Group Algebras for Groups which are not Locally Compact.

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Abstract *We generalise the definition of a group algebra so that it makes sense for non–locally compact topological groups, in particular, we require that the representation theory of the group algebra is isomorphic (in the sense of Gelfand–Raikov) to the continuous representation theory of the group, or to some other important subset of representations. We prove that a group algebra if it exists, is always unique up to isomorphism. From examples, group algebras do not always exist for non–locally compact groups, but they do exist for some. We define a convolution on the dual of the Fourier–Stieltjes algebra making it into a Banach *-algebra, we prove that a group algebra if it exists, can always be embedded in this convolution algebra, and we find sufficient conditions for a subalgebra to be a group algebra. When the group is locally compact, we obtain a new characterisation of its group algebra which does not involve the Haar measure, nor behaviour of measures on compact sets.*

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Introduction.

The Gelfand–Raikov theorem [GR] proved that the continuous (unitary) representation theory of any locally compact group, is isomorphic in a natural sense to the (nondegenerate Hilbert space) representation theory of a C*–algebra. The proof is constructive, in that it explicitly constructs the group algebra as the enveloping C*-algebra of the convolution algebra $L^1(G)$, and faithfully embeds the group as unitaries in the multiplier algebra of the group algebra. Subsequently group algebras for locally compact groups have been generalised in many directions (e.g. twisted group algebras, groupoid group algebras, some semigroup group algebras and cross–products of a C*-algebra by a group action), and has been a central component of harmonic analysis. The question naturally arises as to whether the Gelfand–Raikov theorem can be extended to topological groups which are NOT locally compact, and this problem will be at the focus of our investigation here. This question has attained some urgency, due to the fact that such groups regularly and naturally arise in physics and mathematics (e.g. gauge groups, diffeomorphism groups, symplectomorphism groups, Banach Lie groups, inductive limit groups, Fréchet-Lie groups etc.) and a substantial body of work is currently developing on the continuous representation theory of these groups. Group algebras are very useful to have, e.g. they allow one to use the topological analysis of the spectrum of C*–algebras to analyze group representations, use direct integral decomposition theory for representations of (separable) C*–algebras to decompose continuous group representations into other continuous group representations, and use Rieffel induction to induce group representations from subgroups to larger groups.

In its full generality, the question has a negative answer, i.e. it is not true for all topological groups that their continuous representation theory is isomorphic (in the sense of Gelfand–Raikov) to the representation theory of a C*–algebra. For instance, there are Abelian groups with NO nontrivial continuous unitary representations, cf. Banaszczyk [Ban], and there are Abelian groups with continuous representations, but no irreducible ones cf. Example 5.2 in Pestov [Pes]. However, we can still ask the question of whether some useful subset of representations of a topological group is isomorphic to the representation theory of a C*–algebra. Of course one would also like to characterize those topological groups for which the Gelfand–Raikov theorem holds, i.e. for which their continuous unitary representation theory is isomorphic to the representation theory of a C*–algebra.

To make the discussion more precise, let us fix notation. For a C*-algebra $\mathcal{A}$ denote its set of nondegenerate Hilbert space representations by $\mathrm{Rep}\,\mathcal{A}$. For a topological group G with a fixed nondiscrete Hausdorff topology, let $\sigma : G\times G \to \mathbb{T}$ be a 2–cocycle $\sigma \in Z^2(G,\,\mathbb{T})$ which is jointly continuous and normalised, i.e. $1 = \sigma(x,\,e) = \sigma(e,\,x) = \sigma(x,\,x^{-1})$ and $\overline{\sigma(x,\,y)} = \sigma(y^{-1},\,x^{-1})$ for all $x,\,y \in G$. Let $\mathrm{Rep}_\sigma G$ denote the set of strong operator continuous unitary σ–representations on Hilbert space, and in the case $\sigma = 1$, we simplify the notation to $\mathrm{Rep}\,G$. Let G_d denote G with its discrete topology. Clearly $\mathrm{Rep}_\sigma G_d\big(\supseteq \mathrm{Rep}_\sigma G\big)$ is the set of *all* unitary σ–representations of G, not necessarily continuous, and this is isomorphic to to $\mathrm{Rep}\,\mathrm{C}^*_\sigma(G_d)$ where $\mathrm{C}^*_\sigma(G_d)$ is the group algebra of G_d.

Structurally, continuous group representation theory is quite similar to C*-algebra representation theory, but there are also differences. Note first that both of $\mathrm{Rep}\,G$ and $\mathrm{Rep}\,\mathcal{A}$ are:

- closed w.r.t. composition with (continuous) homomorphisms,
- closed with respect to direct sums of representations,
- closed with respect to unitary conjugation,
- closed with respect to subrepresentations, i.e. if for a representation π on $\mathcal{H}_\pi$ there is a closed invariant subspace $\mathcal{K} \subset \mathcal{H}_\pi$, then $\pi(G)$ (resp. $\pi(\mathcal{A})$) restricted to $\mathcal{K}$ produces a representation in $\mathrm{Rep}\,G$ (resp. $\mathrm{Rep}\,\mathcal{A}$).

As for the differences, note that $\mathcal{A}$ is separated by its irreducible representations, but the continuous irreducible representations need not separate G. Secondly, under tensor products $\mathrm{Rep}\,G$ is closed (or more generally, $\mathrm{Rep}_\sigma G \otimes \mathrm{Rep}_\rho G \subseteq \mathrm{Rep}_{\sigma\rho} G$) but the tensor product of two representations of $\mathcal{A}$ is a representation of $\mathcal{A}\otimes\mathcal{A}$, not of $\mathcal{A}$. Our concept of isomorphism between $\mathrm{Rep}_\sigma G$ and $\mathrm{Rep}\,\mathcal{A}$ is:

Def. Let G be a topological group, σ as above, and let $\mathcal{R} \subset \mathrm{Rep}_\sigma G_d$ be a given subset of unitary σ–representations of G. Then a σ–**group algebra** for the pair $(G, \mathcal{R})$ is a C*–algebra $\mathcal{L}$ and a σ–homomorphism $\varphi : G \to UM(\mathcal{L})$ such that the unique extension map $\theta : \mathrm{Rep}\mathcal{L} \to \mathrm{Rep}_\sigma G_d$ is injective, and with image $\theta\big(\mathrm{Rep}\,\mathcal{L}\big) = \mathcal{R}$.

In this case we say that $\mathcal{R}$ is **isomorphic** to $\mathrm{Rep}\,\mathcal{L}$.

Remark. (1) The map φ maps to the unitaries in the multiplier algebra of $\mathcal{L}$ and satisfies $\varphi(g)\varphi(h) = \sigma(g,h)\varphi(gh)$. Moreover, any nondegenerate representation of $\mathcal{L}$ has a unique extension to its multiplier algebra $M(\mathcal{L})$, and this defines the map $\theta : \mathrm{Rep}\mathcal{L} \to \mathrm{Rep}_\sigma G_d$ by $\theta(\pi)(g) := \operatorname*{s\text{–}lim}_{\alpha\to\infty} \pi\big(\varphi(g)E_\alpha\big)$ where $\{E_\alpha\} \subset \mathcal{L}$ is any approximate identity of $\mathcal{L}$. Note that we may have that φ is not injective, e.g. in the case when $\mathcal{R}$ does not separate G.

(2) When G is locally compact, the usual σ–group algebra will be for the case that $\mathcal{R} = \mathrm{Rep}_\sigma G$, and then $\mathcal{L} = \mathrm{C}^*_\sigma(G)$ satisfies the conditions of the definition, where φ is injective. Below we will prove uniqueness. This generalisation of group algebras seem useful even for locally compact groups, because it allows the analysis of representation sets other than $\mathrm{Rep}_\sigma G$.

(3) For a small class of non-locally compact groups, σ–group algebras were constructed for $\mathcal{R} = \mathrm{Rep}_\sigma G$ in [Gr1]. The existence question for group algebras was studied in a more general context in [Gr2].

(4) Note that the map θ preserves direct sums, unitary conjugation, subrepresentations, and (as we will see) irreducibility, so that this notion of isomorphism between $\mathcal{R}$ and $\mathrm{Rep}\,\mathcal{L}$ involves strong structural correspondences, and restricts the class of sets $\mathcal{R}$ for which group algebras exist. However, this isomorphism is obviously not an equivalence relation, since it relates objects in two distinct sets. In the case that $\theta : \mathrm{Rep}\mathcal{L} \to \mathcal{R}$ is surjective but not injective, it is natural to say that $\mathrm{Rep}\,\mathcal{L}$ is *homomorphic* to $\mathcal{R}$, since θ still transfers some structure to $\mathcal{R}$ (though e.g. irreducibility of representations is lost). We will not examine this concept here.

In the rest of this paper, we will develop the concept of group algebras for groups which are not locally compact. Below, in Sect. 1 we consider general theory and prove that group algebras, if they exist, are unique up to isomorphism. In the subsequent sections, we will be concerned with the existence question when $\mathcal{R}$ is a subset of the continuous representations $\text{Rep}_\sigma G$, and in particular we will analyze an important convolution algebra, and develop sufficient conditions to ensure the existence of a group algebra.

1. General structures for group algebras.

Let $\mathcal{L}$ be a C*-algebra, and recall that the **strict topology** of its multiplier algebra $M(\mathcal{L})$ is given by the family of seminorms on $M(\mathcal{L})$:

$$B \to \|BA\| + \|AB\|, \quad A \in \mathcal{L},\ B \in M(\mathcal{L})\ .$$

Then $\mathcal{L}$ is strictly dense in $M(\mathcal{L})$, cf. Prop. 3.5 and 3.6 in [Bus]. Below $\text{Span}\, X$ will denote the space of finite linear combinations of X.

Proposition 1.1. *Let G, σ as above, let $\mathcal{R} \subset \text{Rep}_\sigma G_d$ be given, and let $\mathcal{L}$ be a σ–group algebra for $(G, \mathcal{R})$. Then*

(1) $\text{Span}\,\varphi(G)$ *is a strictly dense *–algebra in* $M(\mathcal{L})$,

(2) *Each* $\pi \in \text{Rep}\,\mathcal{L}$ *is strict–strong operator continuous, and* $\theta(\pi)$ *is the strict extension of* π *to* $\varphi(G)$.

(3) *Each* $\pi \in \mathcal{R}$ *has a unique extension to* $\text{Span}\,\varphi(G)$ *which is strict–strong operator continuous, and conversely* $\mathcal{R}$ *is exactly the restrictions to* $\varphi(G)$ *of the set of strict–strong operator continuous representations of* $\text{Span}\,\varphi(G)$.

(4) *The inverse map of the bijection* θ, *is the map* $\theta^{-1} : \mathcal{R} \to \text{Rep}\,\mathcal{L}$ *obtained by* $\theta^{-1}(\pi)(A) := \text{s–}\lim\limits_\alpha \widetilde{\pi}(B_\alpha)$ *where* $\widetilde{\pi}$ *is the unique strictly–strong operator continuous extension in (3), and* $\{B_\alpha\} \subset \text{Span}\,\varphi(G)$ *is a net strictly converging to* $A \in \mathcal{L}$.

(5) *If* G *is Abelian and* $\sigma = 1$, *then* $\mathcal{L}$ *is commutative.*

Proof: (1) That $\text{Span}\,\varphi(G)$ is a *–algebra is obvious from the fact that φ is a σ–homomorphism. Let $\mathcal{Q}$ be the strict closure of $\text{Span}\,\varphi(G)$. This is a *-algebra, so since $\varphi(G)$ separates $\mathcal{R} = \theta(\text{Rep}\,\mathcal{L})$, it follows that $\mathcal{Q}$ separates $\text{Rep}\,\mathcal{L}$. Thus by Prop. 2.2 in [Wor], we have that $\mathcal{Q} = M(\mathcal{L})$.

(2) Let $\pi \in \operatorname{Rep}\mathcal{L}$, which is a *–homomorphism $\pi : \mathcal{L} \to \pi(\mathcal{L}) =: \mathcal{C} \subset \mathcal{B}(\mathcal{H}_\pi)$, and by Prop. 3.8 and 3.9 in [Bus], this extends uniquely to a *–homomorphism $\pi : M(\mathcal{L}) \to M(\mathcal{C}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$ which is strict–strict continuous (using nondegeneracy of π). Since on $M(\mathcal{C}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$ the strong operator topology is coarser than the strict topology, it follows that $\pi : M(\mathcal{L}) \to \mathcal{B}(\mathcal{H}_\pi)$ is strict–strong operator continuous. If $\{E_\alpha\} \subset \mathcal{L}$ is an approximate identity of $\mathcal{L}$, then for each $B \in M(\mathcal{L})$ the net $\{BE_\alpha\}$ strictly converges to B, hence $\pi(BE_\alpha)$ converges in strong operator topology to $\pi(B)$, and by definition this is $\theta(\pi)(g)$ when $B = \varphi(g)$.

(3) By the bijection $\theta : \operatorname{Rep}\mathcal{L} \to \mathcal{R}$, for each $\pi \in \mathcal{R}$ there is a $\rho \in \operatorname{Rep}\mathcal{L}$ such that its strict extension to $M(\mathcal{L})$ produces $\pi \in \mathcal{R}$ by (2). Hence each $\pi \in \mathcal{R}$ has a strictly continuous extension $\widetilde{\pi} = \rho{\restriction}\operatorname{Span}\varphi(G)$ to $\operatorname{Span}\varphi(G)$. If $\widehat{\pi}$ is another strictly continuous extension of π to $\operatorname{Span}\varphi(G)$, then since $\operatorname{Span}\varphi(G)$ is strictly dense in $M(\mathcal{L})$, it extends uniquely to $\mathcal{L}$, so by definition we get $\theta(\widetilde{\pi}) = \pi = \theta(\widehat{\pi})$. Since θ is injective, we have that $\widetilde{\pi}{\restriction}\mathcal{L} = \widehat{\pi}{\restriction}\mathcal{L}$ and as $\mathcal{L}$ is strictly dense we have that $\widehat{\pi} = \widetilde{\pi}$. Conversely, if π is a (bounded) *-representation of $\operatorname{Span}\varphi(G)$ which is strictly continuous, then it extends uniquely to $M(\mathcal{L})$, in which case $\theta\big(\pi{\restriction}\mathcal{L}\big) = \pi{\restriction}\varphi(G) \in \mathcal{R}$.

(4) This is clear from the previous parts.

(5) The strict topology on $M(\mathcal{L}) \subset \mathcal{L}''$ is finer than the weak operator topology of $\mathcal{L}''$ on $M(\mathcal{L})$. Thus the strict closure of $\operatorname{Span}\varphi(G)$ (i.e. $M(\mathcal{L})$) is contained in its weak operator closure, and this is the double commutant $\operatorname{Span}\varphi(G)''$ since $\varphi(G)$ contains the identity. Now if G is Abelian and $\sigma = 1$ we have that $\operatorname{Span}\varphi(G)$ is commutative, hence $\operatorname{Span}\varphi(G)'' \supset \mathcal{L}$ is commutative. ∎

Since $\operatorname{Span}\varphi(G)$ and $\mathcal{L}$ are both strictly dense in $M(\mathcal{L})$, and the strictly continuous representations on $M(\mathcal{L})$ are the extensions of representations in $\operatorname{Rep}\mathcal{L}$, it follows that all the properties of these representations are determined by their restrictions to either $\varphi(G)$ or $\mathcal{L}$. Below we will use this to take sufficient analytical structure from $\mathrm{Rep}\,\mathcal{L}$ to $\mathcal{R}$, so that we can characterize $\mathcal{L}$ directly on $\mathcal{R}$ in our uniqueness theorem.

Let $\mathcal{L}$ be a C*-algebra, then we topologize $\mathrm{Rep}\,\mathcal{L}$ according to Takesaki and Bichteler [Bic, Tak]. Let $\pi_{\mathcal{L}}$ be the universal representation of $\mathcal{L}$ on the universal Hilbert space $\mathcal{H}$. Then we can identify $\mathrm{Rep}\,\mathcal{L}$ with the set of all (including degenerate) representations of $\mathcal{L}$ on $\mathcal{H}$, and we denote this by $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$. Then we equip $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ with the pointwise strong operator topology, i.e. a net $\{\pi_\nu\} \subset \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ converges to $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ iff $\pi_\nu(A) \to \pi(A)$ in the strong operator topology of $\mathcal{B}(\mathcal{H})$ for all $A \in \mathcal{L}$.

From Prop. 1.1 we obtain:

Corollary 1.2. *Let $\mathcal{L}$ be a σ–group algebra for $(G,\,\mathcal{R})$. Then $\pi \in \mathrm{Rep}\,\mathcal{L}$ is cyclic (resp. irreducible) iff $\theta(\pi) \in \mathcal{R}$ is cyclic (resp. irreducible).*

From this fact we get that

$$\theta(\pi_{\mathcal{L}}) = \oplus\left\{\,\pi \in \mathcal{R} \;\middle|\; \pi \text{ has a cyclic vector } \Omega \text{ with } \|\Omega\| = 1\,\right\}$$

since $\pi_{\mathcal{L}}$ is the direct sum of GNS–representations of the states of $\mathcal{L}$. This characterises $\theta(\pi_{\mathcal{L}})$ directly on G, so we can consider $\mathcal{R} \subset \mathrm{Rep}_\sigma(G_d,\,\mathcal{H}) \equiv$ σ–representations of G on $\mathcal{H}$ which are unitary on their essential subspaces, and hence we topologise $\mathcal{R}$ also with the pointwise (on G) strong operator topology. From Prop. 1.1 we also get:

Corollary 1.3. *Let $\mathcal{L}$ be a σ–group algebra for $(G,\,\mathcal{R})$, and $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ as above. Denote the essential subspace of $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ by $\mathcal{H}_\pi$, with essential projection $P_\pi : \mathcal{H} \to \mathcal{H}_\pi$.*

(1) *If $\pi,\,\pi' \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ satisfy $\mathcal{H}_\pi \perp \mathcal{H}_{\pi'}$, then $\theta(\pi \oplus \pi') = \theta(\pi) \oplus \theta(\pi')$. Conversely, if $\pi,\,\pi' \in \mathcal{R} \subset \mathrm{Rep}_\sigma(G_d,\,\mathcal{H})$ with $\mathcal{H}_\pi \perp \mathcal{H}_{\pi'}$, then $\theta^{-1}(\pi \oplus \pi') = \theta^{-1}(\pi) \oplus \theta^{-1}(\pi')$.*

(2) *The essential projections of $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ and $\theta(\pi) \in \mathcal{R} \subset \mathrm{Rep}_\sigma(G_d,\,\mathcal{H})$ are the same*

(3) *for $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ let $U \in \mathcal{B}(\mathcal{H})$ be any partial isometry with initial projection $U^*U \geq P_\pi$, and define $\pi^U(A) :=$*

$U\pi(A)U^*$, $A \in \mathcal{L}$. *Then* $\theta(\pi^U) = U\theta(\pi)U^* =: \theta(\pi)^U$, *and conversely* $\theta^{-1}(\pi^U) = \theta^{-1}(\pi)^U$ *for all* $\pi \in \mathcal{R}$.

Thus by these two corollaries, θ preserves much of the structure of $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$. In fact, it also preserves the topology:

Proposition 1.4. *Let* $\mathcal{L}$ *be a* σ*–group algebra for* $(G,\,\mathcal{R})$, *then*

$\theta : \mathrm{Rep}\,(\mathcal{L},\mathcal{H}) \to \mathcal{R}$ *is a homeomorphism.*

Proof: From Prop. 1.1, it suffices to show that for the strict extensions of $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ to $M(\mathcal{L})$, for a net $\{\pi_\nu\}$ we have the convergence $\pi_\nu(A) \to \pi(A)$ for all $A \in \mathcal{L}$ in strong operator topology iff $\pi_\nu(B) \to \pi(B)$ for all $B \in \varphi(G)$ in strong operator topology.

Assume that π and $\{\pi_\nu\}$ are strict–strong operator continuous representations in $\mathrm{Rep}\,(M(\mathcal{L}),\mathcal{H})$ such that $\pi_\nu(A) \to \pi(A)$ for all $A \in \mathcal{L}$ in strong operator topology. For any $B \in \varphi(G)$, let $\{A_\alpha\} \subset \mathcal{L}$ be a net strictly converging to B. Then for all $\psi \in \mathcal{H}$ we have:

$$\begin{aligned}\left\|(\pi_\nu(B) - \pi(B))\psi\right\| \leq \left\|\pi_\nu(B - A_\alpha)\psi\right\| + \left\|(\pi_\nu(A_\alpha) - \pi(A_\alpha))\psi\right\| \\ + \left\|\pi(A_\alpha - B)\psi\right\| . \qquad -(1)\end{aligned}$$

Since $A_\alpha \to B$ strictly, we have for the extension of the universal representation $\pi_{\mathcal{L}}$ to $M(\mathcal{L})$ that $\left\|\pi_{\mathcal{L}}(B - A_\alpha)\psi\right\| \to 0$ for all $\psi \in \mathcal{H}$. Since π and π_ν are subrepresentations of $\pi_{\mathcal{L}}$, we also have that

$$\left\|\pi_\nu(B - A_\alpha)\psi\right\| \leq \left\|\pi_{\mathcal{L}}(B - A_\alpha)\psi\right\| \geq \left\|\pi(B - A_\alpha)\psi\right\| .$$

Thus for each $\varepsilon > 0$ there is an α_1 such that for all ν

$$\left\|\pi_\nu(B - A_\alpha)\psi\right\| + \left\|\pi(B - A_\alpha)\psi\right\| \leq \varepsilon \quad \forall\, \alpha > \alpha_1 .$$

Thus from (1) we get for all $\alpha > \alpha_1$ that

$$\begin{aligned}\lim_{\nu\to\infty} \left\|(\pi_\nu(B) - \pi(B))\psi\right\| &\leq \varepsilon + \lim_{\nu\to\infty} \left\|(\pi_\nu(A_\alpha) - \pi(A_\alpha))\psi\right\| \\ &= \varepsilon .\end{aligned}$$

So, since $\varepsilon > 0$ is arbitrary, we have for all $\psi \in \mathcal{H}$ that $\lim\limits_{\nu\to\infty} \left\|(\pi_\nu(B) - \pi(B))\psi\right\| = 0$.

Conversely, assume that for π and $\{\pi_\nu\}$ strict–strong operator continuous representations in $\mathrm{Rep}\,(M(\mathcal{L}),\mathcal{H})$, that $\pi_\nu(B)\to\pi(B)$ for all $B\in\varphi(G)$ in strong operator topology. By triangle inequalities, we then get that $\pi_\nu(B)\to\pi(B)$ for all $B\in\mathrm{Span}\,\varphi(G)$ in strong operator topology. For any $A\in\mathcal{L}$, let $\{B_\alpha\}\subset\mathrm{Span}\,\varphi(G)$ be a strictly convergent net to $A\in\mathcal{L}$. As above, we get that for any $\varepsilon>0$ and $\psi\in\mathcal{H}$ there is an α_1 such that for all ν

$$\big\|\pi_\nu(A-B_\alpha)\psi\big\|+\big\|\pi(A-B_\alpha)\psi\big\|\leq\varepsilon\quad\forall\,\alpha>\alpha_1\,.$$

Thus:
$$\begin{aligned}\big\|(\pi_\nu(A)-\pi(A))\psi\big\| &\leq \big\|\pi_\nu(A-B_\alpha)\psi\big\|+\big\|(\pi_\nu(B_\alpha)-\pi(B_\alpha))\psi\big\| \\ &\quad+\big\|\pi(B_\alpha-A)\psi\big\| \\ &\leq\varepsilon+\big\|(\pi_\nu(B_\alpha)-\pi(B_\alpha))\psi\big\|\end{aligned}$$

for all $\alpha>\alpha_1$. Take the limit $\nu\to\infty$ on both sides, and use the fact that $\varepsilon>0$ is arbitrary to find that $\lim\limits_{\nu\to\infty}\big\|(\pi_\nu(A)-\pi(A))\psi\big\|=0$. ∎

Following Takesaki and Bichteler [Bic, Tak], we define:

Def. An **admissible operator field** on $\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ (resp. $\mathcal{R}$) is a map $T:\mathrm{Rep}\,(\mathcal{L},\mathcal{H})\to\mathcal{B}(\mathcal{H})$ (resp. $T:\mathcal{R}\to\mathcal{B}(\mathcal{H})$) such that:

(i) $\|T\|=\sup\big\{\,\|T(\pi)\|\;\big|\;\pi\in\mathrm{Rep}\,(\mathcal{L},\mathcal{H})\,\big\}<\infty$ (resp. $\|T\|=\sup\big\{\,\|T(\pi)\|\;\big|\;\pi\in\mathcal{R}\,\big\}<\infty$),

(ii) $T(\pi)=P_\pi T(\pi)=T(\pi)P_\pi$ for all $\pi\in\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ (resp. $\pi\in\mathcal{R}$) where P_π denotes the essential projection of π,

(iii) $T(\pi\oplus\pi')=T(\pi)\oplus T(\pi')$ whenever $\mathcal{H}_\pi\perp\mathcal{H}_{\pi'}$ in $\mathcal{H}$,

(iv) $T(\pi^U)=UT(\pi)U^*$ for all $\pi\in\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ (resp. $\pi\in\mathcal{R}$) where $U\in\mathcal{B}(\mathcal{H})$ is a partial isometry with $U^*U>P_\pi$.

The set of admissible operator fields form a C*–algebra under pointwise operations and the sup–norm, and we denote the two resultant C*-algebras by $\mathcal{A}(\mathcal{L},\mathcal{H})$ and $\mathcal{A}(\mathcal{R},\mathcal{H})$ respectively. In particular $\mathcal{A}(\mathcal{L},\mathcal{H})$ contains the C*-algebra

$$\widetilde{\mathcal{L}}:=\big\{\,T_A:\mathrm{Rep}\,(\mathcal{L},\mathcal{H})\to\mathcal{B}(\mathcal{H}),\;A\in\mathcal{L}\;\big|\;T_A(\pi):=\pi(A)\quad\forall\,\pi\in\mathrm{Rep}\,(\mathcal{L},\mathcal{H})\,\big\}$$

which is obviously isomorphic to $\mathcal{L}$. Then we have the Takesaki–Bichteler duality theorem:

$$\widetilde{\mathcal{L}}\cong\big\{\,T\in\mathcal{A}(\mathcal{L},\mathcal{H})\;\big|\;T\quad\text{is strong–operator continuous}\,\big\}$$

where $\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ has the defined topology. That is, $\mathcal{L}$ is isomorphic to the algebra of continuous admissible operator fields on $\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$. Using this, it is now easy to prove:

Theorem 1.5. *Let G, σ be as above and let $\mathcal{R}\subset\mathrm{Rep}_\sigma G_d$ be given. If $(G,\mathcal{R})$ has a σ–group algebra $\mathcal{L}$, then up to isomorphism it is unique.*

Proof: Define a map $\widetilde{\theta}:\mathcal{A}(\mathcal{L},\mathcal{H})\to\mathcal{A}(\mathcal{R},\mathcal{H})$ by $\widetilde{\theta}(T):=T\circ\theta^{-1}$. That $\widetilde{\theta}$ takes admissible operator fields to admissible operator fields follows from Corollary 1.3. Since θ is bijective, $\widetilde{\theta}$ is a *-isomorphism of C*-algebras. Since θ is a homeomorphism, it maps the strong operator continuous fields to the strong operator continuous fields on $\mathcal{R}$, i.e.

$$\widetilde{\theta}(\widetilde{\mathcal{L}})=\big\{\,T\in\mathcal{A}(\mathcal{R},\mathcal{H})\;\big|\;T:\mathcal{R}\to\mathcal{B}(\mathcal{H})\quad\text{is strong–operator continuous}\,\big\}\,.$$

But now since we have defined $\mathcal{L}\cong\widetilde{\theta}(\widetilde{\mathcal{L}})$ intrinsically on G, i.e. involving only G and $\mathcal{R}$, it follows that all σ–group algebras $\mathcal{L}$ are isomorphic. ∎

Remark. (1) This uniqueness theorem for group algebras generalises previous uniqueness theorems for (twisted) group algebras of locally compact groups e.g. the one by Packer and Raeburn [PR].

(2) Note that the proof above provides a method for constructing a group algebra, i.e. *if we know* that $(G,\mathcal{R})$ has a σ–group algebra, then we can construct it as the set of (strong operator) continuous admissible operator fields on $\mathcal{R}$. Then we obtain a σ–embedding of G in the multiplier algebra of this algebra through pointwise multiplication of the operator fields $T_g(\pi):=\pi(g)$, $\pi\in\mathcal{R}$, $g\in G$.

(3) One can conjecture an existence theorem; e.g. if $\mathcal{R}\subset\mathrm{Rep}_{\,\sigma}(G_d,\mathcal{H})$ is closed under direct sums, subrepresentations, and the equivalence in Cor. 1.3(3), and if $\mathcal{R}$ has “enough” irreducible representations (e.g. each $\pi\in\mathcal{R}$ can be written as a direct integral of irreducible representations in $\mathcal{R}$) then the set of (strong operator) continuous admissible operator fields on $\mathcal{R}$ is a σ–group algebra $\mathcal{L}$ for $(G,\mathcal{R})$. An encouraging fact for the proof of this, is that the embedding $\varphi:G\to M(\mathcal{L})$

is easy to obtain because the operator fields T_g preserve the continuous admissible fields under multiplication.

The difficult part of the problem is of course the existence question for group algebras, in particular to characterize those topological groups for which $\mathcal{R} = \mathrm{Rep}_\sigma G$ has a σ–group algebra. It is also of interest to find subsets $\mathcal{R} \subset \mathrm{Rep}_\sigma G_d$ for which $(G, \mathcal{R})$ has a σ–group algebra. Apart from the known case $(G, \mathrm{Rep}_\sigma G)$ with G locally compact, here is a class of easy examples of such pairs.

Exmp. Let G be a nonabelian topological group, and let π be an irreducible representation of G on a Hilbert space $\mathcal{H}_\pi$ of dimension higher than one. So $\pi : G \to UM(\mathcal{K}(\mathcal{H}_\pi))$ since $\mathcal{B}(\mathcal{H}_\pi) = M(\mathcal{K}(\mathcal{H}_\pi))$ where $\mathcal{K}(\mathcal{H}_\pi)$ denotes the compact operators on $\mathcal{H}_\pi$. Recall that the strict topology of $M(\mathcal{K}(\mathcal{H}_\pi))$ coincides with the strong operator topology. Thus the *–algebra $\mathrm{Span}\,\pi(G)$ is strictly dense (by irreducibility) in $M(\mathcal{K}(\mathcal{H}_\pi))$. Let $\mathcal{R}_\pi := \theta(\mathrm{Rep}\,\mathcal{K}(\mathcal{H}_\pi)) \subset \mathrm{Rep}\,G$, then it is obvious that $\mathcal{K}(\mathcal{H}_\pi)$ is a group algebra for $(G, \mathcal{R}_\pi)$, and that $\mathcal{R}_\pi$ is isomorphic to the set of normal representations of $\mathcal{B}(\mathcal{H}_\pi)$. In the case that π is (strong operator) continuous, all the elements of $\mathcal{R}_\pi$ will also be continuous because they are restrictions of strictly continuous representations of $M(\mathcal{K}(\mathcal{H}_\pi))$, and $\pi = \varphi$.

Whilst group algebras may not exist for a given pair $(G, \mathcal{R})$, unitary embeddings into multiplier algebras $\varphi : G \to UM(\mathcal{L})$ are not hard to find, as the previous example and remark (3) demonstrate. Given such an embedding, here is a construction by which one can obtain related group algebras. Let $\mathcal{N}$ be the strict closure of $\mathrm{Span}\,\varphi(G)$ in $M(\mathcal{L})$. Define $\mathcal{S}$ to be all $N \in \mathcal{N}$ such that $\big\|(B_\lambda - B)N\big\| + \big\|N(B_\lambda - B)\big\| \to 0$ for all strictly convergent nets $\{B_\lambda\} \subset \mathrm{Span}\,\varphi(G)$, $B_\lambda \to B \in \mathcal{N}$. Clearly $\mathcal{L} \cap \mathcal{N} \subset \mathcal{S}$. Then $\mathcal{S}$ is a C*-algebra, and since products of strictly convergent nets with fixed elements of $M(\mathcal{L})$ are strictly convergent, by the definition we have that

$$\varphi(G)\mathcal{S} \subset \mathcal{S} \supset \mathcal{S}\varphi(G)\ .$$

If $S \in \mathcal{S}$ then by definition there is a net $\{B_\lambda\} \subset \mathrm{Span}\,\varphi(G)$ such that $\big\|(B_\lambda - S)N\big\| + \big\|N(B_\lambda - S)\big\| \to 0$ for all $N \in \mathcal{S}$, hence there is a homomorphism $\psi : \mathrm{Span}\,\varphi(G) \to M(\mathcal{S})$ such that $\psi\big(\mathrm{Span}\,\varphi(G)\big)$ is strictly dense in $M(\mathcal{S})$. Let $\widetilde{\theta}$ denote the extension map of $\mathrm{Rep}\,\mathcal{S}$ from $\mathcal{S}$ to $\varphi(G)$, and $\widetilde{\theta}(\mathrm{Rep}\,\mathcal{S}) =: \widetilde{\mathcal{R}} \subset \mathrm{Rep}_\sigma G_d$. Then $\mathcal{S}$ is a σ–group algebra for the pair $(G, \widetilde{\mathcal{R}})$.

Group algebras do not behave naturally w.r.t. containment, i.e. if $\mathcal{L}_i$ is a σ–group algebra for $(G, \mathcal{R}_i)$, $i = 1, 2$ where $\mathcal{R}_1 \subset \mathcal{R}_2$, then it does not always follow that $\mathcal{L}_1 \subset \mathcal{L}_2$ with $\varphi_2(G)\restriction\mathcal{L}_1 = \varphi_1(G)$. This is because:

Proposition 1.6. *Let $\mathcal{L}_i$ be a σ–group algebra for $(G, \mathcal{R}_i)$, $i = 1, 2$ such that $\mathcal{L}_1 \subset \mathcal{L}_2$, and such that $\varphi_1(g)A = \varphi_2(g)A$ for all $g \in G$, $A \in \mathcal{L}_1$. Then $\mathcal{L}_1$ is a closed two-sided ideal of $\mathcal{L}_2$, and hence $\mathrm{Rep}\,\mathcal{L}_2 = \mathrm{Rep}\,\mathcal{L}_1 \oplus \mathrm{Rep}\,\big(\mathcal{L}_2/\mathcal{L}_1\big)$ where $\mathrm{Rep}\,\mathcal{L}_1$ is identified in $\mathrm{Rep}\,\mathcal{L}_2$ by unique extensions, and $\mathrm{Rep}\,\big(\mathcal{L}_2/\mathcal{L}_1\big)$ corresponds to those representations which vanish on $\mathcal{L}_1$.*

Proof: Recall that $M(\mathcal{L}_1) \subset \mathcal{L}_1'' \subset \mathcal{L}_2'' \supset M(\mathcal{L}_2)$. Recall that $\mathrm{Span}\,\varphi_i(G)$ is $\mathcal{L}_i$–strictly dense in $M(\mathcal{L}_i)$. Since the actions of both $\varphi_i(G)$, $i = 1, 2$ coincide on $\mathcal{L}_1$, it follows that $\mathcal{A} := \mathrm{Span}\,\varphi_2(G)$ is $\mathcal{L}_i$–strictly dense in $M(\mathcal{L}_i)$, $i = 1, 2$ by Prop. 1.1. Since $\mathcal{L}_1 \subset \mathcal{L}_2$ it now follows from the definition of strict topologies that the $\mathcal{L}_1$–strict closure of $\mathcal{A}$ contains the $\mathcal{L}_2$–strict closure of $\mathcal{A}$. Thus $M(\mathcal{L}_1) \supseteq M(\mathcal{L}_2) \supset \mathcal{L}_2$, and hence $\mathcal{L}_1$ is an ideal of $\mathcal{L}_2$. The direct sum decomposition of $\mathrm{Rep}\,\mathcal{L}_2$ follows from the ideal property, cf. [Di]. ∎

Thus we can have natural containment of group algebras only for direct summands.

2. A basic function space.

Our aim in the rest of this paper is to develop a "universal convolution algebra" in which we are guaranteed to find a σ–group algebra if it exists, for $(G, \mathcal{R})$, $\mathcal{R} \subset \mathrm{Rep}_\sigma G_d$, with G non-locally compact. It then makes sense to study suitable subalgebras of it to analyze the existence question. In Sect. 4 we will consider in the context of these convolution algebras the existence of σ–group algebras for $(G, \mathrm{Rep}_\sigma G)$, i.e. the classical Gelfand–Raikov question for continuous group representation theory.

Let $\mathrm{C}^*_\sigma(G_d)$ denote the σ–twisted discrete group algebra, i.e. the C*–algebra generated by unitaries $\left\{\, \delta_x \;\middle|\; x \in G \,\right\}$ such that $\delta_x \cdot \delta_y = \sigma(x,\, y)\, \delta_{xy}$. There is a bijection between the nondegenerate representations of $\mathrm{C}^*_\sigma(G_d)$ and the unitary σ–representations of G, and it is given by $\widetilde{\pi}(x) := \pi(\delta_x)$, $x \in G$ for $\pi \in \mathrm{Rep}\,\mathrm{C}^*_\sigma(G_d)$. Let $R_0 \subset \mathrm{Rep}\,\mathrm{C}^*_\sigma(G_d)$ denote the subset in bijection with $\mathrm{Rep}_\sigma G$.

When the group G is locally compact and nondiscrete, we have the Haar measure μ, a notion of how a nonzero continuous function goes to zero at infinity, hence the function space $C_0(G)$, and the Riesz–Markov theorem which identifies the dual space $C_0(G)^*$ with regular Borel measures on G. There is then a decomposition of spaces (cf. [HR1])

$$C_0(G)^* = M_c(G) \oplus M_s(G) \oplus M_d(G)$$

where $M_d(G)$ (resp. $M_c(G)$, $M_s(G)$) denotes the space of discrete measures (resp. continuous measures absolutely continuous w.r.t. μ, continuous measures singular w.r.t. μ). Then $C_0(G)^*$ is endowed with σ–convolution and involution, w.r.t. which $M_c(G) \cong L^1(G)$ is a *–Banach subalgebra, and its C*–envelope is the usual group algebra $\mathrm{C}^*_\sigma(G)$, which is nonunital. Then $\mathrm{C}^*_\sigma(G)$ contains $\mathrm{C}^*_\sigma(G_d)$(= C*–envelope of $M_d(G)$) in its multiplier algebra i.e. $\mathrm{C}^*_\sigma(G_d) \subset M\big(\mathrm{C}^*_\sigma(G)\big)$, and via the unique extension of a representation from $\mathrm{C}^*_\sigma(G)$ to $M\big(\mathrm{C}^*_\sigma(G)\big)$ we obtain a bijection between $\mathrm{Rep}\,\mathrm{C}^*_\sigma(G)$ and R_0, hence $\mathrm{Rep}_\sigma G$.

Another important algebra which a locally compact G has associated to it, is its Fourier–Stieltjes algebra $B(G)$ which is a complete invariant for G, cf. [Wa]. Specifically, $B(G)$ is the space of finite spans of the continuous positive definite functions on G, (or equivalently the set of coefficient functions for all the continuous unitary representations of G). Now $B(G)$ is a commutative algebra w.r.t. pointwise multiplication, and as $B(G)$ has a canonical identification with the dual space $C^*(G)^*$, it is a Banach space and in fact a Banach algebra. Indeed, by this canonical identification of $B(G)$ we know that we can identify the group algebra (as a space) with a subspace of $B(G)^*$, and hence this will be a good place to look for our generalised group algebra when G is not locally compact. We will below endow $B(G)^*$ with a convolution product and see that the inclusion of the group algebra in $B(G)^*$ also holds for algebras. More concretely, note that $B(G) \subset L^\infty(G)$ is in general not complete w.r.t. the supremum norm over G. In fact, by the proof of Corollary 13.6.5 [Di], the uniform closure of the set of functions of compact support in $B(G)$ contains all of $C_c(G)$, and hence the uniform closure of $B(G)$ (denoted $K(G)$ henceforth) contains $C_0(G)$. Realize $L^1(G) \subset L^\infty(G)^*$ by $\omega_f(h) := \int f(x)h(x)\,d\mu(x)$, $h \in L^\infty(G)$, $f \in L^1(G)$, then ω_f is uniquely determined by its restriction to $C_0(G) \subset K(G)$. Thus we can identify $L^1(G)$ with a subspace of $K(G)^*$, a fact which we will exploit below for more general G.

In the case when G is not locally compact, we lose the Haar measure μ, the space $C_0(G) = \{0\}$, and the Riesz–Markov theorem does not apply. If G is totally regular (hence $C_b(G) \cong C(\beta G)$ with βG the Stone–Čech compactification of G), it is possible to define convolution for the functionals $C_b(G)^* = C(\beta G)^*$ (hence a "generalised" group algebra, cf. [Do]). However on G these functionals are only finitely additive, hence correspond to charges, not measures, cf. Alexandroff theorem [Wh]. Thus there is no Fubini theorem for these charges, and so there are two inequivalent convolutions, cf. [Py], which do not intertwine correctly with the natural involution to produce a *–algebra structure. Furthermore, due to the absence of the Haar measure μ, there is no way to select an analogue of $M_c(G)$ amongst these functionals. (There are alternative characterizations of $L^1(G) = M_c(G)$ which do not use the Haar measure, cf. [DvR, Dz, Gre, BB], but all need some condition on measures involving compact sets, so are not particularly useful for us). However $B(G)$ and $K(G)$ still makes sense even when G is not locally compact, and so, given the discussion in the previous paragraph, we will below consider $K(G)^*$ as the appropriate universe in which to locate generalised group algebras.

For the rest of this section, G need not be locally compact. Define first left and right σ–translations:

$$(\lambda_x f)(y) := \sigma(x,\, y)\, f(xy) =: (\rho_y f)(x)$$

for a function $f : G \to \mathbb{C}$ and the involution $f^*(x) := \overline{f(x^{-1})}$. For a given set $\mathcal{R} \subset \mathrm{Rep}_\sigma G_d$, which is closed w.r.t. direct sums, define the set of coefficient functions:

$$B_\sigma(\mathcal{R}) := \left\{\, f \in C_b(G_d) \;\middle|\; f(x) = (\psi,\, \pi(x)\varphi),\ \pi \in \mathcal{R};\ \psi,\, \varphi \in \mathcal{H}_\pi \,\right\}$$

which is clearly a linear space, and if $\mathcal{R} \subseteq \mathrm{Rep}_\sigma G$ these functions are also continuous. There are two natural norms on $B_\sigma(\mathcal{R})$, the uniform norm $\|f\|_\infty = \sup\limits_{x\in G} |f(x)|$ and the norm on the dual space of $\mathrm{C}^*_\sigma(G_d)$, where we identify $B_\sigma(\mathcal{R})$ with a subspace of $\mathrm{C}^*_\sigma(G_d)^*$ by the bijection between $\mathrm{Rep}\,\mathrm{C}^*_\sigma(G_d)$ and $\mathrm{Rep}_\sigma G_d$, $\widetilde{\pi}(x) := \pi(\delta_x)$ mentioned above, i.e. $f(x) = (\psi,\, \widetilde{\pi}(x)\varphi)$ is identified with $h(A) = (\psi,\, \pi(A)\varphi)$. Denote the latter norm by $\|\cdot\|_*$ then it is obvious that $\|f\|_\infty \leq \|f\|_* = \sup\left\{\, |h(A)| \;\middle|\; A \in \mathrm{C}^*_\sigma(G_d),\ \|A\| \leq 1 \,\right\}$ because all δ_x are in the unit ball of $\mathrm{C}^*_\sigma(G_d)$. Let $K_\sigma(\mathcal{R})$ (resp. $J_\sigma(\mathcal{R})$) denote the closure of $B_\sigma(\mathcal{R})$ in the norm $\|\cdot\|_\infty$ (resp. $\|\cdot\|_*$), then clearly $J_\sigma(\mathcal{R}) \subseteq K_\sigma(\mathcal{R})$. When the context makes clear the set $\mathcal{R}$ under investigation, we will simplify the notation to B_σ, K_σ, and J_σ and when $\sigma = 1$ we will omit the subscript. We will also not distinguish between π and $\widetilde{\pi}$. We collect a few easy facts.

Theorem 2.1. *Let $\mathcal{R} \subset \mathrm{Rep}_\sigma G_d$ be closed w.r.t. direct sums, then*

(i) For a fixed x, the maps λ_x, ρ_x and the involution preserve B_σ, J_σ and K_σ, and are isometries w.r.t. both the norms $\|\cdot\|_\infty$ and $\|\cdot\|_*$.

(ii) $\lambda_x\lambda_y = \sigma(y,\, x)\,\lambda_{yx}$, $\rho_x\rho_y = \sigma(x,\, y)\,\rho_{xy}$, * is indeed an involution on $C_b(G_d)$ and $(\lambda_x f)^* = \rho_{x^{-1}} f^*$.

(iii) With respect to pointwise multiplication we have $X_{\sigma_1}(\mathcal{R}_1)\cdot X_{\sigma_2}(\mathcal{R}_2)\subseteq X_{\sigma_1\sigma_2}(\mathcal{R}_1\otimes\mathcal{R}_2)$, *where* X *can be either of* B_σ *or* K_σ, *and if* $\mathcal{R}\subset\mathrm{Rep}\,G_d$ *is closed w.r.t tensor products, then* $K(\mathcal{R})$ *is a Banach *-algebra.*

(iv) If $\mathcal{R}$ *has a faithful representation, then* B_σ *separates* G,

(v) every $f\in B_\sigma(\mathrm{Rep}_\sigma G)$ *is left and right* $\|\cdot\|_*$*-continuous, (hence uniformly continuous) i.e.* $\lim\limits_{x\to e}\|\lambda_x f-f\|_* = 0 = \lim\limits_{x\to e}\|\rho_x f-f\|_*$ *so we obtain the appropriate norm continuity for functions in* J_σ *and* K_σ *.*

(vi) If G *is locally compact, then* $C_0(G)\subset K_\sigma(\mathrm{Rep}_\sigma G)$.

Proof: (i) It is obvious that $\|\lambda_x f\|_\infty=\|f\|_\infty=\|\rho_x f\|_\infty=\|f^*\|_\infty$ for bounded f. For a coefficient function $f(y)=(\psi,\pi(y)\varphi)$, $\pi\in\mathcal{R}$ we have

$$\begin{aligned}(\lambda_x f)(y)&=\sigma(x,\,y)\,f(xy)=\sigma(x,\,y)(\psi,\pi(xy)\varphi)=(\psi,\,\pi(x)\pi(y)\varphi)\\&=(\pi(x)^*\psi,\,\pi(y)\varphi),\end{aligned}\qquad(*)$$

hence $\lambda_x f\in B_\sigma$. Likewise $(\rho_x f)(y)=(\psi,\,\pi(y)\pi(x)\varphi)$, hence $\rho_x f\in B_\sigma$. Furthermore

$$f^*(x)=\overline{f(x^{-1})}=\overline{(\psi,\,\pi(x^{-1})\varphi)}=(\varphi,\,\pi(x^{-1})^*\psi)=(\varphi,\,\pi(x)\psi),$$

hence $f^*\in B_\sigma$ and hence * preserves B_σ as well. Now from equation (*) above, $\lambda_x f$ corresponds to the functional $h(A)=(\psi,\,\pi(x)\pi(A)\varphi)=(\psi,\,\pi(\delta_x A)\varphi)$ on $\mathrm{C}^*_\sigma(G_d)$, hence since multiplication by δ_x maps the unit ball onto the unit ball, it follows that $\|\lambda_x f\|_*=\sup\big\{\,|h(A)|\;\big|\;A\in\mathrm{C}^*_\sigma(G_d),\,\|A\|\le 1\,\big\}=\|f\|_*$. Likewise we get that $\|\rho_x f\|_*=\|f\|_*=\|f^*\|_*$. It now follows from continuity of these maps w.r.t. the two norms that they also preserve the two closures J_σ and K_σ.

(ii) $(\lambda_x\lambda_y f)(z)=\sigma(x,\,z)\,(\lambda_y f)(xz)=\sigma(x,\,z)\,\sigma(y,\,xz)\,f(yxz)=\sigma(y,\,x)\,\sigma(yx,\,z)\,f(yxz)=\sigma(y,\,x)(\lambda_{yx}f)(z)$ and similarly for ρ. That $f\to f^*$ is antilinear and satisfies $f^{**}=f$, is obvious. Moreover

$$(\lambda_x f)^*(y)=\overline{(\lambda_x f)(y^{-1})}=\overline{\sigma(x,y^{-1})\,f(xy^{-1})}$$

$$= \sigma(y, x^{-1})\, \overline{f((yx^{-1})^{-1})} = \sigma(y, x^{-1})\, f^*(yx^{-1})$$
$$= \rho_{x^{-1}} f^*(y)\ .$$

(iii) By continuity of the pointwise product w.r.t. the uniform norm, it suffices to prove that: $B_{\sigma_1}(\mathcal{R}_1) \cdot B_{\sigma_2}(\mathcal{R}_2) \subseteq B_{\sigma_1\sigma_2}(\mathcal{R}_1 \otimes \mathcal{R}_2)$. Let $f_i \in B_{\sigma_i}(\mathcal{R}_i)$, $i = 1, 2$, i.e. $f_i(x) = \big(\varphi_i,\, \pi_i(x)\psi_i\big)$ where $\pi_i \in \mathcal{R}_i$, $\varphi_i,\ \psi_i \in \mathcal{H}_{\pi_i}$. Then

$$f_1(x)f_2(x) = \big(\varphi_1 \otimes \varphi_2,\, (\pi_1 \otimes \pi_2)(x)\, \psi_1 \otimes \psi_2\big)$$

and since $\pi_1 \otimes \pi_2 \in \mathcal{R}_1 \otimes \mathcal{R}_2$, it follows that $f_1 \cdot f_2 \in B_{\sigma_1\sigma_2}(\mathcal{R}_1 \otimes \mathcal{R}_2)$. If $\mathcal{R}$ is closed under tensor products (as well as direct sums), then $K_1(\mathcal{R})$ is closed under pointwise multiplication, the involution, and is uniformly closed, so it follows that it is a Banach *-algebra.
(iv) If there is a faithful representation $\pi \in \mathcal{R}$, the functions $f(x) = (\psi,\, \pi(x)\varphi)$ must separate G.
(v) It is only necessary to establish continuity w.r.t. $\|\cdot\|_*$ for the elements of $B_\sigma(\mathrm{Rep}_\sigma G)$. For left continuity of $f(x) = (\psi,\, \pi(x)\varphi)$:

$$\begin{aligned}
\|\lambda_x f - f\|_* &= \sup\big\{\, \big|\big(\psi,\, \big(\pi(x) - \mathbb{I}\big)\pi(A)\varphi\big)\big| \;\big|\; A \in \mathrm{C}^*_\sigma(G_d),\ \|A\| \leq 1 \,\big\} \\
&\leq \big\{\, \big\|(\pi(x^{-1}) - \mathbb{I})\psi\big\| \cdot \|\pi(A)\varphi\| \;\big|\; A \in \mathrm{C}^*_\sigma(G_d),\ \|A\| \leq 1 \,\big\} \\
&\leq \big\|\big(\pi(x^{-1}) - \mathbb{I}\big)\psi\big\| \cdot \|\varphi\| \xrightarrow[x]{e} 0
\end{aligned}$$

from which it is clear, and similarly we get right continuity.
(vi) Let G be locally compact, then construct the usual group extension G_σ and recall the bijection between $\mathrm{Rep}_\sigma G$ and $\mathrm{Rep}\, G_\sigma$ (cf. [Ma]), given by $\pi_\sigma(x,\, t) := \pi(x)t$, $\pi \in \mathrm{Rep}_\sigma G$, $x \in G$, $t \in \mathbb{T}$. From the argument at the start of this section (using the proof of Corollary 13.6.5 in [Di]), we have that $C_0(G_\sigma) \subset K(G_\sigma)$. Now since for a coefficient function on G_σ we have $(\psi,\, \pi_\sigma(x,\, t)\varphi) = t(\psi,\, \pi(x)\varphi)$, it is clear that the restriction of $K(\mathrm{Rep}\, G_\sigma)$ to $G \subset G_\sigma$ is just $K_\sigma(\mathrm{Rep}_\sigma G)$. Thus restriction to G produces $C_0(G) \subset K_\sigma(\mathrm{Rep}_\sigma G)$, as required.

∎

In the rest of this paper we will always assume that the sets $\mathcal{R} \subseteq \mathrm{Rep}_\sigma G_d$ under consideration (for construction of group algebras) are closed with respect to direct sums.

3. Convolution Algebras.

Fix a set $\mathcal{R} \subseteq \mathrm{Rep}_\sigma G_d$ (closed with respect to direct sums). Here we want to make the dual spaces $J_\sigma(\mathcal{R})^*$ and $K_\sigma(\mathcal{R})^*$ into convolution algebras, following the method of Def. 19.1 [HR1]. Since we can identify $J_\sigma(\mathcal{R})^*$ (resp. $K_\sigma(\mathcal{R})^*$) with the functionals on $B_\sigma(\mathcal{R})$ which are $\|\cdot\|_*$–continuous (resp. $\|\cdot\|_\infty$–continuous), it follows from $\|\cdot\|_\infty \le \|\cdot\|_*$ that $K_\sigma^* \subseteq J_\sigma^*$. Before considering the convolutions, we need notation and a preparatory lemma.

Notation: Let $f \in C_b(G\times\cdots\times G)$ (n factors) and let $L \subseteq C_b(G)$ be a closed subspace. Assume that the function $x_1 \to f(x_1,\ldots,x_n)$ for fixed $x_j,\ j\neq 1$ is in L. Now, given functionals $\omega^{(1)},\ldots,\omega^{(n)} \in L^*$, the expression:

$$\omega^{(n)}_{x_n}\Big(\omega^{(n-1)}_{x_{n-1}}\Big(\cdots\Big(\omega^{(1)}_{x_1}\big(f(x_1,\ldots,x_n)\big)\Big)\cdots\Big)\Big)$$

means the following. Starting with the innermost functional $\omega^{(1)}$, first evaluate $\omega^{(1)}$ of the function $x_1 \to f(x_1, x_2,\ldots,x_n)$ obtaining a function $F(x_2,\ldots,x_n)$. Next evaluate $\omega^{(2)}$ of the function $x_2 \to F(x_2,\ldots,x_n)$ to get $G(x_3,\ldots,x_n)$. Continue until all functionals are evaluated. For this to make sense, we need to have that all subsequent functions $x_2 \to F(x_2,\ldots,x_n)$ etc., to be evaluated are in L. This brings the functional notation closer to integral notation, e.g.

$$\omega_x(f(x)) = \omega(f) = \int f(x)\,d\mu(x)$$

for a functional ω given by a measure.

Lemma 3.1. *For each $\omega \in J_\sigma^*$ and $\pi \in \mathcal{R}$, there is a unique operator $\pi(\omega) \in \mathcal{B}(\mathcal{H}_\pi)$ such that $\|\pi(\omega)\| \le \|\omega\|$ and $\omega_x\big((\psi,\,\pi(x)\,\varphi)\big) = (\psi,\,\pi(\omega)\,\varphi)$ for all $\psi,\ \varphi \in \mathcal{H}_\pi$. Moreover $\pi(\omega)$ preserves each cyclic component of G in $\mathcal{H}_\pi$, i.e. $\pi(\omega)\mathcal{H}_\psi \subseteq \mathcal{H}_\psi$ for all $\psi \in \mathcal{H}_\pi$, where $\mathcal{H}_\psi := [\pi(G)\psi]$.*

Proof: Let $\omega \in J_\sigma^*$, then $\omega_x\big((\psi,\,\pi(x)\,\varphi)\big)$ exists since $x \to (\psi,\,\pi(x)\,\varphi)$ is in B_σ. Now the map $\psi \to \omega_x\big((\psi,\,\pi(x)\,\varphi)\big)$ is conjugate linear, and bounded as

$$\left|\omega_x\big((\psi,\,\pi(x)\,\varphi)\big)\right| \leq \|\omega\|\cdot\sup\left\{\,|(\psi,\,\pi(A)\,\varphi)|\;\;\middle|\;\; A\in \mathrm{C}^*_\sigma(G_d),\,\|A\|\leq 1\,\right\}$$
$$\leq \|\omega\|\cdot\|\psi\|\cdot\|\varphi\| \qquad (*)$$

hence it is a conjugate linear functional on $\mathcal{H}_\pi$. Thus by the Riesz representation theorem, there is a vector $\varphi_\omega\in\mathcal{H}_\pi$ such that

$$\omega_x\big((\psi,\,\pi(x)\,\varphi)\big) = (\psi,\,\varphi_\omega)\quad \forall\,\psi\in\mathcal{H}_\pi \qquad (+)$$

Denote φ_ω by $\pi(\omega)\varphi$, then by $(*)$ we see $\|\pi(\omega)\varphi\|\leq\|\omega\|\cdot\|\varphi\|$, hence by linearity of $\varphi\to\pi(\omega)\varphi$ (clear from $(+)$), we have defined a bounded operator $\pi(\omega):\mathcal{H}_\pi\to\mathcal{H}_\pi$. Uniqueness comes from the fact that $\pi(\omega)$ is fully determined by the coefficients $(\psi,\,\pi(\omega)\varphi)$ as ψ and φ ranges over $\mathcal{H}_\pi$.
Finally, fix $\psi\in\mathcal{H}_\pi\backslash 0$, then the functional $\xi\to\omega_x\big((\xi,\,\pi(x)\,\varphi)\big)$ for $\varphi\in\mathcal{H}_\psi$ is zero on the orthogonal complement of $\mathcal{H}_\psi$, hence the vector $\varphi_\omega=\pi(\omega)\varphi$ from the Riesz theorem must be in $\mathcal{H}_\psi$. ∎

Note that when ω is associated to a measure: $\omega(f)=\int f\,d\mu$, then

$$\pi(\omega)=\int\pi(x)\,d\mu(x)\,.$$

In the case that G is locally compact, the convolution of two measures $\mu,\,\nu\in M(G)=C_0(G)^*$ is given by:

$$\int f(x)\,d(\mu*\nu)(x) = \int\int\sigma(x,\,y)\,f(xy)\,d\mu(x)\,d\nu(y) \qquad (1)$$
$$= \int\int\sigma(x,\,y)\,f(xy)\,d\nu(y)\,d\mu(x)\quad\forall\,f\in C_0(G), \qquad (2)$$

by Fubini. (When we generalise to non-locally compact groups or other types of functionals, these two formulii will give different convolutions [Py].) Let us first rewrite these in terms of the associated functionals $\omega(f):=\int f d\mu$ and $\xi(f):=\int f\,d\nu$. For $f\in C_0(G)$, $\psi\in C_0(G)^*$, define

$$f^\psi(x):=\psi(\lambda_x f)=\psi_y(\sigma(x,y)f(xy)) \qquad\text{and}\qquad f_\psi(x):=\psi(\rho_x f)=\psi_y(\sigma(y,x)f(yx))$$

Then by Lemma 19.5 [HR1], $f^\psi \in C_0(G) \ni f_\psi$, hence we can write the convolution formulii as

$$\begin{aligned}(\omega * \xi)(f) &:= \int f(x)\, d(\mu * \nu)(x) = \int\int (\rho_y f)(x)\, d\mu(x)\, d\nu(y) \\ &= \xi(f_\omega) = \xi_y(\omega_x(\sigma(x,y)\, f(xy)))\end{aligned} \tag{3}$$

and

$$\begin{aligned}(\omega * \xi)(f) &= \int\int (\lambda_x f)(y)\, d\nu(y)\, d\mu(x) \\ &= \omega(f^\xi) = \omega_x(\xi_y(\sigma(x,y)\, f(xy)))\end{aligned} \tag{4}$$

Henceforth, let G be non–locally compact. In order to generalise (3) and (4) to G, we first need the lemma:

Lemma 3.2. *Let X denote either J or K. Let $f \in X_\sigma(\mathcal{R})$, $\omega \in X_\sigma^*$ and define as above $f^\omega(x) := \omega(\lambda_x f)$ and $f_\omega(x) := \omega(\rho_x f)$. Then $f^\omega \in X_\sigma(\mathcal{R}) \ni f_\omega$.*

Proof: Consider $f \in B_\sigma$ of the form $f(x) = (\psi,\, \pi(x)\varphi)$, $\pi \in \mathcal{R}$. Let $\omega \in X_\sigma^*$ then

$$\begin{aligned}f^\omega(x) = \omega(\lambda_x f) &= \omega_y\big(\sigma(x,\, y)\,\big(\psi,\, \pi(xy)\varphi\big)\big) \\ &= \omega_y\big((\psi,\, \pi(x)\pi(y)\varphi)\big) = \omega_y\big((\pi(x)^*\psi,\, \pi(y)\varphi)\big) \\ &= \big(\pi(x)^*\psi,\, \pi(\omega)\varphi\big) = \big(\psi,\, \pi(x)\pi(\omega)\varphi\big)\end{aligned}$$

making use of Lemma 3.1. Thus $f^\omega \in B_\sigma$. For fixed ω the map $f \to f^\omega$ is linear and norm–continuous by

$$\|f^\omega\|_\infty = \sup_x \big|\omega(\lambda_x f)\big| \le \|\omega\| \cdot \sup_x \|\lambda_x f\|_\infty = \|\omega\| \cdot \|f\|_\infty$$

in the case $X = K$. For the case $X = J$, consider an $f \in B_\sigma$ as above, then

$$\begin{aligned}\|f^\omega\|_* &= \sup\big\{\, |\omega_y\big((\psi,\, \pi(A)\pi(y)\varphi)\big)| \;\big|\; A \in \mathrm{C}_\sigma^*(G_d),\, \|A\| \le 1 \,\big\} \\ &\le \sup\big\{\, \|\omega\| \cdot \|h_A\|_* \;\big|\; A \in \mathrm{C}_\sigma^*(G_d),\, \|A\| \le 1 \,\big\}\end{aligned}$$

where $h_A(y) := (\psi,\, \pi(A)\pi(y)\varphi)$. Now

$$\|h_A\|_* = \sup\big\{\, |(\psi,\, \pi(A)\pi(B)\varphi)| \;\big|\; B \in \mathrm{C}_\sigma^*(G_d),\, \|B\| \le 1 \,\big\} \le \|f\|_*$$

because $\|AB\| \leq \|A\|\|B\| \leq 1$. Thus $\|f^\omega\|_* \leq \|\omega\|\|f\|_*$, and hence we get that $f^\omega \in X_\sigma$ for all $f \in X_\sigma$. Likewise we have:

$$\begin{aligned} f_\omega(x) &= \omega_y((\rho_x f)(y)) = \omega_y\big((\psi,\, \pi(y)\pi(x)\varphi)\big) \\ &= (\psi,\, \pi(\omega)\pi(x)\varphi) = (\pi(\omega)^*\psi,\, \pi(x)\varphi) \end{aligned}$$

from which it is clear that $f_\omega \in B_\sigma$, and hence by a similar argument as above, $f \to f_\omega$ is norm continuous, so $f_\omega \in X_\sigma$ for all $f \in X_\sigma$. ∎

Now as remarked before, there are two possible convolutions one can define on J_σ^* given by (3) and (4), but surprisingly they are the same:

Theorem 3.3. *(i) Given* $\omega,\, \xi \in J_\sigma^*,\ f \in J_\sigma$, *we have* $\xi(f^\omega) = \omega(f_\xi)$, *i.e.*

$$\xi_x\big(\omega_y\big(\sigma(x,\,y)\,f(xy)\big)\big) = \omega_y\big(\xi_x\big(\sigma(x,\,y)\,f(xy)\big)\big),$$

and thus we can define convolution in J_σ^* *by*

$$(\xi * \omega)(f) := \xi(f^\omega) = \omega(f_\xi).$$

Moreover, $K_\sigma^* \subseteq J_\sigma^*$ *is closed under this convolution.*

(ii) Both J_σ^* *and* K_σ^* *are Banach *–algebras w.r.t. convolution, the involution* $\omega^*(f) := \overline{\omega(f^*)}$, *and their usual norms.*

(iii) Let $\pi \in \mathcal{R}$, *and let* $\omega \to \pi(\omega)$ *be the map from* J_σ^* *to* $\mathcal{B}(\mathcal{H}_\pi)$ *given in Lemma 3.1. Then this map is a continuous *–homomorphism of the Banach algebra* J_σ^*, *i.e. a representation and hence it restricts to a continuous representation of* K_σ^*.

Proof: (i) By Lemma 3.2, both of $\xi(f^\omega)$ and $\omega(f_\xi)$ exist. Let $f \in B_\sigma$ be of the form $f(x) = (\psi,\, \pi(x)\varphi)$, $\pi \in \mathcal{R}$, then

$$\begin{aligned} \xi(f^\omega) &= \xi_x\big(\omega_y\big(\sigma(x,\,y)\,f(xy)\big)\big) \\ &= \xi_x\big(\omega_y\big((\psi,\, \pi(x)\pi(y)\varphi)\big)\big) = \xi_x\big((\pi(x)^*\psi,\, \pi(\omega)\varphi)\big) \\ &= (\psi,\, \pi(\xi)\pi(\omega)\varphi) = (\pi(\xi)^*\psi,\, \pi(\omega)\varphi) \\ &= \omega_y\big((\psi,\, \pi(\xi)\pi(y)\varphi)\big) = \omega_y\big(\xi_x\big((\psi,\, \pi(x)\pi(y)\varphi)\big)\big) \\ &= \omega_y\big(\xi_x\big(\sigma(x,\,y)\,f(xy)\big)\big) = \omega(f_\xi). \end{aligned} \tag{5}$$

By the norm–continuity found in the proof of Lemma 3.2:

$$\|\xi(f^\omega)\| \leq \|\xi\| \cdot \|\omega\| \cdot \|f\|_* \geq \|\omega(f_\xi)\|,$$

the equality (5) extends to all of J_σ, thus establishing the claim.
If $\omega, \xi \in K_\sigma^*$, then so is $\xi * \omega$, by the continuity $\|\xi(f^\omega)\| \leq \|\xi\| \cdot \|\omega\| \cdot \|f\|_\infty$ encountered above.
(ii) Linearity of $\xi * \omega$ in $\xi, \omega \in J_\sigma^*$ is clear from the definition. For associativity:

$$\begin{aligned}
\big((\xi * \beta) * \omega\big)(f) &= (\xi * \beta)_x \big(\omega_y \big(\sigma(x,\, y)\, f(xy)\big)\big) \\
&= \xi_z \big(\beta_v \big(\sigma(z,\, v)\omega_y \big(\sigma(zv,\, y)\, f(zvy)\big)\big)\big) \\
&= \xi_z \big(\beta_v \big(\omega_y \big(\sigma(v,\, y)\, \sigma(z,\, vy)\, f(zvy)\big)\big)\big) \\
&= \xi_z \big((\beta * \omega)_x \big(\sigma(z,\, x)\, f(zx)\big)\big) \\
&= \big(\xi * (\beta * \omega)\big)(f)
\end{aligned}$$

making use of the two–cocycle relation for σ. Norm continuity of course follows from $|(\xi * \omega)(f)| \leq \|\xi\|\|\omega\|\|f\|_*$ or $|(\xi * \omega)(f)| \leq \|\xi\|\|\omega\|\|f\|_\infty$ if instead $\xi, \omega \in K_\sigma^*$. Thus J_σ^* and K_σ^* are Banach algebras. Concerning the involution, it is clear that $\omega \to \omega^*$ is antilinear and that $\omega^{**} = \omega$. Now

$$\begin{aligned}
\|\omega^*\| &= \sup \big\{ \big|\omega^*(f)\big| \;\big|\; f \in K_\sigma,\ \|f\| \leq 1 \big\} \\
&= \sup \big\{ \big|\omega(f^*)\big| \;\big|\; f \in K_\sigma,\ \|f\| \leq 1 \big\} = \|\omega\|
\end{aligned}$$

using $\|f^*\| = \|f\|$ where here $\|f\|$ denoted $\|f\|_\infty$ if $\omega \in K_\sigma^*$, and $\|f\|_*$ otherwise. Furthermore, for $\omega,\ \xi \in J_\sigma^*$,

$$\begin{aligned}
(\xi * \omega)^*(f) = \overline{(\xi * \omega)(f^*)} &= \overline{\xi_x \big(\omega_y \big(\sigma(x,\, y)(f^*)(xy)\big)\big)} \\
&= \overline{\xi_x \big(\omega_y \big(\sigma(x,\, y)\overline{f(y^{-1}x^{-1})}\big)\big)} \\
&= \overline{\xi_x \big(\omega_y \big(\overline{\sigma(y^{-1}, x^{-1})\, f(y^{-1}x^{-1})}\big)\big)} \\
&= \overline{\xi_x \big(\overline{\omega_y^* \big(\sigma(y, x^{-1})\, f(yx^{-1})\big)}\big)} \\
&= \xi_x^* \big(\omega_y^* \big(\sigma(y, x)\, f(yx)\big)\big) = (\omega^* * \xi^*)(f)
\end{aligned}$$

where we made use of $\overline{\sigma(x,\, y)} = \sigma(y^{-1},\, x^{-1})$. Thus J_σ^* and K_σ^* are Banach *–algebras.

(iii) That $\omega \to \pi(\omega)$ for $\pi \in \mathcal{R}$ is linear is easy to see. From Lemma 3.1 we also have that $\|\pi(\omega)\| \leq \|\omega\|$, hence the map is continuous. We show that $\omega \to \pi(\omega)$ is a homomorphism.

$$\begin{aligned}
(\psi,\, \pi(\omega * \xi)\varphi) &= (\omega * \xi)_x\big((\psi,\, \pi(x)\varphi)\big) \\
&= \omega_x\big(\xi_y\big(\sigma(x,\, y)\,(\psi,\, \pi(xy)\varphi)\big)\big) \\
&= \omega_x\big(\xi_y\big((\pi(x)^*\psi,\, \pi(y)\varphi)\big)\big) \\
&= \omega_x\big((\psi,\, \pi(x)\pi(\xi)\varphi)\big) = (\psi,\, \pi(\omega)\pi(\xi)\varphi)\, .
\end{aligned}$$

Finally, to establish that π is a *–homomorphism of J_σ^*,

$$\begin{aligned}
(\psi,\, \pi(\omega^*)\varphi) &= (\omega^*)_x\big((\psi,\, \pi(x)\varphi)\big) \\
&= \overline{\omega_x\big((\psi,\, \pi(x)\varphi)^*\big)} = \overline{\omega_x\big((\pi(x^{-1})\varphi,\, \psi)\big)} \\
&= \overline{\omega_x\big((\varphi,\, \pi(x)\psi)\big)} = \overline{(\varphi,\, \pi(\omega)\psi)} \\
&= (\psi,\, \pi(\omega)^*\varphi)
\end{aligned}$$

hence $\pi(\omega^*) = \pi(\omega)^*$. ∎

There are several distinguished subalgebras of J_σ^*, but one in particular which we'll find useful, is the algebra of point measures, i.e. $\delta_x \in K_\sigma^* \subseteq J_\sigma^*$, $x \in G$ defined by $\delta_x(f) := f(x)$, $f \in K_\sigma$.

Theorem 3.4. *For any* $\omega \in J_\sigma^*$, *we have that* $\delta_x * \omega = \omega \circ \lambda_x$ *and* $\omega * \delta_x = \omega \circ \rho_x$, *hence* δ_e *is the identity* $\mathbb{I}$ *of* J_σ^*. *Moreover* $\delta_x^* = \delta_{x^{-1}} = (\delta_x)^{-1}$, *and* $\delta_x * \delta_y = \sigma(x,\, y)\,\delta_{xy}$.

Proof: $(\delta_x * \omega)(f) = (\delta_x)_z\big(\omega_y\big(\sigma(z,\, y)\, f(zy)\big)\big) = \omega_y\big(\sigma(x,\, y)\, f(xy)\big) = \omega(\lambda_x f)$.
Likewise $\omega * \delta_x = \omega \circ \rho_x$.
$(\delta_x^*)_y(f(y)) = \overline{(\delta_x)_y(\overline{f(y^{-1})})} = f(x^{-1}) = \delta_{x^{-1}}(f)$ i.e. $\delta_x^* = \delta_{x^{-1}}$.
Furthermore,

$$\begin{aligned}
(\delta_x * \delta_y)(f) &= (\delta_x)_z\big((\delta_y)_v(\sigma(z,\, v)\, f(zv))\big) = (\delta_x)_z\big(\sigma(z,\, y)\, f(zy)\big) \\
&= \sigma(x,\, y)\, f(xy) = \big(\sigma(x,\, y)\,\delta_{xy}\big)(f),
\end{aligned}$$

i.e. $\delta_x * \delta_y = \sigma(x,\, y)\, \delta_{xy}$ and hence $\delta_{x^{-1}} = (\delta_x)^{-1} = \delta_x^*$. ∎

In the case that $K_\sigma(\mathcal{R})$ separates G, we get that the σ–homomorphism $\delta : G \to J_\sigma^*$ is injective, and then the C*–enveloping algebra of the *-algebra $\operatorname{Span} \delta_G$ is the (twisted) discrete group algebra $C^*(G_d)$.

The next theorem establishes that the convolution algebra J_σ^* is the appropriate setting in which to locate group algebras, if they exist.

Theorem 3.5. *Let $\mathcal{L}$ be a σ–group algebra for the pair $(G,\, \mathcal{R})$ with associated σ–homomorphism $\varphi : G \to UM(\mathcal{L})$, then there is a *-isomorphism $\Psi : M(\mathcal{L}) \to J_\sigma(\mathcal{R})^*$ (into) such that $\Psi(\varphi(x)) = \delta_x$ for all $x \in G$.*

Proof: Given the group algebra $\mathcal{L}$, recall from Proposition 1.1 that each $\pi \in \mathcal{R}$ has a unique strict–strong operator continuous extension from $\varphi(G)$ to a representation $\widehat{\pi} \in \operatorname{Rep} M(\mathcal{L})$, hence each $f \in B_\sigma$ has a canonical extension to $M(\mathcal{L})$ by $\widehat{f}(A) := (\psi,\, \widehat{\pi}(A)\xi)$, $A \in \mathcal{L}$ when $f(x) = (\psi,\, \pi(x)\xi)$. Now $\widehat{\pi}\,(C^*(\varphi(G))) = \pi\,(\mathrm{C}^*_\sigma(G_d))$, so by definition $\|f\|_* = \left\|\widehat{f}\restriction C^*(\varphi(G))\right\|$ so since by Prop. 1.1 $C^*(\varphi(G))$ is strictly dense in $M(\mathcal{L})$, it follows by the strict-strong operator continuity of $\widehat{\pi}$ that $\|\widehat{f}\| \leq \|h\|$ where h is the functional $h(A) := (\psi,\, (A)\xi)$ on $\pi(G)''$. By a Kaplansky density argument we know that $\|h\| = \big\|h\restriction\widehat{\pi}\big(C^*(\varphi(G))\big)\big\| = \|f\|_*$ and thus $\|\widehat{f}\| = \|f\|_*$. Hence the map $f \to \widehat{f}$ is a linear isometry, $\Phi : J_\sigma \to M(\mathcal{L})^*$, and in particular by $M(\mathcal{L}) \subset M(\mathcal{L})^{**}$ we obtain a linear map $\Psi : M(\mathcal{L}) \to J_\sigma^*$ by $\Psi(A)(f) := \Phi(f)(A)$ for $f \in J_\sigma$, $A \in M(\mathcal{L})$, i.e. on B_σ

$$\Psi(A)_x\big((\psi,\, \pi(x)\xi)\big) = (\psi,\, \widehat{\pi}(A)\xi) = \big(\psi,\, \pi(\Psi(A))\xi\big)$$

by Lemma 3.1, hence $\widehat{\pi}(A) = \pi(\Psi(A))$. This establishes also that $\Psi(A)$ is $\|\cdot\|_*$–continuous on B_σ, hence confirms that $\Psi(A) \in J_\sigma^*$. In particular if $A = \varphi(x)$ and $f(x) = (\psi,\, \pi(x)\xi)$ then

$$\Psi(A)(f) = (\psi,\, \widehat{\pi}(\varphi(x))\xi) = (\psi,\, \pi(x)\xi) = (\delta_x)_y\big((\psi,\, \pi(y)\xi)\big)$$

and by letting $\psi,\, \xi$ range over $\mathcal{H}_\pi$ we get that $\Psi(\varphi(x)) = \delta_x$. To see that Ψ is a homomorphism:

$$\begin{aligned}\Psi(A) * \Psi(B)(f) &= (\Psi(A) * \Psi(B))_x \left((\psi,\, \pi(x)\xi)\right) = \big(\psi,\, \pi\big(\Psi(A) * \Psi(B)\big)\xi\big) \\ &= \big(\psi,\, \pi\big(\Psi(A)\big)\pi\big(\Psi(B)\big)\xi\big) \qquad\qquad \text{by Theorem 3.3(iii)} \\ &= (\psi,\, \widehat{\pi}(AB)\xi) = \Psi(AB)(f)\,.\end{aligned}$$

The adjoint is preserved because

$$\begin{aligned}\Psi(A)^*(f) = \overline{\Psi(A)(f^*)} = \overline{\Psi(A)_x\big(\xi,\, \pi(x)\psi\big)} &= \overline{\big(\xi,\, \widehat{\pi}(A)\psi\big)} \\ = \big(\psi,\, \widehat{\pi}(A^*)\xi\big) = \Psi(A^*)(f)\,.&\end{aligned}$$

Thus $\Psi : M(\mathcal{L}) \to J_\sigma^*$ is a *-homomorphism of a C*-algebra into a Banach *-algebra, and hence $\|\Psi(A)\| \leq \|A\|$ for all $A \in M(\mathcal{L})$. In fact Ψ is isometric on $\mathcal{L}$ because for all $A \in \mathcal{L}$:

$$\begin{aligned}\big\|\Psi(A)\big\| &= \sup\big\{\, |\Psi(A)(f)| \;\big|\; f \in J_\sigma,\; \|f\|_* \leq 1 \,\big\} \\ &= \sup\big\{\, |\Psi(A)(f)| \;\big|\; f \in B_\sigma,\; \|f\|_* \leq 1 \,\big\} \quad \text{as } B_\sigma \text{ is dense in } J_\sigma. \\ &= \sup\big\{\, \big|\big(\psi,\, \widehat{\pi}(A)\xi\big)\big| \;\big|\; \pi \in \mathcal{R};\; \psi, \xi \in \mathcal{H}_\pi,\; \|\psi\| \leq 1 \geq \|\xi\| \,\big\} \\ &= \sup\big\{\, \big|\big(\psi,\, \pi(A)\xi\big)\big| \;\big|\; \pi \in \operatorname{Rep}\mathcal{L};\; \psi, \xi \in \mathcal{H}_\pi,\; \|\psi\| \leq 1 \geq \|\xi\| \,\big\} \\ &= \|A\|\,.\end{aligned}$$

where in the penultimate step we used the fact that $\widehat{\pi}\restriction\mathcal{L} = \theta^{-1}(\pi)$ and that $\theta : \operatorname{Rep}\mathcal{L} \to \mathcal{R}$ is a bijection. Thus Ψ is an isomorphism on $\mathcal{L}$, and as multipliers $B \in M(\mathcal{L})$ are uniquely determined by their action on $\mathcal{L}$, it follows that Ψ must also be an isomorphism on $M(\mathcal{L})$. ∎

Note that in the theorem we established that $\|\Psi(A)\| = \|A\|$ on $\mathcal{L}$, hence on the image of the group algebra in J_σ^*, the norm is already a C*-norm.

By this theorem, one should look for group algebras for a given pair $(G, \mathcal{R})$ in the subalgebras of J_σ^* which are stable under multiplication by δ_G.

4. Conditions for group algebras.

Inspired by Theorem 3.5, we now want to characterise the properties which a subalgebra of J_σ^* should satisfy in order to be a σ–group algebra for $(G,\,\mathcal{R})$. We will specialise to the case $\mathcal{R} = \mathrm{Rep}_\sigma G$ (the Gelfand–Raikov question) at the end of this section.

Def. A *d–ideal* $\mathcal{A}$ of $J_\sigma(\mathcal{R})^*$ is a nonzero norm–closed *–subalgebra such that $\delta_x * \mathcal{A} \subseteq \mathcal{A} \supseteq \mathcal{A} * \delta_x$ for all $x \in G$, (i.e. δ_G is in the relative multiplier algebra of $\mathcal{A}$.)

From any nonzero $A \in J_\sigma^*$ we can generate a d-ideal by just taking the closed *-algebra generated by the set $\delta_G * A$. For any d-ideal we have the usual map $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}_\sigma G_d$ by $\theta(\pi)(x) = \text{s–}\lim_\alpha \pi(\delta_x * E_\alpha)$ where $\{E_\alpha\} \subset \mathcal{A}$ is any approximate identity of $\mathcal{A}$. Obviously a d-ideal must satisfy $\theta(\mathrm{Rep}\,\mathcal{A}) \subseteq \mathcal{R}$ if it is to be a σ–group algebra for $(G,\,\mathcal{R})$. So we denote

$$\mathcal{I}(\mathcal{R}) := \big\{\, \mathcal{A} \subset J_\sigma^* \;\big|\; \mathcal{A} \text{ is a d-ideal and } \theta(\mathrm{Rep}\,\mathcal{A}) \subseteq \mathcal{R} \,\big\}\;.$$

The natural map which we will want to be inverse to θ, is the map $\pi \in \mathrm{Rep}_\sigma G \to \pi_{\mathcal{A}} \in \mathrm{Rep}\,\mathcal{A}$ defined by

$$\omega_x\big((\psi,\,\pi(x)\varphi)\big) = (\psi,\,\pi_{\mathcal{A}}(\omega)\varphi) \qquad \forall\,\psi,\,\varphi \in \mathcal{H}_\pi,\;\omega \in \mathcal{A}\;,$$

via Lemma 3.1 and Theorem 3.3(iii). In the case that G is locally compact and $\mathcal{R} = \mathrm{Rep}_\sigma G$, we realize $L^1(G)$ in J_σ^* as usual by integrals: $\omega_h(f) := \int_G f(x)h(x)\,d\mu(x)$, $f \in B_\sigma$, $h \in L^1(G)$ with μ the Haar measure (then $\omega_h \in K_\sigma^* \subset J_\sigma^*$). Then the closure of $L^1(G)$ is a d-ideal $\mathcal{A}$, and the map $\pi \to \pi_{\mathcal{A}}$ is the usual one given by $\pi_{\mathcal{A}}(f) = \int f(x)\,\pi(x)\,d\mu(x)$.

Theorem 4.1. *If a d-ideal $\mathcal{A} \in \mathcal{I}(\mathcal{R})$ separates B_σ, then $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathcal{R}$ is surjective.*

Proof: Let $\mathcal{A}$ separate B_σ. We first show that $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ is nondegenerate for any $\pi \in \mathcal{R}$. If $\pi_{\mathcal{A}}$ were degenerate, there would be a nonzero $\varphi \in \mathcal{H}_\pi$ such that $\pi_{\mathcal{A}}(\mathcal{A})\varphi = 0$, i.e. $\omega_x\big((\psi,\,\pi(x)\varphi)\big) = 0$ for all $\psi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$. Now $\pi \in \mathcal{R}$ is nondegenerate, hence there is a vector $\psi \in \mathcal{H}_\pi$ such that the function $x \to (\psi,\,\pi(x)\varphi)$ is nonzero, and by the previous sentence this is in $\operatorname{Ker}\omega$ for all $\omega \in \mathcal{A}$. This contradicts the hypothesis that $\mathcal{A}$ separates B_σ, and thus $\pi_{\mathcal{A}}$ is nondegenerate. We will now show that $\pi = \theta(\pi_{\mathcal{A}})$, which establishes surjectivity of θ. For all $\psi, \varphi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$ we have:

$$\begin{aligned}\big(\varphi,\, \theta(\pi_{\mathcal{A}})(x)\, \pi_{\mathcal{A}}(\omega)\, \psi\big) &= \big(\varphi,\, \pi_{\mathcal{A}}(\delta_x * \omega)\psi\big) \\ = (\delta_x * \omega)_y\big((\varphi,\, \pi(y)\, \psi)\big) &= \omega_y\big((\varphi,\, \sigma(x,y)\, \pi(xy)\, \psi)\big) \\ = \omega_y\big((\pi(x)^*\varphi,\, \pi(y)\psi)\big) &= \big(\varphi,\, \pi(x)\, \pi_{\mathcal{A}}(\omega)\psi\big)\ ,\end{aligned}$$

i.e. $\theta(\pi_{\mathcal{A}})(x)\cdot\pi_{\mathcal{A}}(\omega)\psi = \pi(x)\cdot\pi_{\mathcal{A}}(\omega)\psi$ for all $\psi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$. Since $\pi_{\mathcal{A}}$ is nondegenerate, $\pi_{\mathcal{A}}(\mathcal{A})\mathcal{H}_\pi$ is dense, hence $\theta(\pi_{\mathcal{A}})(x) = \pi(x)$ for all $x \in G$, which proves that θ is surjective. ∎

Recall that we have the canonical isometry $\iota : J_\sigma \to J_\sigma^{**}$ by $\iota(f)(\omega) := \omega(f)$ for $\omega \in J_\sigma^*$, $f \in J_\sigma$, and that J_σ is reflexive if $\iota(J_\sigma) = J_\sigma^{**}$. If $\mathcal{A} \subset J_\sigma^*$ is a d-ideal, we denote the restriction of ι by $j : J_\sigma \to \mathcal{A}^*$ where $j(f)(\beta) := \beta(f)$, $\beta \in \mathcal{A}$, $f \in J_\sigma$. Note that j is injective if $\mathcal{A}$ separates J_σ. Now even if J_σ is not reflexive, there may still be d-ideals $\mathcal{A}$ such that $j(J_\sigma) = \mathcal{A}^*$, and we need these because:

Theorem 4.2. *For a d-ideal* $\mathcal{A} \in \mathcal{I}(\mathcal{R})$, *the map* $\theta : \operatorname{Rep}\mathcal{A} \to \mathcal{R}$ *is injective with inverse map* $\pi \in \mathcal{R} \to \pi_{\mathcal{A}} \in \operatorname{Rep}\mathcal{A}$ *iff* $j(J_\sigma) = \mathcal{A}^*$. *In this case, the C*-envelope of* $\mathcal{A}$ *is a group algebra for* $(G,\, \theta(\operatorname{Rep}\mathcal{A}))$.

Proof: We need to prove that $\theta(\pi)_{\mathcal{A}}(\omega) = \pi(\omega)$ for all $\pi \in \operatorname{Rep}\mathcal{A}$, $\omega \in \mathcal{A}$ iff $j(J_\sigma) = \mathcal{A}^*$. Assume that $\theta(\pi)_{\mathcal{A}} = \pi$. Let $f(x) := (\varphi,\, \theta(\pi)(x)\, \psi)$, then

$$j(f)(\omega) = \omega(f) = \big(\varphi,\, \theta(\pi)_{\mathcal{A}}(\omega)\psi\big) = (\varphi, \pi(\omega)\psi)$$

for all $\varphi, \psi \in \mathcal{H}_\pi$, $\pi \in \operatorname{Rep}\mathcal{A}$, $\omega \in \mathcal{A}$. By varying the rhs over $\pi \in \operatorname{Rep}\mathcal{A}$, $\varphi = \psi \in \mathcal{H}_\pi$, we obtain all states of $\mathcal{A}$, and since these span $\mathcal{A}^*$ and j is linear, it means any functional of $\mathcal{A}$ can be expressed as an element of $j(J_\sigma)$, i.e. $j(J_\sigma) = \mathcal{A}^*$.

Conversely, let $j(J_\sigma) = \mathcal{A}^*$. Now observe that

$$j(f)(\omega * \beta) = (\omega * \beta)(f) = \omega_x\big(\beta(\lambda_x f)\big) = \omega_x\big((\delta_x * \beta)(f)\big)$$

$$= \omega_x\big(j(f)(\delta_x * \beta)\big) \qquad \forall\, \omega,\, \beta \in \mathcal{A},\ f \in J_\sigma\ .$$

Thus, since $j(J_\sigma) = \mathcal{A}^*$, we have:

$$\xi(\omega * \beta) = \omega_x\big(\xi(\delta_x * \beta)\big) \qquad \forall\, \xi \in \mathcal{A}^*,\ \omega,\, \beta \in \mathcal{A}\ .$$

In particular, choose $\xi(\omega) = (\varphi,\, \pi(\omega)\psi)$, $\pi \in \operatorname{Rep}\mathcal{A}$, φ, $\psi \in \mathcal{H}_\pi$, then

$$\begin{aligned}\big(\varphi,\, \pi(\omega * \beta)\psi\big) &= \omega_x\big((\varphi,\, \pi(\delta_x * \beta)\psi)\big) \\ &= \omega_x\big((\varphi,\, \theta(\pi)(x)\, \pi(\beta)\psi\big) \\ &= \big(\varphi,\, \theta(\pi)_{\mathcal{A}}(\omega)\, \pi(\beta)\psi\big)\end{aligned}$$

for all $\pi \in \operatorname{Rep}\mathcal{A}$, φ, $\psi \in \mathcal{H}_\pi$, ω, $\beta \in \mathcal{A}$. Thus

$$\pi(\omega) \cdot \pi(\beta)\psi = \theta(\pi)_{\mathcal{A}}(\omega) \cdot \pi(\beta)\psi\ .$$

By nondegeneracy of $\pi \in \operatorname{Rep}\mathcal{A}$ we get $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ for all $\omega \in \mathcal{A}$. ∎

The condition $j(J_\sigma) = \mathcal{A}^*$ is quite natural, if we keep in mind that if $\mathcal{A}$ is a group algebra, then its dual is the coefficient space of its representation space $\mathcal{R}$, and the latter is B_σ.

Corollary 4.3. *The C*–envelope of any d-ideal $\mathcal{A} \in \mathcal{I}(\mathcal{R})$ which separates B_σ and satisfies $j(J_\sigma) = \mathcal{A}^*$ is a group algebra for $(G,\, \mathcal{R})$.*

Proof: By Theorems 4.1 and 4.2, $\theta : \operatorname{Rep}\mathcal{A} \to \mathcal{R}$ is bijective, so when we pass to the C*–envelopes to get $C^*_\sigma(G_d) \subset M(C^*(\mathcal{A}))$, then $\theta : \operatorname{Rep}\, C^*(\mathcal{A}) \to \operatorname{Rep}\, C^*_\sigma(G_d)$ is injective with image $\big\{\pi \;\big|\; x \to \pi(\delta_x) \text{ in } \mathcal{R}\ \big\}$ since the Hilbert space representations of a Banach *–algebra are the same as those of its C*–envelope. ∎

A natural class of functionals in $J^*_\sigma(\mathcal{R})$ to consider, are those associated with finite (σ-additive, signed) Borel measures on G according to $\omega_\mu(f) = \int_G f(x)\, d\mu(x)$, with f any bounded Borel function. Note that these functionals are continous w.r.t. the supremum norm, i.e. $|\omega_\mu(f)| \le \|\omega_\mu\| \cdot \|f\|_\infty$

for f bounded and Borel. Since we need integrable maps to define such functionals, denote by $\mathrm{Rep}_\sigma^B G$ those representations whose coefficient functions are Borel. So for any $\mathcal{R} \subset \mathrm{Rep}_\sigma^B G$, we can restrict the functionals ω_μ to B_σ, and find $\omega_\mu \restriction B_\sigma \in K_\sigma^* \subset J_\sigma^*$. Denote the set of these functionals by $\mathcal{M}(G) \subset K_\sigma^*(\mathcal{R})$. Then Lemma 3.1 has a well-known extension: given ω_μ as above, and a $\pi \in \mathrm{Rep}_\sigma^B G$, then there is a unique operator $\pi(\omega_\mu) \in \mathcal{B}(\mathcal{H}_\pi)$ such that $\|\pi(\omega)\| \leq \|\omega\|$ and

$$\int (\psi,\, \pi(x)\, \varphi) d\mu(x) = (\psi,\, \pi(\omega_\mu)\, \varphi)$$

for all $\psi,\, \varphi \in \mathcal{H}_\pi$.

We will also need to integrate the map $x \to \delta_x * \beta =: h_\beta(x) \in J_\sigma^*$, so recall the two conditions of measurability for a Banach space–valued function w.r.t. a Borel measure μ, cf. Lemma 9, Sect III.6.7 of Dunford and Schwartz [DS]: (i) inverse images of Borel sets are Borel, (ii) on the complement of a null set, the range of the function must be separable. So define for a given $\mathcal{R} \subset \mathrm{Rep}_\sigma^B G$:

$$F_B(\mathcal{R}) := \Big\{ \beta \in J_\sigma^* \;\Big|\; h_\beta^{-1}(S) \text{ is Borel when } S \subset J_\sigma^* \text{ is Borel}, \\ h_\beta(G) \text{ is separable} \Big\}.$$

If h_β is continuous and G is separable, then $\beta \in F_B(\mathcal{R})$.

Theorem 4.4. *Let* $\mathcal{R} \subset \mathrm{Rep}_\sigma^B G$, *let* $\mathcal{A} \subseteq F_B(\mathcal{R})$ *be a d-ideal, and let* $\omega \in \mathcal{M}(G) \cap \mathcal{A}$. *Then* $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ *for all* $\pi \in \mathrm{Rep}\,\mathcal{A}$, *and hence* θ *is injective on* $(\mathrm{Rep}\,\mathcal{A}) \restriction (\mathcal{M}(G) \cap \mathcal{A})$. *In particular, if* $\mathcal{A} \subset \mathcal{M}(G) \cap F_B(\mathcal{R})$, *then* $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}_\sigma G_d$ *is injective.*

Proof: Let $\omega \in \mathcal{M}(G) \cap \mathcal{A}$ with associated Borel measure μ. Now for any $\beta \in \mathcal{A}$, the function $x \to \delta_x * \beta \in \mathcal{A}$ is measurable by definition of $F_B(\mathcal{R})$, and bounded by $\|\beta\|$. Thus, the Bochner integral $B := \int_G \delta_x * \beta \, d\mu(x)$ is well–defined (cf. Chapter III [DS]), and $B \in \mathcal{A}$. Then

$$\xi(B) = \int_G \xi(\delta_x * \beta)\, d\mu(x) \qquad \forall\, \xi \in \mathcal{A}^* \tag{1}$$

and in particular for $\xi = j(f)$, $f \in J_\sigma$, we have

$$\begin{aligned} j(f)(B) = B(f) &= \int_G (\delta_x * \beta)(f)\, d\mu(x) = \omega_x\big((\delta_x * \beta)(f)\big) \\ &= (\omega * \beta)(f) \qquad \forall\, f \in J_\sigma\,. \end{aligned}$$

Thus $B = \omega * \beta = \int_G \delta_x * \beta \, d\mu(x)$, and so, using Eq. (1) again:

$$\xi(\omega * \beta) = \int_G \xi(\delta_x * \beta) \, d\mu(x) = \omega_x\big(\xi(\delta_x * \beta)\big) \tag{2}$$

for all $\xi \in \mathcal{A}^*$, $\beta \in \mathcal{A}$, $\omega \in \mathcal{M}(G) \cap \mathcal{A}$. Now choose $\xi(\omega) = (\varphi, \pi(\omega)\psi)$, $\pi \in \operatorname{Rep} \mathcal{A}$, $\varphi, \psi \in \mathcal{H}_\pi$, then by Eq (2) we find:

$$\begin{aligned}\xi(\omega * \beta) = \big(\varphi, \pi(\omega * \beta)\psi\big) &= \omega_x\big((\varphi, \pi(\delta_x * \beta)\psi)\big) \\ &= \omega_x\big((\varphi, \theta(\pi)(x)\, \pi(\beta)\psi\big) \\ &= \big(\varphi, \theta(\pi)_{\mathcal{A}}(\omega)\, \pi(\beta)\psi\big)\end{aligned}$$

for all $\pi \in \operatorname{Rep} \mathcal{A}$, $\varphi, \psi \in \mathcal{H}_\pi$, $\omega, \beta \in \mathcal{A}$. By nondegeneracy of $\pi \in \operatorname{Rep} \mathcal{A}$ we get $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ for all $\omega \in \mathcal{A}$. Hence $\pi = \theta(\pi)_{\mathcal{A}}$ for all $\pi \in \operatorname{Rep} \mathcal{A}$. ∎

Note that we did not require that $\mathcal{A} \in \mathcal{I}(\mathcal{R})$, and so $\theta(\operatorname{Rep} \mathcal{A})$ need not have anything to do with $\mathcal{R}$. In fact, if we only want to study convolution algebras of measures, then it is natural to take $\mathcal{R} = \operatorname{Rep}^B_\sigma G$, and to analyze these algebras in $J^*_\sigma(\mathcal{R})$.

Corollary 4.5. *Let $\mathcal{R} \subseteq \operatorname{Rep}^B_\sigma G$ and $\mathcal{M}(G) \cap F_B(\mathcal{R}) \neq \emptyset$. For any subset $X \subseteq \mathcal{M}(G) \cap F_B(\mathcal{R})$ let $\mathcal{A}(X)$ be the the d-ideal generated by X with C*–envelope $\mathcal{D}(X)$, and let $\mathcal{R}_X := \theta\big(\operatorname{Rep} \mathcal{D}(X)\big)$. Then*

(i) $\mathcal{D}(X)$ is a σ–group algebra for the pair $\big(G, \mathcal{R}_X\big)$.

(ii) If $\mathcal{A}(X)$ is in $\mathcal{I}(\mathcal{R})$ and separates B_σ then $\mathcal{D}(X)$ is a σ–group algebra for $(G, \mathcal{R})$.

Proof: Observe that since all σ–translations of Borel measures are Borel measures, and all σ–convolutions of Borel measures are Borel measures, $\mathcal{A}(X) \cap \mathcal{M}(G) \subset F_B(\mathcal{R})$ is dense in $\mathcal{A}(X)$ (the convolutions defined here coincide with the usual ones for measures). By Theorem 4.4 the map $\theta : \operatorname{Rep} \mathcal{A}(X) \to \mathcal{R}_X \subset \operatorname{Rep}_\sigma G_d$ is bijective, hence since the Hilbert space representation theory of a Banach *–algebra is the same as that of its enveloping C*–algebra, it follows that $\mathcal{D}(X)$ is a σ–group algebra for the pair $\big(G, \mathcal{R}_X\big)$. By Theorem 4.1, if $\mathcal{A}(X) \in \mathcal{I}(\mathcal{R})$ separates B_σ, then $\mathcal{R}_X = \mathcal{R}$. ∎

The subsets of $\mathcal{M}(G) \cap F_B(\mathcal{R})$ behave very well, e.g. if $X \subset Y \subseteq \mathcal{M}(G) \cap F_B(\mathcal{R})$ such that for their d-ideals $\mathcal{D}(X) \subset \mathcal{D}(Y)$, then by Corollary 4.5 and Proposition 1.6, $\mathcal{R}_X$ is a direct summand of $\mathcal{R}_Y$.

Next we would like to return to the Gelfand-Raikov problem, i.e. to consider the existence of a σ–group algebra for $\mathcal{R} = \mathrm{Rep}_\sigma G$. For the rest of this section we will maintain this choice for $\mathcal{R}$.

Let us start by listing relevant structure of the embedding $C^*(G_d) \subset M(C^*(G))$ for a locally compact group G.

(i) The map $G \to UM(C^*(G))$ by $x \to \delta_x$ is continuous w.r.t. the strict topology, i.e. if $x_\nu \to x$ is a convergent net in G, then $\|(\delta_{x_\nu} - \delta_x)A\| \to 0 \leftarrow \|A(\delta_{x_\nu} - \delta_x)\|$ for all $A \in C^*(G)$.

(ii) The action of $C^*(G_d)$ on $C^*(G)$ has cyclic elements, in the sense that $C^*(G) = C^*\left\{\, \delta_x A \;\middle|\; x \in G \,\right\}$ for some $A \in C^*(G)$. For example we can take for A any nonzero element of $C_c(G)$.

(iii) The inverse of the extension map $\theta : \mathrm{Rep}\, C^*(G) \to \mathrm{Rep}\, G$ is given by $\pi \in \mathrm{Rep}\, G \to \pi_{\mathcal{A}} \in \mathrm{Rep}\, C^*(G)$ where $\pi_{\mathcal{A}}(f) := \int_G f(x)\, \pi(x)\, d\mu(x)$, $f \in L^1(G)$ (cf. Lemma 3.1 for generalisation of this representation to functionals).

Returning now to the situation where G is not locally compact, in the light of property (i), we want to select a subalgebra of J_σ^* with good continuity properties w.r.t. translations. So, for any topological group G we define:

Def. $\mathcal{L}_0(G) := \left\{\, \omega \in J_\sigma^* \;\middle|\; \big\|(\delta_x - \mathbb{I}) * \delta_y * \omega\big\| \to 0 \quad \text{as} \quad x \to e \quad \forall\, y \in G \,\right\}$

$\mathcal{L}(G) := \mathcal{L}_0(G) \cap \mathcal{L}_0(G)^*$

Thus by Theorem 3.4, $\omega \in \mathcal{L}_0(G)$ iff $\big\|\omega \circ \big(\sigma(x, y)\, \lambda_{xy} - \lambda_y\big)\big\| \to 0$ as $x \to e$ for all $y \in G$. Note that it is the *uniform* continuity which distinguishes $\mathcal{L}_0$ because we always have pointwise continuity $\lim\limits_{x \to e}((\delta_x - \mathbb{I}) * \omega)(f) = 0$ for all $\omega \in J_\sigma^*$, $f \in J_\sigma$ by Theorem 2.1(v).

Theorem 4.6. *(i)* $\mathcal{L}_0(G)$ *is a closed space, hence so is* $\mathcal{L}(G)$.

(ii) If there is a representation in $\mathrm{Rep}_\sigma G$ *which is not uniformly*

continuous, then $\delta_x \notin \mathcal{L}_0(G)$ *for any* $x \in G$.

(iii) $\mathcal{L}_0(G)$ *is a right ideal in* J_σ^*, *hence a Banach algebra. Thus* $\mathcal{L}(G)$ *is a Banach *–algebra.*

(iv) $\delta_x * \mathcal{L}(G) \subseteq \mathcal{L}(G) \supseteq \mathcal{L}(G) * \delta_x$ *for all* $x \in G$, *hence* δ_G *is in the relative multiplier algebra of* $\mathcal{L}(G)$.

(v) If G *is locally compact, then* $L^1(G) \subset \mathcal{L}(G)$, *where as usual we identify* $h \in L^1(G)$ *with* $\omega_h \in J_\sigma^*$ *by* $\omega_h(f) := \int h(x)\, f(x)\, d\mu(x)$, $f \in J_\sigma$ *and* μ *the Haar measure.*

Proof: (i) Consider a sequence $\{\omega_n\} \subset \mathcal{L}_0(G)$ which converges to $\omega \in J_\sigma^*$. Then

$$\begin{aligned}\|(\delta_x - \mathbb{I}) * \delta_y * \omega\| &\leq \|\delta_x * \delta_y * (\omega - \omega_n)\| + \|\delta_x * \delta_y * \omega_n - \delta_y * \omega_n\| \\ &\qquad + \|\delta_y * (\omega_n - \omega)\| \\ &\leq 2\|\omega - \omega_n\| + \|(\delta_x - \mathbb{I}) * \delta_y * \omega_n\| \\ &\xrightarrow[e]{x} 2\|\omega - \omega_n\| \xrightarrow[\infty]{n} 0\end{aligned}$$

i.e. $\omega \in \mathcal{L}_0(G)$.

(ii) Let $\pi \in \mathrm{Rep}_\sigma G$, $\|\pi(x) - \mathbb{I}\| \not\to 0$ as $x \to e$. Then

$$\begin{aligned}\Big\|(\delta_x - \mathbb{I}) * \delta_y * \delta_z\Big\| &= \|\delta_x - \mathbb{I}\| = \sup\big\{\, |f(x) - f(e)| \;\big|\; f \in J_\sigma,\ \|f\|_* \leq 1 \,\big\} \\ &\geq \sup\big\{\, \big|(\psi,\, \pi(x)\xi) - (\psi,\, \xi)\big| \;\big|\; \psi, \xi \in \mathcal{H}_\pi,\ \|\psi\| \leq 1 \geq \|\xi\| \,\big\} \\ &= \sup\big\{\, \big|(\psi,\, (\pi(x) - \mathbb{I})\xi)\big| \;\big|\; \psi, \xi \in \mathcal{H}_\pi,\ \|\psi\| \leq 1 \geq \|\xi\| \,\big\} \\ &= \|\pi(x) - \mathbb{I}\|\end{aligned}$$

and this does not go to zero as $x \to e$. Thus $\delta_z \notin \mathcal{L}_0(G)$ for any z.

(iii) Let $\omega \in \mathcal{L}_0(G)$ and $\varphi \in J_\sigma^*$, then

$$\Big\|(\delta_x - \mathbb{I}) * \delta_y * (\omega * \varphi)\Big\| \leq \Big\|(\delta_x - \mathbb{I}) * \delta_y * \omega\Big\| \cdot \|\varphi\| \xrightarrow[e]{x} 0$$

for all $y \in G$. Thus $\omega * \varphi \in \mathcal{L}_0(G)$. To show that $\mathcal{L}(G)$ is a Banach *–subalgebra of J_σ^*, note that we already have norm–closure, and that it is closed under involution, so it only remains to check that it is an algebra. Let $\omega,\ \varphi \in \mathcal{L}(G)$, hence $\omega,\ \omega^* \in \mathcal{L}_0(G) \ni \varphi,\ \varphi^*$. Since

$\mathcal{L}_0(G)$ is a right ideal, it contains $\omega * \varphi$, as well as $\varphi^* * \omega^* = (\omega * \varphi)^*$. Thus $\omega * \varphi \in \mathcal{L}(G)$.

(iv) By (iii) we already know that $\mathcal{L}_0(G) * \delta_x \subseteq \mathcal{L}_0(G)$. Let $\omega \in \mathcal{L}_0(G)$, $z \in G$, then

$$\begin{aligned}\big\|(\delta_x - \mathbb{I}) * \delta_y(\delta_z * \omega)\big\| &= \big\|(\delta_x - \mathbb{I}) * \big(\sigma(y,z)\,\delta_{yz}\big) * \omega\big\| \\ &= \big\|(\delta_x - \mathbb{I}) * \delta_{yz} * \omega\big\| \xrightarrow[e]{x} 0\end{aligned}$$

for all $y \in G$. Thus $\delta_z * \omega \in \mathcal{L}_0(G)$, i.e. $\delta_x * \mathcal{L}_0(G) \subseteq \mathcal{L}_0(G)$ for all $x \in G$. Now let $\omega \in \mathcal{L}(G) \subset \mathcal{L}_0(G)$, hence $\delta_x * \omega \in \mathcal{L}_0(G)$, and also $(\delta_x * \omega)^* = \omega^* * \delta_{-x} \in \mathcal{L}_0(G)$ because $\omega^* \in \mathcal{L}_0(G)$. Thus $\delta_x * \omega \in \mathcal{L}(G)$, and likewise $\omega * \delta_x \in \mathcal{L}(G)$, hence $\delta_x * \mathcal{L}(G) \subseteq \mathcal{L}(G) \supseteq \mathcal{L}(G) * \delta_x$.

(v) Here G is locally compact. Now

$$\begin{aligned}\|\omega_h\| &= \sup\big\{\,|\omega_h(f)| \;\big|\; f \in J_\sigma,\ \|f\|_* \le 1\,\big\} \\ &\le \sup\big\{\,|\omega_h(f)| \;\big|\; f \in J_\sigma,\ \|f\|_\infty \le 1\,\big\}\end{aligned}$$

since $\|f\|_\infty \le \|f\|_*$. Then by $|\omega_h(f)| \le \|f\|_\infty \|h\|_1$ (where $\|\cdot\|_1$ denotes the L^1–norm), it follows that $\|\omega_h\| \le \|h\|_1$, and hence that the norm closure of a set in J_σ^* contains its L^1–closure. Since $\mathcal{L}(G)$ is norm–closed, it thus suffices to show that $C_c(G) \subset \mathcal{L}(G)$. Let $h \in C_c(G)$, then

$$\big\|(\delta_x - \mathbb{I}) * \delta_y * \omega_h\big\| = \sup\big\{\,\big|\big((\delta_x - \mathbb{I}) * \delta_y * \omega_h\big)(f)\big| \;\big|\; \|f\|_* \le 1\,\big\}\ . \quad \text{–}(*)$$

Now
$$\begin{aligned}\big|\big((\delta_x - \mathbb{I}) * \delta_y * \omega_h\big)(f)\big| &= \big|\omega_h(\lambda_y(\lambda_x - 1)f)\big| \\ &= \Big|(\omega_h)_z\big(\big(\sigma(x,y)\,\lambda_{xy} - \lambda_y\big)f(z)\big)\Big| \\ &= \Big|(\omega_h)_z\big(\sigma(x,y)\,\sigma(xy,z)\,f(xyz) - \sigma(y,z)\,f(yz)\big)\Big| \\ &= \Big|\int_G \sigma(y,z)\Big(\sigma(x,yz)\,f(xyz) - f(yz)\Big)h(z)\,d\mu(z)\Big| \\ &= \Big|\int_G f(s)\Big(\sigma(y,y^{-1}x^{-1}s)\,\sigma(x,x^{-1}s)\,h(y^{-1}x^{-1}s) - \sigma(y,y^{-1}s)\,h(y^{-1}s)\Big)\,d\mu(s)\Big| \\ &\le \|f\|_\infty \cdot \int_G \Big|\overline{\sigma(y^{-1},x^{-1}s)} \cdot \overline{\sigma(x^{-1},s)}\,h(y^{-1}x^{-1}s) - \overline{\sigma(y^{-1},s)}\,h(y^{-1}s)\Big|\,d\mu(s) \\ &\le \|f\|_* \cdot \int_G \Big|\overline{\sigma(y^{-1},x^{-1}s)} \cdot \overline{\sigma(x^{-1},s)}\,h(y^{-1}x^{-1}s) - \overline{\sigma(y^{-1},s)}\,h(y^{-1}s)\Big|\,d\mu(s)\end{aligned}$$

where we made use of $\sigma(a, a^{-1}b) = \overline{\sigma(a^{-1}, b)}$. The integrand is bounded by $2\|h\|$, of compact support contained in $xy\,\mathrm{supp}(h) \cap y\,\mathrm{supp}(h)$ and goes pointwise to zero as x approaches e, independently of f. Thus the Lebesgue dominated convergence theorem applies, and we get that the last integral goes to zero as x approaches e, independently of f. Thus from $(*)$ we get that $\big\|(\delta_x - \mathbb{I}) * \delta_y * \omega_h\big\| \xrightarrow[e]{x} 0$ for all $y \in G$, i.e. $\omega_h \in \mathcal{L}_0(G)$. Now

$$\begin{aligned}\omega_h^*(f) = \overline{\omega_h(\gamma f)} &= \overline{\int_G h(x)\,\overline{f(x^{-1})}\,d\mu(x)} = \int_G \overline{h(x)}\,f(x^{-1})\,d\mu(x)\\ &= \int_G \overline{h(x^{-1})}\,f(x)\,\Delta(x)\,d\mu(x) = \omega_{\widetilde{h}}(f)\end{aligned}$$

where Δ is the modular function of G and $\widetilde{h}(x) := \overline{h(x^{-1})}\Delta(x)$. As $\widetilde{h} \in C_c(G)$, it follows that $\omega_h^* \in \mathcal{L}_0(G)$ and hence $\omega_h \in \mathcal{L}(G)$ for all $h \in C_c(G)$. ∎

To show that $\mathcal{L}(G)$ is not an empty construct, we need to give some examples of groups G which are not locally compact, with $\mathcal{L}(G) \neq \{0\}$. In the next section is an example for groups which are amenable but not locally compact (which is a large class).

One might be tempted to regard $\mathcal{L}(G)$ as a possible candidate for a group algebra, however, in general it is too large. For example, if G is amenable, $\sigma = 1$, then all its (two-sided) invariant means are in $\mathcal{L}(G)$. When G is locally compact Abelian, but not compact, we know it has uncountably many invariant means not identified with elements of $L^1(G)$, cf. [CG]. We are therefore interested in the d-ideals contained in $\mathcal{L}(G)$. By Theorem 4.6(iv), $\mathcal{L}(G)$ is itself a d-ideal, and it is easy to see that its closed two-sided ideals are also d-ideals. In the case that G is locally compact, of course the closure of $L^1(G)$ is a d-ideal in $\mathcal{L}(G)$.

Let us now consider the continuous representations. Recall first that for each $\pi \in \mathrm{Rep}_\sigma G$ we have by Theorem 3.3(iii) a *–representation $\pi : J_\sigma^* \to \mathcal{B}(\mathcal{H})$ such that

$$\big(\psi,\, \pi(\omega)\xi\big) = \omega_x\big((\psi,\, \pi(x)\xi)\big) \qquad \forall\, \psi,\, \xi \in \mathcal{H}\,.$$

From this it follows immediately that $\pi(\delta_x) = \pi(x)$, and thus the map $\mathrm{Rep}_\sigma G \to \mathrm{Rep}\, J_\sigma^*$ is injective. It need not be surjective, i.e. for a general $\pi \in \mathrm{Rep}\, J_\sigma^*$ the map $x \to \pi(\delta_x)$ need not be strong operator continuous. Consider a d-ideal $\mathcal{A}$ of $\mathcal{L}(G)$. By $M_d(G) \subset M(\mathcal{A})$, we can use the canonical extension of representations to define a map θ from $\mathrm{Rep}\,\mathcal{A}$ to σ–representations of G, by $\theta(\pi)(x) := \pi(\delta_x)) = \operatorname*{s-lim}_\alpha \pi(\delta_x E_\alpha)$ for any approximate identity $\{E_\alpha\}$ of $\mathcal{A}$. (Equivalently, $\theta(\pi)$ is uniquely determined by the equation $\theta(\pi)(x) \cdot \pi(\omega)\psi = \pi(\delta_x * \omega)\psi$ for all $\omega \in \mathcal{A}$, $\psi \in \mathcal{H}_\pi$.) We now show that $\theta(\pi)$ is strong operator continuous.

Theorem 4.7. *For a d-ideal $\mathcal{A} \subset \mathcal{L}(G)$, we have that $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}_\sigma G$, i.e. $x \to \theta(\pi)(x)$ is strong operator continuous for all $\pi \in \mathrm{Rep}\,\mathcal{A}$ for all nonzero d-ideals $\mathcal{A}$ of $\mathcal{L}(G)$, i.e. $\mathcal{A} \in \mathcal{I}(\mathrm{Rep}_\sigma G)$.*

Proof: Let $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\omega \in \mathcal{A}$ and $\psi \in \mathcal{H}_\pi$, then

$$\begin{aligned}\big\|\big(\theta(\pi)(x) - \mathbb{I}\big)\pi(\omega)\psi\big\| &= \big\|\pi(\delta_x * \omega - \omega)\psi\big\| \\ &\leq \big\|\delta_x * \omega - \omega\big\| \cdot \|\psi\| \xrightarrow[e]{x} 0\,.\end{aligned}$$

Since π is nondegenerate, $\pi(\mathcal{A})\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$. Let $\{\psi_n\} \subset \pi(\mathcal{A})\mathcal{H}_\pi$ be a sequence converging to $\psi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}\big\|\big(\theta(\pi)(x) - \mathbb{I}\big)\psi\big\| &\leq \big\|\theta(\pi)(x)(\psi - \psi_n)\big\| + \big\|\big(\theta(\pi)(x) - \mathbb{I}\big)\psi_n\big\| + \|\psi_n - \psi\| \\ &\leq 2\|\psi_n - \psi\| + \big\|\big(\theta(\pi)(x) - \mathbb{I}\big)\psi_n\big\| \\ &\xrightarrow[e]{x} 2\|\psi_n - \psi\| \xrightarrow[\infty]{n} 0\,.\end{aligned}$$

∎

Now in the light of Theorem 4.7, a d-ideal $\mathcal{A} \subset \mathcal{L}(G)$ will be an adequate group algebra for G if we can show that θ is bijective. Clearly we can now use Theorems 4.1–4.4 to find sufficient conditions.

Note that at this point, we have obtained a distinguished subset of representations $\theta(\mathrm{Rep}\,\mathcal{L}(G)) \subseteq \mathrm{Rep}_\sigma G$ which is the homomorphic image of the representation theory of a C*-algebra. By the construction preceding Prop. 1.6, we can also obtain from $M_d(G) \subset M(\mathcal{L}(G))$ another distinguished set of representations isomorphic to the representation theory of a C*-algebra.

Corollary 4.8. *Let $\mathcal{A} \subset \mathcal{L}(G)$ be a d-ideal with C*-envelope $\mathcal{D}$, then*

(i) If $\mathcal{A}$ separates B_σ and satisfies $j(J_\sigma) = \mathcal{A}^*$, then $\mathcal{D}$ is a group algebra for $(G, \mathrm{Rep}_\sigma G)$.

(ii) If G is separable and $\mathcal{A} \subseteq \mathcal{M}(G) \cap \mathcal{L}(G)$, then $\mathcal{D}$ is a group algebra for $(G, \theta(\mathrm{Rep}\,\mathcal{A}))$, (note that $\theta(\mathrm{Rep}\,\mathcal{A})) \subseteq \mathrm{Rep}_\sigma G)$. If in addition $\mathcal{A}$ separates B_σ, then $\mathcal{D}$ is a group algebra for $(G, \mathrm{Rep}_\sigma G)$.

(iii) If G is locally compact, then $\mathrm{C}^*_\sigma(G) \cong \mathcal{D}$ where $\mathcal{D}$ is the enveloping C-algebra of $\mathcal{A} = \overline{\mathcal{M}(G) \cap \mathcal{L}(G)}$.*

Proof: (i) This follows from Corollary 4.3 and Theorem 4.7.

(ii) Note that $\mathcal{R} = \mathrm{Rep}_\sigma G \subset \mathrm{Rep}_\sigma^B G$, and that if $\omega \in \mathcal{M}(G) \cap \mathcal{L}(G)$ then $h_\omega(x) := \delta_x * \omega$ is norm continuous in x by definition of $\mathcal{L}(G)$. Thus since G is separable, $\omega \in F_B(\mathcal{R})$, i.e. $\mathcal{A} \subset F_B(\mathcal{R})$. Thus we can apply Theorem 4.4 and Corollary 4.5 to $\mathcal{A}$. By Corollary 4.5 and Theorem 4.7 the claim above follows.

(iii) Observe that since all σ–translations of Borel measures are Borel measures, and all σ–convolutions of Borel measures are Borel measures, $\mathcal{M}(G) \cap \mathcal{L}(G)$ is a *-algebra stable under multiplication by δ_G, and so $\mathcal{A}$ is a d-ideal. (The convolutions defined here coincide with the usual ones for measures). By Theorem 4.6 $L^1(G) \subseteq \mathcal{M}(G) \cap \mathcal{L}(G)$, and as it separates B_σ, it follows from (ii) that $\mathcal{D}$ is a group algebra for $(G, \mathrm{Rep}_\sigma G)$. By uniqueness (Theorem 1.5) the isomorphism $\mathcal{D} \cong \mathrm{C}^*_\sigma(G)$ follows. ∎

The characterization of $\mathrm{C}^*_\sigma(G)$ in 4.8 (iii) above, is interesting because it uses neither the Haar measure nor the behaviour of measures w.r.t. compact sets, and it seems to improve the criterion in [DvR].

Since we have an example of a group with a faithful continuous representation, but no irreducible ones (cf. Exmp 5.2 in [Pes]), for such a group we know that it has no nonzero d-ideals satisfying the conditions in Corollary 4.8.

5. Examples.

We want to show that $\mathcal{L}(G) \neq \emptyset$ for some groups which are not locally compact. Let G be amenable but not locally compact with a faithful continuous representation. This is a large class, e.g. any Abelian group is amenable (and there are many examples of these with faithful continuous representations). For a non-abelian example, take the unitary group of any nuclear C*-algebra (cf. [Pa]) with the relative weak topology which obviously has a faithful continuous representation since the C*-algebra has.

For our purposes we will take the definition of "amenable group" to mean that there is a left–invariant mean n on $K_\sigma \supset J_\sigma$. (This is weaker than the usual definition, since K_σ is uniformly continuous cf Theorem 2.1(v)).

Lemma 5.1. *Let G be amenable and $\sigma = 1$.*

(i) There is a two-sided invariant mean $m \in K^* \subset J^*$, i.e. $m = m \circ \lambda_a = m \circ \rho_a$ for all $a \in G$.

(ii) Let $m \in K^*$ be a two-sided invariant mean, then $m \in \mathcal{L}(G)$.

Proof: **(i)** We adapt the usual proof, cf. 17.10 in [HR1].

Let $n \in K^*$ be a left-invariant mean and recall that

$$n^*(f) := \overline{n(f^*)} = \overline{n_x\big(\overline{f(x^{-1})}\big)} \quad \text{where} \quad f^*(x) := \overline{f(x^{-1})}\,.$$

Clearly $n^*(\mathbb{I}) = 1$ and $n^*(f) \geq 0$ when $f \geq 0$. Moreover

$$\begin{aligned}\overline{\big(n^* \circ \rho_a\big)(f)} &= \overline{n_x^*\big(f(xa)\big)} = n_x\big(\overline{f(x^{-1}a)}\big) = n_x\big(\overline{f((a^{-1}x)^{-1})}\big) \\ &= n_x(f^*(a^{-1}x)) = n_x(f^*(x)) = \overline{n^*(f)}\;,\end{aligned}$$

i.e. n^* is right–invariant. Define $m := n * n^*$, i.e. $m(f) := n_x(n_y^*(f(xy))) = n_y^*(n_x(f(xy)))$ for $f \in K$ (cf. Theorem 3.3(i)). Then $m \in K^*$ is positive and normalised and

$$\begin{aligned}(m \circ \lambda_a)(f) &= n_y^*(n_x((\lambda_a f)(xy))) = n_y^*(n_x(\lambda_a(\rho_y f)(x))) \\ &= n_y^*(n_x(\rho_y f(x))) = m(f)\;,\end{aligned}$$

$$(m \circ \rho_a)(f) = n_x(n_y^*((\rho_a f)(xy))) = n_x(n_y^*(\rho_a(\lambda_x f)(y))) = n_x(n_y^*((\lambda_x f)(y))) = m(f)\,.$$

Thus m is two-sided invariant.

(ii) For $m \in K^*$ a two-sided invariant mean as above,

$(\delta_x - \mathbb{I}) * \delta_y * m(f) = m\left(\lambda_{xy}(f) - \lambda_y(f)\right) = m(f) - m(f) = 0,$ and

$$\begin{aligned}(\delta_x - \mathbb{I}) * \delta_y * m^*(f) &= m^*\left(\lambda_{xy}(f) - \lambda_y(f)\right) \\ &= \overline{m\big((\lambda_{xy} - \lambda_y)(f)^*\big)} = \overline{m_z\left(\overline{(\lambda_{xy} - \lambda_y)(f)(z^{-1})}\right)} \\ &= \overline{m_z\left(\overline{f(xyz^{-1})} - \overline{f(yz^{-1})}\right)} = \overline{m_z\left(\rho_{y^{-1}x^{-1}} f^*(z) - \rho_{y^{-1}} f^*(z)\right)} \\ &= \overline{m(f^*)} - \overline{m(f^*)} = 0\,.\end{aligned}$$

Thus $m \in \mathcal{L}(G)$. ∎

These two–sided invariant means do not by themselves seem particularly useful elements of $\mathcal{L}(G)$, because by the invariance they cannot separate B_σ (cf. Theorem 4.3), and the only representation they can produce on G via θ is the identity representation. We define a more promising set of functionals.

Def. Let $\sigma \neq 1$, and recall that $f \cdot h \in K_1$ if $f \in K_{\overline{\sigma}}$ and $h \in K_\sigma$. Let $m \in K_1^*$ be a two–sided invariant mean of the amenable group G, and define a functional $m^f \in K_\sigma^*$ by $m^f(h) := m(f \cdot h) = m_x(f(x)h(x))$.

Proposition 5.2. *Let G be amenable, and $f \in J_{\overline{\sigma}}$. Then $m^f \in \mathcal{L}(G)$ for all two-sided invariant means $m \in K_1^*$.*

Proof: It suffices to prove the theorem for $f(x) = (\varphi, \pi(x)\psi)$, $\pi \in \mathrm{Rep}_{\overline{\sigma}} G$, $\varphi, \psi \in \mathcal{H}_\pi$. Now

$$\left\|((\delta_x - \mathbb{I}) * \delta_y * m^f\right\| = \sup\left\{\, \left|\left((\delta_x - \mathbb{I}) * \delta_y * m^f\right)(h)\right| \;\middle|\; h \in J_\sigma,\ \|h\|_* \leq 1 \,\right\}$$

so $\left((\delta_x - \mathbb{I}) * \delta_y * m^f\right)(h) = m\left(f \cdot \big(\sigma(x,y)\lambda_{xy} - \lambda_y\big)h\right)$

$$\begin{aligned}&= m_z\left(f(z) \cdot \big(\sigma(x,y)\,\sigma(xy,z)\,h(xyz) - \sigma(y,z)\,h(yz)\big)\right) \\ &= m_z\left(f(y^{-1}x^{-1}z)\sigma(x,y)\,\sigma(xy, y^{-1}x^{-1}z)\,h(z) - f(y^{-1}z)\sigma(y, y^{-1}z)\,h(z)\right) \\ &= m_z\left(h(z)\Big(f(y^{-1}x^{-1}z)\sigma(x,y)\,\overline{\sigma(y^{-1}x^{-1}, z)} - f(y^{-1}z)\overline{\sigma(y^{-1}, z)}\Big)\right) \quad \text{–}[\mathbf{1}]\end{aligned}$$

using $\sigma(a, a^{-1}b) = \overline{\sigma(a^{-1}, b)}$. Now

$$\begin{aligned}H_{x,y}(z) &:= f(y^{-1}x^{-1}z)\sigma(x,y)\,\overline{\sigma(y^{-1}x^{-1}, z)} - f(y^{-1}z)\overline{\sigma(y^{-1}, z)} \\ &= \sigma(x,y)\,\overline{\sigma(y^{-1}x^{-1}, z)}\big(\varphi,\, \pi(y^{-1}x^{-1}z)\psi\big) - \overline{\sigma(y^{-1}, z)}\big(\varphi,\, \pi(y^{-1}z)\psi\big)\end{aligned}$$

$$= \sigma(x,y)\big(\pi(xy)\varphi,\,\pi(z)\psi\big) - \big(\pi(y)\varphi,\,\pi(z)\psi\big) \qquad\qquad \text{by } \pi\in\mathrm{Rep}_{\overline{\sigma}}G$$

$$= \big(\big(\sigma(x,y)\pi(xy)-\pi(y)\big)\varphi,\,\pi(z)\psi\big)$$

so $\|H_{x,y}\|_* = \sup\left\{\,\big|\big(\big(\sigma(x,y)\pi(xy)-\pi(y)\big)\varphi,\,\pi(A)\psi\big)\big| \;\big|\; A\in \mathrm{C}^*_\sigma(G_d),\ \|A\|\le 1\,\right\}.$

$$\le \big\|\big(\sigma(x,y)\pi(xy)-\pi(y)\big)\varphi\big\|\cdot\|\psi\|$$

$$\le \big|\sigma(x,y)-1\big|\cdot\|\varphi\|\cdot\|\psi\| + \big\|\big(\pi(xy)-\pi(y)\big)\varphi\big\|\cdot\|\psi\|$$

$$\xrightarrow[e]{x} 0 \qquad \forall\, y$$

Thus $\|H_{x,y}\|_* \xrightarrow[e]{x} 0$ for all y, and so by equation [1],
$\left|\,\big((\delta_x-\mathbb{I})*\delta_y*m^f\big)\,(h)\right| \le \|m\|\cdot\|hH_{x,y}\|_* \le \|m\|\cdot\|h\|_*\|H_{x,y}\|_* \xrightarrow[e]{x} 0$
for all y. Thus $\big\|(\delta_x-\mathbb{I})*\delta_y*m^f\big\| \xrightarrow[e]{x} 0$ for all y, i.e. $m^f\in\mathcal{L}_0(G)$.
To prove the same for $(m^f)^*$,

$$\overline{((\delta_x-\mathbb{I})*\delta_y*(m^f)^*)\,(h)} = \overline{(m^f)^*\,\big(\big(\sigma(x,y)\lambda_{xy}-\lambda_y\big)h\big)}$$

$$= m^f\,\big(\big(\sigma(x,y)\lambda_{xy}-\lambda_y\big)h^*\big) = m_z\left(f(z)\cdot\overline{\big(\sigma(x,y)\lambda_{xy}-\lambda_y\big)h(z^{-1})}\right)$$

$$= m_z\left(f(z)\cdot\big(\overline{\sigma(x,y)\,\sigma(xy,z^{-1})\,h(xyz^{-1})} - \overline{\sigma(y,z^{-1})\,h(yz^{-1})}\big)\right)$$

$$= m_z\left(f(zxy)\,\overline{\sigma(x,y)}\,\overline{\sigma(xy,y^{-1}x^{-1}z^{-1})h(z^{-1})} - f(zy)\overline{\sigma(y,y^{-1}z^{-1})h(z^{-1})}\right)$$

$$= m_z\left(h^*(z)\Big(f(zxy)\overline{\sigma(x,y)}\,\sigma(y^{-1}x^{-1},z^{-1}) - f(zy)\sigma(y^{-1},z^{-1})\Big)\right) \qquad \text{–[}\mathbf{2}\text{]}$$

by cocycle identities. Now

$$\begin{aligned} F_{x,y}(z) &:= (f(zxy)\overline{\sigma(x,y)}\,\sigma(y^{-1}x^{-1},z^{-1}) - f(zy)\sigma(y^{-1},z^{-1}) \\ &= \overline{\sigma(x,y)\sigma(z,xy)}\big(\varphi,\,\pi(zxy)\psi\big) - \overline{\sigma(z,y)}\big(\varphi,\,\pi(zy)\psi\big) \\ &= \overline{\sigma(x,y)}\,\big(\pi(z)^*\varphi,\,\pi(xy)\psi\big) - \big(\pi(z)^*\varphi,\,\pi(y)\psi\big) \end{aligned}$$

Thus similar to above, we find

$$\begin{aligned} \|F_{x,y}\|_* &\le \|\varphi\|\cdot\left\|\big(\overline{\sigma(x,y)}\,\pi(xy)-\pi(y)\big)\psi\right\| \\ &\le \|\varphi\|\cdot\big\|\big(\pi(x)-\mathbb{I}\big)\pi(y)\psi\big\| \xrightarrow[e]{x} 0 \qquad \forall\, y \end{aligned}$$

So by equation [2]:

$$\left|\,\big((\delta_x-\mathbb{I})*\delta_y*(m^f)^*\big)\,(h)\right| \le \|m\|\cdot\|h\|_*\|F_{x,y}\|_* \xrightarrow[e]{x} 0 \qquad \forall\, y\,.$$

and thus $\left\|(\delta_x - \mathbb{I}) * \delta_y * (m^f)^*\right\| \xrightarrow[e]{x} 0 \qquad \forall y$, i.e. $(m^f)^* \in \mathcal{L}_0(G)$
and hence $m^f \in \mathcal{L}(G) = \mathcal{L}_0(G) \cap \mathcal{L}_0(G)^*$. ∎

In general there are $x \in G$, and $f \in J_{\overline{\sigma}}$ such that $\delta_x * m^f \neq m^f$. Thus G has sets of continuous representations which are homomorphic images of the representation theory of C*-algebras. By the example in [Pes] for some G, none of these sets are isomorphic to the representation theory of a C*-algebra.

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