Nonrelativistic QED at large momentum of photons

Fumio Hiroshima*

Dedicated to Professor Hiroshi Ezawa on the occasion of his seventieth birthday

Abstract

An interacting system of one spinless electron minimally coupled to a quantized radiation field is considered. On the Hamiltonian, H_Λ, of this system, an ultraviolet cutoff $\Lambda > 0$ and an infrared cutoff $\lambda > 0$ are imposed. We do *not* assume the existence of a ground state of the decoupled Hamiltonian. It is proven, however, that there exists Λ_* such that for all $\Lambda > \Lambda_*$, H_Λ has a unique ground state. Furthermore we investigate the asymptotic behavior of $g(\Lambda) = \inf \sigma(H_\Lambda)$ without external potentials as $\Lambda \to \infty$.

1 Introduction

It is a pleasure to dedicate this article to Professor Hiroshi Ezawa on the occasion of his seventieth birthday. He was a source of encouragement in mathematical physics. Especially he suggested physical inspirations to myself in seminars in the Gakushuin University, in ICMP '97 held in Australia, and in Les Houches '98 "Ecole de Physique des Houches" in France. He also invited me as one of lecturers in the summer school of mathematical physics 2000 organized by H. Araki and himself. I wish to express my deep thanks to him for stimulating my works and for innumerable, unbounded and helpful conversations.

*Department of Mathematics and Physics, Setsunan University, Osaka 572-8508, Japan. email: hiroshima@mpg.setsunan.ac.jp

The aim of this paper is to consider a system of one spinless electron minimally coupled to a quantized radiation field. It is the so-called Pauli-Fierz model [24] in nonrelativistic quantum electrodynamics. The Pauli-Fierz Hamiltonian is defined as a self-adjoint operator acting in the tensor product of $L^2(\mathbb{R}^3)$ and a boson Fock space $\mathcal{F}$, $L^2(\mathbb{R}^3)\otimes\mathcal{F}$, which is given by

$$\left(\frac{1}{2m}p^2+V\right)\otimes 1+1\otimes H_{\mathrm{f}}$$

with $p\otimes 1=-i\nabla_x\otimes 1$ replaced by $p\otimes 1+A_\Lambda(x)$, i.e.,

$$\frac{1}{2m}(p\otimes 1+A_\Lambda(x))^2+V\otimes 1+1\otimes H_{\mathrm{f}},$$

where V denotes an external potential, H_{f} the free Hamiltonian in a Fock space and $A_\Lambda(x)$ a quantized radiation field with a ultraviolet cutoff $\Lambda>0$. Details will be seen later. Under some suitable conditions, it is established that a ground state of the Pauli-Fierz Hamiltonian exists if $\frac{1}{2m}p^2+V$ has a ground state. See [3, 8, 9, 10, 13, 18].

In this paper we take the dipole approximation, i.e., we replace as

$$A_\Lambda(x)\longrightarrow 1\otimes A_\Lambda(0),$$

and assume that on the quantized radiation field an infrared cutoff is imposed. Then the Hamiltonian under consideration is

$$H_\Lambda=\frac{1}{2m}(p\otimes 1+1\otimes A_\Lambda(0))^2+V\otimes 1+1\otimes H_{\mathrm{f}}.$$

It is proven in [12] that H_Λ also has a ground state even when no infrared cutoff is imposed, if $\frac{1}{2m}p^2+V$ has a ground state.

A more difficult, but physically reasonable, question is about enhanced binding. One may expect that a coupling with a quantized radiation field enhances binding. In this paper, we suppose that the electron is governed by a sufficiently shallow nonpositive external potential V, and then $\frac{1}{2m}p^2+V$ has *no ground states*. We prove, however, that H_Λ has a ground state φ_{g} for sufficiently large Λ. The physical reasoning behind such a result is simple. On a heuristic level one argues that a coupling with a quantized radiation field amounts to renormalizing bare mass m to effective mass:

$$m \to m_{\text{eff}} = m + \delta m(\Lambda)$$

Since effective mass $m(\Lambda)$ is increasing in Λ, a bound state could be produced for a sufficiently large Λ. Roughly speaking, H_Λ may be replaced by

$$H_\Lambda \sim H_{\text{eff}} = \left(\frac{p^2}{2m_{\text{eff}}} + V\right) \otimes 1 + 1 \otimes H_{\text{f}}, \tag{1.1}$$

which binds for sufficiently large Λ. Needless to say (1.1) has no mathematical meaning, we show, however, the associated phenomena in this paper.@

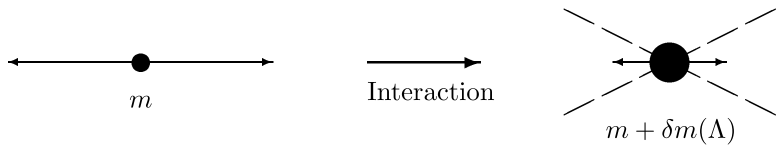

Figure 1: Effective mass

This kind of investigation has been done in Hiroshima and Spohn [20], and developed by e.g., [2, 4, 5, 11]. Arai and Kawano [2] proved the similar result as ours in a general framework. Chen, Vougalter and Vugalter [5], Catto and Hainzl [4], Hainzl, Vougalter and Vugalter [11] study the case of no dipole approximation under some conditions.

This paper is organized as follows. In Section 2 we define the Pauli-Fierz Hamiltonian and its self-adjointness is stated. In Section 3 we diagonalize the Pauli-Fierz Hamiltonian. In Section 4 we prove the main theorem. Section 5 is devoted to showing the multiplicity and localization of a ground state. In Section 6 we investigate the asymptotic behavior of the ground state energy of H_Λ with no external potential.

2 Nonrelativistic QED

2.1 The Pauli-Fierz Hamiltonian

The boson Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3 \times \{1,2\})$ is defined by

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^3 \times \{1,2\})) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(L^2(\mathbb{R}^3 \times \{1,2\})),$$

where

$$\mathcal{F}^{(n)}(L^2(\mathbb{R}^3 \times \{1,2\})) = \otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})$$

denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1,2\})$ with

$$\otimes_s^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}.$$

The Fock vacuum is defined by

$$\Omega = \{1, 0, 0, ...\} \in \mathcal{F}.$$

$\{a^*(k,j), a(k,j)\}_{j=1,2}$ are bose fields with commutation relations

$$[a^*(k,j), a(k',j')] = \delta_{jj'}\delta(k-k'),$$

$$[a^*(k,j), a^*(k',j')] = 0,$$

$$[a(k,j), a(k',j')] = 0.$$

The quantized radiation field, $A_\Lambda(x), x \in \mathbb{R}^3$, is defined by

$$A_\Lambda(x) = \sum_{j=1,2} \int \frac{\chi_\Lambda(k)}{\sqrt{2\omega(k)}} e_j(k) \left\{ a^*(k,j)e^{-ik\cdot x} + a(k,j)e^{ik\cdot x} \right\} d^3k,$$

where χ_Λ denotes the momentum cutoff given by

$$\chi_\Lambda(k) = \begin{cases} 0, & |k| < \lambda, \\ 1, & \lambda \le |k| \le \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

We fix an infrared cutoff, $\lambda > 0$, and assume that

$$\lambda \le \Lambda.$$

$e_j(k) = (e_j^1(k), e_j^2(k), e_j^3(k)), j = 1, 2$, are polarization vectors, i.e.,

$$\{e_1(k), e_2(k), k/|k|\}$$

forms a right-handed system, i.e.,

$$e_j \cdot k = 0, \quad e_i \cdot e_j = \delta_{ij}, \quad e_1 \times e_2 = k/|k|.$$

The dispersion relation is given by

$$\omega(k) = |k|$$

and therefore the free Hamiltonian of $\mathcal{F}$ is defined by

$$H_{\rm f} = \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) d^3k.$$

Let us denote the spectrum (resp. discrete spectrum, essential spectrum) of T by $\sigma(T)$ (resp. $\sigma_{\rm disc}(T)$, $\sigma_{\rm ess}(T)$). It is well known that

$$\sigma(H_{\rm f}) = [0, \infty), \quad \sigma_{\rm disc}(H_{\rm f}) = \{0\},$$

and

$$H_{\rm f}\Omega = 0.$$

The Hilbert space of our system is

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

The decoupled Hamiltonian is given by

$$H_0 = \left(\frac{1}{2m} p^2 + V\right) \otimes 1 + 1 \otimes H_{\rm f}.$$

Here m denotes the bare mass of an electron, $p = -i\nabla_x$ the momentum operator canonically conjugate to the position operator x in $L^2(\mathbb{R}^3)$ and V an external potential. Thus

$$\sigma(H_0) = \{\lambda_1 + \lambda_2 | \lambda_1 \in \sigma(\frac{1}{2m} p^2 + V), \ \lambda_2 \in \sigma(H_{\rm f})\}.$$

We take the dipole approximation, i.e., we replace as

$$A_\Lambda(x) \longrightarrow 1 \otimes A_\Lambda(0).$$

Set

$$A_\Lambda = A_\Lambda(0).$$

The total Hamiltonian, H_Λ, is defined by H_0 with the replacement

$$p \otimes 1 \longrightarrow p \otimes 1 + 1 \otimes A_\Lambda,$$

i.e.,

$$H_\Lambda = \frac{1}{2m}\left(p \otimes 1 + 1 \otimes A_\Lambda\right)^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}}.$$

For notational convenience we mostly omit the tensor notation $\otimes$ unless confusions may arise, e.g., we simply write as p, V, A_Λ, H_{f}, instead of $p \otimes 1$, $V \otimes 1$, $1 \otimes A_\Lambda$ and $1 \otimes H_{\mathrm{f}}$, respectively. Then H_Λ is simply written as

$$H_\Lambda = \frac{1}{2m}(p + A_\Lambda)^2 + V + H_{\mathrm{f}}.$$

Proposition 2.1 *Suppose that $D(p^2) \subset D(V)$ and*

$$\|Vf\| \leq a\|\frac{1}{2m}p^2 f\| + b\|f\|$$

with some $0 \leq a < 1$ and $b \geq 0$. Then H_Λ is self-adjoint on $D(p^2) \cap D(H_{\mathrm{f}})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $\frac{1}{2m}p^2 + H_{\mathrm{f}}$.

Proof: See [15, 16]. □

Remark 2.2 *The Hamiltonian without the dipole approximation, i.e.,*

$$\frac{1}{2m}(p + A_\Lambda(x))^2 + V + H_{\mathrm{f}},$$

is also self-adjoint on $D(p^2) \cap D(H_{\mathrm{f}})$. See [15, 16].

2.2 Enhanced binding and the main theorem

In this paper we consider the case where the external potential V is sufficiently shallow. By the Lieb-Thirring inequality [23],

$$\#\{\text{bound states of } \frac{1}{2m}p^2 + V\} \leq L_{0,3} \int |mV_-(x)|^{3/2} d^3x$$

$$(V_- : \text{ the negative part of } V)$$

with some constant $L_{0,3}$ independent of V, we see that for a sufficiently shallow nonpositive potential V, $\frac{1}{2m}p^2 + V$ has no bound states. In particular it has no ground state. Thus the decoupled Hamiltonian $\frac{1}{2m}p^2 + V + H_{\rm f}$ also has no ground state. Our contribution is to show, however, that $H_\Lambda = \frac{1}{2m}(p + A_\Lambda)^2 + V + H_{\rm f}$ has a ground state for sufficiently large Λ.

Throughout this paper we assume Hypothesis (V).

Hypothesis (V)

(a) $V \in C_0^\infty(\mathbb{R}^3)$.

(b) $V \leq 0$.

(c) *There exists $\mu_0 > 1$ and $r > 0$ such that for $\mu > \mu_0$,*

$$\inf \sigma(\frac{1}{2m}p^2 + \mu V) < -r.$$

Since V is relatively compact with respect to $\frac{1}{2m}p^2$, it holds that

$$\sigma_{\rm ess}(\frac{1}{2m}p^2 + \mu V) = [0, \infty).$$

Hence $\frac{1}{2m}p^2 + \mu V$, $\mu > \mu_0$, has a ground state.

Remark 2.3 *We do not assume the existence of ground states of $\frac{1}{2m}p^2 + V$.*

The main result of this paper is as follows.

Theorem 2.4 *There exists Λ_* such that for $\Lambda > \Lambda_*$, H_Λ has a ground state.*

3 Canonical transformations

The following proposition is a key in this paper.

Proposition 3.1 *There exists a unitary operator U such that U maps $D(p^2)\cap D(H_{\rm f})$ onto itself and*

$$U^{-1}H_\Lambda U = \frac{1}{2m_{\rm eff}}p^2 + V(\cdot - K/m_{\rm eff}) + H_{\rm f} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2\|\chi_\Lambda/(t^2+\omega^2)\|^2}{m+\frac{2}{3}\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2}dt,$$

where

$$m_{\rm eff} = m + \frac{8\pi}{3}(\Lambda - \lambda),$$

$$K = \sum_{j=1,2}\int \frac{\chi_\Lambda(k)e_j(k)}{\sqrt{2\omega(k)}}\left\{\frac{a^*(k,j)}{\omega(k)D_-(\omega(k)^2)} + \frac{a(k,j)}{\omega(k)D_+(\omega(k)^2)}\right\}d^3k, \qquad (3.1)$$

and

$$D_\pm(s) = m - \frac{4\pi}{3}\left(\lim_{\epsilon\downarrow 0}\int_{|s-x|<\epsilon}\frac{\sqrt{x}\chi_\Lambda(\sqrt{x})}{s-x}dx \mp 2\pi i\chi_\Lambda(\sqrt{s})\sqrt{s}\right) \neq 0.$$

Proof: This is proven by a slight modification of [20, Appendix]. □

Remark 3.2 *In [20] it is obtained that*

$$m_{\rm eff} = m + \frac{2}{3}\|\chi_\Lambda/\omega\|^2.$$

Remark 3.3 *We see that $D_\pm(s) = m$ for $0 \le s < \lambda^2$ or $\Lambda^2 < s$, and $\Im D_\pm(s) = 2\pi\sqrt{s}$ for $\lambda^2 \le s \le \Lambda^2$. In particular $D_\pm(s) \neq 0$ for $s \ge 0$.*

We set

$$\varrho(\cdot,j) = (\varrho_1(\cdot,j),\varrho_2(\cdot,j),\varrho_3(\cdot,j)),$$

where

$$\varrho_\mu(k,j) = \frac{e_j^\mu(k)\chi_\Lambda(k)}{\omega^{3/2}(k)D_+(\omega(k)^2)}, \quad \mu = 1,2,3.$$

Then we rewrite (3.1), $K = (K_1,K_2,K_3)$, as

$$K_\mu = \sum_{j=1,2}\frac{1}{\sqrt{2}}\int\{\varrho_\mu(k,j)a^*(k,j) + \varrho_\mu(k,j)^*(k)a(k,j)\}d^3k$$

and $\varrho_\mu(\cdot, j)$ satisfies that

$$\|\omega^{n/2}\varrho_\mu(\cdot, j)\| \le C_1\|\omega^{(n-3)/2}\chi_\Lambda\| \tag{3.2}$$

with some constant C_1. Let

$$\phi(f) = \sum_{j=1,2}\int \{f(k)a^*(k,j) + f^*(k)a(k,j)\}\, d^3k.$$

Then the following inequality is well known for $\Psi \in D({H_\mathrm{f}}^{1/2})$,

$$\|\phi(f)\Psi\| \le (\|f/\omega^{1/2}\| + \|f\|)(2\|{H_\mathrm{f}}^{1/2}\Psi\| + \|\Psi\|).$$

From (3.2) it follows that

$$\|K_\mu\Psi\| \le C_2(\|\chi_\Lambda/\omega^2\| + \|\chi_\Lambda/\omega^{3/2}\|)(\|{H_\mathrm{f}}^{1/2}\Psi\| + \|\Psi\|) \tag{3.3}$$

with some constant C_2. We set

$$\delta V = V(\cdot - K/m_\mathrm{eff}) - V,$$

$$H_\mathrm{eff} = \frac{1}{2m_\mathrm{eff}}p^2 + V,$$

$$g(\Lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2\|\chi_\Lambda/(t^2+\omega^2)\|^2}{m + \frac{2}{3}\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2}dt,$$

and

$$\widehat{H}_\Lambda = U^{-1}H_\Lambda U = H_\mathrm{eff} + \delta V + H_\mathrm{f} + g(\Lambda). \tag{3.4}$$

We investigate a spectral property of $\widehat{H}_\Lambda$ instead of H_Λ in what follows.

Lemma 3.4 *Let Λ be such that*

$$\Lambda > \frac{3}{8\pi}(\mu_0 - 1)m.$$

Then H_eff has a ground state, and moreover

$$\#\{\textit{bound states of } H_\mathrm{eff}\} \le L_{0,3}\left(m + \frac{8}{3}\pi(\Lambda-\lambda)\right)^{3/2}\int |V(x)|^{3/2}d^3x. \tag{3.5}$$

Proof: By Hypothesis (V),

$$H_{\text{eff}} = \frac{1}{2m_{\text{eff}}}p^2 + V = \frac{m}{m_{\text{eff}}}\left(\frac{1}{2m}p^2 + \frac{m_{\text{eff}}}{m}V\right)$$

implies that if

$$\frac{m_{\text{eff}}}{m} > \mu_0, \tag{3.6}$$

then H_{eff} has a ground state. (3.6) is identical with $\Lambda > \frac{3}{8\pi}(\mu_0 - 1)m$. (3.5) follows from the Lieb-Thirring inequality. Then the lemma follows. □

4 Existence of ground states

4.1 Lattice approximation

We introduce an artificial mass of photon, $\nu > 0$, and define

$$\widehat{H}_\Lambda^\nu = H_{\text{eff}} + \delta V^\nu + H_{\text{f}}^\nu + g(\Lambda),$$

where δV^ν and H_{f}^ν are defined by δV and H_{f} with ω replaced by $\omega + \nu$, respectively. Using a momentum lattice approximation we will prove that $\widehat{H}_\Lambda^\nu$ has a ground state. Let $\Gamma(l, a)$, $l = (l_1, l_2, l_3) \in \mathbb{Z}^3$, $a > 0$, be the momentum lattice with spacing $1/a$, i.e.,

$$\Gamma(l, a) = [\frac{l_1}{a}, \frac{(l_1+1)}{a}) \times [\frac{l_2}{a}, \frac{(l_2+1)}{a}) \times [\frac{l_3}{a}, \frac{(l_3+1)}{a}),$$

and

$$\chi_{\Gamma(l,a)}(k) = \begin{cases} 0, & k \notin \Gamma(l, a), \\ a^{3/2}, & k \in \Gamma(l, a). \end{cases}$$

For $L > 0$ we define the momentum-lattice-approximated Hamiltonian by

$$\widehat{H}_\Lambda^\nu(a, L) = H_{\text{eff}} + H_{\text{f}}^\nu(a, L) + \delta V^\nu(a, L) + g(\Lambda),$$

where

$$H_{\text{f}}^\nu(a, L) = \sum_{j=1,2} \int \sum_{|l| \le L} \chi_{\Gamma(l,a)}(k)(\omega(l) + \nu)a^*(k, j)a(k, j)d^3k,$$

$$\delta V^\nu(a, L) = V(\cdot - K^\nu(a, L)/m_{\text{eff}})$$

and $K^\nu(a,L) = (K_1^\nu(a,L), K_2^\nu(a,L), K_3^\nu(a,L))$ with

$$K_\mu^\nu(a,L) = \sum_{j=1,2} \frac{1}{\sqrt{2}} \int \sum_{|l|\le L} \chi_{\Gamma(l,a)}(k)(\varrho_\mu^\nu(l,j)a^*(k,j) + \varrho_\mu^\nu(l,j)^* a(k,j))d^3k.$$

In the similar way as (3.3) we can show that

$$\|K_\mu^\nu(a,L)\Psi\| \le C_2' \left(\left\| \frac{\sum_{|l|\le L} \chi_{\Gamma(l,a)}\chi_\Lambda(l)}{\sqrt{\omega}(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}} \right\| + \left\| \frac{\sum_{|l|\le L} \chi_{\Gamma(l,a)}\chi_\Lambda(l)}{(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}} \right\| \right)$$
$$\times (\|H_{\rm f}^\nu(a,L)^{1/2}\Psi\| + \|\Psi\|) \tag{4.1}$$

with some constant C_2'. Here we used

$$c_1\|H_{\rm f}^\nu(a,L)^{1/2}\Psi\| \le \|H_{\rm f}^{\nu 1/2}\Psi\| \le c_2\|H_{\rm f}^\nu(a,L)^{1/2}\Psi\|$$

with some constants c_1 and c_2.

Lemma 4.1 *We have*

$$\lim_{L\to\infty}\lim_{a\to\infty} \widehat{H}_\Lambda^\nu(a,L) = \widehat{H}_\Lambda^\nu$$

in the operator norm in the sense of resolvent.

Proof: It is well known that there exists $\epsilon(a,L)$ such that

$$\|(H_{\rm f}^\nu(a,L) - H_{\rm f}^\nu)\Psi\| \le \epsilon(a,L)\|H_{\rm f}^\nu\Psi\|$$

and

$$\lim_{L\to\infty}\lim_{a\to\infty}\epsilon(a,L) = 0.$$

Moreover

$$\|(\delta V^\nu(a,L) - \delta V^\nu)\Psi\| = \|(V(\cdot - K^\nu(a,L)/m_{\rm eff}) - V(\cdot - K^\nu/m_{\rm eff}))\Psi\|$$
$$\le \sum_{\mu=1}^3 \frac{\|\nabla_\mu V\|_\infty}{m_{\rm eff}} \|(K_\mu^\nu - K_\mu^\nu(a,L))\Psi\|.$$

Since

$$\|(K_\mu^\nu - K_\mu^\nu(a,L))\Psi\| \le \sum_{j=1,2} C \left(\left\| \frac{1}{\sqrt{\omega}} \left\{ \frac{\varrho_\mu^\nu(\cdot,j)}{\omega^{3/2}} - \frac{\sum_{|l|\le L}\chi_{\Gamma(l,a)}\varrho_\mu^\nu(l,j)}{(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}} \right\} \right\| \right.$$

$$+\left\|\frac{\varrho_\mu^\nu(\cdot,j)}{\omega^{3/2}}-\frac{\sum_{|l|\le L}\chi_{\Gamma(l,a)}\varrho_\mu^\nu(l,j)}{(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}}\right\|\Big)(\|H_{\rm f}^{\nu 1/2}\Psi\|+\|\Psi\|)$$

$$=\epsilon(a,L)'(\|H_{\rm f}^{\nu 1/2}\Psi\|+\|\Psi\|)$$

with some constant C, and $\epsilon(a,L)'$ satisfies that

$$\lim_{L\to\infty}\lim_{a\to\infty}\epsilon(a,L)'=0,$$

we have

$$\|(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi-(\widehat{H}_\Lambda^\nu(a,L)-z)^{-1}\Psi\|$$

$$\le\|(\widehat{H}_\Lambda^\nu(a,L)-z)^{-1}\|\|(\widehat{H}_\Lambda^\nu(a,L)-\widehat{H}_\Lambda^\nu)(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi\|$$

$$\le\frac{(\max_\mu\|\nabla_\mu V\|_\infty)\epsilon(a,L)'}{|\Im z|m_{\rm eff}}\Big(\|H_{\rm f}^{\nu 1/2}(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi\|+\|(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi\|\Big)$$

$$+\frac{\epsilon(a,L)}{|\Im z|}\|H_{\rm f}^{\nu 1/2}(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi\|.$$

Since

$$\|H_{\rm f}^\nu(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi\|\le C'\|\Psi\|$$

with some constant C', we have

$$\|(\widehat{H}_\Lambda^\nu-z)^{-1}\Psi-(\widehat{H}_\Lambda^\nu(a,L)-z)^{-1}\Psi\|\le\epsilon''(a,L)\|\Psi\|$$

with $\epsilon''(a,L)$ such that

$$\lim_{L\to\infty}\lim_{a\to\infty}\epsilon''(a,L)=0.$$

Hence the lemma follows. □

4.2 Artificial mass

Let $f\in L^2(\mathbb{R}^3)$. We identify

$$\{f(l)\}_{l\in\mathbb{Z}^3}\in\ell^2(\mathbb{Z}^3)$$

with

$$\sum_{l\in\mathbb{Z}^3}f(l)\chi_{\Gamma(l,a)}(\cdot)\in L^2(\mathbb{R}^3).$$

By this identification we regard $\ell^2(\mathbb{Z}^3)$ as

$$\ell^2 = \ell^2(\mathbb{Z}^3) \subset L^2(\mathbb{R}^3).$$

Let

$$\mathcal{H}_a = L^2(\mathbb{R}^3) \otimes \mathcal{F}(\ell^2 \otimes \mathbb{C}^2).$$

By the following fundamental identification

$$\begin{aligned}
\mathcal{F}(L^2(\mathbb{R}^3 \times \{1,2\})) &\cong \mathcal{F}(L^2(\mathbb{R}^3)\otimes\mathbb{C}^2) = \mathcal{F}([\ell^2 \oplus \ell^{2\perp}]\otimes\mathbb{C}^2) \\
&\cong \mathcal{F}(\ell^2\otimes\mathbb{C}^2)\otimes\mathcal{F}(\ell^{2\perp}\otimes\mathbb{C}^2) \cong \mathcal{F}(\ell^2\otimes\mathbb{C}^2)\otimes\left[\bigoplus_{n=0}^{\infty}\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\right] \\
&= \mathcal{F}(\ell^2\otimes\mathbb{C}^2)\otimes\left[\bigoplus_{n=1}^{\infty}\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\oplus\mathbb{C}\right] \\
&\cong \left(\mathcal{F}(\ell^2\otimes\mathbb{C}^2)\otimes\left[\bigoplus_{n=1}^{\infty}\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\right]\right)\oplus\mathcal{F}(\ell^2\otimes\mathbb{C}^2),
\end{aligned}$$

we have

$$\mathcal{H} \cong \left(\mathcal{H}_a\otimes\left[\bigoplus_{n=1}^{\infty}\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\right]\right)\oplus\mathcal{H}_a.$$

In particular we see that

$$\mathcal{H}_a^{\perp} \cong \mathcal{H}_a\otimes\left[\bigoplus_{n=1}^{\infty}\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\right]. \tag{4.2}$$

The following lemma is easily proven.

Lemma 4.2 *$\widehat{H}_\Lambda^\nu(a,L)$ is reduced by $\mathcal{H}_a$ and, under identification (4.2), we have*

$$\widehat{H}_\Lambda^\nu(a,L)\Big\lceil_{\mathcal{H}_a^\perp} \cong \left(\widehat{H}_\Lambda^\nu(a,L)\Big\lceil_{\mathcal{H}_a}\right)\otimes 1 + 1\otimes H_{\mathrm{f}}^\nu(a,L)\lceil_{\left[\oplus_{n=1}^\infty\mathcal{F}^{(n)}(\ell^{2\perp}\otimes\mathbb{C}^2)\right]}.$$

In particular

$$\inf\sigma(\widehat{H}_\Lambda^\nu(a,L)\Big\lceil_{\mathcal{H}_a^\perp}) \geq \inf\sigma(H_{\mathrm{f}}^\nu(a,L)) + \nu.$$

Note that $H_{\mathrm{f}}^\nu(a,L)\lceil_{\mathcal{H}_a}$ has a purely discrete spectrum, i.e.,

$$\sigma(H_{\mathrm{f}}^\nu(a,L)\lceil_{\mathcal{H}_a}) = \sigma_{\mathrm{disc}}(H_{\mathrm{f}}^\nu(a,L)\lceil_{\mathcal{H}_a}).$$

Lemma 4.3 *Let* $\Psi \in D(p^2) \cap D({H_{\rm f}}^{1/2})$. *Then*

(1) $\Psi \in D(\delta V^\nu)$ *and*

$$\|\delta V^\nu \Psi\| \le \theta(\Lambda)(\|H_{\rm f}^{\nu\,1/2}\Psi\| + \|\Psi\|), \tag{4.3}$$

where

$$\theta(\Lambda) = \frac{C\|\nabla V\|_\infty}{m_{\rm eff}}(\|\chi_\Lambda/\omega^2\| + \|\chi_\Lambda/\omega^{3/2}\|)$$

with some constant C.

(2) $\Psi \in D(\delta V^\nu(a,L))$ *and*

$$\|\delta V^\nu(a,L)\Psi\| \le \theta(\Lambda,a,L)(\|H_{\rm f}^\nu(a,L)^{1/2}\Psi\| + \|\Psi\|), \tag{4.4}$$

where

$$\theta(\Lambda,a,L) = \frac{C'\|\nabla V\|_\infty}{m_{\rm eff}}$$

$$\times\left(\left\|\frac{1}{\sqrt{\omega}}\left\{\frac{\sum_{|l|\le L}\chi_{\Gamma(l,a)}\chi_\Lambda(l)}{(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}}\right\}\right\| + \left\|\frac{\sum_{|l|\le L}\chi_{\Gamma(l,a)}\chi_\Lambda(l)}{(\sum_{|l|\le L}\chi_{\Gamma(l,a)}\omega(l))^{3/2}}\right\|\right)$$

with some constant C'.

Proof: We have

$$\|\delta V^\nu\Psi\| \le \sum_{\mu=1}^3 \frac{\|\nabla_\mu V\|_\infty}{m_{\rm eff}}\|K_\mu\Psi\|.$$

(4.3) follows from (3.3). (4.4) is similarly proven. $\square$

Remark 4.4 $\theta(\Lambda)$ *and* $\theta(\Lambda,a,L)$ *are independent of* ν.

We write $A \le B$, if $D(B) \subset D(A)$ and $(\psi, A\psi) \le (\psi, B\psi)$ for $\psi \in D(B)$.

We have

$$|(\Psi, \delta V^\nu\Psi)| \le \theta(\Lambda)\left\{\|\Psi\|(\|H_{\rm f}^{\nu\,1/2}\Psi\| + \|\Psi\|)\right\}$$

$$\le \theta(\Lambda)\left(\|\Psi\|^2 + \frac{1}{2}\|\Psi\|^2 + \frac{1}{2}\|H_{\rm f}^{\nu\,1/2}\Psi\|^2\right) = (\Psi, \theta(\Lambda)(\frac{3}{2} + \frac{1}{2}H_{\rm f}^\nu)\Psi).$$

Thus we conclude

$$-\theta(\Lambda)\left(\frac{1}{2}H_{\mathrm{f}}^{\nu}+\frac{3}{2}\right)\leq \delta V^{\nu}\leq \theta(\Lambda)\left(\frac{1}{2}H_{\mathrm{f}}^{\nu}+\frac{3}{2}\right). \tag{4.5}$$

Hence for $f\in C_0^\infty(\mathbb{R}^3)$,

$$\inf\sigma(\widehat{H}_\Lambda^\nu)\leq (f\otimes\Omega, \widehat{H}_\Lambda^\nu f\otimes\Omega)\leq (f,(H_{\mathrm{eff}}+g(\Lambda)+\frac{3}{2}\theta(\Lambda))f).$$

In particular, since $C_0^\infty(\mathbb{R}^3)$ is a core of H_{eff}, we have

$$\inf\sigma(\widehat{H}_\Lambda^\nu)\leq \inf\sigma(H_{\mathrm{eff}})+\frac{3\theta(\Lambda)}{2}+g(\Lambda).$$

Similarly we have

$$-\theta(\Lambda,a,L)\left(\frac{1}{2}H_{\mathrm{f}}^{\nu}(a,L)+\frac{3}{2}\right)\leq \delta V^{\nu}(a,L)\leq \theta(\Lambda,a,L)\left(\frac{1}{2}H_{\mathrm{f}}^{\nu}(a,L)+\frac{3}{2}\right) \tag{4.6}$$

and hence

$$\inf\sigma(\widehat{H}_\Lambda^\nu(a,L))\leq \inf\sigma(H_{\mathrm{eff}})+\frac{3\theta(\Lambda,a,L)}{2}+g(\Lambda).$$

We summarize above statements in the following lemma.

Lemma 4.5 *We have*

$$\inf\sigma(\widehat{H}_\Lambda^\nu)\leq \inf\sigma(H_{\mathrm{eff}})+\frac{3\theta(\Lambda)}{2}+g(\Lambda)$$

and

$$\inf\sigma(\widehat{H}_\Lambda^\nu(a,L))\leq \inf\sigma(H_{\mathrm{eff}})+\frac{3\theta(\Lambda,a,L)}{2}+g(\Lambda).$$

We see that as $\Lambda\to\infty$

$$\|\chi_\Lambda/\omega^l\|\sim\Lambda^{-l+\frac{3}{2}},\quad l\neq\frac{3}{2},$$

$$\|\chi_\Lambda/\omega^l\|\sim\sqrt{\log\Lambda},\quad l=\frac{3}{2}.$$

From this it follows that as $\Lambda\to\infty$

$$m_{\mathrm{eff}}=m+\frac{8\pi}{3}(\Lambda-\lambda)\sim\Lambda, \tag{4.7}$$

$$\|\chi_\Lambda/\omega^2\|\sim\Lambda^{-1/2}, \tag{4.8}$$

$$\|\chi_\Lambda/\omega^{3/2}\|\sim\sqrt{\log\Lambda}. \tag{4.9}$$

Lemma 4.6 *It follows that*

$$\lim_{\Lambda\to\infty}\theta(\Lambda)=0$$

and

$$\lim_{\Lambda\to\infty}\theta(\Lambda,a,L)=0.$$

Proof: These are follows from (4.7), (4.8), (4.9) and the definitions of $\theta(\Lambda)$ and $\theta(\Lambda,a,L)$. $\square$

We set

$$\Sigma=\Sigma(\Lambda)=\inf\sigma(H_{\mathrm{eff}}).$$

It is easily seen that

$$\lim_{\Lambda\to\infty}\Sigma=\inf_x V(x)<0. \tag{4.10}$$

For a self-adjoint operator M, the spectral projection of M on a Borel set $\mathcal{B}\subset\mathbb{R}$ is denoted by

$$E_{\mathcal{B}}^M.$$

Lemma 4.7 *Let a, L and Λ be sufficiently large such that*

$$\min\{|\Sigma|/3,2\}>\theta(\Lambda,a,L). \tag{4.11}$$

Then for ν such that $|\Sigma|>3\theta(\Lambda,a,L)+\nu$,

$$\begin{aligned}&\widehat{H}_\Lambda^\nu(a,L)\lceil_{\mathcal{H}_a}-\inf\sigma(\widehat{H}_\Lambda^\nu(a,L)\lceil_{\mathcal{H}_a})-\nu\\ &\geq E_{[0,|\Sigma|)}^{H_{\mathrm{eff}}-\Sigma}\otimes\left((1-\frac{\theta(\Lambda,a,L)}{2})H_{\mathrm{f}}^\nu(a,L)-3\theta(\Lambda,a,L)-\nu\right).\end{aligned}$$

Proof: In this proof we set

$$H=\widehat{H}_\Lambda^\nu(a,L)\lceil_{\mathcal{H}_a}$$

as an operator in $\mathcal{H}_a$ and

$$\theta=\theta(\Lambda,a,L).$$

We directly see that

$$H-\inf\sigma(H)-\nu$$

$$
\begin{aligned}
&= H_{\mathrm{eff}} + \delta V^{\nu}(a,L) + H_{\mathrm{f}}^{\nu}(a,L) + g(\Lambda) - \inf\sigma(H) - \nu \\
&\geq H_{\mathrm{eff}} + \delta V^{\nu}(a,L) + H_{\mathrm{f}}^{\nu}(a,L) + g(\Lambda) - g(\Lambda) - \frac{3}{2}\theta - \Sigma - \nu \\
&\geq H_{\mathrm{eff}} + (1-\frac{\theta}{2})H_{\mathrm{f}}^{\nu}(a,L) - \frac{3}{2}\theta - \frac{3}{2}\theta - \Sigma - \nu \\
&= (H_{\mathrm{eff}} - \Sigma) + (1-\frac{\theta}{2})H_{\mathrm{f}}^{\nu}(a,L) - 3\theta - \nu \\
&\geq |\Sigma| E_{[|\Sigma|,\infty)}^{H_{\mathrm{eff}}-\Sigma} \otimes 1 - \theta'(E_{[0,|\Sigma|)}^{H_{\mathrm{eff}}-\Sigma} + E_{[|\Sigma|,\infty)}^{H_{\mathrm{eff}}-\Sigma}) \otimes 1 \\
&\quad + (1-\frac{\theta}{2})(E_{[0,|\Sigma|)}^{H_{\mathrm{eff}}-\Sigma} + E_{[|\Sigma|,\infty)}^{H_{\mathrm{eff}}-\Sigma}) \otimes H_{\mathrm{f}}^{\nu}(a,L),
\end{aligned}
$$

where $\theta' = 3\theta + \nu$. Then

$$
\begin{aligned}
H - \inf\sigma(H) - \nu &\geq (|\Sigma| - \theta')E_{[|\Sigma|,\infty)}^{H_{\mathrm{eff}}-\Sigma} \otimes 1 + (1-\frac{\theta}{2})E_{[|\Sigma|,\infty)}^{H_{\mathrm{eff}}-\Sigma} \otimes H_{\mathrm{f}}^{\nu}(a,L) \\
&\quad + E_{[0,|\Sigma|)}^{H_{\mathrm{eff}}-\Sigma} \otimes \left((1-\frac{\theta}{2})H_{\mathrm{f}}^{\nu}(a,L) - \theta' \right).
\end{aligned}
$$

Since by (4.11),

$$|\Sigma| - \theta' = |\Sigma| - 3\theta - \nu > 0$$

and

$$1 - \frac{\theta}{2} > 0,$$

we have

$$H - \inf\sigma(H) - \nu \geq E_{[0,|\Sigma|)}^{H_{\mathrm{eff}}-\Sigma} \otimes \left((1-\frac{\theta}{2})H_{\mathrm{f}}^{\nu}(a,L) - \theta' \right).$$

Thus the lemma follows. □

Set

$$T = \widehat{H}_{\Lambda}^{\nu}(a,L) - \inf\sigma(\widehat{H}_{\Lambda}^{\nu}(a,L)) - \nu.$$

Define

$$\mathcal{H}_a(+) = E_{[0,\infty)}^{T}\mathcal{H}_a,$$

$$\mathcal{H}_a(-) = E_{[-\nu,0)}^{T}\mathcal{H}_a.$$

Lemma 4.8 *Suppose that* $\min\{|\Sigma|/3, 2\} > \theta(\Lambda, a, L)$. *Then for* ν *such that* $|\Sigma| > 3\theta(\Lambda, a, L) + \nu$, $T\lceil_{\mathcal{H}_a(-)}$ *has a purely discrete spectrum, i.e.,*

$$\sigma(\widehat{H}^\nu_\Lambda(a, L)\Big\lceil_{\mathcal{H}_a}) \cap [\inf \sigma(\widehat{H}^\nu_\Lambda(a, L)\Big\lceil_{\mathcal{H}_a}), \inf \sigma(\widehat{H}^\nu_\Lambda(a, L)\Big\lceil_{\mathcal{H}_a}) + \nu)$$
$$\subset \sigma_{\rm disc}(\widehat{H}^\nu_\Lambda(a, L)\Big\lceil_{\mathcal{H}_a}).$$

Proof: Let $\{\phi_n\}_n$ be a complete orthonormal system of $\mathcal{H}_a(-)$ and $\{\psi_m\}_m$ a complete orthonormal system of $\mathcal{H}_a(+)$. We see that by Lemma 4.7,

$$0 \geq {\rm Tr}\ T\lceil_{\mathcal{H}_a(-)} = \sum_n (\phi_n, T\phi_n) \geq \sum_n (\phi_n, T'\phi_n),$$

where

$$T' = E^{H_{\rm eff}-\Sigma}_{[0,|\Sigma|)} \otimes \left((1 - \frac{\theta(\Lambda, a, L)}{2}) H^\nu_{\rm f}(a, L) - 3\theta(\Lambda, a, L) - \nu \right).$$

Set

$$T'_- = T' E^{T'}_{(-\infty, 0)}.$$

Then

$$0 \geq {\rm Tr}\ T\lceil_{\mathcal{H}_a(-)} \geq \sum_n (\phi_n, T'_-\phi_n)$$
$$\geq \sum_n (\phi_n, T'_-\phi_n) + \sum_m (\psi_m, T'_-\psi_m) = {\rm Tr} T'_-.$$

Hence we obtain that

$$\left| {\rm Tr}\ T\lceil_{\mathcal{H}_a(-)} \right| \leq \left| {\rm Tr} T'_- \right|$$
$$= {\rm Tr} E^{H_{\rm eff}-\Sigma}_{[0,|\Sigma|)} \times \left| {\rm Tr} \left((1 - \frac{\theta(\Lambda, a, L)}{2}) H^\nu_{\rm f}(a, L) - 3\theta(\Lambda, a, L) - \nu \right)_- \Big\lceil_{\mathcal{H}_a} \right|,$$

where $(\cdots)_-$ denotes the negative part of the spectrum of $(\cdots)$. Since

$$\sigma(H^\nu_{\rm f}(a, L)\lceil_{\mathcal{H}_a}) = \sigma_{\rm disc}(H^\nu_{\rm f}(a, L)\lceil_{\mathcal{H}_a})$$

and

$$\left| {\rm Tr} E^{H_{\rm eff}-\Sigma}_{[0,|\Sigma|)} \right| < \infty,$$

it follows that

$$\left| {\rm Tr}\ T\lceil_{\mathcal{H}_a(-)} \right| < \infty$$

Thus the lemma follows. □

Lemma 4.9 *Suppose that* $\min\{|\Sigma|/3, 2\} > \theta(\Lambda, a, L)$. *Then for* ν *such that* $|\Sigma| > 3\theta(\Lambda, a, L) + \nu$,

$$\sigma(\widehat{H}^\nu_\Lambda(a, L)) \cap [\inf \sigma(\widehat{H}^\nu_\Lambda(a, L)), \inf \sigma(\widehat{H}^\nu_\Lambda(a, L)) + \nu) \subset \sigma_{\rm disc}(\widehat{H}^\nu_\Lambda(a, L)).$$

Proof: We have by Lemmas 4.2 and 4.8,

$$\sigma(\widehat{H}^\nu_\Lambda(a, L)) = \sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a^\perp}) \cup \sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a}),$$

$$\sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a^\perp}) \subset [\inf \sigma(\widehat{H}_\Lambda(a, L)) + \nu, \infty),$$

and

$$\sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a}) \cap [\inf \sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a}), \inf \sigma(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a}) + \nu)$$
$$\subset \sigma_{\rm disc}(\widehat{H}^\nu_\Lambda(a, L)\lceil_{\mathcal{H}_a}).$$

Hence the lemma follows. □

Lemma 4.10 *Suppose that* $\min\{|\Sigma|/3, 2\} > \theta(\Lambda)$. *Then*

$$\sigma(\widehat{H}^\nu_\Lambda) \cap [\inf \sigma(\widehat{H}^\nu_\Lambda), \inf \sigma(\widehat{H}^\nu_\Lambda) + \nu) \subset \sigma_{\rm disc}(\widehat{H}^\nu_\Lambda).$$

In particular $\widehat{H}^\nu_\Lambda$ *has a ground state.*

Proof: Note that

$$\lim_{L\to\infty} \lim_{a\to\infty} \theta(\Lambda, a, L) = \theta(\Lambda)$$

Then by Lemmas 4.1 and 4.9, the lemma follows from [21]. □

4.3 Binding

The number operator N in $\mathcal{F}$ is defined by

$$N = \sum_{j=1,2} \int a^*(k, j)a(k, j)d^3k.$$

A ground state of $\widehat{H}^\nu_\Lambda$ is denoted by $\varphi_{\rm g}(\nu)$.

Lemma 4.11 *Suppose that* $\min\{|\Sigma|/3, 2\} > \theta(\Lambda)$. *Then for* ν *such that* $|\Sigma| > 3\theta(\Lambda) + \nu$,

$$\frac{\|{N_{\rm f}}^{1/2}\varphi_{\rm g}(\nu)\|}{\|\varphi_{\rm g}(\nu)\|} \leq \frac{C\|\chi_\Lambda/\omega^{5/2}\|(\max_\mu \|\nabla_\mu V\|_\infty)}{m_{\rm eff}}$$

with some constant C.

Proof: We set

$$E = \inf \sigma(\widehat{H}^\nu_\Lambda).$$

Since

$$[\widehat{H}^\nu_\Lambda, a(k,j)] = -\omega(k) - \nu + [\delta V^\nu, a(k,j)],$$

we have

$$\widehat{H}^\nu_\Lambda a(k,j)\varphi_{\rm g}(\nu) = -(\omega(k)+\nu)a(k,j)\varphi_{\rm g}(\nu)$$
$$+Ea(k,j)\varphi_{\rm g}(\nu) + [\delta V^\nu, a(k,j)]\varphi_{\rm g}(\nu).$$

Hence

$$(\widehat{H}^\nu_\Lambda - E + \omega(k) + \nu)a(k,j)\varphi_{\rm g}(\nu) = [\delta V^\nu, a(k,j)]\varphi_{\rm g}(\nu).$$

Here we see that

$$[\delta V^\nu, a(k,j)] = [V(\cdot - K^\nu/m_{\rm eff}), a(k,j)] = [e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} V e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}, a(k,j)]$$
$$= e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}[V, e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}} a(k,j) e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}] e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}.$$

Since

$$e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}} a(k,j) e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}} = a(k,j) - \frac{i}{\sqrt{2}m_{\rm eff}} p\cdot\varrho^\nu(k,j),$$

it follows that

$$[\delta V^\nu, a(k,j)] = e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}[V, -\frac{i}{\sqrt{2}m_{\rm eff}} p\cdot\varrho^\nu(k,j)] e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}$$
$$= e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}\left(\frac{1}{\sqrt{2}m_{\rm eff}}(\nabla V)\cdot\varrho^\nu(k,j)\right) e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}.$$

Thus we obtain a pull-through formula:

$$a(k,j)\varphi_{\rm g}(\nu) = (\widehat{H}^\nu_\Lambda - E + \omega(k) + \nu)^{-1}\times$$

$$\times e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}\left(\frac{1}{\sqrt{2}m_{\rm eff}}(\nabla V)\cdot\varrho^\nu(k,j)\right)e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}\varphi_{\rm g}(\nu). \tag{4.12}$$

Using identity (4.12) we see that

$$(N^{1/2}\varphi_{\rm g}(\nu), N^{1/2}\varphi_{\rm g}(\nu)) = \sum_{j=1,2}\int \|a(k,j)\varphi_{\rm g}(\nu)\|^2 d^3k$$

$$= \sum_{j=1,2}\int\Bigg\|(\widehat{H}^\nu_\Lambda - E + \omega(k)+\nu)^{-1}\times$$

$$\times e^{-i\frac{p\cdot K^\nu}{m_{\rm eff}}}\left(\frac{1}{\sqrt{2}m_{\rm eff}}(\nabla V)\cdot\varrho^\nu(k,j)\right)e^{i\frac{p\cdot K^\nu}{m_{\rm eff}}}\varphi_{\rm g}(\nu)\Bigg\|^2 d^3k$$

$$\leq 3\sum_{\mu=1}^3\sum_{j=1,2}\int\left(\frac{1}{\omega(k)}\|\nabla_\mu V\|_\infty\right)^2\left|\frac{1}{\sqrt{2}m_{\rm eff}}\varrho^\nu_\mu(k,j)\right|^2 d^3k\|\varphi_{\rm g}(\nu)\|^2$$

$$\leq \frac{3}{2}\sum_{\mu=1}^3\left(\frac{\|\nabla_\mu V\|_\infty}{m_{\rm eff}}\right)^2\sum_{j=1,2}\|\varrho^\nu(\cdot,j)/\omega\|^2\|\varphi_{\rm g}(\nu)\|^2$$

$$\leq C\left(\frac{(\max_\mu\|\nabla_\mu V\|_\infty)}{m_{\rm eff}}\right)^2\|\chi_\Lambda/\omega^{5/2}\|^2\|\varphi_{\rm g}(\nu)\|^2.$$

Hence the lemma follows. □

Lemma 4.12 *Let P_Ω be the projection onto $\{\alpha\Omega \mid \alpha\in\mathbb{C}\}$ and*

$$Q = E^{H_{\rm eff}}_{[\Sigma+\delta,\infty)}\otimes P_\Omega$$

with some $\delta>0$ such that

$$\delta > \frac{3}{2}\theta(\Lambda).$$

Suppose that $\min\{|\Sigma|/3, 2\}>\theta(\Lambda)$. Then for ν such that $|\Sigma|>3\theta(\Lambda)+\nu$,

$$\frac{\|Q\varphi_{\rm g}(\nu)\|}{\|\varphi_{\rm g}(\nu)\|}\leq\sqrt{\frac{\theta(\Lambda)}{\delta-\frac{3}{2}\theta(\Lambda)}}.$$

Proof: Since $(\varphi_{\rm g}(\nu), Q(\widehat{H}^\nu_\Lambda - \inf\sigma(\widehat{H}^\nu_\Lambda))\varphi_{\rm g}(\nu))=0$, we have

$$(\varphi_{\rm g}(\nu), Q(H_{\rm eff}-\inf\sigma(\widehat{H}^\nu_\Lambda)+g(\Lambda))\varphi_{\rm g}(\nu)) = -(\varphi_{\rm g}(\nu), Q\delta V^\nu\varphi_{\rm g}(\nu)).$$

The left-hand side above is estimated as

$$(\varphi_{\rm g}(\nu), Q(H_{\rm eff} - \inf\sigma(\widehat{H}^\nu_\Lambda) + g(\Lambda))\varphi_{\rm g}(\nu))$$

$$\geq (\Sigma + \delta - \inf\sigma(\widehat{H}^\nu_\Lambda) + g(\Lambda))(\varphi_{\rm g}(\nu), Q\varphi_{\rm g}(\nu)).$$

Note that

$$\Sigma + \delta + g(\Lambda) - \inf\sigma(\widehat{H}^\nu_\Lambda) \geq \Sigma + \delta + g(\Lambda) - \Sigma - \frac{3}{2}\theta(\Lambda) - g(\Lambda)$$

$$= \delta - \frac{3}{2}\theta(\Lambda) > 0.$$

Then

$$(-\inf\sigma(\widehat{H}^\nu_\Lambda) + g(\Lambda))(\varphi_{\rm g}(\nu), Q\varphi_{\rm g}(\nu)) \geq (\delta - \frac{3}{2}\theta(\Lambda))\|Q\varphi_{\rm g}(\nu)\|^2 > 0.$$

Moreover

$$|(\varphi_{\rm g}(\nu), Q\delta V^\nu\varphi_{\rm g}(\nu))| = |(\delta V^\nu Q\varphi_{\rm g}(\nu), \varphi_{\rm g}(\nu))| \leq \|\delta V^\nu Q\varphi_{\rm g}(\nu)\|\|\varphi_{\rm g}(\nu)\|$$

$$\leq \theta(\Lambda)\left(\|{H_{\rm f}}^{1/2}Q\varphi_{\rm g}(\nu)\| + \|Q\varphi_{\rm g}(\nu)\|\right)\|\varphi_{\rm g}(\nu)\|$$

$$= \theta(\Lambda)\|Q\varphi_{\rm g}(\nu)\|\|\varphi_{\rm g}(\nu)\| \leq \theta(\Lambda)\|\varphi_{\rm g}(\nu)\|^2.$$

Hence we have

$$0 < (\delta - \frac{3}{2}\theta(\Lambda))\|Q\varphi_{\rm g}(\nu)\|^2 \leq \theta(\Lambda)\|\varphi_{\rm g}(\nu)\|^2.$$

The lemma follows. □

4.4 Proof of the main theorem

We normalize $\varphi_{\rm g}(\nu)$, i.e., $\|\varphi_{\rm g}(\nu)\| = 1$. Take a subsequence ν' such that $\varphi_{\rm g}(\nu')$ weakly converges to a vector $\varphi_{\rm g}$ as $\nu' \to \infty$. It is established in [1] that if $\varphi_{\rm g} \neq 0$, then $\varphi_{\rm g}$ is a ground state of H_Λ.

Proof of Theorem 2.4

It is enough to prove $\varphi_{\rm g} \neq 0$. Note that on $\mathcal{F}$

$$N + P_\Omega \geq 1.$$

Then on $\mathcal{H}$

$$1 \otimes N + 1 \otimes P_\Omega \geq 1.$$

Hence

$$\begin{aligned}
&1 \otimes N + (E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} + E^{H_{\rm eff}}_{[\Sigma+\delta,\infty)}) \otimes P_\Omega \\
&= 1 \otimes N + E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega + Q \geq 1,
\end{aligned}$$

and

$$E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \geq 1 - 1 \otimes N - Q. \tag{4.13}$$

Suppose that $\min\{|\Sigma|/3, 2\} > \theta(\Lambda)$ and $\delta > \frac{3}{2}\theta(\Lambda)$. Then for ν' such that $|\Sigma| > 3\theta(\Lambda) + \nu'$, we have by (4.13),

$$\begin{aligned}
&(\varphi_{\rm g}(\nu'), E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \varphi_{\rm g}(\nu')) \\
&\geq \|\varphi_{\rm g}(\nu')\|^2 - (\varphi_{\rm g}(\nu'), N\varphi_{\rm g}(\nu')) - (\varphi_{\rm g}(\nu'), Q\varphi_{\rm g}(\nu')) \\
&\geq \|\varphi_{\rm g}(\nu')\|^2 - \frac{C\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}}(\max_\mu \|\nabla_\mu V\|_\infty)\|\varphi_{\rm g}(\nu')\|^2 - \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}\|\varphi_{\rm g}(\nu')\|^2 \\
&= 1 - \frac{C\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}}(\max_\mu \|\nabla_\mu V\|_\infty) - \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}.
\end{aligned}$$

Note that

$$\lim_{\Lambda\to\infty} \frac{\|\chi_\Lambda/\omega^{5/2}\|}{m_{\rm eff}} = 0,$$

$$\lim_{\Lambda\to\infty} \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)} = 0.$$

Hence for sufficiently large Λ,

$$(\varphi_{\rm g}(\nu'), E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \varphi_{\rm g}(\nu')) > \epsilon$$

uniformly in ν' with some $\epsilon > 0$. Take $\nu' \to \infty$ on the both sides above. Since $E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega$ is a finite rank operator, we have

$$(\varphi_{\rm g}, E^{H_{\rm eff}}_{[\Sigma,\Sigma+\delta)} \otimes P_\Omega \varphi_{\rm g}) > \epsilon.$$

In particular $\varphi_{\rm g} \neq 0$. Then $\varphi_{\rm g}$ is a ground state of $\widehat{H}_\Lambda$. $\square$

Remark 4.13 *$\varphi_{\rm g}$ satisfies that $\varphi_{\rm g} \in D(N^{1/2})$. Actually in the similar manner as (4.12) we have*

$$a(k,j)\varphi_{\rm g} = (H_\Lambda - \inf\sigma(H_\Lambda) + \omega(k))^{-1}$$
$$\times e^{-i\frac{p\cdot K}{m_{\rm eff}}}\left(\frac{1}{\sqrt{2}m_{\rm eff}}(\nabla V)\cdot\varrho(k,j)\right)e^{i\frac{p\cdot K}{m_{\rm eff}}}\varphi_{\rm g}.$$

Using this we can obtain that

$$(N^{1/2}\varphi_{\rm g}, N^{1/2}\varphi_{\rm g}) = \sum_{j=1,2}\int \|a(k,j)\varphi_{\rm g}\|^2 d^3k < \infty.$$

5 Remarks

5.1 Multiplicity and localization

In Section 3 we established that H_Λ has a ground state for $\Lambda > \Lambda_*$ with some Λ_* even in the case where the decoupled Hamiltonian $\frac{1}{2m}p^2 + V + H_{\rm f}$ has no ground state. The uniqueness of ground states is established by means of a functional integral representation of e^{-tH_Λ}. There exists a unitary operator such that

$$\mathcal{U} : \mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathcal{F} \longrightarrow L^2(\mathbb{R}^3\times\mathbf{S}, dx\otimes d\mu),$$

where $d\mu$ is a Gaussian probability measure on the following direct sum of the real-valued Schwartz distribution,

$$\mathbf{S} = \mathcal{S}'_{\rm real}(\mathbb{R}^3)\times\mathcal{S}'_{\rm real}(\mathbb{R}^3)\times\mathcal{S}'_{\rm real}(\mathbb{R}^3).$$

Using the identification implemented by $\mathcal{U}$,

$$\mathcal{H} \cong L^2(\mathbb{R}^3\times\mathbf{S}, dx\otimes d\mu),$$

we established in [14] that

$$\Theta(t) = e^{i\frac{\pi}{2}N}e^{-tH_\Lambda}e^{-i\frac{\pi}{2}N}$$

is positivity improving, i.e.,

$$(\Psi, \mathcal{U}\Theta(t)\mathcal{U}^{-1}\Phi) > 0$$

for $\Psi, \Phi \in L^2(\mathbb{R}^3 \times \mathbf{S}, dx \otimes d\mu)$ with $\Psi, \Phi \geq 0$ but $\Psi \not\equiv 0, \Phi \not\equiv 0$. This implies that a ground state (if it exists) of H_Λ is unique up to a phase factor.

In Section 3 we also see that $\varphi_{\rm g} \in D(N^{1/2})$. Modifying a method in [19] slightly, we can also establish that

$$\varphi_{\rm g} \in \bigcap_{k=1}^{\infty} D(N^{k/2}). \tag{5.1}$$

Proposition 5.1 *Let $\Lambda > \Lambda_*$. Then the ground state, $\varphi_{\rm g}$, of H_Λ is unique and (5.1) is satisfied.*

5.2 A_Λ^2 term

A_Λ^2 term in H_Λ is a subjective part for large Λ. It is established in [17, Remark 3.10] that the Hamiltonian without A_Λ^2,

$$H'_\Lambda = \frac{1}{2m} p^2 + V + \frac{1}{m} p \cdot A_\Lambda + H_{\rm f},$$

is unbounded from below for Λ such that

$$\|\chi_\Lambda / \omega\|^2 > m.$$

In particular H'_Λ has no ground state for such Λ.

Proposition 5.2 *H'_Λ has no ground state for $\Lambda > m/(4\pi) + \lambda$.*

6 Stability of matter

Stability of matter investigated in this section are pointed out in e.g., Lieb and Loss [22] and references therein. See also Fefferman, Fröhlich and Graf [7].

6.1 Ultraviolet cutoffs

In this subsection we estimate the asymptotics of $g(\Lambda)$ in (3.4) as $\Lambda \to \infty$.

Remark 6.1 *In the case of $V = 0$, from Proposition 6.3 it follows that*

$$g(\Lambda) = \inf \sigma(H_\Lambda).$$

It is seen that $g(\Lambda)$ is monotonously increasing in Λ.

Since

$$\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2 = 4\pi\left\{(\Lambda-\lambda)+t\left(\arctan\frac{\lambda}{t}-\arctan\frac{\Lambda}{t}\right)\right\},$$

and

$$\|\chi_\Lambda/(t^2+\omega^2)\|^2 = 4\pi\left\{\frac{1}{2t}\left(\arctan\frac{\Lambda}{t}-\arctan\frac{\lambda}{t}\right)+\frac{1}{2}\left(\frac{\lambda}{t^2+\lambda^2}-\frac{\Lambda}{t^2+\Lambda^2}\right)\right\},$$

we have

$$g(\Lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2\pi t\left\{(\arctan\frac{\Lambda}{t}-\arctan\frac{\lambda}{t})+\left(\frac{\lambda/t}{1+(\lambda/t)^2}-\frac{\Lambda/t}{1+(\Lambda/t)^2}\right)\right\}}{m+\frac{8\pi t}{3}\left\{\left(\frac{\Lambda}{t}-\frac{\lambda}{t}\right)+\left(\arctan\frac{\lambda}{t}-\arctan\frac{\Lambda}{t}\right)\right\}}dt.$$

Change variable t to $r=\Lambda/t$. We have

$$g(\Lambda)=4\Lambda^2\int_0^\infty\frac{(\arctan r-\frac{r}{1+r^2})-\left(\arctan r\left(\frac{\lambda}{\Lambda}\right)-\frac{r\left(\frac{\lambda}{\Lambda}\right)}{1+r^2\left(\frac{\lambda}{\Lambda}\right)^2}\right)}{mr+\frac{8\pi}{3}\Lambda\left\{(r-\arctan r)-(r\left(\frac{\lambda}{\Lambda}\right)-\arctan r\left(\frac{\lambda}{\Lambda}\right))\right\}}\frac{dr}{r^2}.$$

Theorem 6.2 *Assume that*

$$m>8\pi\lambda/3. \tag{6.1}$$

Then

$$\frac{8}{3}\left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}\le\lim_{\Lambda\to\infty}\frac{g(\Lambda)}{\Lambda^{3/2}}\le\frac{8}{3}\left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Proof: This proof is due to Ezawa [6]. We decompose

$$\frac{g(\Lambda)}{4\Lambda}=\int_0^\infty\frac{(\arctan r-\frac{r}{1+r^2})-\left(\arctan r\left(\frac{\lambda}{\Lambda}\right)-\frac{r\left(\frac{\lambda}{\Lambda}\right)}{1+r^2\left(\frac{\lambda}{\Lambda}\right)^2}\right)}{\frac{m}{\Lambda}r+\frac{8\pi}{3}\left\{(r-\arctan r)-(r\left(\frac{\lambda}{\Lambda}\right)-\arctan r\left(\frac{\lambda}{\Lambda}\right))\right\}}\frac{dr}{r^2}$$

such as

$$\frac{g(\Lambda)}{4\Lambda}=\int_0^{1/\Lambda^{1/4}}+\int_{1/\Lambda^{1/4}}^\infty=I_1(\Lambda)+I_2(\Lambda).$$

It is enough to show that

$$\frac{2}{3}\left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}\le\lim_{\Lambda\to\infty}\frac{I_1(\Lambda)}{\sqrt{\Lambda}}\le\frac{2}{3}\left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}, \tag{6.2}$$

and that

$$\lim_{\Lambda\to\infty}\frac{I_2(\Lambda)}{\sqrt{\Lambda}}=0. \tag{6.3}$$

Note that

$$\arctan x=\frac{x}{1+x^2}+\frac{2}{3}\frac{x^3}{(1+x^2)^2}+\frac{5}{4}\frac{3}{2}\frac{x^5}{(1+x^2)^3}+\cdots.$$

We define f and h by

$$\arctan r-\frac{r}{1+r^2}=\frac{2}{3}\frac{r^3}{(1+r^2)^2}+f(r)$$

and

$$r-\arctan r=\frac{r^3}{1+r^2}-h(r).$$

It is satisfied that $f(r)\geq 0$, $h(r)\geq 0$,

$$\lim_{r\to 0}\frac{f(r)}{r^3}=0,\quad \lim_{r\to 0}\frac{h(r)}{r^3}=\frac{2}{3}$$

and

$$h(r)=\frac{2}{3}\frac{r^3}{(1+r^2)^2}+f(r)=\arctan r-\frac{r}{1+r^2}.$$

Let us set

$$f_\Lambda(r)=\arctan r\left(\frac{\lambda}{\Lambda}\right)-\frac{r\left(\frac{\lambda}{\Lambda}\right)}{1+r^2\left(\frac{\lambda}{\Lambda}\right)^2}>0$$

and

$$h_\Lambda(r)=r\left(\frac{\lambda}{\Lambda}\right)-\arctan r\left(\frac{\lambda}{\Lambda}\right)>0.$$

Then $I_1(\Lambda)$ is written as

$$I_1(\Lambda)=\int_0^{1/\Lambda^{1/4}}\frac{\frac{2}{3}+\frac{(1+r^2)^2}{r^3}(f(r)-f_\Lambda(r))}{\frac{m}{\Lambda}(1+r^2)+\frac{8\pi}{3}r^2-\frac{8\pi}{3}\frac{1+r^2}{r}(h(r)+h_\Lambda(r))}\frac{dr}{1+r^2}.$$

It follows that for $0\leq r\leq 1/\Lambda^{1/4}$,

$$\begin{aligned}\frac{8\pi}{3}\frac{1+r^2}{r}(h(r)+h_\Lambda(r))&=\frac{8\pi}{3}r^2(1+r^2)\frac{h(r)+h_\Lambda(r)}{r^3}\\&\leq\frac{8\pi}{3}\left(1+\frac{1}{\sqrt{\Lambda}}\right)r^2\theta(\Lambda):=r^2\delta(\Lambda),\end{aligned}$$

where

$$\theta(\Lambda) := \sup_{0\le r\le 1/\Lambda^{1/4}} \frac{h(r)+h_\Lambda(r)}{r^3}.$$

Note that

$$\lim_{r\downarrow 0} \frac{h(r)+h_\Lambda(r)}{r^3} = \frac{2}{3}+\frac{1}{3}\left(\frac{\lambda}{\Lambda}\right)^3$$

and

$$\lim_{\Lambda\to\infty} \delta(\Lambda) = \frac{8\pi}{3}\frac{2}{3}. \tag{6.4}$$

Moreover we have

$$\frac{(1+r^2)^2}{r^3}(f(r)-f_\Lambda(r)) \le \sup_{0\le r\le (1/\Lambda^{1/4})} \frac{(1+r^2)^2}{r^3} f(r)$$

and

$$\frac{(1+r^2)^2}{r^3}(f(r)-f_\Lambda(r)) \ge -\sup_{0\le r\le (1/\Lambda^{1/4})} \frac{(1+r^2)^2}{r^3} f_\Lambda(r).$$

Set

$$\epsilon(\Lambda) := \max\left\{ \sup_{0\le r\le (1/\Lambda^{1/4})} \frac{(1+r^2)^2}{r^3} f(r), \sup_{0\le r\le (1/\Lambda^{1/4})} \frac{(1+r^2)^2}{r^3} f_\Lambda(r) \right\}.$$

It is trivial that

$$\lim_{\Lambda\to\infty} \epsilon(\Lambda) = 0. \tag{6.5}$$

Hence we have

$$\frac{(2/3)-\epsilon(\Lambda)}{1+1/\sqrt{\Lambda}} \int_0^{1/\Lambda^{1/4}} \frac{1}{\frac{m}{\Lambda}+\left(\frac{m}{\Lambda}+\frac{8\pi}{3}\right) r^2} dr \le I_1(\Lambda)$$

$$\le \frac{(2/3)+\epsilon(\Lambda)}{1-1/\sqrt{\Lambda}} \int_0^{1/\Lambda^{1/4}} \frac{1}{\frac{m}{\Lambda}+\left(\frac{m}{\Lambda}+\frac{8\pi}{3}-\delta(\Lambda)\right) r^2} dr.$$

Together with (6.4) a direct calculation yields that

$$\lim_{\Lambda\to\infty} \frac{1}{\sqrt{\Lambda}} \int_0^{1/\Lambda^{1/4}} \frac{1}{\frac{m}{\Lambda}+\left(\frac{m}{\Lambda}+\frac{8\pi}{3}-\delta(\Lambda)\right) r^2} dr$$

$$= \lim_{\Lambda\to\infty} \frac{1}{\sqrt{m(\frac{m}{\Lambda}+\frac{8\pi}{3}-\delta(\Lambda))}} \arctan \sqrt{\frac{\frac{m}{\Lambda}+\frac{8\pi}{3}-\delta(\Lambda)}{m/\sqrt{\Lambda}}}$$

$$= \left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Similarly we have

$$\lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}\int_0^{1/\Lambda^{1/4}}\frac{1}{\frac{m}{\Lambda}+\left(\frac{m}{\Lambda}+\frac{8\pi}{3}\right)r^2}dr = \left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Thus by (6.5) we have

$$\frac{2}{3}\left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2} \le \lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}I_1(\Lambda) \le \frac{2}{3}\left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Hence (6.2) follows. Next we show (6.3). Since

$$(\arctan r - \frac{r}{1+r^2}) - \left(\arctan r\left(\frac{\lambda}{\Lambda}\right) - \frac{r\left(\frac{\lambda}{\Lambda}\right)}{1+r^2\left(\frac{\lambda}{\Lambda}\right)^2}\right)$$

$$\le \frac{2}{3}\frac{r^3}{(1+r^2)^2} + \frac{5}{4}\frac{3}{2}\frac{r^5}{(1+r^2)^2}$$

and by the assumption $m > 8\pi\lambda/3$,

$$\frac{m}{\Lambda}r + \frac{8\pi}{3}\left\{(r-\arctan r) - \left(r\left(\frac{\lambda}{\Lambda}\right) - \arctan r\left(\frac{\lambda}{\Lambda}\right)\right)\right\}$$

$$> \frac{m}{\Lambda}r - \frac{\lambda}{\Lambda}\frac{8\pi}{3}r + \frac{8\pi}{3}\left\{\frac{r^3}{1+r^2} - \frac{2}{3}\frac{r^3}{1+r^2} + \arctan(r\left(\frac{\lambda}{\Lambda}\right))\right\}$$

$$> \frac{8\pi}{9}\frac{r^3}{1+r^2}.$$

Then

$$\lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}I_2(\Lambda) \le \lim_{\Lambda\to\infty}\frac{1}{\sqrt{\Lambda}}\int_{1/\Lambda^{1/4}}^{\infty}\frac{\frac{2}{3}\frac{r^3}{(1+r^2)^2}+\frac{5}{4}\frac{3}{2}\frac{r^5}{(1+r^2)^2}}{\frac{8\pi}{9}\frac{r^3}{1+r^2}}\frac{dr}{r^2}$$

$$= \lim_{\Lambda\to\infty}\frac{9}{8\pi}\frac{1}{\sqrt{\Lambda}}\int_{1/\Lambda^{1/4}}^{\infty}\frac{1}{1+r^2}\left(\frac{2}{3}+\frac{15}{8}r^2\right)\frac{dr}{r^2} = 0.$$

Thus (6.3) follows, and then the theorem follows. □

6.2 N particle system

We consider an N particle system. We assume simply that each particle has mass m and there is no external potential. The Hamiltonian, H_Λ^N, is defined as a self-adjoint operator acting on

$$L^2(\mathbb{R}^{3N}) \otimes \mathcal{F},$$

and is given by

$$H_\Lambda^N = \sum_{j=1}^N \frac{1}{2m}(p_j + A_{j\Lambda}(0))^2 + H_{\mathrm{f}},$$

where

$$A_{j\Lambda}(0) = \sum_{j'=1,2} \int \frac{\chi_{j\Lambda}(k)}{\sqrt{2\omega(k)}} e_{j'}(k) \left\{a^*(k,j') + a(k,j')\right\} d^3k.$$

Let

$$\inf \sigma(H_\Lambda^N) = g(\Lambda, N).$$

We consider the two cases such as

(1) $\chi_{j\Lambda}(k) = \chi_\Lambda(k), \quad j = 1, ..., N,$

(2) $\chi_{j\Lambda}$, $j = 1, ..., N$, are characteristic functions on closed sets in $\mathbb{R}^3$ such as

$$\mathrm{supp}\chi_{j\Lambda} \cap \mathrm{supp}\chi_{i\Lambda} \cap \{0\} = \emptyset, \quad (i \neq j).$$

We will see below that the asymptotic behavior of $g(\Lambda, N)$ as $N \to \infty$ depends on ultraviolet cutoffs. In the case of (2) we intuitively expect that

$$g(\Lambda, N) \approx N,$$

since N particles may have no interaction through quantized radiation fields. In [17] we established the following proposition.

Proposition 6.3 *In the case of (1),*

$$g(\Lambda, N) = \frac{N}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_\Lambda/(t^2+\omega^2)\|^2}{m + \frac{2}{3}N\|\chi_\Lambda/\sqrt{t^2+\omega^2}\|^2} dt,$$

in the case of (2),

$$g(\Lambda, N) = \sum_{j=1}^N \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_{j\Lambda}/(t^2+\omega^2)\|^2}{m + \frac{2}{3}\|\chi_{j\Lambda}/\sqrt{t^2+\omega^2}\|^2} dt.$$

In the case of (1) the following theorem holds.

Theorem 6.4 *We assume case (1) and*

$$m > 8\pi\lambda/3. \tag{6.6}$$

Then

$$\frac{8}{3}\left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2} \le \lim_{\Lambda,N\to\infty}\frac{g(\Lambda,N)}{\sqrt{N}\Lambda^{3/2}} \le \frac{8}{3}\left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2}.$$

Proof: By Proposition 6.3 we have

$$\frac{g(\Lambda,N)}{4\Lambda} = \int_0^\infty \frac{(\arctan r - \frac{r}{1+r^2}) - \left(\arctan r\left(\frac{\lambda}{\Lambda}\right) - \frac{r\left(\frac{\lambda}{\Lambda}\right)}{1+r^2\left(\frac{\lambda}{\Lambda}\right)^2}\right)}{\frac{m}{N\Lambda}r + \frac{8\pi}{3}\left\{(r-\arctan r) - (r\left(\frac{\lambda}{\Lambda}\right) - \arctan r\left(\frac{\lambda}{\Lambda}\right))\right\}}\frac{dr}{r^2}.$$

Then in the similar way as the proof of Theorem 6.4 we decompose it such as

$$\frac{g(\Lambda,N)}{4\Lambda} = \int_0^{1/(\Lambda N)^{1/4}} + \int_{1/(\Lambda N)^{1/4}}^\infty = I_1(\Lambda,N) + I_2(\Lambda,N),$$

and it can be seen that

$$\frac{2}{3}\left(\frac{3}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2} \le \lim_{\Lambda,N\to\infty}\frac{I_1(\Lambda,N)}{\sqrt{N\Lambda}} \le \frac{2}{3}\left(\frac{9}{8\pi}\frac{1}{m}\right)^{1/2}\frac{\pi}{2},$$

and that

$$\lim_{\Lambda,N\to\infty}\frac{I_2(\Lambda,N)}{\sqrt{N\Lambda}} = 0.$$

Then the theorem follows. □

Remark 6.5 *In the case of (2), if we adjust $\chi_{j\Lambda}$ such as*

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{t^2\|\chi_{j\Lambda}/(t^2+\omega^2)\|^2}{m+\frac{2}{3}\|\chi_{j\Lambda}/\sqrt{t^2+\omega^2}\|^2}dt = g$$

with some constant g independent of j. Then

$$g(\Lambda,N) = Ng.$$

Acknowledgment I thank H. Ezawa for sending his proof of Theorems 6.2 and 6.4 to myself. I thank Grant-in-Aid 13740106 for Encouragement of Young Scientists from the Ministry of Education, Science, Sports and Culture for financial support.

References

[1] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151** (1997), 455–503.

[2] A. Arai and H. Kawano, Enhanced binding in a general class of quantum field models, mp-arc 02-508, preprint, 2002.

[3] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207** (1999), 249–290.

[4] I. Catto and C. Hainzl, Self-energy of one electron in non-relativistic QED, math-ph/0207036, preprint, 2002

[5] T. Chen, V. Vougalter and S. Vugalter, The increase of binding energy and enhanced binding in non-relativistic QED, math-ph/0209062, preprint, 2002.

[6] H. Ezawa, private communications.

[7] C. Fefferman, J. Fröhlich, G. M. Graf, Stability of ultraviolet-cutoff quantum electrodynamics with non-relativistic matter, *Commun. Math. Phys.* **190** (1997), 309–330.

[8] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. Henri Poincaré* **19** (1973), 1–103.

[9] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159–198.

[10] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* **145** (2001), 557–595.

[11] C. Hainzl, V. Vougalter and S. A. Vugalter, Enhanced binding in non-relativistic QED, mp-arc 01-455, preprint, 2001.

[12] M. Hirokawa, Divergence of soft photon number and absence of ground state for Nelson's model, math-ph/0211051, preprint, 2002.

[13] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics I, *J. Math. Phys.* **40** (1999), 6209–6222.

[14] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661-674.

[15] F. Hiroshima, Essential self-adjointness of translation invariant quantum filed models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585-613.

[16] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian in quantum electrodynamics for arbitrary values of coupling constants, *Ann. H. Poincaré* **3** (2002), 171–201.

[17] F. Hiroshima, Observable effect and parameterized scaling limits of a model in nonrelativistic electrodynamics, *J. Math. Phys.* **43** (2002), 1775-1795.

[18] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation field, preprint, 2003.

[19] F. Hiroshima, Localization of the number of photons of nonrelativistic QED, math-ph/0211009, to be published in *Rev. Math. Phys.*

[20] F. Hiroshima and H. Spohn, Enhanced binding through coupling to a quantum field, *Ann. H. Poincaré* **2** (2001), 1159–1187.

[21] R. Høegh-Krohn and B. Simon, Hypercontractive semigroup and two dimensional self-coupled Bose fields, *J. Funct. Anal.* **9** (1972), 121–180.

[22] E. Lieb and M. Loss, Bound on binding energies and mass renormalization in models of quantum electrodynamics, preprint, 2001.

[23] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, *Studies in Mathematical Physics*, Princeton Univ. Press, 269-303, 1976.

[24] W. Pauli and M. Fierz, Zur Theorie der Emission langwelliger Lichtquanten, *Nuovo Cimento* **15** (1938), 167–188, Theory of the emission of long-wave light quanta, *Early Quantum Electrodynamics*, ed. by I. Miller, Cambridge Univ. Press, 1995, 227–243.