much beyond the work [7]. Our approach however has the following two main merits. First: It identifies the concept, the restricted quasifreeness, to replace the independence in the classical setting so that the structure of the collision term, i.e., the quartic nonlinearity and the product of the factors F and $1 - F$ in the equation (10), appears as a simple consequence of this assumption. Second: Unlike Hugenholtz's approach which is tied to the commutator expansion, the restricted quasifreeness can now be verified using other methods such as field–theoretical techniques.

Recent work by Benedetto, Castella, Esposito and Pulvirenti [2] has given an interesting different derivation. We had learned this work in a recent meeting and had subsequently sent them an early version of this manuscript.

2 Definitions of the Dynamics

We describe the quantum dynamics in the second–quantized formulation. For definiteness, we shall restrict ourselves to a fermion system. Our derivation is valid for bosons as well with the only difference being that some $\pm$ signs change along the derivation and in the final quantum Boltzmann equation (the terms $(1 - F)$ change to $(1 + F)$ in (10)). It should be noted that many-boson systems are in general more difficult to control rigorously than many-fermion systems. The quantum Boltzmann equation for fermions also preserves the property $0 \le F \le 1$. On the contrary, the equation for bosons may blow up in finite time.

Most of our setup is fairly standard and we recall the details briefly here. For background, see [3, 8]. The configuration space is a discrete torus $\Lambda = \mathbb{Z}^d / L\mathbb{Z}^d$, of a very large sidelength, $L \in \mathbb{N}$, which is kept finite throughout the argument. The Hilbert space for the fermions is the standard Fock space $\mathcal{F}_\Lambda := \bigoplus_{n \ge 0} \bigwedge^n \mathcal{H}_\Lambda$, where $\mathcal{H}_\Lambda := \ell^2(\Lambda, \mathbb{C})$. [1] Because $\mathcal{H}_\Lambda$ is finite–dimensional, the same holds for $\mathcal{F}_\Lambda$.

We shall work in momentum space, which for our finite lattice is the discrete torus $\Lambda^* := \frac{2\pi}{L}\mathbb{Z}^d / 2\pi\mathbb{Z}^d$. Let $w_p(x) := \mathrm{e}^{\mathrm{i}p\cdot x}$ where $p \cdot x := \sum_{i=1}^d p_i x_i$. The set of functions $\{w_p : p \in \Lambda^*\}$ is an orthogonal basis of $\mathcal{H}$ (the normalization is $\|w_p\|^2 = L^d$). Therefore the annihilation operators $a_p = a(w_p)$ that are associated to this basis in the standard way (see [3] or [8]) obey the canonical anticommutation relations (CAR)

$$a_p\, a_q^+ + a_p^+ a_q = \delta(p,q) := \begin{cases} L^d & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If F is continuous on $\mathcal{B} := \mathbb{R}^d / 2\pi\mathbb{Z}^d$, then $L^{-d} \sum_{p \in \Lambda^*} F(p) \to \int_{\mathcal{B}} \frac{\mathrm{d}^d p}{(2\pi)^d}\, F(p)$ as $L \to \infty$. Since we are ultimately interested in the limit $L \to \infty$, we will use the continuum notation even for the finite sums, i.e. we write $\int_{\Lambda^*} \mathrm{d}p\, F(p)$ for $L^{-d} \sum_{p \in \Lambda^*} F(p)$, etc.

Let $\mathcal{A}$ be the C^* algebra generated by $\{a_p^+, a_p : p \in \Lambda^*\}$. For a selfadjoint element $H \in \mathcal{A}$, consider the time evolution given by H in the Heisenberg picture, i.e. $A_t := \mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}$ for all $A \in \mathcal{A}$. Given a state ρ on the algebra, we define for $A \in \mathcal{A}$, $\rho_t(A) := \rho(A_t)$. For all $t \in \mathbb{R}$, ρ_t is again a state on $\mathcal{A}$, and its time evolution is given by the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\rho_t(A) = \rho_t([H, A]). \tag{2}$$

We take the Hamiltonian $H := H_0 + \lambda\Phi$ where

$$H_0 := \int \mathrm{d}p\; e(p)\, a_p^+ a_p \tag{3}$$

[1] Thus the fermions are spinless. We could also choose $\mathcal{H}'_\Lambda = \ell^2(\Lambda, \mathbb{C}^2)$ as the one–particle Hilbert space, to allow for a fermion spin 1/2, without changing the derivation in an essential way.

is the kinetic energy and

$$\Phi := \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \; \langle k_1 \; k_2 \mid \Phi \mid k_3 \; k_4 \rangle a^+_{k_1} a^+_{k_2} a_{k_3} a_{k_4} \tag{4}$$

is the interaction. The coefficient function $\langle k_1 \; k_2 \mid \Phi \mid k_3 \; k_4 \rangle$ is antisymmetric under exchange of k_1 and k_2 and under exchange of k_3 and k_4, and it contains the momentum conservation delta function. For an interaction generated by a two-body potential $v(x-y)$ we have

$$\begin{aligned} \langle k_1 \; k_2 \mid \Phi \mid k_3 \; k_4 \rangle \;\; = \;\; & \delta(k_1 + k_2, k_3 + k_4) \\ & \frac{1}{4} \left(\hat{v}(k_1 - k_4) - \hat{v}(k_2 - k_4) - \hat{v}(k_1 - k_3) + \hat{v}(k_2 - k_3) \right). \end{aligned} \tag{5}$$

We assume that $\langle k_4 \; k_3 \mid \Phi \mid k_2 \; k_1 \rangle = \overline{\langle k_1 \; k_2 \mid \Phi \mid k_3 \; k_4 \rangle}$. Then $\Phi = \Phi^+$. In terms of v this condition means that v is real. For simplicity, we shall assume that v is symmetric, i.e.,

$$v(x) = v(-x) \; . \tag{6}$$

We shall call a polynomial F in the creation and annihilation operators *quartic* if it is homogeneous of degree four and contains exactly two creation and two annihilation operators. Any such quartic F has a representation similar to (5) with a coefficient $\langle k_1 \; k_2 \mid F \mid k_3 \; k_4 \rangle$, and we shall always assume that the coefficient is given in the properly antisymmetrized form so that we can compare coefficients.

The state ρ_t is determined by its values on monomials in the creation and annihilation operators. The two point function in Fourier space is defined by

$$\nu_{pq}(t) := \rho_t(a^+_p a_q). \tag{7}$$

We are interested in the Euler scaling limit of the two point function. In configuration space it amounts to the rescaling

$$x = X/\varepsilon, \; t = T/\varepsilon, \qquad \varepsilon \to 0.$$

Recall the *Wigner transform* of a function $\psi \in L^2(\mathbb{R}^d)$ is defined as

$$W_\psi(x,v) := \int e^{i\eta x} \overline{\widehat{\psi}\Big(v - \frac{\eta}{2}\Big)} \widehat{\psi}\Big(v + \frac{\eta}{2}\Big) d\eta \; = \int e^{ivy} \overline{\psi\Big(x + \frac{y}{2}\Big)} \psi\Big(x - \frac{y}{2}\Big) dy \; .$$

Define the rescaled Wigner distribution as

$$W^{\varepsilon}_{\psi}(X,V) := \varepsilon^{-d} W_{\psi}\Big(\frac{X}{\varepsilon}, V\Big).$$

Its Fourier transform in X is given by

$$\widehat{W}^{\varepsilon}_{\psi}(\xi,V) = \overline{\widehat{\psi}\Big(V - \frac{\varepsilon\xi}{2}\Big)}\widehat{\psi}\Big(V + \frac{\varepsilon\xi}{2}\Big)\ .$$

We can easily extend this notions to the two point function ν_{pq}. In particular, we can define the rescaled Wigner distribution $W^{\varepsilon}_{\rho}(X,V)$ through its Fourier transform:

$$\widehat{W}^{\varepsilon}_{\rho}(\xi,V) = \rho(a^+_{V-\frac{\varepsilon\xi}{2}}\ a_{V+\frac{\varepsilon\xi}{2}})\ .$$

Assume that

$$W^{\varepsilon}(X,V,T) := W^{\varepsilon}_{\rho_{T/\varepsilon}}(X,V) \to F(X,V,T)$$

as $\varepsilon \to 0$. Under the weak coupling scaling assumption, i.e.,

$$x = X/\varepsilon,\ t = T/\varepsilon,\ \lambda = \sqrt{\varepsilon} \tag{8}$$

one expects that $F(X,V,T)$ satisfies the nonlinear Boltzmann equation

$$\begin{aligned}
&\frac{\partial F(X,V,T)}{\partial t} + V\cdot\nabla_X F(X,V,T)\\
=\ & 4\pi \int \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\ \delta(k_1+k_2,k_3+k_4)\ \delta(E_1+E_2-E_3-E_4) \qquad (9)\\
&\times|\hat{v}(k_1-k_4)-\hat{v}(k_1-k_3)|^2\\
&\times\Big[F_{k_1}F_{k_2}(1-F_{k_3})(1-F_{k_4}) - F_{k_4}F_{k_3}(1-F_{k_2})(1-F_{k_1})\Big]
\end{aligned}$$

where F_{k_j} is short notation for $F(X,k_j,T)$ and $E_i = e(k_i)$ with $k_1 = V$. If the state is homogeneous in space (i.e. translation invariant) at time zero, then $\nu_{pq} = \delta(p,q)F_p(t)$ for all later times t as well and the Boltzmann equation is reduced to

$$\begin{aligned}
\frac{\partial F_{k_1}}{\partial T} &= 4\pi \int \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\ |\hat{v}(k_1-k_4)-\hat{v}(k_1-k_3)|^2 \qquad (10)\\
&\times\delta(k_1+k_2,k_3+k_4)\ \delta(E_1+E_2-E_3-E_4)\\
&\times[F_{k_1}F_{k_2}(1-F_{k_3})(1-F_{k_4}) - F_{k_4}F_{k_3}(1-F_{k_2})(1-F_{k_1})].
\end{aligned}$$

Our goal is to give a heuristic derivation of this equation.

Remark 1. Without the symmetry assumption (6) one has to replace the collision kernel $|\hat{v}(k_1 - k_4) - \hat{v}(k_1 - k_3)|^2$ in (10) with its symmetrized version

$$\frac{1}{4}|\hat{v}(k_1 - k_4) - \hat{v}(k_2 - k_4) - \hat{v}(k_1 - k_3) + \hat{v}(k_2 - k_3)|^2$$

and our derivation remains valid.

The quartic structure of the collision is due to the quantum nature and the weak coupling limit. Instead of the weak coupling limit, one can take the low density limit ($x = X/\varepsilon,\ t = T/\varepsilon, \lambda = 1$ and the density of the particles is ε). The resulting equation will be the standard nonlinear Boltzmann equation where collision term is quadratic with full quantum scattering kernel and not just its Born approximation in the weak coupling limit. Technically, the emergence of the full quantum scattering kernel can be seen from resumming the Born series, see [4] for the simpler case of the Lorentz gas. From our experience working on the weak coupling [5] and low density limits in random environments, we believe that a rigorous derivation of the low density limit will be somewhat more complicated than in the weak coupling limit. However, the key difficulties arising from many-body quantum dynamics are already present in the weak coupling limit which we shall focus on.

3 The equation for the two point function

The Schrödinger equation can be written as

$$(\mathrm{i}\partial_t - e(p) + e(q))\nu_{pq}(t) = \lambda\rho_t(F_{pq}), \tag{11}$$

where $F_{pq} := [\Phi, a_p^+ a_q]$ is quartic with

$$\begin{aligned} \langle k_1\ k_2 \mid F_{pq} \mid k_3\ k_4\rangle = &- \delta(q,k_4)\langle k_1\ k_2 \mid \Phi \mid p\ k_3\rangle \\ &+ \delta(q,k_3)\langle k_1\ k_2 \mid \Phi \mid p\ k_4\rangle \\ &+ \delta(p,k_1)\langle k_2\ q \mid \Phi \mid k_3\ k_4\rangle \\ &- \delta(p,k_2)\langle k_1\ q \mid \Phi \mid k_3\ k_4\rangle. \end{aligned} \tag{12}$$

Therefore

$$\nu_{pq}(t) = \nu_{pq}(0)\mathrm{e}^{-\mathrm{i}t(e(p)-e(q))} - \mathrm{i}\lambda\int_0^t \mathrm{d}s\ \mathrm{e}^{-\mathrm{i}(t-s)(e(p)-e(q))}\rho_s(F_{pq}). \tag{13}$$

Thus we need expectation values of quartic monomials, whose evolution equation analogous to (13) involves ones of degree six, etc. This system of equations "hierarchy") is similar to the Schwinger–Dyson equations, but the commutator structure implies that in an expansion in Feynman graphs only connected graphs contribute.

Let $\varphi_{r_1,\dots,r_4}(t) := \rho_t(a^+_{r_1} a^+_{r_2} a_{r_3} a_{r_4})$ and

$$\Delta e(r_1, \dots, r_4) := e(r_1) + e(r_2) - e(r_3) - e(r_4). \tag{14}$$

The Schrödinger equation for φ gives

$$(\mathrm{i}\partial_t - \Delta e(r_1, \dots, r_4))\varphi_{r_1,\dots,r_4}(t) = \lambda \rho_t([\Phi, a^+_{r_1} a^+_{r_2} a_{r_3} a_{r_4}]) \tag{15}$$

which integrates to

$$\begin{aligned} \varphi_{r_1,\dots,r_4}(t) &= \varphi_{r_1,\dots,r_4}(0)\mathrm{e}^{-\mathrm{i}t\Delta e(r_1,\dots,r_4)} \\ &\quad -\mathrm{i}\lambda \int_0^t \mathrm{d}s\ \mathrm{e}^{-\mathrm{i}(t-s)\Delta e(r_1,\dots,r_4)} \rho_s([\Phi, a^+_{r_1} a^+_{r_2} a_{r_3} a_{r_4}]). \end{aligned} \tag{16}$$

Thus

$$\rho_t(F_{pq}) = \rho_0(F_{pq}) - \mathrm{i}\lambda \int_0^t \mathrm{d}s\ \rho_s([\Phi, G_{pq}(t-s)]) \tag{17}$$

where $G_{pq} = G_{pq}(t-s)$ is quartic with

$$\begin{aligned} \langle k_1\ k_2 \mid G_{pq} \mid k_3\ k_4 \rangle &= \mathrm{e}^{-\mathrm{i}(t-s)\Delta e(k_1,\dots,k_4)} \\ &\quad \Big[\langle k_1\ k_2 | \Phi | k_3\ p \rangle \delta(k_4, q) - \langle k_1\ k_2 | \Phi | k_4\ p \rangle \delta(k_3, q) \\ &\quad - \langle q\ k_2 | \Phi | k_3\ k_4 \rangle \delta(k_1, p) + \langle q\ k_1 | \Phi | k_3\ k_4 \rangle \delta(k_2, p) \Big]. \end{aligned} \tag{18}$$

Thus the equation (11) for $\nu_{pq}(t)$ can be written as

$$(\mathrm{i}\partial_t - e(p) + e(q))\nu_{pq}(t) = \lambda \rho_0(F_{pq}) - \mathrm{i}\lambda^2 \int_0^t \mathrm{d}s\ \rho_s([\Phi, G_{pq}(t-s)]). \tag{19}$$

4 Restricted Quasifreeness

Up to this point, everything was exact; a heuristic derivation of the Boltzmann equation now begins by treating the state ρ_s as quasifree in (19), i.e. expressing the term on the right hand side of (19) as a product over $\nu_{pq}(s)$.

To this end, it is advantageous to avoid any contractions in the commutator, i.e. simply leave it in the form $[\Phi, G] = \Phi G - G\Phi$, which gives

$$\begin{aligned} [\Phi, G_{pq}(t-s)] &= \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \mathrm{d}l_1 \ldots \mathrm{d}l_4 a^+_{k_1} a^+_{k_2} a_{l_4} a_{l_3} a^+_{k_3} a^+_{k_4} a_{l_2} a_{l_1} \\ &\quad M_{pq}(k_1, k_2, k_3, k_4, l_1, l_2, l_3, l_4) \end{aligned} \tag{20}$$

with

$$\begin{aligned} M_{pq}(k_1, \ldots, l_4) &= \Big[\langle k_1\ k_2 | \Phi | l_4\ l_3 \rangle \langle k_3\ k_4 | G_{pq}(t-s) | l_2\ l_1 \rangle \\ &\quad - \langle k_3\ k_4 | \Phi | l_2\ l_1 \rangle \langle k_1\ k_2 | G_{pq}(t-s) | l_4\ l_3 \rangle \Big]. \end{aligned} \tag{21}$$

We recall that expectation values of higher order monomials in a quasifree state ρ_s can be expressed by the two-point functions (see the Appendix). In particular, the four-point function is given by the following determinant

$$\rho_s(a^+_{k_1} a^+_{k_2} a_{l_2} a_{l_1}) = \begin{vmatrix} \nu_{k_1 l_1} & \nu_{k_1 l_2} \\ \nu_{k_2 l_1} & \nu_{k_2 l_2} \end{vmatrix} \tag{22}$$

and the eight-point function appearing in (20) is

$$\rho_s(a^+_{k_1} a^+_{k_2} a_{l_4} a_{l_3} a^+_{k_3} a^+_{k_4} a_{l_2} a_{l_1}) = \begin{vmatrix} \nu_{k_1 l_1} & \nu_{k_1 l_2} & \nu_{k_1 l_3} & \nu_{k_1 l_4} \\ \nu_{k_2 l_1} & \nu_{k_2 l_2} & \nu_{k_2 l_3} & \nu_{k_2 l_4} \\ \nu_{k_3 l_1} & \nu_{k_3 l_2} & \tilde{\nu}_{k_3 l_3} & \tilde{\nu}_{k_3 l_4} \\ \nu_{k_4 l_1} & \nu_{k_4 l_2} & \tilde{\nu}_{k_4 l_3} & \tilde{\nu}_{k_4 l_4} \end{vmatrix} \tag{23}$$

Here each ν_{kl} stands for $\nu_{kl}(s)$ and $\tilde{\nu}_{kl} = -\delta(k,l) + \nu_{kl}(s)$ appears in the lower right block because the monomial is not normal ordered. *We shall call a state ρ_s restricted quasifree if both* (22) *and* (23) *are satisfied. We shall assume this condition in the limit $\lambda \to 0$.*

Return to the derivation of the Boltzmann equation. A Laplace expansion of the determinant gives

$$
\begin{aligned}
\rho_t(a^+_{k_1}\dots a_{l_1}) \;=\; & \begin{vmatrix} \nu_{k_1l_1} & \nu_{k_1l_2} \\ \nu_{k_2l_1} & \nu_{k_2l_2} \end{vmatrix} \begin{vmatrix} \tilde\nu_{k_3l_3} & \tilde\nu_{k_3l_4} \\ \tilde\nu_{k_4l_3} & \tilde\nu_{k_4l_4} \end{vmatrix} \\
- & \begin{vmatrix} \nu_{k_1l_1} & \nu_{k_1l_2} \\ \nu_{k_3l_1} & \nu_{k_3l_2} \end{vmatrix} \begin{vmatrix} \nu_{k_2l_3} & \nu_{k_2l_4} \\ \tilde\nu_{k_4l_3} & \tilde\nu_{k_4l_4} \end{vmatrix} \\
+ & \begin{vmatrix} \nu_{k_1l_1} & \nu_{k_1l_2} \\ \nu_{k_4l_1} & \nu_{k_4l_2} \end{vmatrix} \begin{vmatrix} \nu_{k_2l_3} & \nu_{k_2l_4} \\ \tilde\nu_{k_3l_3} & \tilde\nu_{k_3l_4} \end{vmatrix} \\
+ & \begin{vmatrix} \nu_{k_2l_1} & \nu_{k_2l_2} \\ \nu_{k_3l_1} & \nu_{k_3l_2} \end{vmatrix} \begin{vmatrix} \nu_{k_1l_3} & \nu_{k_1l_4} \\ \tilde\nu_{k_4l_3} & \tilde\nu_{k_4l_4} \end{vmatrix} \\
- & \begin{vmatrix} \nu_{k_2l_1} & \nu_{k_2l_2} \\ \nu_{k_4l_1} & \nu_{k_4l_2} \end{vmatrix} \begin{vmatrix} \nu_{k_1l_3} & \nu_{k_1l_4} \\ \tilde\nu_{k_3l_3} & \tilde\nu_{k_3l_4} \end{vmatrix} \\
+ & \begin{vmatrix} \nu_{k_3l_1} & \nu_{k_3l_2} \\ \nu_{k_4l_1} & \nu_{k_4l_2} \end{vmatrix} \begin{vmatrix} \nu_{k_1l_3} & \nu_{k_1l_4} \\ \nu_{k_2l_3} & \nu_{k_2l_4} \end{vmatrix} .
\end{aligned}
\tag{24}
$$

Noting that $M_{pq}(k_1,\dots,l_4)$ is antisymmetric under exchange of l_1 and l_2 and under exchange of l_3 with l_4, and that the same is true for each of the six summands in the Laplace expansion, we see that we may replace every 2×2 determinant by the product of the diagonal elements if we include a symmetry factor 4. Moreover, $M_{pq}(k_1,\dots,l_4)$ is antisymmetric under $(k_1,k_2,l_4,l_3)\to(k_3,k_4,l_2,l_1)$ (this is just the antisymmetry of the commutator in its two arguments), but the last of the six summands, $\nu_{k_3l_1}\nu_{k_4l_2}\nu_{k_1l_3}\nu_{k_2l_4}$, is symmetric, so it cancels out. Graphically, this is the cancellation of the disconnected term. Thus the term multiplying $M_{pq}(k_1,\dots,l_4)$ is

$$
\begin{aligned}
4 \quad (\quad & \nu_{k_1l_1}\nu_{k_2l_2}\tilde\nu_{k_3l_3}\tilde\nu_{k_4l_4} \\
- \quad & \nu_{k_1l_1}\nu_{k_3l_2}\nu_{k_2l_3}\tilde\nu_{k_4l_4} \\
+ \quad & \nu_{k_1l_1}\nu_{k_4l_2}\nu_{k_2l_3}\tilde\nu_{k_3l_4} \\
+ \quad & \nu_{k_2l_1}\nu_{k_3l_2}\nu_{k_1l_3}\tilde\nu_{k_4l_4} \\
- \quad & \nu_{k_2l_1}\nu_{k_4l_2}\nu_{k_1l_3}\tilde\nu_{k_3l_4}) \\
= 4 \quad \{ \quad & \nu_{k_1l_1}\nu_{k_2l_2}\tilde\nu_{k_3l_3}\tilde\nu_{k_4l_4} \\
+ \quad & (\nu_{k_3l_2}\tilde\nu_{k_4l_4} - \nu_{k_4l_2}\tilde\nu_{k_3l_4})\,(\nu_{k_1l_3}\nu_{k_2l_1} - \nu_{k_1l_1}\nu_{k_2l_3})\}.
\end{aligned}
\tag{25}
$$

Again, the last factor is antisymmetric with respect to an exchange of k_1 and

k_2 and an exchange of k_3 and k_4, so there is another symmetry factor 4, and

$$\begin{aligned} \varrho_s([\Phi, G_{pq}(t-s)]) &= \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \mathrm{d}l_1 \ldots \mathrm{d}l_4 M_{pq}(k_1, \ldots, l_4) \\ & \quad 4(\nu_{k_1 l_1}\nu_{k_2 l_2}\tilde{\nu}_{k_3 l_3}\tilde{\nu}_{k_4 l_4} + 4\nu_{k_1 l_1}\nu_{k_2 l_3}\nu_{k_4 l_2}\tilde{\nu}_{k_3 l_4}). \end{aligned} \tag{26}$$

We remark that if we had used the commutator contraction to express $[\Phi, G]$ in (20), then we would have needed to evaluate only monomials of degree six on the state ϱ_s. The calculation would have been longer because certain cancellations would be less transparent. However this approach has the advantage that it requires the quasifree factorization property of ϱ_s only for degree six monomials instead of degree eight.

5 Spatial homogeneity

If we assume that the distribution is homogeneous in space (i.e. translation invariant) at time zero, then $\nu_{pq} = \delta(p, q) f_p(t)$ for all later times t as well by the translation invariance of H. In this case there are further simplifications: the term $\rho_0(F_{pp})$ vanishes and the $e(p) - e(q)$ term in the differential equation also drops out. Moreover, for $p = q$

$$\begin{aligned} \langle k_1\ k_2 \mid G_{pq} \mid k_3\ k_4 \rangle &= \mathrm{e}^{-\mathrm{i}(t-s)\Delta e(k_1, \ldots, k_4)} \langle k_1\ k_2 | \Phi | k_3\ k_4 \rangle \\ & \quad \times (\delta(p, k_4) + \delta(p, k_3) - \delta(p, k_2) - \delta(p, k_1)) \end{aligned} \tag{27}$$

so we can compute the contribution of the first term on the right hand side of (26) as

$$\begin{aligned} \int \mathrm{d}k_1 \ldots \mathrm{d}l_4 M_{pq}(k_1, \ldots, l_4) 4\nu_{k_1 l_1}\nu_{k_2 l_2}\tilde{\nu}_{k_3 l_3}\tilde{\nu}_{k_4 l_4} & \\ = \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \mathrm{e}^{-\mathrm{i}(t-s)\Delta e(k_1, \ldots, k_4)} & \\ \times 8 |\langle k_1\ k_2 | \Phi | k_3\ k_4 \rangle|^2\ (\delta(k_4, p) - \delta(k_1, p)) & \\ \times (f_{k_1} f_{k_2} \tilde{f}_{k_3} \tilde{f}_{k_4} - f_{k_4} f_{k_3} \tilde{f}_{k_2} \tilde{f}_{k_1}) & \end{aligned} \tag{28}$$

with $\tilde{f}_p = 1 - f_p$. The second term $16 M_{pq}(k_1, \ldots, l_4)\nu_{k_1 l_1}\nu_{k_2 l_3}\nu_{k_4 l_2}\tilde{\nu}_{k_3 l_4}$ in (26) drops out because with the assignment of momenta it is equal to

$$32(\delta(k_1, p) - \delta(k_3, p))\langle k_1 k_2 | \Phi | k_3 k_2 \rangle \langle k_3 k_4 | \Phi | k_4 k_1 \rangle \cos[(t-s)(e(k_3) - e(k_1))],$$

and this quantity vanishes because the delta functions in both Φ factors are $\delta(k_1, k_3)$ and thus $\delta(k_1, p) - \delta(k_3, p) = 0$.

Inserting (5) into (28), recalling that in our finite volume, $\delta(p, p) = L^d$, so that $\delta(p, q)^2 = L^d \delta(p, q)$, and using (6) and symmetry arguments as above, we get

$$\begin{aligned}\partial_t f_p(t) \quad &= \quad -\lambda^2 \int_0^t \mathrm{d}s \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \; \delta(k_1 + k_2, k_3 + k_4) \mathrm{e}^{-\mathrm{i}(t-s)\Delta e(k_1, \ldots, k_4)} \\ &\qquad \times 2(\delta(k_4, p) - \delta(k_1, p)) |\hat{v}(k_1 - k_4) - \hat{v}(k_1 - k_3)|^2 \\ &\qquad \times (f_{k_1}(s) f_{k_2}(s) \tilde{f}_{k_3}(s) \tilde{f}_{k_4}(s) - f_{k_4}(s) f_{k_3}(s) \tilde{f}_{k_2}(s) \tilde{f}_{k_1}(s)).\end{aligned} \tag{29}$$

6 Local approximation in time

We rewrite the equation as

$$-\lambda^{-2} \partial_t f_p(t) = \int_{-\infty}^{\infty} \mathrm{d}E \int_0^t \mathrm{d}s \; \mathrm{e}^{-\mathrm{i}E(t-s)} \; \beta(E, p, s) \tag{30}$$

with

$$\begin{aligned}\beta(E, p, s) \quad &= \quad \int \mathrm{d}k_1 \ldots \mathrm{d}k_4 \; \delta(k_1 + k_2, k_3 + k_4) \; 2(\delta(k_4, p) - \delta(k_1, p)) \\ &\qquad \times |\hat{v}(k_1 - k_4) - \hat{v}(k_1 - k_3)|^2 \delta(E - \Delta e(k_1, \ldots, k_4)) \\ &\qquad \times (f_{k_1}(s) f_{k_2}(s) \tilde{f}_{k_3}(s) \tilde{f}_{k_4}(s) - f_{k_4}(s) f_{k_3}(s) \tilde{f}_{k_2}(s) \tilde{f}_{k_1}(s)),\end{aligned} \tag{31}$$

Since v is symmetric (6), β is a symmetric function of E.

Notice that f and β are λ dependent and we shall denote them by f^λ and β^λ. We now assume that the limits

$$\lim_{\lambda \to 0} f_p^\lambda(T/\lambda^2) = F(T, p), \qquad \lim_{\lambda \to 0} \beta^\lambda(E, p, T/\lambda^2) = B(E, p, T) \tag{32}$$

exist and the relation (31) continues to hold in the limit. We can take the limit $\lambda \to 0$ in (30) and this yields

$$-\partial_T F(T, p) = \lim_{\lambda \to 0} \int_{-\infty}^{\infty} \mathrm{d}E \int_0^T \frac{\mathrm{d}S}{\lambda^2} \; \mathrm{e}^{-\mathrm{i}E(T-S)/\lambda^2} \; \beta^\lambda(E, p, S/\lambda^2) \; . \tag{33}$$

We now assume that we can replace the function β^λ by its limit B. Thus we have

$$-\partial_T F(T,p) = \lim_{\lambda\to 0} \int_{-\infty}^{\infty} \mathrm{d}E \int_0^T \frac{\mathrm{d}S}{\lambda^2}\ \mathrm{e}^{-\mathrm{i}E(T-S)/\lambda^2}\ B(E,p,S/\lambda^2)\ . \tag{34}$$

Interchanging the the integration and performing the E integration, we have

$$-\partial_T F(T,p) = \lim_{\lambda\to 0} \int_0^T \frac{\mathrm{d}S}{\lambda^2}\ \widehat{B}\big(\,(T-S)/\lambda^2, p, S/\lambda^2\,\big)\ .$$

Let $u = (T-S)/\lambda^2$. We can rewrite the last equation as

$$-\partial_T F(T,p) = \lim_{\lambda\to 0} \int_0^T \mathrm{d}u\ \ \widehat{B}\big(\,u, p, T+u\lambda^2\,\big)\ .$$

In the limit $\lambda \to 0$, the right side converges to

$$\int_0^{\infty} \mathrm{d}u\ \ \widehat{B}\big(\,u,p,T\,\big) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}u\ \ \widehat{B}\big(\,u,p,T\,\big) = \pi B(0,p,T)\ .$$

where we have used the symmetry of β in E. Combining the last two equations, we have derived the Boltzmann equation (10).

In this derivation, we used the restricted quasifreeness assumption, spatial homogeneity and the existence of the limit for the the two point function ν_{pq} (*cf*: (32)). We have not made precise the meaning of the limit and we have freely interchanged limits with differentiations and integrations etc. This suggests that for a rigorous proof the two point function has to be controlled precisely, perhaps through some expansion method.

The spatial homogeneity of the initial state can be replaced with the assumption that the point function at any time scales as

$$\nu_{pq}(t) = R(\varepsilon t, \tfrac{p+q}{2}, \tfrac{p-q}{2\varepsilon}).$$

Our derivation above can easily be extended to this case to give the spatially inhomogeneous Boltzmann equation (9).

A Quasifree states and determinants

For finite L, the observable algebra is finite–dimensional, so a state ρ is quasifree if and only if it is given by density matrix coming from a quadratic Hamiltonian (see, e.g. [1]). That is,

$$\rho(A) = \frac{1}{Z}\ \mathrm{tr}\ (\mathrm{e}^{-H_0} A)\ , \tag{35}$$

where $Z = \mathrm{tr}\ \mathrm{e}^{-H_0}$. We restrict to states which are invariant under the transformations $a_p \to \mathrm{e}^{\mathrm{i}\alpha} a_p$ for all $\alpha \in \mathbb{R}$. For this case we prove below that the expectation value of any normal ordered monomial can be computed with the following formula:

$$\rho\left(\prod_{n=1}^{m} a^+_{p_n} \prod_{n'=1}^{m'} a_{q_{n'}}\right) = \delta_{mm'} (-1)^{m(m-1)/2} \det\Big(\rho(a^+_{p_n} a_{q_{n'}})\Big)_{1\le n,n'\le m} . \tag{36}$$

To simplify notation, we enumerate our finite set Λ^* in some way so that we can replace the subscript $p \in \Lambda^*$ by a number $i \in \{1, \dots, N\}$, $N = L^d$. Moreover, because (36) is homogeneous, we may rescale the creation and annihilation operators by $L^{-d/2}$, so that they obey the CAR $a_i\, a^+_j + a^+_j a_i = \delta_{ij}$ with δ_{ij} the Kronecker delta. With these conventions, and by the just stated $U(1)$ invariance,

$$H_0 = \sum_{i,j} a^+_i Q_{ij} a_j\ . \tag{37}$$

Positivity of ρ requires H_0 to be hermitian, so $\bar{Q}_{ij} = Q_{ji}$. Thus there is $U \in U(N)$ such that $Q = UEU^*$ with $E =\ \mathrm{diag}\,\{E_1, \dots, E_N\}$. The operators $b_k = \sum_j U_{kj} a_j$ have canonical anticommutation relations $b_k\, b^+_l + b^+_l b_k = \delta_{kl}$, so that $n_k = b^+_k b_k$ satisfies $n_k^2 = n_k$ and $n_k n_l = n_l n_k$. Thus e^{-H_0} is the product of commuting factors

$$\mathrm{e}^{-H_0} = \prod_k \big(1 + (\mathrm{e}^{-E_k} - 1) n_k\big)\,, \tag{38}$$

hence $Z = \prod_k (1 + \mathrm{e}^{-E_k})$, and $\rho(b^+_k b_l\ ) = \delta_{kl}(1 + \mathrm{e}^{E_k})^{-1}$. This implies

$$\rho(a^+_i a_{i'}) = U_{i'l} \frac{1}{1 + \mathrm{e}^{E_l}} \bar{U}_{il} = (1 + \mathrm{e}^{Q})^{-1}{}_{i'i}. \tag{39}$$

Because H_0 is diagonal when expressed in terms of b^+ and b ,

$$\rho \left(\prod_{k=1}^{m} b^+_{u_k} \prod_{k'=1}^{m'} b_{v'_k} \right) \tag{40}$$

vanishes unless $m' = m$ and $(v_1, \ldots, v_m)$ is a permutation of $(u_1, \ldots, u_m)$: $v_k = u_{\pi(k)}$. In that case, by the CAR, (40) equals

$$(-1)^{m(m-1)/2} \text{ sign } (\pi) \, \rho \left(\prod_{k=1}^{m} (b^+_{u_k} b_{u_k}) \right) \ . \tag{41}$$

Eq. (36) now follows straightforwardly by expressing the product of a^+ and a in terms of b^+ and b and using the definition of the determinant.

In (22) and (23), the indices of the annihilation operators are ordered downwards, so the factor $(-1)^{m(m-1)/2}$ is absent. The procedure of commuting a monomial that is not normal ordered to its normal ordered form corresponds to successive row expansions of the determinant in (23).

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