Spectral Singularities and Asymptotics of Contractive Semigroups. I.

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Abstract

The problem of localization of spectral singularities of dissipative operators in terms of the asymptotic of the corresponding exponential function is studied. We give a solution to this problem for the singularities of higher orders in the frame of the perturbation theory.

Introduction. In this paper we continue the study of relationships between spectral singularities of dissipative operators and the asymptotic of the corresponding semigroup started in [1]. For a dissipative operator, L, with purely absolutely

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continuous spectrum spectral singularities [2] are the points on the real axis at vicinities of which the estimate

$$\left\|(L-z)^{-1}\right\| \leq \frac{C}{\operatorname{Im} z}, \qquad \operatorname{Im} z > 0, \tag{1}$$

fails to hold. The role of spectral singularities in analysis of operators is revealed by the Sz.-Nagy - Foias criterion [3] according to which

A maximal dissipative operator L such that $\sigma(L) \subset \mathbb{R}$ is similar to a self-adjoint one[1] *if and only if its resolvent satisfies the estimate (1).*

The criterion implies that if a maximal dissipative operator L with purely absolutely continuous spectrum has no spectral singularities at finite distance and satisfies (1) in a neighborhood of infinity then the operator $Z_t = e^{iLt}$ is invertible for $t \geq 0$ and $\left\|Z_t^{-1}\right\| \leq C_0$. In fact, according to a theorem of Sz.-Nagy, L is similar to a self-adjoint operator iff L generates a group $Z_t = e^{iLt}$ of bounded operators, and $\sup_{t\in\mathbb{R}} \|Z_t\| < \infty$. Our general task is to analyze in detail the impact of spectral singularities on asymptotic behaviour of Z_t as $t \to -\infty$. As the first step, in [1] we suggested the problem of localization of spectral singularities in terms of the asymptotic in the "simplest" case of finitely many singularities each of finite power order, that is, of calculating their orders and locations from the asymptotic.

In the self-adjoint theory, if a self-adjoint operator, A, has an absolutely continuous spectrum and finitely many eigenvalues, $\{\lambda_j\}$, then the similar problem of calculating the λ_j's from the asymptotic of e^{iAt} is solved trivially because of the spectral theorem. Namely, we fix arbitrary u and v such that $(u, e_j) \neq 0$, $(v, e_j) \neq 0$ for all e_j, the eigenvectors of A, and consider $(e^{iAt}u, v)$. Then

$$(e^{iAt}u, v) = \int e^{i\lambda t} d\rho(t) + \sum_{j=0}^{N} c_j e^{i\lambda_j t},$$

where ρ is a measure absolutely continuous with respect to the Lebesgue measure and $c_j = (u, e_j)(e_j, v) \neq 0$ for all j. The first term is $o(1)$ as $t \to \pm\infty$ by the Riemann-Lebesgue lemma, and one can determine the λ_j's from the finite sum in the right hand side, provided that we know $f(t) = (e^{iAt}u, v)$ at large t.

The nonself-adjoint problem above is considerably more complicated since for an operator with spectral singularities we do not have a spectral decomposition *converging in the topology of the original Hilbert space* [4]. Also, notice that even the self-adjoint problem gets complicated if we allow the singular part of the spectrum to have rich structure, in particular, because of the existence of singular measures satisfying the Riemann-Lebesgue property [5]. These make it natural to consider first the case of finitely many spectral singularities. The analysis of an operator with isolated power singularities still constitutes a highly nontrivial problem, which has

[1] that is, there exists a bounded operator U with bounded inverse such that $L = UAU^{-1}$ for some self-adjoint A.

important applications. For instance, the Schrödinger operator on the real axis with complex potential decreasing like $O\left(\exp(-C\sqrt{x})\right)$ studied in [6], is of the class under consideration.

Results in the present paper are aimed towards the solution of the above problem of localization. In [1] we have shown that the power estimate of the resolvent of arbitrary real positive order when approaching the real axis implies the corresponding power estimate for $\left\|Z_t^{-1}\right\|$ for $t > 0$. We now show that in the situation under consideration this upper estimate is sharp in the power scale. The precise definition of what we call a spectral singularity of the real order p is given in terms of the boundary behaviour of the characteristic function [3] of the operator. In terms of the localization, one can say that this result answers the question of calculating the maximal order of spectral singularities. A similar result is obtained for logarithmic singularities. As an application we consider the three - dimensional Boltzmann operator studied in [7].

The proof of this result basically amounts to an explicit construction of the Cauchy data realizing the estimate with arbitrarily close norm growth exponent in terms of the functional model of Sz.-Nagy and Foias. We then derive an asymptotic for the solution of the Cauchy problem for this data and show that the improper eigenfunction of the spectral singularity appears naturally as a leading term coefficient in the asymptotic expansion outside arbitrary small vicinity of the singularity in the spectral representation of the real part of the operator. This matches the expectation coming from consideration of the discrete spectrum case. This observation also suggests that the improper eigenfunctions corresponding to spectral singularities, first introduced by Ljance [8] when attempting to construct a spectral decomposition for the Schrödinger operator with exponentially decreasing complex potential, make a meaningful object, despite the abstract fact [4] that a system of generalized eigenfunctions of the absolutely continuous spectrum is complete in the absolutely continuous subspace. Then we study the asymptotic behaviour of the local norms for the evolution operator and obtain an upper estimate for the norm of $P_\omega Z_{-t}$ where P_ω is the spectral projection of the real part of the operator L corresponding to the interval $\omega \subset \mathbb{R}$. These results allow us to conclude that the spectral singularities of higher orders can indeed be localized "algorithmically" in terms of the asymptotic of Z_t as $t \to -\infty$. More precisely, one can localize all the singularities of the orders greater than $p - 1/2$ where p is the maximal order.

The main new results are formulated in theorems 3 (exactness in the power case), 6 and 6a (upper estimate for local norms), 8 (asymptotic of the nonsmooth vector) and corollary 9 (algorithm for localization). The structure of the paper is as follows. In §1 we give a brief description of the functional model and present some necessary preliminaries. We use the symmetric form of the model suggested by Pavlov [2]. §2 contains the abstract results, and the application to the Boltzmann operator is given in §3.

Throughout the paper the notion almost everywhere (a.e.) refers to the Lebesgue measure on the real axis. For a set $\omega \subset \mathbb{R}$ we denote by χ_ω the indicator function of this set. By the essential spectrum of a closed operator we mean the subset of its spectrum obtained by removing the isolated points such that the corresponding invariant subspaces are finite dimensional.

§1. Let L be a closed dissipative operator in a Hilbert space H with bounded imaginary part $V = \operatorname{Im} L$ such that $\sigma_{ess}(L) \subset \mathbb{R}$. Denote by $E \subset H$ the subspace $E = \overline{\operatorname{Ran} V}$. By the *characteristic function* of L we call the contractive analytic function $S(z)\colon E \to E$, $z \in \mathbb{C}_+$, defined by the formula

$$S(z) = I + 2i\sqrt{V}\,(L^* - z)^{-1}\sqrt{V}, \quad z \in \mathbb{C}_+.$$

This function has boundary values on the real axis, $S(k) \equiv S(k + i0)$, in the strong sense for a.e. $k \in \mathbb{R}$. Let L_{pure} be the completely nonself-adjoint part of L. Then the set $\sigma(L_{pure}) \bigcap \mathbb{R}$ coincides with the complement (in $\mathbb{R}$) of the union of open intervals on which the function S is analytic and unitary. The set $\mathbb{C}_+ \bigcap \rho(L)$ coincides with the set of such $z \in \mathbb{C}_+$ that the operator $S(z)$ has bounded inverse defined on the whole of E. Denote by $\mathcal{X} = L^2 \begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$ the Hilbert space obtained by the closure of the linear set $L^2(\mathbb{R}, E \oplus E)$ in the metric given by the weight $\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$, after factoring modulo the set of elements of zero norm. Let $H^2_\pm(E)$ be the Hardy classes of E-valued functions f analytic in $\mathbb{C}_\pm$, respectively, and satisfying $\sup_{\varepsilon>0} \int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2_E\, dk < \infty$. Define for a representative $(\tilde{g}, g) \in L^2(\mathbb{R}, E \oplus E)$ from the equivalence class of an element of $\mathcal{X}$

$$P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+ (\tilde{g} + S^* g) \\ g - P_- (S\tilde{g} + g) \end{pmatrix} \tag{2}$$

where $P_\pm$ are the Riesz projections on $H^2_\pm(E)$ in $L^2(E)$. Then [2, 4], $P_{\mathcal{K}}$ is a correctly defined operator in $\mathcal{X}$ and its closure in $\mathcal{X}$ which will be denoted with the same letter, is an orthogonal projection. Let $\mathcal{K}$ be the range of $P_{\mathcal{K}}$ in $\mathcal{X}$. Define a unitary group $\mathcal{U}_t$ of operators in $\mathcal{X}$ by the formula $(\mathcal{U}_t f)(k) = e^{ikt} f(k)$. Then the completely nonself-adjoint part of L is unitarily equivalent to the generator of the contraction semigroup $B_t = P_{\mathcal{K}} \mathcal{U}_t \mid_{\mathcal{K}}$, $t \geq 0$. This generator is called the functional model of the operator L_{pure}.

By the general theory (see [3]), there exists a contractive outer function in the upper half plane, S_e, such that for a.e. $k \in \mathbb{R}$ we have $S(k) = \Xi(k) S_e(k)$ where $\Xi(k)$ is an isometric operator. The function S_e is called the *outer factor of S*. Recall that a bounded analytic function $\Theta : \mathbb{C}_+ \to \mathbf{B}(E)$ is called *outer* if $\overline{\Theta H^2_+(E)} = H^2_+(E)$.

We now define the absolutely continuous subspace of the completely nonself-adjoint part of L. Let H' be the minimal reducing subspace of L containing E. Then *the absolutely continuous subspace* $\mathcal{N}_e \subset H$ of the operator L_{pure} [2, 9] is defined as the closure of the set $\widetilde{\mathcal{N}_e}$ of smooth vectors

$$\mathcal{N}_e = \overline{\widetilde{\mathcal{N}_e}}, \qquad \widetilde{\mathcal{N}_e} \equiv \left\{ u \in H' : \sqrt{V}\,(L - z)^{-1} u \in H^2_+(E) \right\}.$$

The subspace $\mathcal{N}_e$ is a regular invariant subspace of the operator L, that is, for all $\lambda \in \rho(L)$ we have $\overline{(L-\lambda)^{-1}\mathcal{N}_e} = \mathcal{N}_e$. It is easy to show that $e^{iLt}\mathcal{N}_e \subset \mathcal{N}_e$ for all $t \in \mathbb{R}$, and hence the restriction $L|_{\mathcal{N}_e}$ with the domain $\mathcal{D}(L) \bigcap \mathcal{N}_e$ is defined as a closed operator. The characteristic function of $L|_{\mathcal{N}_e}$ coincides (see [2]) with the pure part [3] of the outer factor S_e. We say that the completely nonself-adjoint part of L has *absolutely continuous spectrum* if $H_0 = H \ominus \mathcal{N}_e$ is a reducing subspace of L such that $L|_{H_0}$ is a self-adjoint operator.

The set of smooth vectors admits the following description in terms of the functional model [2]. In what follows, to keep notation at minimum, we omit in formulas the operator that accomplishes the unitary equivalence between L_{pure} and its functional model. Define an a.e. nonnegative function Δ by $\Delta(k) = I - S^*(k)S(k)$ and the space $L^2(\mathbb{R};\Delta)$ to be the closure of $L^2(\mathbb{R};E)/\{f\colon\ \Delta f = 0\}$ in the metric given by the weight Δ. Then, for any $\tilde{g} \in L^2(\mathbb{R};\Delta)$ the vector $\begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix}$ defines an element of $\mathcal{X}$ of the norm $\|\tilde{g}\|_{L^2(\mathbb{R};\Delta)}$. Let us define a bounded operator $W : L^2(\mathbb{R};\Delta) \to \mathcal{K}$ by setting $W : \tilde{g} \mapsto P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix}$. Then $\widetilde{\mathcal{N}_e} = \operatorname{Ran} W$.

We shall not distinguish in our notation an element $f \in L^2(\mathbb{R}, E)$ and the corresponding equivalence class in $L^2(\mathbb{R};\Delta)$. Note that $\sqrt{\Delta}u$ for $u \in L^2(\mathbb{R};\Delta)$ is a well - defined element of $L^2(\mathbb{R}, E)$. Then, the norm of a smooth vector $W\tilde{g}$ is given by[2] [4]

$$\|W\tilde{g}\|^2 = \left\|S\sqrt{\Delta}\tilde{g}\right\|^2 + \|P_-\Delta\tilde{g}\|^2 , \tag{3}$$

and the following intertwining relations hold [9, 10]

$$(L-z)^{-1}W \;=\; W(A_0-z)^{-1}, \quad z \in \rho(L) \tag{4}$$

$$e^{itL}W \;=\; We^{itA_0}, \quad t \in \mathbb{R}, \tag{5}$$

where A_0 is the operator of multiplication by the independent variable in $L^2(\mathbb{R};\Delta)$.

Remark that it is obvious from (3) that W is invertible if S, or, equivalently, S_e, is boundedly invertible on the real axis. In this case, the operator $L|_{\mathcal{N}_e}$ is similar to a self-adjoint operator (A_0) which is manifestly absolutely continuous in the sense of the self-adjoint theory. Further motivations of the definition of the absolutely continuous subspace can be found in [2, 4, 9, 10].

Let $\omega_\delta(k) = \{z \in \mathbb{C}_+ :\ |z-k| < \delta\}$. A point $k \in \mathbb{R}$ is called *a proper point* of the operator L if $\sup_{z\in\omega_\delta(k)} \|S_e^{-1}(z)\| < \infty$ for a $\delta > 0$. Following [2] we call a point $k \in \mathbb{R}$ *a spectral singularity* if it is not proper. In the case when the operator L has absolutely continuous spectrum, and thus $S = S_e$, this definition is equivalent to the one described in Introduction, which is seen from the fact [3] that for any dissipative operator L

$$\|S^{-1}(z)\| \asymp \operatorname{Im} z \left\|(L-z)^{-1}\right\| . \tag{6}$$

[2]Throughout the paper $\|\cdot\|$ for E-valued functions on the real axis means the norm in $L^2(\mathbb{R}, E)$.

We refer to the papers [1, 4, 11] and references therein for a discussion of spectral projections for operators with spectral singularities.

§2. We first establish a general assertion giving an upper estimate for $\left\|e^{iLt}\right\|$, $t<0$, provided that we have an upper estimate for the norm of the resolvent by a function of $\operatorname{Im} z$. In the following theorem L is a maximal dissipative operator with real spectrum satisfying $\left\|(L-z)^{-1}\right\| \le (\operatorname{Im} z-\omega)^{-1}$ for some $\omega>0$ and all z with $\operatorname{Im} z>\omega$. The latter condition guarantees that the operator $Z_t=e^{iLt}$, $t\ge 0$, has bounded inverse.

Theorem 1. *Assume that for a locally bounded function $\rho:[t_0,+\infty)\to\mathbb{R}_+$, $t_0<\omega^{-1}$, the estimate*

$$\left\|(L-z)^{-1}\right\| \le \rho\left((\operatorname{Im} z)^{-1}\right) \tag{7}$$

holds for all $z\in\Pi$ where $\Pi=\left\{z:\ 0<\operatorname{Im} z<t_0^{-1}\right\}$. Then for $t\ge t_0$

$$\left\|Z_t^{-1}\right\| \le e+2e\frac{\rho(t)}{t}. \tag{8}$$

Proof of the theorem is based on the following inequality which is valid for any maximal dissipative operator D [9]:

$$\int_{\mathbb{R}} \left\|R_D(k-i\varepsilon)u\right\|^2 dk \le \frac{\pi}{\varepsilon}\left\|u\right\|^2,\ \varepsilon>0, \tag{9}$$

where $R_D(z)=(D-z)^{-1}$. This inequality is deduced in [9] from the existence of a self-adjoint dilation of D. An elementary derivation of it is outlined in [12].

By the Hille - Yosida theorem one has for all u from the domain of L and $t>0$

$$Z_t^{-1}u=-\frac{1}{2\pi i}\lim_{N\to\infty}\int_{-N+i\varepsilon}^{N+i\varepsilon} e^{-izt}R_L(z)u\,dz \tag{10}$$

where $\varepsilon>0$ is large enough. We first show that (10) holds for any $\varepsilon>0$. Indeed, the assumptions made about L imply that $R_L(z)$ is uniformly bounded in each half plane $\operatorname{Im} z\ge\nu>0$. It then follows from the identity

$$R_L(z)u=-\frac{1}{z}u+\frac{1}{z}R_L(z)Lu$$

that $R_L(z)=O\left(z^{-1}\right)$ as z ranges over any such a half plane. This allows us to shift the contour of integration in (10), ensuring that the integrals over vertical segments vanish in the limit $N\to\infty$. It also shows that $\|R_L(z)u\|$ is square summable over any line $\mathbb{R}+i\varepsilon$, $\varepsilon>0$.

Notice that

$$\int_{-N+i\varepsilon}^{N+i\varepsilon} e^{-izt}\left\langle R_L(\overline{z})u,v\right\rangle dz=e^{\varepsilon t}\int_{-N}^{N} e^{-ikt}\left\langle R_L(k-i\varepsilon)u,v\right\rangle dk \underset{N\to\infty}{\longrightarrow} 0$$

for all $t > 0$ by the Jordan lemma. We then have for $t > 0$

$$\begin{aligned}\left\langle Z_t^{-1}u, v\right\rangle = -\frac{1}{2\pi i}\lim_{N\to\infty}\int_{[-N,N]+i\varepsilon} e^{-izt}\left\langle R_L(z)u, v\right\rangle dz = \\ -\frac{1}{2\pi i}\lim_{N\to\infty}\int_{[-N,N]+i\varepsilon} e^{-izt}\left\langle (R_L(z)-R_L(\overline{z}))u, v\right\rangle dz = \\ -\frac{\varepsilon}{\pi}\lim_{N\to\infty}\int_{[-N,N]+i\varepsilon} e^{-izt}\left\langle R_L(z)u, R_L^*(\overline{z})v\right\rangle dz.\end{aligned} \tag{11}$$

The last integral admits the following estimate,

$$\left|\int_{[-N,N]+i\varepsilon}\cdots\right| \le e^{\varepsilon t}\left(\int_{\mathbb{R}+i\varepsilon}\|R_L(z)u\|^2\,dz\right)^{1/2}\left(\int_{\mathbb{R}+i\varepsilon}\|R_L^*(\overline{z})v\|^2\,dz\right)^{1/2} \le \\ e^{\varepsilon t}\left(\int_{\mathbb{R}+i\varepsilon}\|R_L(z)u\|^2\,dz\right)^{1/2}\sqrt{\frac{\pi}{\varepsilon}}\,\|v\|.$$

On the last step we have applied the inequality (9) to the dissipative operator $-L^*$. To estimate the remaining integral in the right hand side, we observe that from the resolvent identity

$$R_L(z) = [I + 2i\varepsilon R_L(z)]R_L(\overline{z}),\ \operatorname{Im} z = \varepsilon,$$

it follows that

$$\|R_L(z)u\| \le \left(1 + 2\varepsilon\rho\left(\varepsilon^{-1}\right)\right)\|R_L(\overline{z})u\|$$

for $z \in \Pi$. This gives for $\varepsilon < t_0^{-1}$

$$\int_{\mathbb{R}+i\varepsilon}\|R_L(z)u\|^2\,dz \le \left(1 + 2\varepsilon\rho\left(\varepsilon^{-1}\right)\right)^2\int_{\mathbb{R}+i\varepsilon}\|R_L(\overline{z})u\|^2\,dz \le \\ \frac{\pi}{\varepsilon}\left(1 + 2\varepsilon\rho\left(\varepsilon^{-1}\right)\right)^2\|u\|^2.$$

Thus,

$$\left|\int_{[-N,N]+i\varepsilon}\cdots\right| \le \frac{\pi}{\varepsilon}e^{\varepsilon t}\left(1 + 2\varepsilon\rho\left(\varepsilon^{-1}\right)\right)\|u\|\,\|v\|.$$

Inserting this into (11) we obtain

$$\left|\left\langle Z_t^{-1}u, v\right\rangle\right| \le \frac{\varepsilon}{\pi}\frac{\pi}{\varepsilon}e^{\varepsilon t}\left(1 + 2\varepsilon\rho\left(\varepsilon^{-1}\right)\right)\|u\|\,\|v\|.$$

Now setting $\varepsilon = t^{-1}$ for t large enough one gets the result. ■

Note that the above proof of the theorem does not hint on sharpness of the assertion. Most of the results in this paper refer to the important partial case when ρ is a power function. For this choice of ρ theorem 1 was first proven in [1]. From now on we assume that L is an operator of the type considered in §1, that is, a maximal dissipative operator with a bounded imaginary part and such that $\sigma_{ess}(L) \subset \mathbb{R}$. Introduce the notation $I_\delta(k) = [k-\delta, k+\delta]$.

Corollary 2. *Assume L has at most finitely many spectral singularities such that for a certain real $p > 0$ the estimate*

$$\left\|S^{-1}(k)\right\| \le C\,|k-k_j|^{-p}$$

holds for each singularity, k_j, for a. e. $k \in I_\delta(k_j)$, $\delta > 0$, and ess $sup_{|k|>M}\|S^{-1}(k)\|$ is finite for some M. Then for all $t > 0$

$$\left\| Z_t^{-1}\big|_{\mathcal{N}_e} \right\| \le C\,(1+t^p)$$

with some $C > 0$.

Proof. Put $\{k_j\}$ be the set of spectral singularities, and let m be an arbitrary scalar H^∞ - function in $\mathbb{C}_+$ such that $|m(z)| \sim c_j\,|z-k_j|^p$ for $z \in \omega_\delta(k_j)$ for all j, for instance, $m(z) = \prod_j \left(\frac{z-k_j}{z+i}\right)^p$ will do. Then, under the assumption of the corollary, the function $\Omega = mS_e^{-1}$ belongs to $L^\infty(\mathbb{R}, \mathbf{B}(E))$. Now, $\Omega H_2^+ \subset H_2^+$ since $\Omega\mathcal{D} \subset H_2^+$ for $\mathcal{D} = S_e H_2^+$, and $\mathcal{D}$ is dense in H_2^+. Thus, Ω is a bounded operator in $L^2(\mathbb{R}, E)$ commuting with multiplication and preserving H_2^+. Therefore, [3] there exists a function $\Omega' \in H^\infty(\mathbf{B}(E))$ such that $\Omega'(k) = \Omega(k)$ for a.e. $k \in \mathbb{R}$. Multiplying this by S_e and applying the uniqueness theorem one gets that $\Omega'(z)S_e(z) = m(z)I$ for all $z \in \mathbb{C}_+$. Hence, $mS_e^{-1} \in H^\infty(\mathbf{B}(E))$, and the assumption of theorem 1 is fulfilled with $\rho(t) = Ct^{1+p}$ in view of (6). ■

Let us remark that the condition of "absence of the spectral singularity at infinity", that is, of finiteness of $\operatorname{ess\,sup}_{|k|>M}\|S^{-1}(k)\|$, in this corollary holds true for the wide class of operators satisfying the condition $S(k) \to I$ as $k \to \infty$, often fulfilled in applications. In fact, under the assumptions of the corollary $L|_{\mathcal{N}_e}$ is a linear sum of a bounded operator and an operator similar to a self-adjoint one.

A situation covered by corollary 2 arises, for instance, if we consider the Schrödinger operator with exponentially decreasing potential [8, 13]. Another example is given by the Friedrichs model with analytic perturbation: consider the Hilbert space $H = L^2(I)$ where I is a compact interval, and define L to be $L = A + i\langle\cdot,\varphi\rangle\varphi$ where $(Af)(s) = sf(s)$ and $\varphi \in L^2(I)$ is a function analytic in a neighborhood of I.

In both these examples the characteristic function is analytic on the real axis, and $S(k) - I$ is compact, which implies that the spectral singularities are exactly the real poles of S^{-1}. The following definition provides the simplest natural abstract generalization of this situation.

A point $k_0 \in \mathbb{R}$ is said to be a *spectral singularity of the order $p > 0$, $p \in \mathbb{R}$, in the strong sense*, if for some nonzero $e_0 \in E$

$$\|S(k)e_0\|_E \le C\,|k-k_0|^p\,, \tag{12}$$

$$\|S^{-1}(k)\| \le C\,|k-k_0|^{-p} \tag{13}$$

for a. e. k in a vicinity of k_0 on the real axis.

Theorem 3. *Assume L has a spectral singularity of order p in the strong sense. Then for any sufficiently small $\varepsilon > 0$ there exists such a $u \in \mathcal{N}_e$ that*

$$\left\|Z_t^{-1}u\right\| = t^{p-\varepsilon}(1+o(1)), \quad t \to +\infty. \tag{14}$$

Proof. We construct the required u for the functional model of the operator. Without loss of generality one can assume that the spectral singularity is located at the point 0.

Observe first that if for a sequence of functions $\tilde{g}_\delta \in H^2_+(E)$ the sequence $S\tilde{g}_\delta$ converges in $L^2(\mathbb{R}, E)$ as $\delta \to 0$, then so does $W\tilde{g}_\delta$ in $\mathcal{N}_e$. Indeed, then $S\sqrt{\Delta}\tilde{g}_\delta$ converges in $L^2(\mathbb{R}, E)$, since for all $\widetilde{v} \in L^2(\mathbb{R}, E)$ we have

$$\left\|S\sqrt{\Delta}\widetilde{v}\right\| = \left\|\sqrt{\Delta}\sqrt{S^*S}\widetilde{v}\right\| \leq \left\|S\widetilde{v}\right\|,$$

and the fact follows from (3) since $P_-\Delta\tilde{g}_\delta = -P_-S^*S\tilde{g}_\delta$ obviously converges.

Define the function $h_\delta(k) = (k+i\delta)^{-1/2-p+\varepsilon}$, $\delta \geq 0$, where the cut is drawn along the negative imaginary axis, and let $\tilde{g}_\delta = h_\delta e_0$, $\tilde{g} = \tilde{g}_0$. Obviously, $\tilde{g}_\delta \in H^2_+(E)$ for all $\delta > 0$. One then has

$$\left\|S(k)\tilde{g}_\delta(k)\right\|_E \leq C\frac{|k|^p}{|k+i\delta|^{1/2+p-\varepsilon}} \leq C\,|k|^{\varepsilon-1/2},$$

and therefore $S\tilde{g}_\delta$ converge in $L^2(\mathbb{R}, E)$ as $\delta \to 0$. From this we infer that there exists the limit $u = \lim\limits_{\delta\to 0} W\tilde{g}_\delta$.

According to (3) and (5), for all $\widetilde{v} \in L^2(\mathbb{R}; \Delta)$ we have

$$\left\|Z_t^{-1}W\widetilde{v}\right\|^2 = \left\|S\sqrt{\Delta}\widetilde{v}\right\|^2 + \left\|P_-e^{-ikt}\Delta\widetilde{v}\right\|^2,$$

hence

$$\left\|Z_t^{-1}u\right\|^2 = \left\|S\sqrt{\Delta}\tilde{g}\right\|^2 + \lim_{\delta\to 0}\left\|P_-e^{-ikt}\Delta\tilde{g}_\delta\right\|^2. \tag{15}$$

Since $\tilde{g}_\delta - \Delta\tilde{g}_\delta = S^*S\tilde{g}_\delta$ converges in $L^2(\mathbb{R}, E)$, one sees that (here the O - symbolic refers to the variable $t \in [0, +\infty)$)

$$\lim_{\delta\to 0}\left\|P_-e^{-ikt}\Delta\tilde{g}_\delta\right\| = \lim_{\delta\to 0}\left\|P_-e^{-ikt}\tilde{g}_\delta\right\| + O(1).$$

Assume first $p \in (0, 1]$. Considering that $P_-\tilde{g}_\delta = 0$, one has the identity

$$P_-e^{-ikt}\tilde{g}_\delta = P_-\left(e^{-ikt}-1\right)\tilde{g}_\delta.$$

By majorized convergence theorem, one can now pass to the limit $\delta \to 0$ in the right hand side to get

$$\lim_{\delta\to 0}\left\|P_-e^{-ikt}\Delta\tilde{g}_\delta\right\| = \left\|P_-\left(e^{-ikt}-1\right)\tilde{g}\right\| + O(1).$$

The projection P_- commutes with the dilations, $(P_- f)_a = P_- f_a$ where $f_a(x) = f(ax)$, which gives

$$\left\|P_-\left(e^{-ikt}-1\right)\tilde{g}\right\| = \left\|\left(P_-\left[\left(e^{-ik}-1\right)h_{t^{-1}}e_0\right]\right)_t\right\| = t^{p-\varepsilon}\left\|P_-\left[\left(e^{-ik}-1\right)h\,e_0\right]\right\|.$$

It remains to notice that $f = \left(e^{-ik}-1\right)h \notin H^2_+$ since $f(is) \to \infty$ as $s \to +\infty$, and therefore the constant $C_\varepsilon = \|P_-[fe_0]\| \neq 0$.

In the case of integer $p > 1$, we use a formula for $P_- e^{-ikt}\tilde{g}_\delta$ obtained by subtraction of the first p terms of the Taylor expansion for the function e^{-ikt}. Since $k^j\tilde{g}_\delta \in H^2_+(E)$ for any nonnegaitve integer $j \leq p-1$, we have

$$P_- e^{-ikt}\tilde{g}_\delta = P_-\left[\left(e^{-ikt} - \sum_{j=0}^{p-1}\frac{(-ikt)^j}{j!}\right)\tilde{g}_\delta\right].$$

The function in square brackets is $O\left(|k|^{-1/2+\varepsilon}\right)$ in a neighborhood of 0 uniformly in δ, which makes it possible to pass to the limit $\delta \to 0$ in the right hand side to obtain that

$$\lim_{\delta\to 0}\left\|P_- e^{-ikt}\tilde{g}_\delta\right\| = \left\|P_-\left[\left(e^{-ikt} - \sum_{j=0}^{p-1}\frac{(-ikt)^j}{j!}\right)\tilde{g}\right]\right\|.$$

The rest of the proof coincides with that in the case $p = 1$. The case of non-integer $p > 1$ is considered in a similar way by subtraction of the first $[p]+1$ terms of the Taylor expansion for the exponent. ■

Another partial case of theorem 1 important in applications is obtained if we take $\rho(t) = \ln t$. This situation arises, for instance, when studying the neutron transport operator for slab geometry [7, 14]. The following assertion is analogous to corollary 2.

Corollary 4. *Assume L is bounded and has at most finitely many spectral singularities such that the estimate*

$$\left\|S^{-1}(k)\right\| \leq -C' \ln|k-k_j|$$

holds for each singularity, k_j, for a. e. $k \in I_\delta(k_j)$, $\delta > 0$. Then

$$\left\| Z_t^{-1}\big|_{\mathcal{N}_e}\right\| \leq C\ln t$$

for all $t \geq 2$.

Proof. This fact follows from theorem 1 with $\rho(t) = \ln t$. The proof can be taken verbatim from that of corollary 2, if we set $m(z) = \prod_j \frac{1}{\ln(z-k_j)}$ where the cut of the logarithm is chosen to be, for instance, the negative real axis, and $\ln 1 = 2\pi i$. ■

A logarithmic spectral singularity in the strong sense is defined in the way analogous to the power case, with (12) replaced by

$$\|S(k)e_0\|_E \leq \frac{C}{|\ln|k-k_0||}.$$

Theorem 5. *Assume L has a logarithmic spectral singularity in the strong sense. Then for any sufficiently small $\varepsilon > 0$ there exists a $u \in \mathcal{N}_e$ such that*

$$\left\|Z_t^{-1}u\right\| = (\ln t)^{1-\varepsilon}\,(1+o(1)), \qquad t \to +\infty.$$

Proof of this theorem is analogous to that of theorem 3. For the function $\tilde{g}$ we choose $\tilde{g}(k) = \frac{(\ln k)^{1/2-\varepsilon}}{(k+i)\sqrt{k}} e_0$ where the cut of the square root is chosen to be the negative imaginary axis, of the divisor to be the positive real axis. Obviously, $S\tilde{g} \in L^2(\mathbb{R}; E)$ since the function $\frac{1}{k|\ln k|^{1+2\varepsilon}}$ is integrable at zero. Also, with $\tilde{g}_\delta$ defined as $\tilde{g}_\delta = h_\delta e_0$, $h_\delta(z) = \frac{(\ln z)^{1/2-\varepsilon}}{(z+i)\sqrt{z+i\delta}}$, we see that $S\tilde{g}_\delta$ converges in $L^2(\mathbb{R}, E)$ as $\delta \downarrow 0$. Since $h_\delta \in H^2_+$ for $\delta > 0$, it follows that there exists the limit $u = \lim\limits_{\delta\to 0} W\tilde{g}_\delta$.

Then,

$$P_- e^{-ikt}\Delta\tilde{g}_\delta = P_- e^{-ikt}P_+\Delta\tilde{g}_\delta + e^{-ikt}P_-\Delta\tilde{g}_\delta.$$

Arguing as in the proof of theorem 3 one concludes that Δ in the right hand side of this equality can be omitted with an accuracy of $O(1)$ in t, and thus

$$\lim_{\delta\to 0}\left\|P_- e^{-ikt}\Delta\tilde{g}_\delta\right\| = \lim_{\delta\to 0}\left\|P_- e^{-ikt}P_+\tilde{g}_\delta\right\| + O(1).$$

Let φ_δ be the Fourier transform of the function h_δ. According to the Paley - Wiener theorem,

$$\left\|P_- e^{-ikt}P_+\tilde{g}_\delta\right\|^2 = \int_0^t |\varphi_\delta(p)|^2\,dp \underset{\delta\to 0}{\longrightarrow} \int_0^t |\varphi(p)|^2\,dp$$

where φ is the Fourier transform of the function $h \in L^1(\mathbb{R})$. The passage to the limit $\delta \to 0$ in the right hand side is justified by the obvious fact that $h_\delta \xrightarrow{L^1(\mathbb{R})} h$, and therefore $\varphi_\delta \longrightarrow \varphi$ uniformly. Calculating the asymptotic of $\varphi(p)$ as $p \to +\infty$ via the stationary phase method, we find

$$\varphi(p) = \int \frac{e^{-ikp}\,(\ln k)^{\frac{1}{2}-\varepsilon}}{(k+i)\sqrt{k}}\,dk = C\frac{(\ln p)^{\frac{1}{2}-\varepsilon}}{\sqrt{p}}(1+o(1)).$$

From this we obtain,

$$\int_0^t |\varphi(p)|^2\,dp = C\,(\ln t)^{2-2\varepsilon}\,(1+o(1)), \quad t \to +\infty.$$

Therefore

$$\lim_{\delta\to 0}\left\|P_- e^{-ikt}\Delta\tilde{g}_\delta\right\| = C\,(\ln t)^{1-\varepsilon}\,(1+o(1)),$$

and the result follows. ∎

One should mention that the definition of a spectral singularity of the order p we use is more restrictive than the requirement of exactness of the estimate (13): the vector e_0 in (12) does not depend on k. This circumstance is reflected by the words "in the strong sense" inserted in the definition. In fact, our definition does not even cover the case of analytic S, unless, for instance, we assume additionally the relative compactness of the perturbation V with respect to the real part of L. The latter condition, however, is often fulfilled in applications. Study of the general case calls for further investigation.

We now proceed to study the local behaviour of the semigroup Z_t with respect to the spectral representation of the real part of the operator L. This approach is natural when an operator can be considered as a perturbation of its real part.

For an interval $\omega \subset \mathbb{R}$ define P_ω to be the spectral projection of $A = \operatorname{Re} L$ for the ω.

Theorem 6. *Let L be a dissipative operator as in corollary 2 with $p = n$ for a certain real $n \geq 1$. Then for any closed interval ω not containing the spectral singularities*

$$\left\| P_\omega Z_{-t}|_{\mathcal{N}_e} \right\| \leq C_\omega \left(1 + t^{n-1/2}\right), \quad t > 0. \tag{16}$$

We shall need the following abstract lemma. Let D be a dissipative operator, S be the characteristic function of D, and $\alpha = \sqrt{2 \operatorname{Im} D}$.

Lemma 7. *The following identity holds for all $t \in \mathbb{R}$,*

$$\alpha Z_t u = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikt} (\Delta \tilde{g})(k)\, dk \tag{17}$$

for any smooth vector $u = W\tilde{g}$ such that $\tilde{g} \in L^1(\mathbb{R}, E)$.

Proof. It is enough to proof the result for a compactly supported $\tilde{g}$. In view of the intertwining relation (5), it suffices to proof the identity for $t = 0$. For any $u = W\tilde{g} \in \widetilde{\mathcal{N}_e}$ we have from (2)

$$u = P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+ \Delta \tilde{g} \\ -S\tilde{g} \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

By [9, Theorem 2] we then have

$$\begin{cases} g_1 + S^* g_2 = \mathcal{F}_+ u \\ S g_1 + g_2 = \mathcal{F}_- u \end{cases} \Leftrightarrow \begin{cases} P_- \Delta \tilde{g} = \mathcal{F}_+ u \\ -S P_+ \Delta \tilde{g} = \mathcal{F}_- u \end{cases} \tag{18}$$

where

$$\mathcal{F}_+ u = -\frac{1}{\sqrt{2\pi}} \alpha \left(D - \cdot + i0\right)^{-1} u,$$
$$\mathcal{F}_- u = -\frac{1}{\sqrt{2\pi}} \alpha \left(D^* - \cdot - i0\right)^{-1} u.$$

Recall that $\alpha \left(D - \cdot\right)^{-1} v \in H^2_-(E)$ for any $v \in H$ [9, Theorem 4, Corollary 1], and therefore $\mathcal{F}_\pm u$ belong to $H^2_\mp(E)$, respectively. Considering the second equality in

(18) as an equality for elements of $H^2_+(E)$ we have for any $z \in \mathbb{C}_+$

$$-S(z)(P_+\Delta\tilde{g})(z) = -\frac{1}{\sqrt{2\pi}}\alpha\left(D^* - z\right)^{-1}u = -\frac{1}{\sqrt{2\pi}}S(z)\alpha\left(D - z\right)^{-1}u.$$

Since $\rho(D)\bigcap\mathbb{C}_+$ is nonempty and $S(z)$ is invertible for $z \in \rho(D)\bigcap\mathbb{C}_+$ one concludes by analyticity that $P_+\Delta\tilde{g} = \frac{1}{\sqrt{2\pi}}\alpha\left(D - \cdot\right)^{-1}u$. We now get from (18)

$$\Delta\tilde{g} = (P_+ + P_-)\Delta\tilde{g} = \frac{1}{\sqrt{2\pi}}\left(\alpha\left(D - \cdot + i0\right)^{-1}u - \alpha\left(D - \cdot - i0\right)^{-1}u\right),$$

the equality is being understood in the sense of the space $L^2(\mathbb{R}, E)$. Let us integrate this equality over an arbitrary open finite interval I such that $\tilde{g}(k) = 0$ for a.e. $k \notin [-A, A] \subset I$. Since $(2\pi i)^{-1}\int_\gamma (D - z)^{-1}u = u$ for any contour γ encircling $[-A, A]$ which follows from the intertwining relation (4), we obtain

$$\int_I (\Delta\tilde{g})(k)\,dk = \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\alpha\int_{\gamma_\varepsilon}(D - z)^{-1}u\,dz = -i\sqrt{2\pi}\,\alpha u.$$

Here γ_ε is the contour consisting of the intervals $I \pm i\varepsilon$ oriented in the opposite directions. ■

Applying now the Paley-Wiener theorem, we arrive at the following

Corollary. *Let $\mathcal{F}$ be the Fourier transform in $L^2(\mathbb{R}, E)$. Then the following equality holds for any u satisfying the assumption of lemma 7 for a.e. $t \geq 0$,*

$$\alpha Z_{-t}u = i\mathcal{F}\left[P_+\Delta\tilde{g}\right](t). \tag{19}$$

For $u \in \widetilde{\mathcal{N}_e}$ introduce the notation $u^+(k)$ for the boundary values of the $H^2_+(E)$ - function from the definition of the set $\widetilde{\mathcal{N}_e}$,

$$u^+(k) = \frac{1}{\sqrt{2\pi}}\alpha\left(L - k - i0\right)^{-1}u.$$

Then (19) can be rewritten as

$$\alpha Z_{-t}u = i\left(\mathcal{F}u^+\right)(t). \tag{20}$$

This formula can, of course, easily be derived directly from the Hille-Yosida theorem for all $u \in \widetilde{\mathcal{N}_e} \cap \mathcal{D}(L)$ without any use of the functional model. We have used the argument above since we shall need lemma 7 anyway in what follows.

Proof of theorem 6. Let $u = W\tilde{g}$ be a smooth vector satisfying the assumption of lemma 7. According to the Doimelle formula one has

$$Z_{-t}u = \int_0^t e^{iA(t'-t)}VZ_{-t'}u\;dt' + e^{-iAt}u. \tag{21}$$

Let us substitute (20) in this formula. We first notice that

$$\alpha Z_{-t}u = i\mathcal{F}[\chi_1 u^+](t) + Q(t,u) \tag{22}$$

where χ_1 is the indicator of an arbitrary neighborhood $U = \bigcup_j I_d(k_j)$ of the set $\{k_j\}$ of spectral singularities such that $U \bigcap \omega = \emptyset$, and Q satisfies $\int_{\mathbb{R}} \|Q(t,u)\|_E^2\, dt \le C\|u\|^2$. Indeed, letting $\chi_2 = 1 - \chi_1$, one has

$$\int_{\mathbb{R}} \|Q(t,u)\|_E^2\, dt = \left\|\mathcal{F}\left[\chi_2 u^+\right]\right\|^2 = \left\|\chi_2 u^+\right\|^2 \le \operatorname*{ess\,sup}_{k\notin U} \left\|S^{-1}(k)\right\|^2 \left\|Su^+\right\|^2 \le C\|u\|^2.$$

Here we have taken into account that $\|Su^+\| \le \|u\|$, which is implied[3] by the identity

$$S(z)u^+(z) = \frac{1}{\sqrt{2\pi}}\alpha\left(L^* - z\right)^{-1} u,\ z \in \mathbb{C}_+,$$

easily verified by direct calculation, and the inequality (see [9])

$$\int_{\mathbb{R}} \left\|\alpha\left(L^* - k - i\varepsilon\right)^{-1} u\right\|^2 dk \le \pi \left\|u\right\|^2. \tag{23}$$

It now follows that the operator norm of the term corresponding to Q in (21) is estimated above by $\sqrt{t}$,

$$\left\|\int_0^t e^{iA(t'-t)}\alpha Q(t,u)\, dt'\right\| \le \left(\int_0^t dt'\right)^{1/2} \|\alpha\| \left(\int_0^t \|Q(t,u)\|_E^2\, dt'\right)^{1/2} \le C\sqrt{t}\|u\|.$$

Thus, it remains to estimate the contribution of the first term in (22) to (21). We have,

$$\int_0^t e^{iA(t'-t)}\alpha\mathcal{F}\left[\chi_1 u^+\right](t')\, dt' = Ce^{-iAt}\int_0^t \int_U e^{i(A-k)t'}\alpha u^+(k) dk\, dt' = Ce^{-iAt}\int_U g_t(A-k)\alpha u^+(k)\, dk,$$

where $g_t(k) = \left(e^{ikt} - 1\right)/k$. Since the projection P_ω commutes with the functions of A, the theorem will be proved if we show that

$$\left\|\int_U g_t(A-k)P_\omega \alpha u^+(k)\, dk\right\| \le C\left(1 + t^{n-1/2}\right)\|u\|.$$

[3]It also easily follows from (3).

First, let us replace the integral over U by the integral over the contour $\Gamma_\delta = \bigcup \Gamma^j$, $\Gamma^j = \gamma_\delta^j \cup \{k : \delta < |k - k_j| < d\}$, $\gamma_\delta^j = \{z \in \mathbb{C}_+ : \ |z - k_j| = \delta\}$, where $\delta > 0$ will be chosen below. Then,

$$\left\| \int_{\Gamma_\delta} g_t(A-k) P_\omega \alpha u^+(k)\, dk \right\|^2 \leq \|V\| \left(\int_{\Gamma_\delta} \|g_t(A-k)P_\omega\|^2 \left\|S_e^{-1}(k)\right\|^2 |dk| \right) \int_{\Gamma_\delta} \left\|S_e(k)u^+(k)\right\|^2 |dk|. \quad (24)$$

We shall now estimate separately the integrals over γ_δ^j and $\varrho_\delta^j = \{k : \ \delta < |k - k_j| < d\}$ in the first factor. To this end, notice that it follows from the proof of corollary 2 that $\|S_e^{-1}(z)\| = O\left(|z-k_j|^{-n}\right)$ in the d-vicinity of the point k_j in $\overline{\mathbb{C}_+}$, provided that d is small enough, since mS_e^{-1} is bounded in $\mathbb{C}_+$ for a scalar function m such that m^{-1} satisfies this in the vicinity of k_j. This gives

$$\int_{\gamma_\delta^j} \|g_t(A-k)P_\omega\|^2 \left\|S_e^{-1}(k)\right\|^2 |dk| \leq C\delta^{-2n} e^{\delta t} \int_{\gamma_\delta^j} |dk| = C\pi \delta^{1-2n} e^{\delta t}$$

since $\|g_t(A-k)P_\omega\| \leq Ce^{\delta t}$ for $k \in \gamma_\delta$ by the functional calculus, and

$$\int_{\varrho_\delta^j} \|g_t(A-k)P_\omega\|^2 \left\|S_e^{-1}(k)\right\|^2 dk \leq \frac{4}{(\mathrm{dist}(\omega, k_j) - d)^2} \int_{\delta<|k-k_j|<d} \frac{dk}{|k-k_j|^{2n}} \leq C_1 \delta^{1-2n} + C_2. \quad (25)$$

Combining these estimates and setting $\delta = t^{-1}$, one obtains that the factor in parentheses in the right hand side of (24) is not greater than $C\left(1 + t^{2n-1}\right)$. Then, applying the Carleson embedding theorem, an appropriate vector version of which can be found e. g. in [2], one concludes that the norm of an H^2_+ - function in $L^2\left(\Gamma_\delta, |dk|\right)$ is estimated[4] from above by the norm in H^2_+ with a constant C independent of δ, hence

$$\int_{\Gamma_\delta} \left\|S_e(k)u^+(k)\right\|^2 |dk| \leq C\left\|S_e u^+\right\|^2_{H^2_+(E)} \leq C\|u\|^2.$$

It now remains to notice that the linear set described in lemma 7 is dense in $\mathcal{N}_e$. ■

One should mention that one can eliminate completely the functional model from the above proof of theorem 6, if we recall that the fundamental inequality (23) can be obtained in an elementary way (see [12] for an outline).

As we shall now show, an additional assumption of relative smoothness of the perturbation V allows to establish a generalization of the estimate (16) valid for any interval ω and all $n \geq 1/2$ (in the case $n < 1$ it only follows from the proof of theorem 6 that the norm in the left hand side of (16) is $O\left(\sqrt{t}\right)$).

[4] For this very special case of the theorem the conclusion can, of course, be verified in an elementary way.

In the following theorem L is a dissipative operator having at most finitely many spectral singularities, k_j, such that for each k_j there exists a $p_j > 0$ such that the estimate

$$\left\|S^{-1}(k)\right\| \leq C\left|k-k_j\right|^{-p_j}$$

holds for a. e. $k \in I_\delta(k_j)$, $\delta > 0$, and $\operatorname{ess\,sup}_{|k|>M}\|S^{-1}(k)\|$ is finite for some M. Let $\sigma_0 = \{k_j\}$ be the set of spectral singularities of L, $p = \max_j p_j$.

Theorem 6a. *Assume that $p \neq 1/2$, and $\sqrt{V}$ is $\operatorname{Re} L$ - smooth in the sense of Kato [15]. Then for any closed interval $\omega \subset \mathbb{R}$ we have*

$$\left\| P_\omega Z_{-t}|_{\mathcal{N}_e}\right\| \leq C_\omega \left(1 + t^{p-1/2} + \sum_{j:k_j\in\omega} t^{p_j}\right), \quad t > 0. \tag{26}$$

Proof. Let $U = \bigcup_{j:k_j\notin\omega} U_j$ with the intervals $U_j = I_d(k_j)$ chosen to satisfy $U_j \cap \omega = \emptyset$, χ_1 be the indicator of U, and $\chi_2 = 1 - \chi_1$. Fix an arbitrary $B > 0$ such that $U \cup \omega \subset (-B, B)$, and define χ_r to be the indicator of $\mathbb{R}\setminus[-B,B]$, χ_3 to be the indicator of $U' = (-B,B)\setminus U$, so $\chi_2 = \chi_3 + \chi_r$. To take into account the contribution of the singularities from ω we observe that

$$\sup_{k\in\mathbb{R}\setminus U}\left\|S_e^{-1}(k+i\delta)\right\| \leq C \max_{j:k_j\in\omega}\delta^{-p_j} \tag{27}$$

for $\delta < 1$, which also follows from the proof of corollary 2.

The estimate of the contribution of the term in (21) corresponding to χ_1 holds in the situation under consideration. Thus the theorem will be proved if we show that

$$\left\|\int_0^t e^{iAt'}\alpha\mathcal{F}\left[\chi_2 u^+\right](t')\,dt'\right\| \leq C\left(1 + \sum_{j:k_j\in\omega} t^{p_j}\right)\|u\|.$$

We have for any $\xi \in H$,

$$\left|\left\langle\int_0^t \cdots, \xi\right\rangle\right| = \left|\int_0^t \left\langle \mathcal{F}\left[\chi_2 u^+\right](t'), \alpha e^{-iAt'}\xi\right\rangle dt'\right| \leq$$
$$\left(\int_0^t \left\|\mathcal{F}\left[\chi_2 u^+\right](t')\right\|^2 dt'\right)^{1/2}\left(\int_0^t \left\|\alpha e^{-iAt'}\xi\right\|^2 dt'\right)^{1/2}. \tag{28}$$

We first observe that

$$\int_0^t \left\|\mathcal{F}\left[\chi_r u^+\right](t')\right\|^2 dt' \leq \left\|\mathcal{F}\left[\chi_r u^+\right]\right\|^2 = \left\|\chi_r u^+\right\|^2 \leq C\left\|S u^+\right\|^2 \leq C\|u\|^2. \tag{29}$$

Then, transforming the contour of integration, we obtain

$$\sqrt{2\pi}\,\mathcal{F}\left[\chi_3 u^+\right](\tau) = \int_{U'} e^{-ik\tau}u^+(k)\,dk = e^{\delta\tau}\int_{U'} e^{-ik\tau}u^+(k+i\delta)\,dk +$$
$$\int_{\rho_\delta} e^{-ik\tau}u^+(k)\,dk,$$

where ρ_δ is the contour consisting of vertical segments of the length $\delta = 1/t$ starting at $\pm B$ and the ends of the intervals U_j. For $0 \le \tau \le t$ the second integral is estimated above by

$$e^{\delta t} \int_0^\delta \left\| u^+(k+is) \right\| \, ds \le C\sqrt{\delta} \left(\int_0^\delta \left\| u^+(k+is) \right\|^2 \, ds \right)^{1/2} \le$$
$$C\sqrt{\delta} \left(\sup_{k\in\rho_\delta} \left\| S_e^{-1}(k) \right\| \right) \left\| S_e u^+ \right\|_{L^2(\rho_\delta, |dk|)} \le C\sqrt{\delta}\|u\|.$$

Here we took into account that the norm of S_e^{-1} is bounded outside an arbitrary neighborhood of the set σ_0 (see the proof of corollary 2). From this we have

$$\left(\int_0^t \left\| \mathcal{F}\left[\chi_3 u^+\right](t') \right\|^2 dt' \right)^{1/2} \le C_1 e^{\delta t} \left\| \mathcal{F}\left[\chi_3 u^+(\cdot + i\delta)\right] \right\| + C_2\sqrt{t}\sqrt{\delta}\|u\| \le$$
$$C\left(\left\| \chi_3 u^+(\cdot + i\delta) \right\| + \|u\| \right) \le C \left(\max_{j:k_j\in\omega} \left(\delta^{-p_j}\right) \left\| \chi_3 (S_e u^+)(\cdot + i\delta) \right\| + \|u\| \right) \le$$
$$C \left(\max_{j:k_j\in\omega} \left(\delta^{-p_j}\right) \left\| S_e u^+ \right\| + \|u\| \right) \le C \left(1 + \sum_{j:k_j\in\omega} t^{p_j} \right) \|u\|.$$

On the last but one step we have used (27) here. Inserting this and (29) into (28) and taking into account that

$$\int_0^t \left\| \alpha e^{-iAt'} \xi \right\|^2 dt' \le C\|\xi\|^2$$

under the imposed assumption about smoothness, we conclude the proof. ■

The results obtained approach the problem of detecting the singularities formulated as follows. Assume we are given an operator with finitely many spectral singularities, each of a finite power order in the strong sense. How to calculate their orders, p_j, and locations, k_j, from the asymptotic of Z_t, $t \to \pm\infty$? In this respect, corollary 2 and theorem 3 answer the question of calculating the maximal order, p, of spectral singularities. Once this is done, theorem 6a provides a sufficient condition for the localization: if the estimate (16) with $n = p \ne 1/2$ is not satisfied, then ω contains a singularity of an order greater than $p - 1/2$.

In the following theorem we give an asymptotic of the nonsmooth vector constructed in the course of the proof of theorem 3.

Theorem 8. *Assume L is a dissipative operator, and $\sqrt{V}$ is $\operatorname{Re} L$ - smooth. Then for any spectral singularity $k_0 \in \mathbb{R}$ of an order $n > 1/2$ and any $\varepsilon > 0$ small enough there exists a $u \in \mathcal{N}_e$ satisfying (14) and such that for any closed interval ω, $k_0 \notin \omega$,*

$$P_\omega Z_{-t} u = C e^{-ik_0 t} t^{n-1/2-\varepsilon} \Psi_{k_0} + O\left(t^{\max\{0, n-3/2-\varepsilon\}}\right) \tag{30}$$

where $\Psi_{k_0} = (A_\omega - k_0)^{-1} P_\omega \alpha e_0$, $A_\omega = AP_\omega$, e_0 *is defined in (12), and the O-symbol refers to the norm in* H.

In the above example of the Friedrichs model the smoothness assumption is equivalent to the essential boundedness of the function φ. The asymptotic (30) takes in this case the form

$$Z_{-t}u = C_0 e^{-ik_0 t} t^{n-1/2-\varepsilon} \Psi_{k_0} + r_t$$

where

$$\Psi_{k_0}(s) = \frac{\varphi(s)}{s - k_0},$$

the function r_t satisfies

$$\left(\int_{|s-k_0|>\delta} |r_t(s)|^2 ds \right)^{1/2} \leq C_\delta \left(1 + t^{n-3/2-\varepsilon}\right)$$

for any $\delta > 0$, and the equality holds for each $t > 0$ for a. e. $s \in I$. The function Ψ_{k_0} makes sense of a regular part of the generalized eigenfunction of L corresponding to the point k_0: any formal solution to $Lu = k_0 u$ coincides with Ψ_{k_0} for $s \neq k_0$. Thus, the asymptotic obtained can be considered as a rigorous justification of the idea that spectral singularity is a kind of resonance which has been drawn into the continuous spectrum.

Proof of theorem 8. We define the u as in the proof of theorem 3. Let us first show that

$$\alpha Z_{-t} u = C e^{-ik_0 t} t^{n-1/2-\varepsilon} e_0 + r(t) \tag{31}$$

with an $r \in L^2(\mathbb{R}, E)$. In notation of the proof of theorem 3, let $\Delta\tilde{g}_\delta = h_\delta e_0 + \rho_\delta$, then ρ_δ converges in $L^2(\mathbb{R}, E)$. Let $\rho = \lim_{\delta\to 0} \rho_\delta$. For a non-half-integer n we define $p_n(s) = \sum_{j=0}^{[n-1/2]} \frac{(-is)^j}{j!}$. By lemma 7 we have for $u_\delta = W\tilde{g}_\delta$ and any $t > 0$

$$\alpha Z_{-t} u_\delta = C \int_{\mathbb{R}} e^{-ikt} (\Delta\tilde{g}_\delta)(k)\, dk = C \left(\int_{\mathbb{R}} e^{-ikt} h_\delta(k)\, dk \right) e_0 + C(\mathcal{F}\rho_\delta)(t) \tag{32}$$

Then,

$$\int_{\mathbb{R}} e^{-ikt} h_\delta(k)\, dk = C e^{-ik_0 t} \int_{\mathbb{R}} \left(e^{-i(k-k_0)t} - p_n((k-k_0)t) \right) h_\delta(k)\, dk +$$
$$e^{-ik_0 t} \int_{\mathbb{R}} p_n((k-k_0)t) h_\delta(k)\, dk.$$

Blowing up the contour of integration in the upper half plane, one verifies that the second term in the right hand side is zero since $n + 1/2 - [n-1/2] - \varepsilon > 1$. Obviously, in the first term one can pass to the limit $\delta \to 0$ for any $t > 0$, which gives after substitution $k \mapsto (k - k_0)t$:

$$\int_{\mathbb{R}} e^{-ikt} h_\delta(k)\, dk \underset{\delta\to 0}{\longrightarrow} C t^{n-1/2-\varepsilon} e^{-ik_0 t}, \quad C = \int_{\mathbb{R}} \left(e^{-is} - p_n(s) \right) h(k)\, dk.$$

Taking into account that $\rho_\delta \longrightarrow \rho$ as $\delta \to 0$ also in $L^1(\mathbb{R}, E)$, one concludes that both terms in the right hand side of (32) converge in E as $\delta \to 0$ to give (31) for all $t > 0$. For n half-integer the proof is the same, except that we have to take $n - 3/2$ for the upper limit in the sum defining p_n.

We now insert (31) into the Doimelle formula (21). By the smoothness assumption, the term corresponding to $r(t)$ is bounded in t (see the estimate of an analogous term in the proof of theorem 6a, formulae (28),(29)). It now remains to consider the contribution of the first term given by

$$e^{-iAt} \int_0^t e^{i(A-k_0)\tau} \tau^{n-1/2-\varepsilon} d\tau. \tag{33}$$

Integrating by parts (once for $n \in (1/2, 3/2]$ and twice for $n > 3/2$), one easily verifies that

$$\int_0^t e^{i(s-k_0)\tau} \tau^{n-1/2-\varepsilon} d\tau = \frac{1}{i(s-k_0)} \left[e^{i(s-k_0)t} t^{n-1/2-\varepsilon} + O\left(t^{\max\{0, n-3/2-\varepsilon\}} \right) \right]$$

uniformly in $s \in \omega$. By the functional calculus, this implies the result. ■

Substituting $\tau \mapsto \tau/t$ in (33) we obtain another asymptotic representation which describes the behaviour of the solution in a vicinity of the spectral singularity.

$$Z_{-t} u = e^{-iAt} t^{n+1/2-\varepsilon} \theta((A - k_0)t) \alpha e_0 + O(1) \tag{34}$$

where θ is an entire function given by the formula[5] $\theta(s) = \int_0^1 e^{is\tau} \tau^{n-1/2-\varepsilon} d\tau$.

This theorem shows, in particular, that the contribution of arbitrary small neighborhood of the spectral singularity in the asymptotic of $Z_{-t}u$ is of the maximal order, $t^{n-\varepsilon}$, in norm. For $n > 1$ this is immediate, and for $n \in [1/2, 1]$ this follows from

Theorem 8a. *Assume L is an operator as in theorem 8. Then for any spectral singularity $k_0 \in \mathbb{R}$ of an order $n' \leq 1/2$ and any $\varepsilon > 0$ small enough there exists a $u \in \mathcal{N}_e$ satisfying (14) and such that for any closed interval ω, $k_0 \notin \omega$,*

$$P_\omega Z_{-t} u = O(1).$$

Proof. The representation (31) holds in the situation under consideration if understood as an equality a. e. in $t > 0$. To verify this, it is enough to notice that for any $t > 0$

$$\int_{\mathbb{R}} e^{-ikt} h_\delta(k)\, dk \underset{\delta \to 0}{\longrightarrow} C t^{n'-1/2-\varepsilon} e^{-ik_0 t}, \quad C = \int_{\mathbb{R}} e^{-ik} k^{-1/2-n'+\varepsilon}\, dk,$$

and this limit is (locally) uniform in t, and apply an approximation argument to (17). The only difference is that r_δ, although converges in L^2, does not belong to

[5] θ is obviously bounded on the real axis, so the operator in the right hand side defined by the functional calculus is bounded.

L^1. Then, the integral $\int_0^t e^{is\tau}\tau^{n'-1/2-\varepsilon}d\tau$ is obviously uniformly bounded in s and t for $t \geq 1$ and $s \geq s_0 > 0$. By the functional calculus, the result follows. ■

Another corollary of theorem 8 is the fact that the estimate (16) with $n = p \geq 1/2$ cannot be improved in the power scale *for any operator* L of the class under consideration. Indeed, otherwise $\Psi_{k_0} = 0$, for a k_0, a singularity of the order p, and thus $P_\omega \alpha e_0 = 0$ for any ω, $\omega \bigcap \sigma_0 = \emptyset$. This implies that $\alpha e_0 \in \operatorname{Ran} P_{\{\sigma_0\}}$, that is, $\alpha e_0 = \sum c_j u_j$, $u_j \in \ker(A - k_j)$. By (34) we obtain that

$$Z_{-t}u = t^{p+1/2-\varepsilon}\sum_j c_j\theta((k_j - k_0)t)e^{-ik_jt}u_j + O(1).$$

Since $\theta(at) = O\left(t^{-1}\right)$ for $a \neq 0$, the leading term at large t is given by the term with $j = 0$, which is $t^{p+1/2-\varepsilon}c_0\theta(0)e^{-ik_0t}u_0$. But then the upper estimate for the norm of Z_{-t} implies that $c_0 = 0$, for $\theta(0) \neq 0$. Thus, we have obtained a contradiction because the terms corresponding to $j \neq 0$ are of the order $t^{p-1/2-\varepsilon}$ which is incompatible with the asymptotic (14).

Combining the results obtained, let us formulate separately the solution to the localization problem for spectral singularities of higher orders in terms of the local norms $\|P_\omega Z_{-t}\|$. First, we determine using corollary 2 and theorem 3 the maximal order, p. Assume first that $p \neq 1/2$. Consider a closed interval $\omega \subset \mathbb{R}$. If $\left\| P_\omega Z_{-t}|_{\mathcal{N}_e}\right\| \leq C\left(1 + t^{p'}\right)$ for a $p' \geq 0$, then the interior of ω does not contain a singularity of an order greater than p'. In the case when $\left\| P_\omega Z_{-t}|_{\mathcal{N}_e}\right\| \leq C_\omega$ $(p' = 0)$ this means that the interior of ω does not contain singularities at all. Then, theorem 6a shows that if the estimate above holds with a positive $p' > p - 1/2$, and this order cannot be made smaller, then ω contains a singularity of the order p'. In the case $p = 1/2$ the only difference is that (26) holds with 1 changed by $\ln t$ in the parentheses (see (25)). We thus have

Corollary 9. *Let a dissipative operator $L = A + iV$, $V \geq 0$, have at most finitely many spectral singularities, each of a finite power order in the strong sense, and p be the maximal order. Suppose that $\sqrt{V}$ is A - smooth. For a closed interval $\omega \subset \mathbb{R}$ define the number $p(\omega) = \inf_{p' \geq 0}\{p' : \left\| P_\omega Z_{-t}|_{\mathcal{N}_e}\right\| \leq C(p')\left(1 + t^{p'}\right)\}$. Then*

(i) ω contains a singularity of the order $p(\omega)$ if $p(\omega) > p - 1/2$, provided that $p(\omega) \neq 0$,

(ii) The interior of ω does not contain a singularity of an order greater than $p(\omega)$.

This result comprises an algorithmic solution to the problem of localization of spectral singularities of orders greater than $\max\{0, p - 1/2\}$. The separation of singularities of lower orders cannot be achieved in this way since, as it is seen from (30), the norm estimate for an interval containing such singularities is determined by the contribution of singularities outside the interval. The appearance of spectral projections for the real part of operator in the procedure for localization, however, suggests that one can localize the singularities of lower orders in terms of time asymptotic if we increase the smoothness of the perturbation. This problem will be considered in the forthcoming papers of the series.

§3. As an example we consider an operator arising from the one - speed transport theory[6]. Let $d\Omega_p$ be the Lebesgue measure on the unit sphere $\mathbb{S}^2$. Given a nonnegative compactly supported function $c \in L^\infty(\mathbb{R}^3)$, define the operator, T, in the Hilbert space $H = L^2(\mathbb{R}^3 \times \mathbb{S}^2)$ by the formula $(q \in \mathbb{R}^3, \widehat{p} \in \mathbb{S}^2)$

$$T = i\widehat{p} \cdot \nabla_q + ic(q)K, \quad K = \frac{1}{4\pi} \int_{\mathbb{S}^2} \cdot \, d\Omega_p$$

on the natural domain of its real part. Then T is a maximal dissipative operator, and the essential spectrum of T coincides with the real axis. The characteristic function, S, of the operator T is analytic on the real axis, and $S(z) - I$ is a compact operator. Let H_{ess} be the invariant subspace of T corresponding to $\sigma_{ess}(T)$. Define the set $\mathcal{E} \subset L^\infty(\mathbb{R}^3)$ as $\mathcal{E} = \{c \,:\, \ker(I + Q(0)) \neq 0\}$ where $Q(0)$ is the integral operator in $L^2(\mathbb{R}^3)$ with the kernel $-\frac{1}{4\pi}\sqrt{c(q)}\frac{1}{|q'-q|^2}\sqrt{c(q')}$.

Theorem. *[7] The completely nonself-adjoint part of the restriction, T_{ess}, of T to H_{ess} has purely absolutely continuous spectrum. The function S satisfies the condition of absence of a spectral singularity at infinity from corollary 2. If $c \notin \mathcal{E}$ then the operator T_{ess} is similar to a self-adjoint operator. If $c \in \mathcal{E}$ then the operator T has the unique spectral singularity of the order* 1 *located at the point* 0.

Applying corollary 2 and theorem 3 one obtains the following result announced in [7].

Corollary. *Let $Z_t^e = \exp(-iTt)|_{H_{ess}}$, $t \geq 0$. If $c \in \mathcal{E}$ then*

$$\|Z_t^e\| \leq C(1+t),$$

and this estimate is exact in the sense given by theorem 3.

References

[1] S. Naboko and R. Romanov, Spectral singularities, Szokefalvi-Nagy - Foias functional model and the spectral analysis of the Boltzmann operator, *Oper. Theory: Adv. Appl.* **127** (2001), 473 - 490.

[2] B.S. Pavlov, On separation conditions for the spectral components of a dissipative operator, *Math. USSR Izvestija* **9**, No. 1, 113 - 137 (1975).

[3] B. Sz.-Nagy and C. Foias, Analyse harmonique des operateurs de l′espase de Hilbert. Masson et C^{ie}, Academiai Kiado, 1967.

[4] B.S. Pavlov, On expansions in eigenfunctions of the absolutely continuous spectrum of a dissipative operator, *Vestnik Leningrad Univ. (Mat. Mech. Astronom)*, vyp. 1, 130 - 137 (1975) (*Russian*); English transl. in: *Vestnik Leningrad Univ. Math.* **8** (1980), 135 - 143.

[6]The detailed exposition of the results listed below can be found in the paper [7].

[5] Y. Meyer, "Algebraic Numbers and Harmonic Analysis", North-Holland, Amsterdam, 1972.

[6] B.S. Pavlov, The nonself-adjoint Schrödinger operator, in *Topics in Mathematical Physics* (M.Birman, ed.), Vol. 1, New York, 1967, 87 - 114.

[7] Yu. Kuperin, S. Naboko and R. Romanov, Spectral analysis of the transport operator: A functional model approach, *Indiana Univ. Math. J.* **51**(2002), No. 6, 1389 - 1425.

[8] V.E. Ljance, On a differential operator with spectral singularities. I. *Mat. Sb.* **64** (106), 4 (1964), 521 -561; II. Mat. Sb. **65** (107), 1 (1964), 47 - 103 (*Russian*); English transl. in: *Amer. Math. Soc. Transl.* (2) **60** (1967).

[9] S.N. Naboko, A functional model of perturbation theory and its application to scattering theory, *Trudy MIAN* **147** (1980), 86 - 114 (*Russian*); English transl. in: *Proc. Steklov Inst. Math.* (1981), No. 2, 85 - 116.

[10] S.N. Naboko, On the conditions for existence of wave operators in the nonselfadjoint case, *Probl. Mat. Fiz.,* 12 (M. Birman ed.), LGU, Leningrad, 1987, 132-155 (*Russian*); English transl. in: *Amer. Math. Soc. Transl.* (2) **157** (1993), 127-149.

[11] A. Rybkin, On the evaluation of spectral projections onto absolutely continuous parts of a contraction, *Proc. Amer. Math. Soc.* **112** (1991), no. 2, 451 - 455.

[12] S.N. Naboko, Conditions for similarity to unitary and selfadjoint operators, *Funct. Anal. Appl.* **18** (1984), No. 1, 13 - 21.

[13] M.A. Naĭmark, Investigation of the spectrum and the expansion in eigenfunctions of a nonself-adjoint operator of the second order on a semi-axis, *Trudy Mosk. Mat. Obsch.* **3** (1954), 181 - 270 (*Russian*); English transl. in: *Amer. Math. Soc. Transl.* (2) **16** (1960), 103 - 193.

[14] J. Lehner and G. Wing, Solution of the linearized Boltzmann equation for the slab geometry, *Duke Math. J.* **23** (1956), 125-142.

[15] M. Reed and B. Simon, "Methods of Modern Mathematical Physics," v. 4, Academic Press, New York, 1978.