ONE CONSEQUENCE OF THE RESULTS OF LAPTEV-NABOKO-SAFRONOV [11]

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The main goal of this short paper is illustrate applicability of the results of Laptev-Naboko-Safronov [11]. It should be mentioned from the very beginning that inspite that the main theorem of the present paper looks more attractive compared to the ones from [11], the main ingredients and ideas were already introduced in [11]. This article should be considered just as its consequence, since the technique used in our paper is a relatively old and usual tool for treating a discrete spectrum (but not the continuous one!!)

Let us state our main assertion.

Theorem 0.1. *Let $d \geq 3$ and let $V \in L^\infty(\mathbb{R}^d)$ be a real valued potential such that $V(x) \to 0$ as $|x| \to \infty$. Assume that $V \in L^{d+1}(\mathbb{R}^d)$ and for some positive number $\delta > 0$ the Fourier transform of V satisfies the estimate*

$$\int_{|\xi|<\delta} |\hat{V}(\xi)|^2 \, d\xi < \infty.$$

Then the absolutely continuous spectrum of the operator $-\Delta + V$ is essentially supported by the positive real line $\mathbb{R}_+$, i.e. the spectral projection corresponding to any set in $\mathbb{R}_+$ of positive Lebesgue measure is different from zero.

This theorem almost coincides with the result of Deift and Killip [3] for the case $d = 1$ (to be precise they do not impose the conditions $V \in L^\infty$ and $V \to 0$). Note that Denissov [5] proved recently some results about the a.c spectrum of Dirac operators. These results look quite optimal for the considered operator. However the same method can not be applied to the operators having a bottom of the spectrum (Probably this explains where the condition on the Fourier transform for small energies comes from !).

Proof. Let $\lambda_j(V)$ be the negative eigenvalues of $-\Delta + V$ and B_r be the ball of radius r in $\mathbb{R}^d$, $d \geq 3$, around the origin. We are going to prove that

$$\begin{aligned}\sum_j \sqrt{|\lambda_j(V)|} \leq C(r)\Big(\int_{B_r} |\hat{V}(\xi)|^2 \, d\xi + \\ [\int_{B_r} |\hat{V}(\xi)|^2 \, d\xi]^2 + [\int_{B_r} |\hat{V}(\xi)|^2 \, d\xi]^{(d+1)/2} \\ + \int_{\mathbb{R}^d} |V(x)|^{d+1} dx\Big), \qquad \forall V \in C_0^\infty(\mathbb{R}^d).\end{aligned} \tag{0.1}$$

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Therefore we can apply the following result of Laptev-Naboko-Safronov [11].

Theorem 0.2. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real valued potential such that $V(x) \to 0$ as $|x| \to \infty$. Assume that the quantity in the right hand side of (0.1) is finite for V. Then the inequality (0.1) implies the fact that the absolutely continuous spectrum of the operator*

$$H + V = -\Delta + V$$

is essentially supported by the positive real line $\mathbb{R}_+$.

First for any self-adjoint operator T and $s > 0$ we define $n_\pm(s, T)$ as

$$n_\pm(s, T) = \operatorname{rank} E_{\pm T}(s, +\infty),$$

where $E_T(\cdot)$ denotes the spectral measure for T. We put also $n(s, T) = n_+(s^2, T^*T)$ if the operator T is not necessarily self-adjoint. Then the following relations hold:

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2);$$

$$n(s_1 s_2, T_1 T_2) \le n(s_1, T_1) + n(s_2, T_2), \qquad s_1, s_2 > 0.$$

By the Birman-Schwinger principle, we can represent the Lieb-Thirring sum as an integral of the function of distribution for the spectrum of a certain compact self-adjoint operator

$$\sum_j \sqrt{|\lambda_j|} = \int_0^{+\infty} \frac{1}{2\sqrt{s}} n_-(s, H + V)\, ds =$$

$$= \int_0^{+\infty} \frac{1}{2\sqrt{s}} n_+(1, (H + s)^{-1/2} V (H + s)^{-1/2})\, ds$$

Let E denote the spectral projection $E = E_H(0, \delta)$, $\delta > 0$ and let $P = I - E$. Then

$$\begin{gathered}(H + s)^{-1/2} V (H + s)^{-1/2} = (H + s)^{-1/2} E V E (H + s)^{-1/2} + \\ (H + s)^{-1/2} E V P (H + s)^{-1/2} + (H + s)^{-1/2} P V E (H + s)^{-1/2} + \\ (H + s)^{-1/2} P V P (H + s)^{-1/2}\end{gathered} \tag{0.2}$$

Now, let us introduce the potential V_0 whose Fourier transform is the characteristic function of the ball $B_{2\delta}$ multiplied by the Fourier transform of V. Note that then

$$EVE = EV_0E.$$

Therefore

$$(H + s)^{-1/2} V (H + s)^{-1/2} = (H + s)^{-1/2} V_0 (H + s)^{-1/2} + T(s) \tag{0.3}$$

where $T(s)$ satisfies the estimate:

$$\int_0^{+\infty} \frac{1}{\sqrt{s}} n(1, (T(s))\, ds \le C(\delta) \Big(\int_{\mathbb{R}^d} |V|^{d+1}(x)\, dx + \int_{\mathbb{R}^d} |V_0|^{d+1}(x)\, dx \Big) \tag{0.4}$$

It must be noted here that

$$\int_{\mathbb{R}^d} |V_0|^p(x)\,dx \le C_{\delta,p}[\int_{|\xi|<2\delta} |\hat{V}(\xi)|^2\,d\xi]^{p/2}$$

for any $2 < p < \infty$.

Indeed, to prove (0.4) we remark that $||(H+s)^{-1/2}P(H+s+\delta)^{1/2}|| < C$ is bounded uniformly in s. Therefore we have the following inequality for any fixed number $t > 0$

$$\int_0^{+\infty} \frac{1}{2\sqrt{s}} n(t^2,(H+s)^{-1/2}PVP(H+s)^{-1/2})\,ds \le \int_0^{+\infty} \frac{1}{2\sqrt{s}} n(C^{-1}t,(H+s+\delta)^{-1/2}|V|^{1/2})\,ds$$

The right hand side can be estimated by $\int |V|^{d+1}\,dx$ due to the Lieb-Thirring estimate

$$\sum \sqrt{|\lambda_j(\delta - |V|)|} \le C \int (|V|-\delta)_+^{(d+1)/2}\,dx \le C_\delta \int |V|^{d+1}\,dx.$$

The estimate for

$$\int_0^{+\infty} \frac{1}{2\sqrt{s}} n(t,(H+s)^{-1/2}EVP(H+s)^{-1/2})\,ds$$

is more straightforward, since $||P(H+s)^{-1/2}|| \le \delta^{-1/2}$. In this case we obtain

$$\int_0^{+\infty} \frac{1}{2\sqrt{s}} n(t,(H+s)^{-1/2}EVP(H+s)^{-1/2})\,ds \le \int_0^{+\infty} \frac{1}{2\sqrt{s}} n(c(\delta)t,(H+s)^{-1/2}EV)\,ds \le$$

$$\le \sum \sqrt{|\lambda_j(-CV^2)|}, \qquad C = C(t,\delta)$$

The latter sum of eigenvalues can be estimated due to the Lieb-Thirring estimate

$$\sum \sqrt{|\lambda_j(-V^2)|} \le C \int |V|^{d+1}(x)\,dx$$

Finally we apply one of the results of [11] to conclude that

$$\int_0^{+\infty} \frac{1}{\sqrt{s}} n_+(1,(H+s)^{-1/2}V_0(H+s)^{-1/2})\,ds \le C(\delta)\Big(\int_{B_{2\delta}} |\hat{V}|^2(x)\,dx + (\int_{B_{2\delta}} |\hat{V}|^2(x)\,dx)^2\Big).$$

Indeed this estimate is a consequence of the following statement of Laptev-Naboko-Safronov [11]

Theorem 0.3. *Let Ω_δ be the domain*

$$\Omega_\delta = \{\, \xi \in \mathbb{R}^d : |(\xi,e)| < \delta \,\}$$

where e is a fixed vector in $\mathbb{R}^d$ of norm 1. It holds for the eigenvalues of the Shcrödinger operator that

$$\sum \sqrt{|\lambda_j(V)|} \le C(\delta)\Big(\int_{\Omega_\delta} |\hat{V}|^2(\xi)\,d\xi + \int |V|^4(x)\,dx\Big), \quad d \ge 3.$$

In particular, if $\hat{V} = 0$ outside of the ball B_δ then

$$\sum \sqrt{|\lambda_j(V)|} \le C(\delta)\Big(\int_{B_\delta} |\hat{V}|^2(x)\,dx + (\int_{B_\delta} |\hat{V}|^2(x)\,dx)^2\Big).$$

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