Gibbs measures on Brownian paths: Theory and applications

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Abstract

We review our investigations on Gibbs measures relative to Brownian motion, in particular the existence of such measures and their path properties, uniqueness, resp. non-uniqueness. For the case when the energy only depends on increments, we present a functional central limit theorem. We also explain connections with other work and state open problems of interest.

1 Introduction

The probability measures studied in Statistical Mechanics have the generic structure

$$\frac{1}{Z}\exp[-\mathcal{E}] \quad \times \quad a\ priori \text{ measure} \tag{1.1}$$

The *a priori* measure is explicit and simple. The energy function $\mathcal{E}$ is defined on the same space as the *a priori* measure and the partition function Z makes (1.1) a probability measure. Of course, it is understood that $\mathcal{E}$ has a natural structure, as dictated by concrete applications.

One much studied class of examples is that of lattice spin systems with finite state space S. Then the *a priori* measure is the product over the lattice points of the counting measure on S. The energy function typically has the form

$$\mathcal{E}_\Lambda = (k_{\mathrm{B}}T)^{-1} \sum_{x,y\in\Lambda} U(\sigma_x, \sigma_y, |x-y|) \tag{1.2}$$

where $U : S^2 \times \mathbb{R}^+ \to \mathbb{R}$ is a pair potential, and $\sigma_x \in S$ is the value of the spin at site x of the finite subset Λ of the lattice. The inverse temperature $(k_{\mathrm{B}}T)^{-1}$ appears as a strength factor multiplying the energy.

The specific expression of the measures as formally given by (1.1) is actually firmly grounded in the experience of rigorous statistical mechanics. At least in the context of

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lattice spin systems with compact state space the so emerging *Gibbs measures* prove to provide a proper mathematical description of thermodynamic equilibrium states and thus they play a fundamental role in the theory of phase transitions. In more specific cases, such as the Potts model, these measures make a strong link between locality properties and memory effects (Markov random fields), variational principles involving the minimization of free energy so that states appear as tangent functionals (large deviation theory), and the understanding in terms of percolation properties of how macroscopic long range order builds up from small scale events governed by chance (stochastic geometry). Although as soon as we leave the class of discrete models these relationships are not as clear any longer, these signposts pinpoint a programme of a general theory of Gibbs measures from which one can take an inspiration. In this paper we present the first steps in developing a theory of Gibbs measures on path space.

We will study the case where the *a priori* measure is Brownian motion in $\mathbb{R}^d$. Let us denote by $t \to X_t \in \mathbb{R}^d$ a Brownian path and by $\mathcal{W}$ the Wiener measure. Since $t \in \mathbb{R}$, in the parlance of Statistical Mechanics our model is one-dimensional with d components. The finite box Λ corresponds to the time interval $[-T, T]$. $\mathcal{W}$ has then to be supplied with appropriate boundary conditions. For example one could pin the path at both endpoints, $X_{-T} = 0$, $X_T = 0$, in which case $\mathcal{W}$ would turn into a Brownian bridge. The simplest energy function is given through an "on site" potential $V : \mathbb{R}^d \to \mathbb{R}$ and takes the form

$$\mathcal{E}_{1,T} = \int_{-T}^{T} V(X_t)dt. \tag{1.3}$$

The analogue of the pair interaction energy (1.2) transcribes as

$$\mathcal{E}_{2,T} = \int_{-T}^{T}\int_{-T}^{T} W(X_t, X_s, t-s)dtds \tag{1.4}$$

with $W : \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}$, $W(x, x', t) = W(x', x, t)$, and $\int |W(x, x', t)|dt < \infty$. In Statistical Mechanics energies are proportional to the volume, i.e. proportional to T in our case. Clearly, in spirit this is satisfied by both energies (1.3) and (1.4). With these preparations a Gibbs measure on path space reads as

$$\frac{1}{Z(T)} \exp\left[- \mathcal{E}_{1,T}(X) - \mathcal{E}_{2,T}(X)\right]\delta(X_{-T})\delta(X_T)d\mathcal{W}(X)\,. \tag{1.5}$$

Of course, there is considerable freedom in how to pick the energy function. (1.3) and (1.4) come up naturally from applications. A further set of examples is obtained by replacing in (1.3), (1.4) the Riemann integrals by stochastic integrals as

$$\widetilde{\mathcal{E}}_{1,T}(X) = \int_{-T}^{T} a(X_t)\cdot dX_t\,, \quad \widetilde{\mathcal{E}}_{2,T}(X) = \int_{-T}^{T}\int_{-T}^{T} dX_s \cdot W(X_t, X_s, t-s)dX_t\,, \tag{1.6}$$

with $a(x)$ a vector field and $W(x, x', t)$ a $d \times d$ matrix. Since our own work is centered more around (1.5), we will concentrate exclusively on this case.

Our plan is first to explore the probabilistic structure. In the final chapter we list various applications for which measures of the form (1.5) with specific choices of $\mathcal{E}_1$ and $\mathcal{E}_2$

appear. From there it will also be apparent that each application poses specific questions not covered by general theory.

Broadly speaking, given the measure in (1.5) there are two limiting procedures of interest.

i) Short distance (ultraviolet) limit. The box $[-T, T]$ is fixed and the interaction is singular on the diagonal. The prototype are polymer measures, where self-crossings are penalized by the energy

$$\mathcal{E}_{T,\mathrm{poly}}(X) = \int_{-T}^{T}\int_{-T}^{T} \delta_n(X_t - X_s)dtds\,. \tag{1.7}$$

Here $\delta_n \geq 0$ with support in a ball of radius $1/n$ centered at the origin. One goal is then to prove that the Gibbs measure in (1.5) with the energy (1.7) has a limit as $n \to \infty$. Problems of these type also come up in proving renormalizability of quantum field theories. They have been studied in considerable detail. We refer to [20, 35, 36, 40, 6] and references therein. A more detailed discussion is outside the scope of the present review and we will always assume that W is locally bounded.

ii) Large distance (infinite volume) limit. The goal is to show that the measure in (1.5) has a limit as $T \to \infty$. The limit measure has then conditional expectations à la Dobrushin, Lanford, and Ruelle. As standard in the theory of Gibbs measures, the issue divides into the existence of a limit measure and the dependence of the limit measure on the choice of boundary conditions.

The infinite volume limit will be discussed in Section 2. A prerequisite is the case $W \equiv 0$, which leads to the theory of $P(\phi)_1$-processes, i.e. reversible diffusion processes with constant diffusion, which will be taken up in Section 2.1. If the interaction W is weak, one expects that the qualitive properties of the stationary $P(\phi)_1$-process remain intact. Technically, a cluster expansion will be used to establish such a result. The basic set-up will be explained in Section 2.2. It differs from the more conventional cluster expansions because the *a priori* measure is not a product measure and the configurations are segments of Brownian paths rather than the better understood $\mathbb{R}$- or $\mathbb{Z}$-valued spins. To prove existence of the limit measure with no restriction on the interaction strength requires other methods. One possibility is domination and monotonicity [32]. In Section 2.3 we explain a more general scheme, which relies on having an essentially bounded interaction energy between the path $\{X_t, t \leq 0\}$ and the path $\{X_t, t \geq 0\}$. Under such a condition we prove that the Gibbs measure is unique, i.e. independent of the choice of boundary conditions within a reasonable class. To have non-uniqueness, the interaction energy must increase at least as $\log T$, or equivalently $W(x, x', t)$ has to decay at least as slow as $|t|^{-2}$ for large $|t|$. In Section 2.4 we discuss a specific example, for which it can be shown that the limit measure depends on the choice of the boundary conditions.

Another case of interest is the energy

$$\mathcal{E}(X) = \int_{-T}^{T}\int_{-T}^{T} W(X_t - X_s, t-s)dtds \tag{1.8}$$

with $\int |t|W(x,t)dt < \infty$, hence zero external potential V. As the expression shows, the energy depends only on path increments. Thus one expects that, under the Gibbs measure (1.5) for $\mathcal{E}_{1,T} + \mathcal{E}_{2,T}$ replaced by $\mathcal{E}$, X_t behaves like Brownian motion with some effective diffusion coefficient. For example, if X is pinned as $X_{-T} = 0 = X_T$, then $\mathbb{E}_T(X_0^2) \simeq T$, at large T. The $T \to \infty$ limit of the measure (1.5) will not exist and the more sensible project is to prove an invariance principle under suitable rescaling. This will be explained in Section 3. Finally, in Section 4 we discuss some specific applications.

At this point we would like to take the opportunity to thank the organizers of the SPP 1033 "Interacting Stochastic Systems of High Complexity" for their initiative. The Schwerpunkt turned out to be a successful enterprise for joint research in the applied areas of probability theory.

2 Gibbs measures

2.1 The case of external potential

First we outline a method on how to represent $P(\phi)_1$-processes (i.e., Brownian motion in the presence of an external potential) in terms of Gibbs measures. Since B_t, the outcomes of Brownian motion, are correlated for different values of t, Wiener measure carries some dependence and is not as simple as a product measure. However, by its Markovianness and since this property survives under the potentials we consider, $P(\phi)_1$-processes are tractable to a fair extent, which is a first step toward understanding more complicated cases, such as (1.4) when also a pair interaction in present. For early results we refer to [37, 38], for details of Gibbsian description as well as proofs and a discussion of the related literature see [3]; the arguments used here are largely based on a spectral theoretic analysis.

Denote $V^+ = \sup\{0, V\}$ and $V^- = \inf\{-V, 0\}$. Two classes of external potential $V : \mathbb{R}^d \to \mathbb{R}$ will be considered:

(V1) *Kato-class.* Here $V^- \in K_d$ and $V^+ \in K_d^{\mathrm{loc}}$, with

$$K_1 = \{V : \sup_{x\in\mathbb{R}} \int_{|x-y|\le 1} |V(y)|\, dy < \infty\},$$
$$K_d = \{V : \lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le r} |V(y)|\, q(|x-y|)\, dy = 0\} \quad \text{if} \quad d \ge 3,$$

with $q(x) = -\log|x|$ for $d = 2$, and $q(x) = 1/|x|^{d-2}$ for $d \ge 3$, and the local Kato-class

$$K_d^{\mathrm{loc}} = \left\{f : f1_A \in K_d \quad \text{for each compact} \quad A \subset \mathbb{R}^d\right\}. \tag{2.1}$$

(V2) *Confining potentials.* V is bounded from below and continuous, moreover $V(x) = a|x|^{2s} + o(|x|^{2s})$, with some $s > 1$ and $a > 0$.

Examples of Kato-class potentials include smooth functions bounded from below, but also some local (e.g. Coulomb) singularities are allowed. In particular, (V2) is a specific case of (V1). The sets K_d can also be characterized in terms of Wiener integrals.

For V having either of the regularity properties above define the Schrödinger operator $H = -1/2\Delta + V(x)$ on $L^2(\mathbb{R}^d, dx)$ as a sum of quadratic forms (V is regarded as a multiplication operator). Then $C_0^\infty(\mathbb{R}^d)$ is a form core on which H is essentially self-adjoint and bounded from below. If the bottom of the spectrum E_0 of H is a simple eigenvalue, then the corresponding eigenfunction ψ_0 (ground state) is strictly positive. The semigroup e^{-tH}, $t \geq 0$, exists on $L^2(\mathbb{R}^d, dx)$, and it is an integral operator with positive, continuous, uniformly bounded kernel $G_t(x,y)$. For (V2)-type potentials the semigroup is moreover intrinsically ultracontractive. That is, with the probability measure $d\nu = \psi_0^2 dx$ on $\mathbb{R}^d$, and isometry $j : L^2(\mathbb{R}^d, d\nu) \to L^2(\mathbb{R}^d, dx)$, $f \mapsto \psi_0 f$, the operator

$$\begin{aligned} H_\nu f &= (j^{-1}(H - E_0)j)f = \frac{1}{\psi_0}(H - E_0)(\psi_0 f) \\ &= -\frac{1}{2}\Delta f - (\nabla \ln \psi_0, \nabla f)_{\mathbb{R}^d}\,, \end{aligned} \tag{2.2}$$

with $\mathrm{Dom}\, H_\nu = j^{-1}(\mathrm{Dom}\, H)$, defines a semigroup e^{-tH_ν} for all $f \in L^2(\mathbb{R}^d, d\nu)$ and $t \geq 0$. Intrinsic ultracontractivity of e^{-tH} means that e^{-tH_ν} is ultracontractive, i.e. it maps $L^2(\mathbb{R}^d, d\nu)$ into $L^\infty(\mathbb{R}^d, d\nu)$ continuously, or equivalently, $||e^{-tH_\nu}||_{2,\infty} < \infty$, $\forall t \geq 0$.

Choose now H to be a Schrödinger operator such that its ground state ψ_0 exists. For convenience and without loss we shift the potential by E_0 so that the bottom of the spectrum of H is 0. For $t_1 < \ldots < t_n \in \mathbb{R}$, $f_1, \ldots, f_n \in L^2(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$, the $P(\phi)_1$*-process* associated with H is the unique probability measure P on path space $C(\mathbb{R}, \mathbb{R}^d)$ defined by

$$\begin{aligned} &\int f_1(X_{t_1}) \ldots f_n(X_{t_n}) dP(X) \\ &= (\psi_0 f_1, e^{-(t_2 - t_1)H} f_2 \ldots e^{-(t_n - t_{n-1})H} f_n \psi_0)_{L^2(\mathbb{R}^d, dx)}. \end{aligned} \tag{2.3}$$

P is indeed a probability measure as $e^{-tH}\psi_0 = \psi_0$ and $\|\psi_0\|_2 = 1$. A $P(\phi)_1$-process is a reversible stationary Markov process with stationary measure $d\nu$ and generator H_ν, and it has almost surely continuous paths. It is moreover the stationary solution of the stochastic differential equation (Itô-diffusion)

$$dX_t = (\nabla \ln \psi_0)(X_t)\, dt + dB_t,$$

where B_t denotes Brownian motion on $\mathbb{R}^d$.

Processes of this type can be given a Gibbsian description. We emphasize that since in the present stage there are no useful relationships available with variational principles etc as discussed in the Introduction, here the basic fact is that there is at all a probability measure associated with the scalar product in (2.3), a consequence of the Riesz-representation theorem, while its Gibbsianness comes second to it. That we are able to identify this measure as a Gibbs measure leads however to further insight.

Denote $\mathcal{X} = C(\mathbb{R}, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, and its σ-field $\mathcal{A} = \sigma(\pi_t : t \in \mathbb{R})$ generated by the point evaluations $\pi_t : \mathcal{X} \to \mathbb{R}^d$, $X \mapsto \pi_t(X) = X_t$. These will be the configuration space and σ-field for the Gibbs measure, respectively.

For $[-T,T] \subset \mathbb{R}$ we denote by $\mathcal{A}_T$ the σ-field $\sigma(\pi_t : t \in [-T,T]) \subset \mathcal{A}$; also, we put $[-T,T]^c = \mathbb{R} \setminus [-T,T]$.

Write as before $\mathcal{W}$ for Wiener measure, and $\mathcal{W}^{\xi,\eta}_{[-T,T]}$ for the Wiener measure conditional on starting in ξ at time $-T$ and ending in η at time T. This Brownian bridge can be extended to a measure on $\mathcal{X}$ by picking an $Y \in \mathcal{X}$ and putting $\mathcal{W}^Y_T = \mathcal{W}^{Y_{-T},Y_T}_{[-T,T]} \otimes \delta^Y_{[-T,T]^c}$, with Dirac measure on $C([-T,T]^c, \mathbb{R}^d)$ concentrated on $Y|_{[-T,T]^c}$. $\mathcal{W}^Y_T$ is thus a finite measure on $(\mathcal{X}, \mathcal{A})$; it will serve as reference measure for the Gibbs measure to be constructed.

Take any $A \in \mathcal{A}$ and consider

$$dP_T(A|Y) = \frac{1}{Z_T(Y)} 1_A(X) e^{-\int_{-T}^T V(X_s)ds} \, d\mathcal{W}^Y_T(X), \tag{2.4}$$

where

$$Z_T(Y) = \int e^{-\int_{-T}^T V(X_s)ds} \, d\mathcal{W}^Y_T(X) \tag{2.5}$$

is the partition function turning P_T into a probability measure.

Definition 2.1 *Let $\mathcal{X}^* \subset \mathcal{X}$. A probability measure $\mathcal{P}$ on $(\mathcal{X}, \mathcal{A})$ is called a* Gibbs measure *for potential V and reference measure $\mathcal{W}^Y$, if for every bounded interval $[-T,T] \subset \mathbb{R}$*

1. *$\mathcal{P}|_{\mathcal{A}_T} \ll \mathcal{W}|_{\mathcal{A}_T}$,*
2. *for every $A \in \mathcal{A}$ the function $Y \mapsto \mathcal{P}_T(A|Y)$ given by the right hand side of (2.4) is a regular version of the conditional probability $\mathcal{P}(A|\, \mathcal{A}_{[-T,T]^c})$.*

A probability measure $\mathcal{P}_T$ on $([-T,T], \mathcal{A}_T)$ is called a finite time interval Gibbs measure *for V and reference measure W^Y_T if for every bounded interval $[-S,S] \subset [-T,T]$ the function $Y \mapsto \mathcal{P}_S(A|Y)$ as above is a regular version of the conditional probability $\mathcal{P}_S(A|\, \mathcal{A}_{[-S,S]^c})$. Furthermore, a Gibbs measure $\mathcal{P}$ is said to be supported by $\mathcal{X}^*$ whenever $\mathcal{P}(\mathcal{X}^*) = 1$.*

This definition rests on the DLR conception of Gibbs measure. In this sense we then have

Theorem 2.1 *A $P(\phi)_1$-measure P corresponding to potential V is a Gibbs measure with respect to V and Wiener measure.*

On the other hand, that $P_T(\cdot|Y)$ are a family of finite time interval Gibbs measures indexed by bounded intervals can be seen in a straightforward way. It can be proven by a monotone class argument that (infinite time interval) Gibbs measures on path space can be obtained by limits of finite interval Gibbs measures, similarly to the case known from lattice spin models. In this limiting procedure thus one must have a control of boundary conditions.

A Gibbs measure associated with a $P(\phi)_1$-process need not be unique. This non-uniqueness appears as a dependence of the Gibbs measure on the boundary conditions. An example showing this is the Ornstein-Uhlenbeck process, in which case uncountably many Gibbs measures can occur for the same potential. This is related with the rate how the boundary paths increase, or in other words, how fast for each T the boundary path on $[-T,T]^c$ has to "forget" that it was free Brownian motion before stepping in $[-T,T]$ where it must "steady down" to conform with the correct distribution prescribed by (2.4). A condition for uniqueness of the Gibbs measure is provided by the following theorem.

Theorem 2.2 *Let H be a Schrödinger operator for a Kato-class potential V such that the spectral gap Λ of H is strictly positive, and let ψ_0 be its ground state. Put*

$$\mathcal{X}^* = \{X \in \mathcal{X} : \lim_{|t| \to \infty} \frac{e^{-\Lambda|t|}}{\psi_0(X_t)} = 0\}. \tag{2.6}$$

Then the $P(\phi)_1$-measure P corresponding to V is the unique Gibbs measure for V supported by $\mathcal{X}^*$. If, furthermore, V is a $(V2)$-type confining potential, then P is the unique Gibbs measure supported on the entire $\mathcal{X}$.

The first part of the statement results from an argument using direct estimates, the second relies on ultracontractivity.

By restricting to (V2)-type potentials and making use of the fact that for this class ψ_0 is bounded both from below and above by $C \exp(-\theta |x|^{s+1})$, with suitable constants $C, \theta > 0$ for the two bounds respectively, we obtain from Theorem 2.2 that those paths are typical for the $P(\phi)_1$-measure that grow asymptotically like $t^{1/(s+1)}$.

2.2 Weak pair potential: cluster expansion

Next we turn to discussing whether Gibbs measures can be defined also for Brownian motion subjected to both an external and a pair interaction potential. Such a process is not Markovian and therefore not accessible to spectral analysis. Instead, we will develop a cluster expansion; for details and proofs see [26].

We use the same set-up as before. The pair interaction potential is a measurable function $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ with the (inessential) symmetry properties $W(\cdot, \cdot, t-s) = W(\cdot, \cdot, |t-s|)$, $W(x, y, \cdot) = W(y, x, \cdot)$, $x, y \in \mathbb{R}^d$, $s, t \in \mathbb{R}$, and satisfying either of the following regularity conditions:

(W1) There is $R > 0$ and $\alpha > 2$ such that

$$|W(x, y, t-s)| \ \leq \ R \, \frac{|x|^2 + |y|^2}{1 + |t-s|^\alpha} \tag{2.7}$$

for every $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$.

(W2) There is $R > 0$ and $\alpha > 1$ such that

$$|W(x, y, t-s)| \ \leq \ \frac{R}{1 + |t-s|^\alpha} \tag{2.8}$$

for every $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$.

For $[-T, T] \subset \mathbb{R}$ write

$$W_{[-T,T]}(X|Y) = W_{[-T,T]}(X) + W^Y_{[-T,T]}(X) \tag{2.9}$$

for the "total energy" associated with configuration $X \in \mathcal{X}_{[-T,T]}$ given the boundary configuration $Y = Y^- \cup Y^+$, with $Y^- \in \mathcal{X}_{(-\infty,-T]}$ resp. $Y^+ \in \mathcal{X}_{[T,\infty)}$. Term by term,

$$W_{[-T,T]}(X) = \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t) ds dt \tag{2.10}$$

is the "internal energy" associated with the path inside $[-T, T]$, and

$$\begin{aligned} W^Y_{[-T,T]}(X) &= 2\int_{-\infty}^{-T} dt \int_{-T}^{T} ds\, W(Y_t^-, X_s, t-s) + \\ &\quad +2\int_{T}^{\infty} dt \int_{-T}^{T} ds\, W(Y_t^+, X_s, t-s) \end{aligned} \tag{2.11}$$

is the "interaction energy" between X and the boundary path Y. We calibrate the interaction energy such that $W^0_{[-T,T]}(X) = 0$. As before, $P_T(\,\cdot\,|Y) = P_{[-T,T]}(\,\cdot\,|Y^-_{-T}, Y^+_T)$ is the conditional distribution of the reference measure for the given boundary condition Y (which by Markovianness obviously depends only on the positions attained at the ends of the interval). It is readily checked that $\mu_{[-T,T]}(\,\cdot\,|Y)$, with $Y \in C([-T,T]^c, \mathbb{R}^d)$, is a family of finite time interval Gibbs measures. We also allow $\lambda \in \mathbb{R}$, a parameter which can be interpreted as the strength of the coupling of the pair interaction to the Brownian paths.

Consider now a $P(\phi)_1$-process with stationary measure $d\nu = \psi_0^2 dx$ and transition probability density

$$g_t(x|y) = \frac{\psi_0(x) G_t(x,y)}{\psi_0(y) e^{-E_0 t}}. \tag{2.12}$$

Denote again the probability distribution of this process by P, and by P_T its restriction to the field $\mathcal{A}_{[-T,T]}$. We take this as reference measure in constructing the finite time interval Gibbs measures on $\mathcal{X}_{[-T,T]}$ for the pair potentials above:

$$d\mu_T(A|Y) = \frac{1}{\mathcal{Z}_T(Y)} 1_A(X) e^{-\lambda W_{[-T,T]}(X|Y)} dP_T(X|Y), \tag{2.13}$$

for any $A \in \mathcal{A}$ and boundary condition Y. Here we speak about Gibbs measure μ in the same sense as in Definition 2.1, now for potential W and reference measure P. The partition function is

$$\mathcal{Z}_T(Y) = \int e^{-\lambda W_{[-T,T]}(X|Y)} dP_T(X|Y). \tag{2.14}$$

As said before, Gibbs measures can be obtained as limits over finite time interval Gibbs measures. Thus it is of interest whether the sequence of Gibbs measures μ_T has a limit as $T \to \infty$; if it does then it provides a Gibbs measure on the full path space as soon as also condition (i) of Definition 2.1 is met.

Theorem 2.3 *Suppose V and W satisfy assumptions (V2), respectively either (W1) or (W2). Take any unbounded increasing sequence $T^{(n)}$ of positive real numbers, and suppose $0 < |\lambda| \leq \lambda^*$ with λ^* small enough. Then the local weak limit $\lim_{n\to\infty} \mu_{T^{(n)}} = \mu$ exists and is a Gibbs probability measure on $(\mathcal{X}, \mathcal{A})$ with respect to W and reference measure P. Moreover, μ does not depend on the choice of sequence $T^{(n)}$.*

In order to prove this convergence we use a cluster expansion controlled by the small parameter λ. Next we sketch the cluster representation of the partition function (2.14) and outline the main steps of the proof. For simplicity we start with free boundary conditions, i.e. $Y = 0$ in (2.13).

Take a division of $[-T, T]$ into disjoint intervals $\tau_k = (t_k, t_{k+1})$, $k = 0, ..., N-1$, with $t_0 = -T$ and $t_N = T$, each of length b, i.e. fix $b = 2T/N$; for convenience we choose N to be an even number so that the origin is endpoint to some intervals. We break up a path X into pieces X_{τ_k} by restricting it to τ_k. The total energy contribution of the pair interaction then becomes

$$W_T := \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t) ds dt = \sum_{0 \leq i < j \leq N-1} W_{\tau_i, \tau_j} \tag{2.15}$$

where with the notation $\mathcal{J}_{ij} = \int_{\tau_i} dt \int_{\tau_j} W(X_s, X_t, s-t) ds$ we have

$$W_{\tau_i,\tau_j} = \begin{cases} \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| \geq 2 \\ \frac{1}{2}(\mathcal{J}_{ii} + \mathcal{J}_{jj}) + \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| = 1, \text{ and } i \neq 0,\ j \neq N-1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\,\mathcal{J}_{00} & \text{if } i = 0 \text{ and } j = 1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\,\mathcal{J}_{N-1\,N-1} & \text{if } i = N-1 \text{ and } j = N-2 \end{cases} \tag{2.16}$$

(For keeping the notation simple we do not make explicit the X dependence of these objects.) By using (2.15) we obtain

$$e^{-\lambda W_T} = \prod_{0 \leq i < j \leq N-1} (e^{-\lambda W_{\tau_i,\tau_j}} + 1 - 1) = 1 + \sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i,\tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i,\tau_j}} - 1). \tag{2.17}$$

Here the summation is performed over all nonempty sets of different pairs of intervals, i.e. $\mathcal{R} = \{(\tau_i, \tau_j) : (\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'}) \text{ whenever } (i,j) \neq (i', j')\}$.

In order to keep this and the forthcoming summations in hand we need a few more notations. Two distinct pairs of intervals (τ_i, τ_j) and $(\tau_{i'}, \tau_{j'})$ will be called *directly connected* and denoted $(\tau_i, \tau_j) \sim (\tau_{i'}, \tau_{j'})$ if one interval of the pair (τ_i, τ_j) coincides with one interval of the pair $(\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is a collection $\{(\tau_{i_1}, \tau_{j_1}), ..., (\tau_{i_n}, \tau_{j_n})\}$ in which each pair of intervals is connected to another through a sequence of directly connected pairs, i.e., for any $(\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'})$ there exists $\{(\tau_{k_1}, \tau_{l_1}), ..., (\tau_{k_m}, \tau_{l_m})\}$ such that $(\tau_i, \tau_j) \sim (\tau_{k_1}, \tau_{l_1}) \sim ... \sim (\tau_{k_m}, \tau_{l_m}) \sim (\tau_{i'}, \tau_{j'})$. A maximal set of connected pairs of intervals is called a *contour* and denoted by γ. We denote by $\bar{\gamma}$ the set of all intervals that are elements of the pairs of intervals belonging to contour γ, and by γ^* the set of time-points of intervals appearing in $\bar{\gamma}$. Clearly, $\mathcal{R}$ can be decomposed into maximal connected components, i.e. contours: $\mathcal{R} = \{\gamma_1, ..., \gamma_r\}$ with $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, $i \neq j$; $i, j = 1, ..., r$.

The sum in (2.17) is then further expanded as

$$\sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i,\tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i,\tau_j}} - 1) = \sum_{r \geq 1} \sum_{\{\gamma_1,...,\gamma_r\}} \prod_{k=1}^{r} \prod_{(\tau_i,\tau_j) \in \gamma_k} (e^{-\lambda W_{\tau_i,\tau_j}} - 1) \tag{2.18}$$

where now summation goes over collections $\{\gamma_1, ..., \gamma_r\}$ of contours such that $\bar{\gamma}_k \cap \bar{\gamma}_{k'} = \emptyset$ unless $k = k'$.

A collection of consecutive intervals $\{\tau_j, \tau_{j+1} ..., \tau_{j+k}\}$, $j \geq 0$, $j + k \leq N - 1$ is called a *chain.* As in the case of contours, $\bar{\varrho}$ and ϱ^* mean the set of intervals belonging to the chain ϱ and the set of time-points in ϱ, respectively. We call two contours γ_1, γ_2 disjoint if they have no intervals in common, i.e. $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$. Two chains ϱ_1, ϱ_2 are called disjoint if they have no common time-points, i.e. $\varrho_1^* \cap \varrho_2^* = \emptyset$. Take now a non-ordered set of disjoint contours and disjoint chains, $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$, with some $r \geq 1$ and $s \geq 0$. Note that such contours and chains may have common time-points. We use the notation $\Gamma^* = (\cup_i \gamma_i^*) \cup (\cup_j \varrho_j^*)$ for the set of all time-points appearing as beginnings or ends of intervals belonging to some contour or chain in Γ. Also, we put $\bar{\Gamma} = (\cup_i \bar{\gamma}_i) \cup (\cup_j \bar{\varrho}_j)$ for the set of intervals appearing in Γ through entering some contours or chains. Denote by $\partial^- \varrho$ resp. $\partial^+ \varrho$ the leftmost resp. rightmost time-points belonging to ϱ. Γ is called a *cluster* if $\{\gamma_1^*, ..., \gamma_r^*; \varrho_1^*, ..., \varrho_s^*\}$ is a connected collection of sets and for every $\varrho \in \Gamma$ we have that $\partial^- \varrho, \partial^+ \varrho \in \cup_{j=1}^r \gamma_j^*$. This means that in a cluster chains have no loose ends.

Next we fix the positions of path X at the time-points of the division, i.e. we put $X_{t_k} = x_k$, for all $k = 0, ..., N$, with $-T = t_0 < t_1 < ... < t_N = T$. The distribution of path X in interval $[-T, T]$ conditional on the positions attained at the fixed times is

$$dP_T(X_{\tau_0}, ..., X_{\tau_{N-1}} | X_{t_0} = x_0, \ldots, X_{t_N} = x_N) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} | x_k, x_{k+1}). \tag{2.19}$$

We use the shorthand at the right hand side for the corresponding conditional probabilities for easing the notation. Let $p_{t_0,...,t_N}(x_0, ..., x_N)$ be the density with respect to $\prod_{k=0}^N d\nu_k(x_k)$ of the joint distribution of positions of path X recorded at the time-points $t_0, ..., t_N$. Here $d\nu_k$ denotes a copy of $d\nu$ for each $k = 0, ..., N$. By Markovianness it then follows that

$$\begin{aligned} p_{t_0,...,t_N}(x_0, ..., x_N) &= \prod_{k=0}^{N-1} g_b(x_{k+1} | x_k) = \prod_{k=0}^{N-1} (g_b(x_{k+1} | x_k) - 1 + 1) \\ &= 1 + \sum_{\mathcal{S}} \prod_{k: \tau_k \in \mathcal{S}} (g_b(x_{k+1} | x_k) - 1). \end{aligned}$$

The summation is extended over all nonempty sets $\mathcal{S} = \{\tau_k = (t_k, t_{k+1})\}$ of different pairs of consecutive time-points. In a similar way as before the latter formula can be recast in the form

$$\sum_{\mathcal{S}} \prod_{k: \tau_k \in \mathcal{S}} (g_b(x_{k+1} | x_k) - 1) = \sum_{s \geq 1} \sum_{\{\varrho_1, ..., \varrho_s\}} \prod_{j=1}^{s} \prod_{k: \tau_k \in \varrho_j} (g_b(x_{k+1} | x_k) - 1) \,. \tag{2.20}$$

Here $\{\varrho_1, ..., \varrho_s\}$ is a collection of disjoint chains, and this formula explains the way we defined them before.

For every cluster $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$ define the function

$$\kappa_\Gamma = \prod_{l=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i \tau_j}} - 1) \prod_{m=1}^{s} \prod_{k: \tau_k \in \varrho_m} (g_b(x_{k+1} | x_k) - 1) \,. \tag{2.21}$$

Also, introduce the auxiliary probability measure on $\mathcal{X}_T$

$$d\,\mathcal{P}_T(X) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k}|x_k, x_{k+1}) \prod_{k=0}^{N} d\nu_k(x_k) \tag{2.22}$$

and look at

$$K_\Gamma = \mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma]. \tag{2.23}$$

Note that $\int (g_b(x_{k+1}|x_k) - 1) d\nu(x_{k+1}) = \int (g_b(x_{k+1}|x_k) - 1) d\nu(x_k) = 0$. This is the reason why from a cluster we rule out chains having loose ends; for any such chain $\mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma] = 0$.

By putting (2.18), (2.19), (2.20), (2.21) and (2.23) together we obtain the cluster representation of the partition function $\mathcal{Z}_T$:

Proposition 2.4 *For every $T > 0$*

$$\mathcal{Z}_T = 1 + \sum_{n \geq 1} \sum_{\{\Gamma_1, \ldots, \Gamma_n\}} \prod_{l=1}^{n} K_{\Gamma_l}. \tag{2.24}$$

Here the summation is performed over all sets of clusters $\{\Gamma_1, ..., \Gamma_n\} \neq \emptyset$ for which $\Gamma_i^* \cap \Gamma_j^* = \emptyset$ whenever $i \neq j$.

As soon as the cluster representation of $\mathcal{Z}_T$ is established, the existence of the weak limit measure $\mu = \lim_{T \to \infty} \mu_T$ follows by the cluster estimates below and the general arguments presented in e.g. [29], Chapter 3.

We conclude the presentation of the expansion by briefly explaining the two crucial cluster estimates. The first one is given by

Proposition 2.5 *For every cluster Γ we have the bound*

$$|K_\Gamma| \;\leq\; \prod_{\varrho \in \Gamma} (c_1 |\lambda|^{1/3})^{|\bar{\varrho}|} \prod_{\gamma \in \Gamma} \prod_{(\tau_i, \tau_j) \in \gamma} \frac{c_2 |\lambda|^{1/3}}{(|i - j - 1|b)^\delta + 1} \tag{2.25}$$

with $|\bar{\varrho}|$ denoting the number of intervals contained in ϱ, constants $c_1, c_2 > 0$ and exponent $\delta > 1$.

In estimate (2.25) the factor accounting for the contribution of chains comes from the uniform upper bound $Ce^{-\Lambda b}$ on $|g_b(x|x') - 1|$ (see second factor in (2.21)), where Λ is the spectral gap of the Schrödinger operator of the underlying $P(\phi)_1$-process, and $C > 0$. This bound, in its turn, is a consequence of the intrinsic ultracontractivity of e^{-tH}, compare Section 2.1. The factor accounting for the contribution of contours comes from an estimate using a generalized variant of the Hölder inequality applied to the products over $e^{-\lambda W_{\tau_i \tau_j}} - 1$ (see first factor in (2.21)). b is finally chosen in such a combination with λ and Λ that the expression (2.25) results.

The second fundamental estimate ensuring the convergence of the cluster expansion is

Proposition 2.6 *There is a constant $c > 0$, independent of λ, and a number $0 < \eta(\lambda) < 1$ with $\eta \to 0$ as $\lambda \to 0$, such that*

$$\sum_{\substack{\Gamma:\Gamma^* \ni 0 \\ |\bar{\Gamma}|=n}} |K_\Gamma| \;\leq\; c\,\eta^n. \tag{2.26}$$

with $|\bar{\Gamma}|$ denoting the number of intervals contained in Γ through some contour or chain.

This estimate follows through a procedure of translating the summation in the left hand side of (2.26) into a combinatorial problem and resumming over (and counting of) first graphs and then trees. The contours are assigned vertices and they are linked into graphs according to the rules connecting them up into clusters.

So far we assumed free boundary conditions. By an extension of the argument sketched above also other boundary conditions can be taken into account, picked from $\mathcal{X}^*$, the subset provided by Theorem 2.2. Then an important question is how the limiting measure depends on the boundary conditions. Uniqueness (in DLR-sense) means that for any increasing sequence of real numbers $\{T_n\}$ and any corresponding sequence $\{Y_n\} \subset \mathcal{X}^*_{[-T_n,T_n]^c}$, $\lim_{n\to\infty} \mathbb{E}_{\mu_{T_n}}[F_B|Y_n] = \mathbb{E}_\mu[F_B]$, for every bounded $B \subset \mathbb{R}$, and each bounded measurable function F_B.

Theorem 2.7 *Suppose V is of class (V2) and W satisfies (W2). Then we have the following cases:*

1. *If $\alpha > 2$, then whenever the Gibbs measure μ exists, it is unique in DLR sense.*
2. *If $\alpha > 1$, then for sufficiently small $|\lambda|$ the limiting Gibbs measure μ is unique in DLR sense whenever the reference measure is unique.*

If $\alpha > 2$, $|W_T^Y(X)|$ (given by (2.11)) is uniformly bounded in T, and in paths X and Y. This implies that only one Gibbs measure can exist, and the argument requires no restriction on the values of λ. For $1 < \alpha \leq 2$ this uniform boundedness does not hold any longer and we once again take recourse to cluster expansion.

Having a Gibbs measure at hand, an important aspect in its understanding is to see what a typical configuration looks like under it. This is answered by

Theorem 2.8 *Under the same conditions as in the previous theorem, with μ-probability 1 we have*

$$|X_t| \leq C\,(\log(|t|+1))^{1/(s+1)} + Q(\{X\}) \tag{2.27}$$

with a suitable number $C > 0$ and a functional Q, independent of t.

The strategy of proving Theorem 2.8 goes by boosting the typical behaviour of the reference process explained above to the level of the Gibbs measure. First it is shown that for any $a > 0$

$$P\left(\{X \in \mathcal{X} : \max_{0\leq t\leq 1} |X_t| \geq a\}\right) \;\leq C\, e^{-\theta a^{s+1}} \tag{2.28}$$

with appropriate $C, \theta > 0$. This can be proven by using Varadhan's Lemma taken together with the upper bound $\exp(-\theta|x|^{s+1})$ for ψ_0 (the ground state of the Schrödinger operator generating the reference process). Then Theorem 2.8 comes about by proving that also $C' > 0$ and $\theta' > 0$ can be found such that

$$\mu\left(\{X \in \mathcal{X} : \max_{0 \leq t \leq 1} |X_t| \geq a\}\right) \leq C' e^{-\theta' a^{s+1}}. \tag{2.29}$$

The proof requires once again the use of cluster expansion.

Finally, we list some additional properties of Gibbs measures for (W2)-type pair potentials, useful in various contexts. This case in particular covers Nelson's scalar field model, see Section 4 below.

Theorem 2.9 *Let μ be a Gibbs measure for W satisfying (W2). Suppose V is of (V2)-type and $|\lambda|$ is small enough. Then the following hold:*

1. [Invariance properties] *μ is invariant with respect to time shift and time reflection:*

$$\mu \circ \tau_t = \mu, \quad \forall t \in \mathbb{R}, \quad \textit{where} \quad (\tau_s X)_t = X_{s+t},$$
$$\mu \circ \vartheta = \mu, \quad \textit{where} \quad (\vartheta X)_t = X_{-t}.$$

2. [Single time distributions] *The distributions φ_T under μ_T of positions x at time $t = 0$ are equivalent to ν, i.e. there exist $C_1, C_2 \in \mathbb{R}$, independent of T and x such that*

$$C_1 \leq \frac{d\varphi_T}{d\nu}(x) \leq C_2 \tag{2.30}$$

for every $x \in \mathbb{R}^d$ and $T > 0$. Moreover $\lim_{T\to\infty}(d\varphi_T/d\nu)(x) = (d\varphi/d\nu)(x)$ exists pointwise.

3. [Single time conditional distributions] *The conditional distributions $\mu_T(\,\cdot\,|X_0 = x)$ converge locally weakly to $\mu(\,\cdot\,|X_0 = x)$, for all $x \in \mathbb{R}^d$.*

4. [Mixing properties] *For any bounded functions F, G on $\mathbb{R}^d$ we have on the covariance the estimate*

$$|\operatorname{cov}_\mu(F_s; G_t)| \leq \text{const} \frac{\sup|F_s| \sup|G_t|}{1 + |t-s|^\beta} \tag{2.31}$$

where $\beta > 0$, $F_s := F(X_s)$, $G_t := G(X_t)$, and the constant prefactor is independent of s, t and F, G.

2.3 Existence for pair potential of arbitrary strength

The main restriction in the previous section was that the pair potential W had to carry a small prefactor λ. This restriction is inherent in the cluster expansion. An alternative route to the existence of Gibbs measures are compactness arguments; the main tool is the concept of uniform domination [21], which in our context reads as follows:

Definition 2.2 *Let P, $(\mu_T)_{T\geq 0}$ be probability measures on $C(\mathbb{R},\mathbb{R}^d)$. We say that the family $(\mu_T)_{T\geq 0}$ is locally uniformly dominated by P if the following holds true: For each $\varepsilon > 0$ and $S > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies $\limsup_{T\to\infty} \mu_T(A) < \varepsilon$ uniformly in sets A depending on $X_s, |s| < S$, $(X_s)_{s\in\mathbb{R}} \in C(\mathbb{R},\mathbb{R}^d)$.*

The important fact is that each family $(\mu_T)_{T\geq 0}$ of probability measures that is locally uniformly dominated by a probability measure P has at least one cluster point as $T \to \infty$ in the topology of local convergence. In order to apply this to Gibbs measures we adopt the general set-up from the previous section. As a first assumption on the potentials we need

(A1) V is Kato-class, i.e. satisfies (V1) from Section 2.1. Moreover, the Schrödinger operator H corresponding to V has a unique, square-integrable ground state ψ_0.

(A2) W is extensive, i.e. there exists $C_\infty > 0$ such that

$$\int_{-\infty}^{\infty} \sup_{x,y\in\mathbb{R}^d} |W(x,y,|s|)|\, ds < C_\infty. \tag{2.32}$$

Comparing with the previous section, we find that (W2) implies ($A2$).

Let P be the measure of the stationary $P(\phi)_1$-process as given in Section 2.1, and let $W_{[-T,T]}$ be given by (2.10). We will use finite time interval Gibbs measures with free boundary conditions as approximants for our infinite time interval Gibbs measures, i.e. we put

$$d\mu_T = \frac{1}{\mathcal{Z}_T} e^{-W_{[-T,T]}(X)}\, dP.$$

Using the concept of local uniform domination, it is now possible to prove

Proposition 2.10 *[2] Assume (A1) and (A2). Suppose that for each $\varepsilon > 0$ there exists $R > 0$ such that*

$$\mu_T(|X_0| > R) < \varepsilon \tag{2.33}$$

uniformly in $T > 0$. Then there exists an (infinite time interval) Gibbs measure for the potentials V and W and the reference measure $\mathcal{W}$ (Wiener measure).

We have thus reduced the problem to proving (2.33). For this we need some further assumptions.

(A1') In addition to (A1) suppose $\psi_0 \in L^1(\mathbb{R}^d)$.

Condition (A1') is not very restrictive; in many cases ψ_0 decays exponentially at infinity. The additional condition on W will be more restrictive and requires some preparations to formulate. Let $C^{(0)}(\mathbb{R},\mathbb{R}^d)$ denote the space of functions which are continuous with the possible exception of the origin but have left and right hand side limits there. For $\tau > 0$ we define the map

$$\theta_\tau : C(\mathbb{R},\mathbb{R}^d) \to C^{(0)}(\mathbb{R},\mathbb{R}^d), \quad (\theta_\tau X)_t = \begin{cases} X_{t+\tau} & \text{if } t \geq 0, \\ X_{t-\tau} & \text{if } t < 0. \end{cases} \tag{2.34}$$

With $E_0 = \inf \mathrm{Spec}(H)$ as before, and H the Schrödinger operator corresponding to the $P(\phi)_1$-process P, put

$$\alpha = \liminf_{|x|\to\infty} V(x) - E_0 \le \infty. \tag{2.35}$$

Our assumption on W now reads

(A2') In addition to (A2), we assume that there exist $D \ge 0$ and $0 \le C < \alpha$ such that

$$-W_{[-T,T]}(X) \le -W_{[-T,T]}(\theta_\tau X) + C\tau + D \tag{2.36}$$

for all $T, \tau > 0$ and all $X \in C(\mathbb{R}, \mathbb{R}^d)$.

In words, (A2') means that we can control, uniformly in T, the change of energy induced by cutting out a piece of the path X around $t = 0$ and gluing the remaining pieces together again. If we have finite interaction energy between the positive and the negative half-line, i.e.

$$\sup_{X\in C(\mathbb{R},\mathbb{R}^d)} \left| \int_{-\infty}^0 ds \int_0^\infty dt\, W(X_t, X_s, |t-s|) \right| < \infty, \tag{2.37}$$

then (A2') holds with $C = 0$. In particular, (2.37) holds when W fulfills (W2) with $\alpha > 2$. (2.37) is, however, not necessary for (A2'), and part of the interest in condition (A2') is that it also covers cases where (2.37) is not met. Some sufficient conditions for (A2') are given in [1].

Theorem 2.11 *Assume (A1') and (A2'). Then (2.33) holds, and consequently an infinite volume Gibbs measure μ for the potentials V and W, and reference measure $\mathcal{W}$ exists.*

The theorem above does not make any statement about uniqueness. However, in conjunction with (2) of Theorem 2.7 it leads to

Corollary 2.12 *Provided (W2) with $\alpha > 2$ holds, and V satisfies (A1') and (A2'), a unique Gibbs measure exists supported by $\mathcal{X}$.*

Hariya [22] arrives at a similar result under different hypotheses.

The proof of (2.33) relies on the equality

$$\mu_T(|X_0| > R) = \frac{1}{\mathcal{Z}_T} \int_{|y|>R} \psi_0^2(y) \mathbb{E}_P\left[e^{-W_{[-T,T]}} \middle| X_0 = y \right] dy. \tag{2.38}$$

We first prove

$$\frac{1}{\mathcal{Z}_T} \mathbb{E}_P\left[e^{-W_{[-T,T]}} \middle| X_0 = y \right] \le \frac{\mathrm{const}}{\psi_0(y)} \tag{2.39}$$

and then use (A1') in order to obtain (2.33). To get an idea about the proof of the latter inequality, note that (2.39) involves expectation with respect to a Markov process conditioned at its 'midpoint' $t = 0$. For making use of the strong Markov property of P, we flip the negative time axis to the right and obtain a Markov process with a doubled state space $\mathbb{R}^{2d}$, now conditioned on its starting point. Now we start the new process in $y \in \mathbb{R}^{2d}$

and stop it when it reaches the ball B_r around zero with radius r. By the properties of the $P(\phi)_1$-process, the stopping time τ_r the process needs to reach B_r is exponentially integrable. More explicitly, $\mathbb{E}_P^x[\exp(\beta\tau_r)] < \infty$ if $\beta < \alpha$, and the expectation value grows with the starting point x like $1/\psi_0(x)$ as $x \to \infty$. Condition (A2') is now tailor-made to ensure that the energy $\tilde{W}_{[-T,T]}$ acquired by a (flipped) path X on its way down to the B_r is no larger than $\exp(C\tau_r + D)$. Together with the strong Markov property and some technical estimates, this yields (2.33).

2.4 Phase transition

In one-dimensional statistical mechanical sytems the entropy increases as $\log T$. To have a phase transition the interaction energy for the paths $\{X_t,\ -T \le t \le 0\}$ and $\{X_t,\ 0 \le t \le T\}$ must be at least comparable. Transcribed to the Gibbs measures under study this means

$$W(x, x', t) \cong |t|^{-\gamma} \quad \text{for large } |t| \tag{2.40}$$

with $1 < \gamma \le 2$. The lower bound on γ is needed for having the energy extensive. To carry out a proof more specific assumptions will be needed. We set $d = 1$. For the external potential we choose a double well potential of the form

$$V(x) = \beta(x^4 - x^2)\,, \quad \beta > 0\,. \tag{2.41}$$

In fact, as long as $V(x) = V(-x)$, a general class of double well type potentials can be handled. The pair interaction is quadratic,

$$W(x, x', t) = \alpha\rho(t)\frac{1}{2}(x - x')^2\,, \alpha > 0\,, \quad \rho(t) = (1 + |t|)^{-\gamma}\,. \tag{2.42}$$

Since we rely on comparison inequalities, the interaction needs to be quadratic, at least at the present stage of understanding. Thus the only non-Gaussian piece of the Gibbs measure is $\exp[-\beta\int_{-T}^{T}(X_t)^4 dt]$. Let $\langle\ \cdot\ \rangle_{b,T}$ be the expectation of the Gibbs measure for the potentials V and W from (2.41), (2.42), with the pinned boundary conditions $X_{-T} = b = X_T$, $b \in \mathbb{R}$. Then, for $b > 0$, $\langle X_0\rangle_{b,T} \ge 0$ and $\langle X_0\rangle_{b,T}$ is decreasing in T. Hence the limit

$$\lim_{T\to\infty}\langle X_0\rangle_{b,T} = \langle X_0\rangle_{b,\infty} \tag{2.43}$$

exists.

Theorem 2.13 *Let V, W be as in (2.41), (2.42) and fix $1 < \gamma \le 2$. If $b > 0$, then there exist $\alpha, \beta, m^* > 0$ such that*

$$\langle X_0\rangle_{b,\infty} \ge m^*. \tag{2.44}$$

By symmetry, $\langle X_0\rangle_{-b,\infty} = -\langle X_0\rangle_{b,\infty}$. Thus there must be at least two distinct extreme Gibbs measures for the same interaction. Most likely there are no others, but this problem has not been approached yet.

The strategy of proof is to reduce the bound in (2.44) to a corresponding one for a one-dimensional Ising spin system with long-range ferromagnetic pair interaction, for which the famous proofs of Dyson [13] and of Fröhlich and Spencer [19] on the existence of long range order are available. The reduction is based on ferromagnetic type inequalities. With the block variables

$$\phi_j = \frac{1}{\delta}\int_{(j-\frac{1}{2})\delta}^{(j+\frac{1}{2})\delta} X_t dt\,, \quad j \in \mathbb{Z}\,, \tag{2.45}$$

by Griffiths II we obtain that $\langle X_0\rangle_{b,\infty} \geq c_{\mathrm{G}}\langle\phi_0\rangle^c_{b,\infty}$, where $\langle\ \rangle^c$ is a Gibbs measure over $\mathbb{Z}$ with long range interaction for the continuous spin variables ϕ_j, and $c_{\mathrm{G}} > 0$. Secondly, the Wells inequality, see [38] in the case of stochastic processes, [8] implies that $\langle\phi_0\rangle^c_{b,\infty} \geq c_{\mathrm{W}}\langle\sigma_0\rangle_{+,\infty}$ with $c_{\mathrm{W}} > 0$. Here $\langle\ \rangle_{+,\infty}$ is an Ising spin system, $\sigma_0 = \pm$, with ferromagnetic interaction which decays as $|i-j|^\gamma$ for large $|i-j|$ and $+$ boundary conditions. The complete proof is given in [32], where also explicit bounds for the phase diagram are discussed.

3 A central limit theorem

In this section we study the case where $V = 0$, i.e. we consider

$$\mu_T = \frac{1}{\mathcal{Z}_T}\exp\left(-\int_{-T}^{T}\int_{-T}^{T} W(X_t - X_s, t-s)\,dt\,ds\right)\mathcal{W}_T^0\,. \tag{3.1}$$

Here, $\mathcal{W}_T^0$ is two-sided Brownian motion in $[-T,T]$ pinned at 0 at $t=0$. The interaction depends only on the increments $X_t - X_s$. Provided W has a decent decay in the t-variable, one would thus expect a functional central limit theorem to hold, i.e. after rescaling the path measure μ_T should look like Brownian motion with some effective diffusion matrix D. Such a general result is not available. In case $t \mapsto W(\cdot,t)$ decays exponentially, one can use Dobrushin's theory of one-dimesional spin systems [10, 11] to establish exponential mixing of the increment process [39]. This implies the central limit theorem for X_t properly rescaled. Our approach is less restrictive in terms of decay conditions, but assumes W to be of the special form

$$W(x,t) = -\frac{1}{2}\int |\widehat{\rho}(k)|^2 e^{ik\cdot x}e^{-\omega(k)|t|}\frac{1}{2\omega(k)}dk \tag{3.2}$$

with

$$\omega(k) \geq 0, \quad \omega(k) = \omega(-k), \quad \text{and} \quad \widehat{\rho}(k) = \widehat{\rho}(-k)^*. \tag{3.3}$$

In addition, we assume

$$\int |\widehat{\rho}(k)|^2(\omega^{-1} + \omega^{-2} + \omega^{-3})dk < \infty. \tag{3.4}$$

(3.4) is in fact a (mild) decay condition. For example, if $d = 3$, $\omega(k) = |k|$ and $\widehat{\rho}$ is compactly supported, then the most stringent condition is $\int |\widehat{\rho}|^2\omega^{-3}d^3k < \infty$, which corresponds to a decay of W as

$$|W(x,t)| \leq c(1 + |t|^{3+\delta})^{-1},$$

for some $\delta > 0$. The above choice of parameters represents a physically relevant model, see (iv) of Section 4.

Theorem 3.1 *Define μ_T as in (3.1) with W given by (3.2).*

(i): μ_T converges to a measure μ as $T \to \infty$ in the topology of local convergence.

(ii): The stochastic process X_t, $t \geq 0$, induced by μ satisfies a functional central limit theorem

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} X_{t/\varepsilon} = \sqrt{D} B_t$$

in distribution, where $0 \leq D \leq 1$ as a $d \times d$ matrix, and B_t is standard Brownian motion.

(iii): In addition to (3.3),(3.4) suppose

$$\int |\widehat{\rho}(k)|^2 |k|^2 \left(\omega^{-2} + \omega^{-4}\right) \, dk < \infty. \tag{3.5}$$

Then $D > 0$.

In the remainder of this section, we will give an outline of the proof of Theorem 3.1. A full account is [5]. We will do the proof in three steps.

(1) We use the special form (3.2) of W in order to linearize the interaction in (3.1) by introducing an auxiliary Gaussian process. As a result, we will prove (i) above, and the stochastic process X_t under μ is driven by a reversible Markov process η_t.

(2) In the so obtained representation, we use the by now well-established technique of Kipnis and Varadhan [24]; we write X_t as the sum of a martingale and an additive functional of η_t. X_t is then the sum of two martingales and a negligible process, and the martingale central limit theorem applies, proving (ii).

(3) In order to show that the diffusion is nondegenrate, we rely on an idea of Brascamp, Lebowitz and Lieb [7], which in the present context has been employed before [39].

To carry out step (1), let $\mathcal{K}_0$ be the real Hilbert space obtained by completing the subspace of $L^2(\mathbb{R}^d)$ on which the inner product given by

$$\langle a, b \rangle_{\mathcal{K}_0} = \int \widehat{a}(k) \frac{1}{2\omega(k)} \widehat{b}(k)^* \, dk \tag{3.6}$$

is finite. Let $\mathcal{G}$ be the path measure of the infinite dimensional Ornstein-Uhlenbeck process with mean 0 and covariance

$$\mathbb{E}_{\mathcal{G}}[\phi_s(a)\phi_t(b)] = \int \widehat{a}(k) \frac{1}{2\omega(k)} e^{-|t-s|\omega(k)} \widehat{b}(k)^* \, dk \qquad (a, b \in \mathcal{K}_0).$$

There exists a Hilbert space $\mathcal{K} \supset \mathcal{K}_0$ such that $\mathcal{G}$ is a reversible Gaussian Markov process with values in $\mathcal{K}$ and continuous paths. The reversible measure G is the Gaussian measure on $\mathcal{K} \ni \phi$ with mean zero and covariance

$$\mathbb{E}_{\mathsf{G}}[\phi(a)\phi(b)] = \langle a, b \rangle_{\mathcal{K}_0}.$$

For $x \in \mathbb{R}^d$, let τ_x be the shift by x on $\mathcal{K}$, i.e. $(\tau_x \phi)(a) = \phi(\tau_x a)$ and $\tau_x a(y) = a(y-x)$. More generally, for $f \in L^2(\mathsf{G})$, $(\tau_x f)(\phi) = f(\tau_x \phi)$

For $T > 0$ we put

$$\mathcal{P}_T = \frac{1}{\mathcal{Z}_T} \exp\left(- \int_{-T}^{T} \tau_{X_s} \phi_s(\rho)\, ds \right) \mathcal{W}^0 \otimes \mathcal{G}. \tag{3.7}$$

With $\mathcal{P}_T$ we achieved our first goal, the linearization of the interaction: Indeed, for functions F depending on x only,

$$\mathbb{E}_{\mathcal{P}_T}[F] = \mathbb{E}_{\mu_T}[F],$$

as can be seen by carrying out the Gaussian integration. $\mathcal{P}_T$ is the measure of a Markov process, more specifically a $P(\phi)_1$-process with state space $\mathbb{R}^d \times \mathcal{K}$. The role of the Schrödinger operator is now played by

$$Hf(x,\phi) = -\frac{1}{2}\Delta f(x,\phi) + H_{\mathrm{f}} f(x,\phi) + V_\rho(x,\phi) f(x,\phi), \tag{3.8}$$

where H_{f} is the generator of $\mathcal{G}$ and $V_\rho(x,\phi) = \tau_x\phi(\rho)$. The semigroup Π_T generated by H is strongly continuous on $C_0(\mathbb{R}^d, L^2(\mathsf{G}))$. More importantly, it is also strongly continuous on the Hilbert space $\mathcal{T}$ of functions that are invariant under shift over the x-variable. Explicitly, $\mathcal{T}$ is the image of $L^2(\mathsf{G})$ under the operator

$$U : L^2(\mathsf{G}) \to C(\mathbb{R}, L^2(\mathsf{G})), \quad Uf(x,\phi) = \tau_x f(\phi),$$

equipped with the scalar product

$$\langle f, g \rangle_{\mathcal{T}} = \mathbb{E}_{\mathsf{G}}[(U^{-1}f)(U^{-1}g)^*] = \left\langle U^{-1}f, U^{-1}g \right\rangle_{L^2(\mathsf{G})}. \tag{3.9}$$

H is self-adjoint on $\mathcal{T}$, and (3.4) implies

$$\|\Pi_T 1\|_{\mathcal{T}}^2 \leq C \, \langle 1, \Pi_T 1 \rangle_{\mathcal{T}}^2 .$$

Now from spectral theory we obtain

Theorem 3.2 *The infimum E_0 of the spectrum of H acting in $\mathcal{T}$ is an eigenvalue of multiplicity one. The corresponding eigenfunction $\Psi \in \mathcal{T}$ can be chosen strictly positive.*

An alternative proof of Theorem 3.2, using a completely different method, can be found in [18].

It is now easy to identify the infinite volume limit of the families $\mathcal{P}_T$ and $\mathcal{N}_T$. Let $\mathcal{P}$ be the probability measure on paths $(X_t, \phi_t)_{t\in\mathbb{R}}$ determined by

$$\mathbb{E}_{\mathcal{P}}(f) = e^{2TE_0} \mathbb{E}_{\mathcal{W}^0 \otimes \mathcal{G}} \left[\Psi(X_{-T}, \phi_{-T}) e^{-\int_{-T}^{T} \tau_{X_s}\phi_s(\rho)\, ds} \Psi(X_T, \phi_T) f \right] \tag{3.10}$$

for functions f depending only on X_t, ϕ_t with $|t| < T$. Above, $\mathcal{W}^0$ is the measure of two-sided Brownian motion or, equivalently, Wiener measure conditioned on $X_0 = 0$. Let

μ be the measure $\mathcal{P}$ when applied to functions of x only. Then $\mathcal{P}$ is the measure of a Markov process with generator L acting as

$$Lf = -\frac{1}{\Psi}(H - E_0)(\Psi f). \tag{3.11}$$

$\mathcal{P}_T \to \mathcal{P}$ in the topology of local convergence, and by integrating out the Gaussian field, $\mu_T \to \mu$. The $\mathcal{K}$-valued process $\eta_t = \tau_{X_t}\phi_t$ is reversible with reversible measure $(U^{-1}\Psi)^2\mathsf{G}$, and its generator is unitarily equivalent to L.

Let $\gamma \in \mathbb{R}^d$ be fixed, and $h_\gamma(x) = \gamma \cdot x$. Then $L(h_\gamma) = j(\eta)$ with

$$j = U^{-1}(\gamma \cdot \nabla_x \ln \Psi) \in L^2(\mathsf{G}). \tag{3.12}$$

Since the result of the generator L of process $\mathcal{P}$ applied to $\gamma \cdot x$ is a function of η, only η_t influences the behavior of $\gamma \cdot X_t$, i.e. X_t is driven by η_t. Step one is completed.

Next we write

$$\gamma \cdot X_t = M_t + \int_0^t Lh_\gamma(X_s, \phi_s))\, ds \tag{3.13}$$

with

$$M_t = \gamma \cdot X_t - \int_0^t Lh_\gamma(X_s, \phi_s)\, ds = \gamma \cdot X_t - \int_0^t j(\eta_s)\, ds.$$

Then M_t is a martingale with stationary increments and quadratic variation $|\gamma|^2 t$, and

$$\int_0^t Lh_\gamma(X_s, \phi_s))\, ds = \int_0^t j(\eta_s)\, ds$$

is an additive functional of η_t satisfying the assumptions of [24]. It is thus the sum of a martingale N_t with stationary increments and a negligible process. Now the martingale central limit theorem proves Theorem 3.1 (ii) and finishes step 2.

In principle, it could happen that M_t and N_t are strongly dependent and cancel each other. Then the diffusion matrix D would be zero and X_t would behave subdiffusively. We already know the central limit theorem holds with diffusion matrix $D \geq 0$. Thus it is enough to investigate

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_\mu[(\gamma \cdot X_t)^2] = \langle \gamma, D\gamma \rangle_{\mathbb{R}^d}\,. \tag{3.14}$$

It turns out that

$$\langle \gamma, D\gamma \rangle_{\mathbb{R}^d} = |\gamma|^2 - 2\left\langle \gamma \cdot \nabla_x \Psi, (H - E_0)^{-1}\gamma \cdot \nabla_x \Psi \right\rangle_{\mathcal{T}}\,. \tag{3.15}$$

The standard technique is to turn (3.15) into a variational problem and find a reasonably explicit lower bound to the variational functional. We did not succeed in carrying out the second step of this procedure. Instead, we show directly that

$$\mathbb{E}_\mu[(\gamma \cdot X_t)^2] \geq c|\gamma|^2|t| \tag{3.16}$$

for some $c > 0$, by using ideas from Brascamp *et al* [7] originally developed to study fluctuations for anharmonic lattices. Together with (3.14) this immediately shows $D \geq c$.

4 Applications and open problems

The scheme outlined so far is a probabilistically natural way of constructing through the limit $T \to \infty$ stationary stochastic processes with continuous sample paths. Moreover, specific choices of V and W correspond to particular applications on which there is already a large body of literature using a variety of methods. Very roughly, and as far as we are aware of, the applications originate from three distinct corners of low energy physics.

i) Self-avoiding random walks. Polymers with interaction due to excluded volume is an important statistical mechanics topic, in particular because of the connections with critical phenomena [14]. It is tempting to model the free polymer as Brownian motion and the excluded volume through an interaction of the form (1.7). Note, however, that by the nature of the interaction there is no decay in t. In particular the energy is not extensive. Thus, while the energy depends only on the increments, for large T the statistical properties of the self-avoiding polymer are qualitatively different from a free random walk. One conjecture is that the self-similar scaling theory is obtained from the ultraviolet limit. So far most of the mathematical effort went into constructing the limit measure [41, 6]. But it is not obvious how to extract scale invariant properties from this measure. In fact, self-similarity is now established through lace expansion and other methods [30]. The link between the two approaches remains unexplored.

ii) Statistical hydrodynamics. There is general agreement that fully developed turbulence should be described by a suitable measure over divergence free vorticity fields. One attempt to write down such a measure is to assume that the velocity field $\omega(x) = \nabla \wedge u(x)$ is concentrated along Brownian curves $X_t \in \mathbb{R}^3$ [9]. Under the Eulerian incompressible flow, the kinetic energy $\frac{1}{2}\int u(x)^2 d^3x$ is conserved. Thus it seems natural to use it as energy in the Gibbs measure. This yields the formal expression

$$\mathcal{E}(X) = \int_{-T}^{T}\int_{-T}^{T} \frac{1}{|X_t - X_s|} dX_t \cdot dX_s\,. \tag{4.1}$$

In order to have $\exp[-\mathcal{E}(X)]$ as a well-defined random variable, [17] required the condition that the Coulomb potential in (4.1) is smoothened such that it has a finite electrostatic energy.

Our own investigations mostly draw on applications in quantum mechanics. Since upon Wick rotation the free Schrödinger equation turns into the diffusion equation, Brownian motion as *a priori* measure is in fact forced by the problem. Several interesting cases can be distinguished.

iii) Electron coupled to the quantized radiation field. Upon Wick rotation the free Maxwell field is isomorphic to a stationary infinite-dimensional Ornstein-Uhlenbeck process, see Section 3, for the transverse vector potential $A(x,t)$. It has the covariance

$$\begin{aligned} \mathbb{E}[A_\alpha(x,t)A_\beta(x',t')] &= \int d^3k \frac{1}{2\omega(k)} e^{-\omega(k)|t-s|} e^{ik\cdot(x-x')} (\delta_{\alpha\beta} - |k|^{-2}k_\alpha k_\beta) \\ &= W_{\alpha\beta}(x-x', t-s) \end{aligned} \tag{4.2}$$

$\alpha, \beta = 1, 2, 3$. The dispersion relation of the Maxwell field is

$$\omega(k) = |k| \,. \tag{4.3}$$

Within the Euclidean framework, the electron is governed by the Hamiltonian

$$H = \frac{1}{2}\big(- i\nabla_x - eA(x,t)\big)^2 \,, \tag{4.4}$$

on ignoring the electron spin. The units are such that $\hbar = 1$, $c = 1$, mass of electron $m = 1$; e is the charge of the electron expressing the strength of coupling to the Maxwell field. We use the Feynman-Kac-Ito formula for the propagator for H [38]. Then the joint X_t and $A(x, t)$ path measure is given by

$$\exp\left(-ie\int_{-T}^{T} A(X_t, t)\cdot dX_t\right)\ \mathcal{W}(X)\otimes\mathcal{G}, \tag{4.5}$$

where $\mathcal{G}$ is the Gaussian measure of the A-field with covariance (4.2). Note that $\nabla\cdot A(x, t) = 0$ almost surely. Since the exponent is linear in A, the averaging over the Ornstein-Uhlenbeck process can be done explicitly. This results in a finite volume Gibbs measure with energy

$$\mathcal{E}(X) = \frac{1}{2}\int_{-T}^{T}\int_{-T}^{T} dX_t\cdot W(X_t - X_s, t-s)dX_s \,. \tag{4.6}$$

This is of the form (1.6) and should be read as a double Ito stochastic integral.

W is singular on the diagonal, roughly $W_{\alpha\beta}(x, t) = \delta_{\alpha\beta}(x^2 + t^2)^{-1}$. Thus it is necessary to smear out the charge distribution which leads to the regularized version

$$W^{\rho}_{\alpha\beta}(x,t) = \int d^3k|\widehat{\rho}(k)|^2\frac{1}{2\omega(k)}e^{-\omega(k)|t|}e^{ik\cdot x}(\delta_{\alpha\beta} - |k|^{-2}k_\alpha k_\beta). \tag{4.7}$$

Here $\widehat{\rho}$ is rotation invariant, decays rapidly for large $|k|$, and $\widehat{\rho}(0) = (2\pi)^{-3/2}$ by charge normalization. A problem which appears to be very challenging, is to establish that, for fixed T and $X_{-T} = 0 = X_T$, the Gibbs measure for the energy (4.6) is well defined. In other words, with a smoothening as in (4.7) we would like to study the sequence of Gibbs measures as $\widehat{\rho}(k) \to (2\pi)^{-3/2}$ pointwise (ultraviolet or point charge limit). In favorable cases the existence of the limit can be shown by suitable centering and by possibly adding other counter terms. Such a procedure seems unlikely to work in the present context. Thus the ultraviolet limit has to be linked with a change of the diffusion coefficient D of the underlying Wiener process $\mathcal{W}$ (= mass renormalization). We expect $D \to \infty$ in this limit.

iv) Quantum particle coupled to a scalar Bose field. This model was studied by Nelson [31] in the context of energy renormalization. The Bose field translates to the scalar field $\phi(x, t)$, which again is an infinite-dimensional Ornstein-Uhlenbeck process this time with covariance

$$\begin{aligned}\mathbb{E}[\phi(x,t)\phi(x',t')] &= \int d^3k|\widehat{\rho}(k)|^2\frac{1}{2\omega(k)}e^{-\omega(k)|t-t'|}e^{ik\cdot(x-x')}\\ &= W(x-x', t-t').\end{aligned} \tag{4.8}$$

The quantum particle "sees" ϕ as a fluctuating electrostatic potential. Thus the Hamiltonian becomes $H = -\frac{1}{2}\Delta + e\phi(x,t)$. Then through the Feynman-Kac formula the path measure, jointly for X_t and $\phi(x,t)$, is given by

$$\exp\left(-e\int_{-T}^{T}\phi(X_t,t)dt\right)\ \mathcal{W}^0(X)\otimes\mathcal{G}, \tag{4.9}$$

which has the structure of a $P(\phi)_1$-process, since the *a priori* measure is Markovian and the energy is local in time. The only difference to our discussion in Section 2.1 is that $\mathbb{R}^d$ is replaced by the state space $\mathbb{R}^d\times\mathcal{K}$, compare with the discussion preceding (3.8).

The exponent in (4.9) is linear in ϕ. Thus we can perform the integration over ϕ resulting in the following path measure for X,

$$\frac{1}{Z(T)}\exp\,[-\int_{-T}^{T}V(X_t)dt+\frac{e^2}{2}\int_{-T}^{T}\int_{-T}^{T}W(X_t-X_s,t-s)dtds]\ \mathcal{W}^0, \tag{4.10}$$

where we added an external potential V. Thus the Nelson model naturally yields Gibbs measures of the form studied in Sections 2 and 3. In fact, the Nelson model was our source of motivation for studying Gibbs measures over Brownian motion. The existence of the infinite volume limit can be deduced from (3.10) which requires that

$$\int dk|\widehat{\rho}(k)|^2(\omega(k)^{-3}+\omega(k)^{-1})<\infty. \tag{4.11}$$

We can also use the cluster expansion which holds provided e^2 is sufficiently small and

$$\int dk|\widehat{\rho}(k)|^2(\omega(k)^{-1}+\omega(k)^{-2-\delta})<\infty \tag{4.12}$$

for some $\delta>0$. Since it is possible to express the ground state of the Nelson model directly in terms of data of these Gibbs measures, given the existence of the infinite time interval measures we have a useful tool at hand for studying qualitative properties of the ground state. We refer to [4] for details.

The Nelson model, in the case of massless bosons $\omega(k)=|k|$, is both ultraviolet and infrared divergent. The ultraviolet divergence is mild and can be handled by energy renormalization. This is the content of the famous work [31], which uses exclusively functional analytic methods. Somewhat surprisingly, no one has succeeded in a proper transcription of Nelson's results into the framework of path measures. The infrared divergence translates into a somewhat unexpected feature of the joint $(X_t,\phi(x,t))$ process. From (4.12) and suitable conditions on V, we infer that the infinite volume Gibbs measure exists. However, the limiting procedure changes the situation seen by the *a priori* measure dramatically. For instance, the $t=0$ joint distribution is not absolutely continuous with respect to the $t=0$ projection of the *a priori* distribution. One way to cope is to introduce a suitable shifted Gaussian measure which takes on the role of a new *a priori* measure making the model infrared regular. We refer for more details to [27, 28].

v) The polaron. Physically the polaron is an electron coupled to the optical mode of an ionic crystal. It can be viewed as a particular case of the Nelson model with the choice $\omega(k) = \omega_0$ and $\widehat{\rho}(k) = |k|^{-1}$. Then

$$W(x,t) = -\frac{\alpha}{|x|} e^{-\omega_0 |t|}. \tag{4.13}$$

Here $\alpha > 0$ and subsumes all dimensional coupling coefficients. The ground state energy of the polaron is defined through

$$E_{\mathrm{g}}(\alpha) = -\lim_{T\to\infty} \frac{1}{T} \log Z(T,\alpha). \tag{4.14}$$

For small α one can use perturbation theory in α. For large α Pekar [33] developed an approximate strong coupling theory. Thus the challenge was to have reliable predictions at moderate values of α, which turned out to be difficult. Feynman [15, 16] had the insight from functional integration and used a quadratic functional as upper variational bound. Optimizing the quadratic form yields $E_{\mathrm{g}}(\alpha)$ roughly 2% away from Pekar's result and even better at smaller values when compared with machine computations. The strong coupling (Pekar) limit of the ground state energy has been established by Donsker and Varadhan [12] using functional integration, and by Lieb and Thomas [25] using functional analytic methods.

A long standing open problem is to obtain a corresponding result for the effective mass $m(\alpha)$. In fact, as shown in [39], $m(\alpha) = D(\alpha)^{-1}$ with $D(\alpha)$ the diffusion coefficient in Section 3 with the specific choice (4.13) for W. On heuristic grounds one can guess the behavior of $D(\alpha)$ for large α and relate it to Pekar's variational problem [39]. A proof is missing with the exception of [34] in the simplification where Brownian motion on $\mathbb{R}^3$ is replaced by Brownian motion on the circle.

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