Anti-integrability for the Logistic Maps

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Abstract

The embedding of the Bernoulli shift into the logistic map $x \mapsto \mu x(1-x)$ for $\mu > 4$ is reinterpreted by the theory of anti-integrability: it is inherited from the anti-integrable limit $\mu \to \infty$.

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1 Introduction

There is no doubt that one of the most extensively studied dynamical systems is the family of logistic maps on the interval $[0, 1]$:

$$x_{i+1} = f_\mu(x_i) = \mu x_i(1 - x_i), \qquad \mu \geq 0, \tag{1}$$

which, as the parameter μ increases, exhibits chaos via the route of period-doubling bifurcation. See for example [7, 10, 12, 17, 18]. When μ exceeds 4 the map restricted to the invariant set $\Lambda_\mu := \cap_{n=0}^{\infty} f_\mu^{-n}([0, 1])$ is topologically conjugate to the Bernoulli shift with two symbols, e.g. [8, 13, 20], namely the following diagram commutes

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \\ h\big\downarrow & & \big\downarrow h \\ \Lambda_\mu & \xrightarrow{f_\mu} & \Lambda_\mu \end{array}$$

where $\Sigma_2 = \{\mathbf{a}|\ \mathbf{a} = \{a_i\}_{i=0}^{\infty},\ a_i = 0 \text{ or } 1\ \forall i\}$, σ is the Bernoulli shift acting in such a way that $\sigma(\mathbf{a}) = \{a_1, a_2, a_3, \cdots\} = \{\sigma(\mathbf{a})_i\}_{i=0}^{\infty}$ with $\sigma(\mathbf{a})_i = a_{i+1}$, and h, called the conjugacy, is a homeomorphism of the compact set Σ_2 (with the product topology). Partition the interval $[0, 1]$ into two parts $[0, 1/2)$ and $(1/2, 1]$, then an orbit $\mathbf{x} = \{x_i\}_{i=0}^{\infty}$ of $x_0 \in \Lambda_\mu$ under the map f_μ gives a sequence $\mathbf{a}$ of symbols by assigning for each $i \geq 0$ that $a_i = 0$ if $x_i < 1/2$ or $a_i = 1$ if $x_i > 1/2$. The conjugacy indicates there is a one to one correspondence between the orbit with initial point x_0 and the sequence $\mathbf{a}$ of symbols. We call $\mathbf{a} = h^{-1}(x_0)$ the *code* of the orbit determined by x_0. The conjugacy also indicates Λ_μ is a Cantor set. (A simple proof of Λ_μ being a Cantor set, without using h, was given by Zeller and Thaler [22].)

In this note we shall present an approach to the embedding of the Bernoulli shift based on the concept of anti-integrability. Our approach is analytical and has an advantage that it manifests a vivid picture on how the map is conjugate to the shift dynamics when $\mu > 4$. The concept was first utilised in 1990 by Aubry and Abramovici [2] to investigate chaotic orbits in the standard map, and has been extended, for example, by MacKay and Meiss [16] to prove the existence of cantori for symplectic maps of arbitrary dimension, by Easton *et al.* [9] to construct a non-zero measure set of drifting orbits in a symplectic map, and by Bolotin and MacKay [4] as well as Chen [6] to the existence of multi-bump trajectories in time-dependent Lagrangian systems, and to the embedding of symbolic dynamics into scattering billiards [5]. We are going to show the family of logistic maps is also anti-integrable in the sense of Aubry [1], and all its dynamics for $\mu > 4$

are inherited from the anti-integrable limit. The philosophy behind is as follows: when μ becomes larger and larger, the invariant set Λ_μ becomes smaller and smaller, and eventually collapses to two points $x = 0$ and 1 when $\mu \to \infty$. One can think that the dynamics of the system also "collapses" at the singular limit $\mu \to \infty$, and "reduces" [1] to the Bernoulli shift.

In the next section, the anti-integrable limit for the logistic maps and how the dynamics of the family collapses are formalised rigorously, then we give a proof for the embedding of the Bernoulli shift.

2 Anti-integrability

Our results are true for all μ greater than 4, but in order to process without encountering any obstacle and to have self-contained proofs, we only consider the case $\mu > 2 + \sqrt{5}$. The reason is because we require a crucial fact that the non-trivial tangent orbits (or linearised orbits) of the system in question are unbounded and this fact would be particularly easy to be obtained in the considered case. We point out in remark 2.5 that the required fact is equivalent to the one that Λ_μ is a repelling hyperbolic invariant set for f_μ. We recall the definition:

Definition 2.1. *An invariant set Λ is repelling hyperbolic for a C^1-function $f : \mathbb{R} \hookleftarrow$ if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 1$.*

The hyperbolicity, which is a key for proving Λ_μ has empty interior, or proving h is a homeomorphism in the commutative diagram, for μ larger than 4 is not trivial. For $\mu > 2 + \sqrt{5}$ it can be achieved easily because all points in $f_\mu^{-1}([0, 1])$ have absolute values of derivatives strictly greater than one, whereas there are points x with $|f_\mu'(x)| \leq 1$ for $4 < \mu \leq 2 + \sqrt{5}$, so some works are needed to get around this problem. See for example the textbook of Robinson [20], where the Poincaré metric and the Schwarz lemma of complex analysis are employed, or the exposition of Kraft [14], in which an idea of the "return map" (cf. [8, 12]) and the fact that f_μ has negative Schwarzian derivatives are used.

A dynamical systems is, in Aubry's sense [1], at the anti-integrable limit if it becomes non-deterministic and virtually a Bernoulli shift. This picture of the limit is particularly clear in our case.

Theorem 2.2. *The logistic map f_u is anti-integrable at the limit $\mu \to \infty$.*

To see this, let $\epsilon = 1/\mu$, then $\{x_i\}$, $i \geq 0$, is an orbit of (1) if and only if

$$x_i(1 - x_i) = \epsilon x_{i+1}.$$

When $\epsilon = 0$, the above difference equation becomes an algebraic one, and we have $x_i = 0$ or 1 for each i. Since that x_i is equal to 0 or 1 does not determine the value of x_{i+1}, the system at $\epsilon = 0$ (or $\mu = \infty$) is non-deterministic; since for any $\{x_i\} \in \Sigma_2$ we have $x_{i+1} = \sigma(\{x_i\})_i$, the system is virtually a Bernoulli shift with two symbols. Hence we have the theorem.

Any anti-integrable orbit of the logistic map is then precisely a point lying in the space Σ_2. In the sequel, we show that all the anti-integrable orbits persist as long as μ is greater than $2 + \sqrt{5}$.

We re-write the map (1) as another map $F(\cdot, \epsilon)$ in the space $l_\infty := \{\mathbf{x}|\ \mathbf{x} = \{x_0, x_1, x_2, \cdots\}, x_i \in \mathbb{R},\ \text{bounded}\}$ of bounded sequences with the sup norm:

$$\begin{aligned} F : l_\infty \times \mathbb{R} &\rightarrow l_\infty, \\ (\mathbf{x}, \epsilon) &\mapsto F(\mathbf{x}, \epsilon) = \{F_0(\mathbf{x}, \epsilon), F_1(\mathbf{x}, \epsilon), F_2(\mathbf{x}, \epsilon), \cdots\} \end{aligned}$$

with $F_i(\mathbf{x}, \epsilon) = -\epsilon x_{i+1} + x_i(1 - x_i)$. It is readily to see that $\mathbf{x}$ is an orbit of (1) if and only if it solves $F(\mathbf{x}, \epsilon) = \mathbf{0}$. Theorem 2.2 can be rephrased as the following.

Proposition 2.3. *$F(\mathbf{x}^\dagger, 0) = \mathbf{0}$ if and only if $x_i^\dagger = 0$ or 1 for every $i \geq 0$.*

We proceed to show, by employing the implicit function theorem, that any $\mathbf{x}^\dagger$ in the proposition can be continued to ϵ less than $1/(2 + \sqrt{5})$.

Theorem 2.4. *Providing $\epsilon < 1/(2 + \sqrt{5})$, then for any anti-integrable orbit $\mathbf{x}^\dagger$, there corresponds a unique C^1-family of points $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger) = \{x_i^*(\epsilon; \mathbf{x}^\dagger)\}$ in l_∞ parametrised by ϵ such that $\mathbf{x}^*(0; \mathbf{x}^\dagger) = \mathbf{x}^\dagger$ and $F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) = \mathbf{0}$. Conversely, if $\mathbf{x}_\epsilon$ is such a point that $F(\mathbf{x}_\epsilon, \epsilon) = \mathbf{0}$, then there exists a unique $\mathbf{x}^\dagger$ such that $F(\mathbf{x}^\dagger, 0) = \mathbf{0}$ and $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger) = \mathbf{x}_\epsilon$.*

The proof is as follows. F certainly is a C^1-map with its derivative at $\mathbf{x}$ a linear map

$$D_\mathbf{x} F(\mathbf{x}, \epsilon) : l_\infty \hookleftarrow, \quad \xi \mapsto \left\{\Sigma_{j=0}^\infty D_{x_j} F_i(\mathbf{x}, \epsilon) \xi_j\right\}_{i=0}^\infty,$$

or in matrix form

$$D_\mathbf{x} F(\mathbf{x}, \epsilon)\xi = \begin{pmatrix} 1 - 2x_0 & -\epsilon & 0 & 0 & \cdots \\ 0 & 1 - 2x_1 & -\epsilon & 0 & \cdots \\ 0 & 0 & 1 - 2x_2 & -\epsilon & \cdots \\ 0 & 0 & 0 & 1 - 2x_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \end{pmatrix}.$$

Thus it is easy to see $D_{\mathbf{x}}F(\mathbf{x}^{\dagger},0)$ is invertible because it is a diagonal matrix with entries ± 1. So, the theorem is true for ϵ sufficiently small.

When ϵ is not zero, $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$ is invertible if and only if

$$-\epsilon\xi_{i+1} + (1-2x_i)\xi_i = \eta_i \tag{2}$$

possesses a unique bounded solution for any given $\eta = \{\eta_i\}_{i\geq 0} \in l_\infty$. But (2) has a solution

$$\begin{aligned} \xi_i &= \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N} (1-2x_{i+k})^{-1} \right) \eta_{i+N} \\ &= \epsilon^{-1} \sum_{N\geq 0} \left(\left(f_\mu^{N+1}\right)'(x_i) \right)^{-1} \eta_{i+N}, \end{aligned} \tag{3}$$

which is bounded for every $i \geq 0$ because (3) is a geometric series coming from the expanding property that $|(1-2x)\epsilon^{-1}| = |f'_\mu(x)| > 1 + C$ for some positive constant C and for all x belonging to $f_\mu^{-1}([0,1])$ (e.g. [8, 20]). Furthermore, notice that

$$\xi_{i+N} = \xi_i \epsilon^{-N} \prod_{k=0}^{N-1} (1-2x_{i+k}) = (f_\mu^N)'(x_i)\xi_i \quad \forall\, i \geq 0,\ N \geq 1$$

is a homogeneous solution of (2), then by the same property, we see $\{\xi_i\}$ is unbounded unless ξ is identical to $\mathbf{0}$. This means solution (3) is the only bounded solution. Therefore $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$ is invertible and the theorem follows from the implicit function theorem.

Remark 2.5. The above proof reveals the fact that an orbit $\mathbf{x}$ is uniformly hyperbolic if and only if the linearised difference equation $\xi_{i+1} - f'_\mu(x_i)\xi_i = 0$ has only the trivial bounded solution if and only if $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$ is invertible (cf. [15, 19], also [3] for the case of symplectic maps).

Since the set $\Lambda_{1/\epsilon}$ consists of those points in the interval $[0,1]$ whose orbits are bounded, in the light of theorem 2.4, we infer that

$$\Lambda_{1/\epsilon} = \bigcup_{\mathbf{x}^{\dagger}\in\Sigma_2} x_0^*(\epsilon;\mathbf{x}^{\dagger})$$

and that the bijectivity of h in the commutative diagram in the introduction section can be constructed by the composition of mappings $\mathbf{x}^{\dagger} \mapsto \mathbf{x}^*(\epsilon;\mathbf{x}^{\dagger}) \mapsto x_0^*(\epsilon;\mathbf{x}^{\dagger})$. The projection $\mathbf{x}^*(\epsilon;\mathbf{x}^{\dagger}) \mapsto x_0^*(\epsilon;\mathbf{x}^{\dagger})$ certainly is continuous, one can therefore accomplish the embedding simply by showing the continuation $\mathbf{x}^{\dagger} \mapsto \mathbf{x}^*(\epsilon;\mathbf{x}^{\dagger})$ is continuous with the product topology.

Then h is a continuous bijection from the compact set Σ_2 into the interval $[0,1]$ hence is a homeomorphism. To show h is a conjugacy, suppose an initial point x_0 which determines a code $h^{-1}(x_0)$, then we must have $\sigma \circ h^{-1}(x_0) = h^{-1}(x_1) = h^{-1} \circ f_\mu(x_0)$.

Lemma 2.6. *With the product topology, the map $\mathbf{x}^\dagger \mapsto \mathbf{x}^*(\epsilon; \mathbf{x}^\dagger)$ is continuous.*

This lemma is a consequence of the following result.

Proposition 2.7. *Suppose $F(\mathbf{x}, \epsilon) = 0$, $F(\mathbf{y}, \epsilon) = 0$ and suppose the corresponding codes are such that $x_i^\dagger = y_i^\dagger$ for $i \leq N$. Providing $\epsilon < 1/(2+\sqrt{5})$ then there exists $\lambda > 1$ such that $|x_i - y_i| < \lambda^{-(N-i)}|x_N - y_N|$ for $i \leq N$.*

To prove the proposition, notice the condition that the codes of $\mathbf{x}$ and $\mathbf{y}$ agree for $i \leq N$ guarantees two facts that $(f_\mu^{N-i})'(z)$ have the same sign for all z lying between x_i and y_i and that $|(f_\mu^{N-i})'(z)| > (1+C)^{N-i}$ for some $C > 0$. The reason is that each x_i and y_i are continued respectively from $x_i^\dagger$ and $y_i^\dagger$, but these continuations cannot cross the middle point $x = 1/2$. Then by invoking the mean value theorem, we get $|f_\mu^{N-i}(x_i) - f_\mu^{N-i}(y_i)| > (1+C)^{N-i}|x_i - y_i|$ and the proposition follows.

Having the proposition, it is easy to derive the lemma. For instance, we suppose the product topology is induced by the metric, e.g. [21],

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i \geq 0} \frac{|x_i - y_i|}{3^i},$$

then the proposition implies

$$d(\mathbf{x}, \mathbf{y}) < \frac{\lambda^{-N-1} - 3^{-N-1}}{\lambda^{-1} - 3^{-1}} + \sum_{i>N} \frac{1}{3^i}$$

but we already presumed that $1/3^{N+1} \leq d(\mathbf{x}^\dagger, \mathbf{y}^\dagger) \leq 1/(2 \cdot 3^N)$.

Theorem 2.4 indicates that $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger)$ depends continuously on ϵ in l_∞ (with the uniform topology) for every $\mathbf{x}^\dagger$, therefore also depends continuously on ϵ with the product topology since the latter topology is weaker then the former one. Hence we conclude Λ_μ is continuously on μ. We write down what we have proved in this section, by exploiting the anti-integrable limit of the logistic maps.

Theorem 2.8. *When μ varies from $2+\sqrt{5}$ to infinity, Λ_μ form a continuous family of Cantor sets. For a given μ, the restriction of f_μ to Λ_μ is topologically conjugate to the Bernoulli shift with two symbols.*

We finish by the following remarks.

Remark 2.9. The proofs of theorem 2.4 and proposition 2.7 rely on the property that $|f'_\mu(x)| > 1$ for all $x \in f_\mu^{-1}([0,1])$ and $\mu > 2+\sqrt{5}$. The case $4 < \mu \leq 2+\sqrt{5}$ is subtler because we need a property that the absolute values of the derivative of f_μ being less than or equal to one do not happen too often. By virtue of the compactness of Λ_μ, it would be sufficient to obtain the desired property if we can show that for every $x \in \Lambda_\mu$ there is an integer $N_x \geq 1$ such that $|(f_\mu^{N_x})'(x)| > 1$. This is because it will then imply that Λ_μ is hyperbolic and vice versa [11, 14, 18].

Remark 2.10. The presented approach can be easily extended to the embedding of the Bernoulli shift with $K+1$ symbols into K-modal maps such as $f_\nu(x) = \nu x(a_1 - x)(a_2 - x)\cdots(a_{K-1} - x)(1-x)$ for sufficiently large ν and $0 < a_1 < a_2 < \cdots < a_{K-1} < 1$.

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