Linear Hamilton Systems without Regular Properties. Solving a Problem Stated by M.G. Krein.

Sergej A. Chorošavin

Keywords: Hamilton dynamical system, Ljapunov exponent, indefinite inner product, linear canonical transformation, Bogolubov transformation
2000 MSC. 37K40, 37K45, 47A10, 47A15, 47B37, 47B50

Email: sergius@pve.vsu.ru

Abstract

We construct linear Hamilton systems without usual dichotomy property. The Ljapunov spectra of these systems are unfamiliar and conflicting, the behaviour of trajectories is very complicated. The paper's subject refers to some problems of indefinite inner product methods in the stability theory of abstract dynamical equation solutions.

1 Introduction

We start briefly describing some basic facts about finite-dimensional linear autonomous invertible dynamical systems, that is mathematically, finite-dimensional systems of linear [1] ordinary differential equations with constant coefficients which are such that each separate system can be written in matrix form as

$$\frac{dx(t)}{dt} = Ax(t) \,.$$

So, when we say "trajectory of the dynamical system", we mean "solution to the corresponding ODE system".

Note, given a trajectory and another trajectory —we say "displaced trajectory"—, then, as we treat *linear* systems, the displacement *of* the former trajectory —or, deviation *from* the former trajectory— is also a solution to the corresponding ODE system.

Now then, suppose one treats trajectories of a linear dynamical systems. Then a standard classification fact is:

Every trajectory $x(t)$ has a decomposition which is of the form

$$(*) \qquad x(t) = x_1(t) + \cdots + x_N(t)$$

where $x_1(t), \ldots, x_N(t)$ are trajectories of the **same** *System, which have usual (standard) exponential-wise behaviour, as $t \to \pm\infty$.*

If the finite-dimensional linear autonomous invertible dynamical system is, in addition, Hamilton, then we observe a property (of the trajectories) of such a system, we name that property "being split", which is of an especial interest, especially in the theory of **non-linear (!!)** dynamical systems [2], and which can be expressed as

Splitting Theorem. *Every trajectory $x(t)$ has a decomposition which is of the form*

$$(**) \qquad x(t) = x_-(t) + x_+(t)\,; \qquad |x_\pm(t)| \le P_\pm(t) \qquad \text{as } t \to \pm\infty$$

[1] when we say "linear" we mean "properly linear", that is,"linear homogeneous"

[2] we will not going into details, how does stability of **non-linear** system relates to the property "being split", which is defined for a **linear** one

where $P_\pm$ are some polynomials and $x_\pm$ are suitable trajectories of the **same** *dynamical system .*

It is while one treats any linear finite-dimensional Hamilton system. **But what about infinite-dimensional systems?**

First of all, we need to explain what we mean by "linear autonomous dynamical system".

Definition of abstract linear autonomous invertible dynamical system.
Let L be a linear space.
Let a one-parameter family of linear operators on L,

$$\{U_t\}_t \qquad (t \in \mathbf{R}),$$

be such that

$$U_{t-r}U_{r-s} = U_{t-s}\,, \qquad \text{if } t, r, s \in \mathbf{R} \qquad (\textbf{ consistency relation })$$

$$U_0 = I = \textit{ identity operator on } L\,.$$

In this case the pair $L, \{U_t\}_t$ is said to be an **abstract linear autonomous invertible dynamical system** *and* two-*parameter family U_{t-s} , $t, s \in \mathbf{R}$ is called a* **propagator***, alias* **evolution operator***.*

Given a $t_0 \in \mathbf{R}$ and an $x_0 \in L$, we say that the one-parameter family

$$x(t) = U_{t-t_0}x_0 \qquad (t \in \mathbf{R})$$

is the **trajectory***. In such a case we say that t_0, x_0 are* **initial data***.*

In this paper we will mostly treat the case where the expression $t, s \in \mathbf{R}$ is replaced by $t, s \in \mathbf{Z}$, i.e., t, s are integers. In this case we say that the dynamics is **discrete** *and write something like N, M or n, m instead of t, s. Note that*

$$U_N = \mathcal{U}^N \qquad (\text{for all integers } N)$$

where $\mathcal{U} := U_1$.

In this paper we will treat only the case where the underlying space L is **Hilbert** (complex or real) and mostly the case where the dynamics is *discrete* [3]. So, we need not immediately give any generalization of the concepts defined above. On the contrary, we shall restrict ourselves by special classes of propagators.

As already noted, we are interested in linear *Hamilton* systems. We adopt two different ways to formulate it in terms of propagator:

1) the underlying space is symplectic (real or complex) and every U_t is a symplectic automorphism;
2) the underlying space is J-space (complex or real) and every U_t is J-unitary.

Commonly, the phrase "a linear operator, call it T, is **J-unitary**" means:

$$T^*JT = J = TJT^*; \quad J^* = J; \quad J^2 = I,$$

whereas if T and a $\mathcal{J}$ are such that

$$T^*\mathcal{J}T = \mathcal{J} = T\mathcal{J}T^*; \quad \mathcal{J}^* = -\mathcal{J}; \quad \mathcal{J}^2 = -I,$$

[3]some more details will be expounded later. As for a possible (and quite standard) generalization of the concept of linear dynamics, see e.g. the section Appendix A.

then $\mathcal{J}$ is an instance of **operator of a symplectic structure** and T is a **symplectic automorphism** [4] , which is also called linear canonical transformation and Bogoliubov transformation. We say "**J-unitary and symplectic automorphism**" instead of "J-unitary operator which is at the same time a symplectic automorphism".

First of all, it is necessesary to recall

Lemma H1-1 *Let V be a linear bounded operator acting on a Hilbert space H. Suppose, V^{-1} exists and is bounded.*

Then $V \oplus V^{*-1}$ (i.e. the Hilbert direct sum of V and V^{*-1}) is J-unitary with respect to J which is defined by

$$J \,:\, x \oplus y \;\mapsto\; y \oplus x \quad (x\,, y \,\in H\,)$$

If one set instead of J the operator which acts by the rule

$$\mathcal{J} \,:\, x \oplus y \;\mapsto\; y \oplus -x \quad (x\,, y \,\in H\,) \qquad ,$$

then one obtains that $\mathcal{J}^* = -\mathcal{J}; \mathcal{J}^2 = -I$, i.e. $\mathcal{J}$ is an operator of a symplectic structure; in this case $V \oplus V^{*-1}$ is a symplectic automorphism .

□

We will systematically exploit the construction $V \oplus V^{*-1}$. In that cases, we will write $\hat{V} := V \oplus V^{*-1}$.

Of course, the formulation of the Property "being split" must be revised. At least two formulations seem to be appropriate:

1) Instead of letting only polynomials be in $(**)$, it is reasonable to allow the functions of "not too rapid increase" to be present there: we will admit "subexponentially increasing" functions; in this case we say about the (regular) splitting (or regular separation) of trajectories and discuss the corresponding **Existence Problem** of such trajectories;
   We specify the notion of "being split" by

   Definition of a **regularly split trajectory.**
   If a trajectory $x(t)$ has a decomposition of the form

   $$(***) \qquad\qquad x(t) = x_-(t) + x_+(t)$$

   where $x_\pm$ are trajectories of the **same** *dynamical system, and where $x_\pm$ are such that*

   $$(\forall \lambda > 0) \quad \|x_+(t)\| e^{-\lambda t} \to 0 \text{ as } t \to +\infty$$
   $$(\forall \lambda > 0) \quad \|x_-(t)\| e^{+\lambda t} \to 0 \text{ as } t \to -\infty\,,$$

   then we say that $x(t)$ is **regularly decomposable** *or* **regularly split** *or, for brief, $x(t)$ is* **regular**.

2) instead of seeking for the set of subexponentially increasing trajectories, one prefers to seek, having ideas of spectral theory in mind, for that invariant subspaces of the propagator of the system, on which the spectral radius of the propagator would be ≤ 1; in this case one says about **Problem of** M.G. KREIN.

[4]linear symplectic automorphism is much more abstract object than that the T which satisfies $T^*\mathcal{J}T = \mathcal{J} = T\mathcal{J}T^*$ for a $\mathcal{J}$ such that $\mathcal{J}^* = -\mathcal{J}; \quad \mathcal{J}^2 = -I$

It is important to take into account, that not only the case of the continuous "time" (i.e. $t \in \mathbf{R}$), but also the discrete "time" case (primarily, $t \in \mathbf{Z}$) or some other abstract "time" cases are interesting: we mean now "symplectic representations of semi-groups". As for phase spaces, i.e. the spaces on which such representations act, we repeat that we must consider not only real, but also complex spaces.

To specify the Problems, introduce a suitable definition.

Definition D1-1

$$S_0(T) := \{x \in H| \quad \|T^N x\| \to 0 \textit{ for } N \to +\infty\},$$

$$S(T) := \{x \in H| \, \exists C \geq 0 \, \forall N \geq 0 \quad \|T^N x\| \leq C\},$$

$$S_+(T) := \{x \in H| \forall a > 1 \, \exists C \geq 0 \, \forall N \geq 0 \quad \|T^N x\| \leq C a^N\},$$

$$r(T) := \textit{spectral radius of } T\,.$$

Remark R1-1 *Given a linear bounded operator T and a T-invariant subspace L such that $r(T|L) \leq c$, then*

$$L \subset S_+(c^{-1}T)$$

Remark R1-2

$$S_\mathrm{x}(T_1 \oplus T_2) = S_\mathrm{x}(T_1) \oplus S_\mathrm{x}(T_2);$$

here S_x stands for S_0 or S or S_+ respectively.

Remark R1-3

$$S_0(T) \perp S(T^{*-1})\,, \qquad S(T) \perp S_0(T^{*-1})\,,$$

which is immediate if one notices that

$$|(x,y)| = |(T^N x, T^{*-N} y)| \leq \|T^N x\| \, \|T^{*-N} y\|\,.$$

So, the first Problem is a problem of describing the structure of the set of the kind $S_\mathrm{x}(T)$. At least, one asks:

Question 1.
Let T be any symplectic or J-unitary.
Is $S_+(T) \neq \{0\}$?

The second Problem looks more traditional: one seek for the T-invariant subspaces, on which the previously given bounds of spectral radius are fulfilled. At least, one asks:

Question 2.
Let T be any symplectic automorphism or J-unitary operator.
Is there a non-trivial T-invariant subspace L, such that $r(T|L) \leq 1$?

Perhaps it is just the time and place to recall the notion of Ljapunov indices.

Let 0 stand for the zero-element of the underlying space, x_0 be an element of the same space. We will refer to x_0 as an instance of **initial displacement** of 0 (or, as an instance of **initial deviation** from 0).
Let $t_0 \in \mathbf{R}$. We will refer to t_0 as an **initial value of the time**, t.

Let $\{x(t)\}_{t\in\mathbf{R}}$ stand for the trajectory such that $x(t_0) = x_0$. Thus, $\{x(t)\}_{t\in\mathbf{R}}$ is the displacement of the zero trajectory starting at $t = t_0$. [5]

The **Ljapunov** upper **indices** of the growth of the initial displacement, x_0, are real numbers, $\lambda_\pm$, defined by

$$\lambda_\pm := \limsup_{t\to\pm\infty} \frac{\ln\|x(t)\|}{|t|} = \inf\{\lambda|\ \|x(t)\| \leq C_\lambda e^{\lambda|t|}\,(t\to\pm\infty)\}\,,$$

Note, since the systems we handle are autonomous, neither λ_+ nor λ_- depend on t_0.

So, the first Problem and Question 1, stated above, concern the associated theory of Ljapunov spectrum, which is sometimes called Floquet-spectrum.

We will not go into historical details and restrict ourselves by a simple enumeration of papers and books we dealt elaborating the theme. They are : [Will36], [Will37] [Kre65], [KL1], [KL2], [IK], [IKL], [DalKre], [DadKul], [Kul], [Ikr89], [Bogn], [DR], [Maj],

[Bog], [BraRob], [Ber], [Rob], [MV], [Emch], [RS2], [RS3], [Oks], [DadKul], [Kul], [Ikr89]. [Ch81], [Ch83], [Ch83T], [Ch84], [ChDTh].

Although the Problem of describing regular split displacements is well-known for many years, it is still open. Even the Questions 1 and 2 have been answeren (in negative) not long ago, by [Ch97], [Ch98].

The present paper is based just on [Ch97], [Ch98].

More precisely, we construct three discrete linear dynamical Hamilton systems (the associated operators are $\hat{U}, \hat{V}, \hat{W}$ in Sections 2, 3) and briefly describe their continuous analogues (Section 4). All that systems have very complicated behaviour.

Naturally, it is hardly worthy to qualify that systems as completely chaotic.

However their Ljapunov spectra seem completely exotic, and the coresponding spectral subspaces and the sets S_0, S, S_+ seem to be strange and surprising. From this point of view we would rather say, that the behaviour of the constructed systems is **pre-chaotic**.

In outline the situation is this:

The first system we have constructed is such that S_+ contains only element: this element is the zero element, certainly. Hence, all Ljapunov indices are strictly positive. Moreover, we take a number $c > 0$ quite arbitrarily, and then we construct a system, such that **both** of the Ljapunov indices $\lambda_\pm$ of **every** non-zero displacement $\geq \ln(2+c)$. And at the same time, there are displacements such that their **lower** index of growth $= -\ln(2+c)$.

In the case of the second system the set S_+ is "rich": There are "many" displacements such that their Ljapunov indices are strictly negative definite, all they $< -\ln 2 - \ln c$, $c > 1$. Nevertheless the closure of that set contains some displacements such that their indices are strictly positive, furthermore, they all $\geq \ln 2 + \ln c$.

As for third system, it is such that S_0 is non-vanishing (so, $\{0\} \neq S_0 \subset S \subset S_+$), and a space to be seeked for exists, i.e. a **maximal** invariant subspace L exists such that $r(\hat{W}|L) = 1$ (hence, $L \subset S_+$). Nevertheless they, L and S_0, are mutually orthogonal, and furthermore, $L \cap \overline{S} = \{0\}$.

Moreover, $L = {S_0}^\perp$, even $L = S^\perp$. Moreover, the spectral radius of the propagator restricted on $L^\perp \equiv \overline{S_0}$ is equal to 2 and the spectrum of the restriction itself is a subset of $\{z|1 \leq |z| \leq 2\}$.

[5] Notice, we now handle **linear** systems

So are the facts that concern the Question 1.

As for Question 2 in itself, for many years there was a suspicion that the answer is positive in every case. This suspicion was suggested by results of various kinds, as by particular existence theorems— original [Kre64], relatively recent [Shk99]—, as by theorems of the sort— " $S_0(T)$ is a subset of the subspace that had been constructed by M.G.Krein [Kre64], [Kre65]" (for details see [Ch89-1], [Ch89-2], [Ch96T], [Ch00]).

In spite of that the answer is negative.

We are now going to the exposition proper of the theme. Troughout the paper, when we use "J-Terminologie " we have [Kre65] in mind. As for general mathematical terminology, we follow [RS].

2 Discrete Dynamical System without Regular Trajectories

Theorem Th2-1 *Let c be any real such that $c > 0$. Then there exists a J-unitary and symplectic automorphism $\mathcal{U}$ such that*

$$S_+(c^{-1}\mathcal{U}) = S_+(c^{-1}\mathcal{U}^{-1}) = \{0\}\,.$$

In particular,

(i) *Let L be a $\mathcal{U}$-invariant subspace such that $L \neq \{0\}$. Then $r(\mathcal{U}|L) > c$;*

(ii) *Let L' be a $\mathcal{U}$-invariant subspace such that $L' \neq \{0\}$. Then $r(\mathcal{U}^{-1}|L) > c$;*

(iii) *there is no $\mathcal{U}$-invariant subspace L'' such that $|spectrum\,(\mathcal{U}|L'')| \geq c^{-1}$.*

Proof

Assume, we have a Hilbert space H_0 and an operator $U : H_0 \to H_0$ such that:

1) U is linear, bijective, bounded; [6]

2) $S_+(c^{-1}U) = S_+(c^{-1}U^{*-1}) = S_+(c^{-1}U^{-1}) = S_+(c^{-1}U^*) = \{0\}$.

Given such an U, let $\mathcal{U} := \hat{U} = U \oplus U^{*-1}$. Then:

a) $S_+(c^{-1}\mathcal{U}) = S_+(c^{-1}\mathcal{U}^{-1}) = \{0\}$ (by Remark **R1-2**);

b) $\mathcal{U}$ is J-unitary and symplectic (by Lemma **H1-1**);

c) items (i), (ii), (iii) are fulfilled (by Remark **R1-1**).

Now we have to construct that U. We will do it in Lemma **L2-1**, but before starting we must introduce some definitions and facts related to the theory of so-called weighted shifts.

Definition D2-1 *Let H_0 be any separable (real or complex) Hilbert space.*

Let $(\,,\,)$ stand for the scalar product in H_0 and let $\{b_n\}_n$ denote an orthonomal basis of H_0, the elements of which are indexed by $n = ..., -1, 0, 1, ...$

Let $\{u_n\}_{n\in\mathbf{Z}}$ be a bilateral sequence; we will suppose that $u_n \neq 0$ for all $n \in \mathbf{Z}$.

Now let U denote the **shift** [7] *that is generated by the formula*

$$U \,:\, b_n \;\mapsto\; \frac{u_{n+1}}{u_n}\, b_{n+1} \qquad . \tag{$*$}$$

The general facts we need are these:

Observation O2-1

One constructs the U as follows:

One starts extending the instruction $(*)$ on the linear span of the $\{b_n\}_{n\in\mathbf{Z}}$ so that the resulted operator becomes linear. That extension is unique and defines a linear densely defined operator, which is here denoted by U_{min}, and which is closable. The closure of U_{min} is just the U.

Now then, this U is closed and at least densely defined and injective; it has dense range and the action of U^N, U^{*-N}, $U^{*N}U^N$, $U^{-N}U^{*-N}$ (for any integer N) is generated by

$$U^N : b_n \mapsto \frac{u_{n+N}}{u_n} b_{n+N}\,; \qquad U^{*-N} : b_n \mapsto \frac{u_n^*}{u_{n+N}^*} b_{n+N}\,;$$

[6] hence U^*, U^{-1}, U^{*-1} are bounded

[7] the full name is: the **bilateral weighted shift of $\{b_n\}_n$, to the right**.

$$U^{*N}U^N : b_n \mapsto |\frac{u_{n+N}}{u_n}|^2 b_n\,; \qquad U^{-N}U^{*-N} : b_n \mapsto |\frac{u_n}{u_{n+N}}|^2 b_n\,.$$

In particular, U^N is bounded just when the number sequence $\{|u_{n+N}/u_n|\}_n$ is bounded. □

The special factors we need are these:

Observation O2-2 *The family $\{b_n\}_n$ is an orthonormal basis. In addition $U^N b_n \perp U^N b_m$ for $n \neq m$. Thus*

$$\|U^N f\|^2 = \sum_n |(b_n, f)|^2 \|U^N b_n\|^2 = \sum_n |(b_n, f)|^2 |\frac{u_{n+N}}{u_n}|^2$$

for every $f \in D_{U^N}$.

In particular,

$$\|U^N f\| \geq |(b_n, f)||u_{n+N}/u_n|$$

for all integers n.

It follows that:

Given $f \in H_0 \setminus \{0\}$ and given some real M, a such that

$$\|U^N f\| \leq Ma^N \text{ for } N = 0, 1, 2, ...,$$

then there exists a real M' such that

$$|u_N| \leq M'a^N \text{ for } N = 0, 1, 2, ...$$

For U^{*-1}, U^{-1}, U^*, we have the similar implications. Stated explicitly, they are:

$$\begin{array}{lcccrcll}
\|U^N f\| & \leq & Ma^N & \Rightarrow & |u_N| & \leq & M'a^N & (N = 0, 1, 2, ...) \\
\|U^{*-N} f\| & \leq & Ma^N & \Rightarrow & |u_N|^{-1} & \leq & M'a^N & (N = 0, 1, 2, ...) \\
\|U^{-N} f\| & \leq & Ma^N & \Rightarrow & |u_{-N}| & \leq & M'a^N & (N = 0, 1, 2, ...) \\
\|U^{*N} f\| & \leq & Ma^N & \Rightarrow & |u_{-N}|^{-1} & \leq & M'a^N & (N = 0, 1, 2, ...)
\end{array}$$

[8] *By proving the next Lemma, we will apply exactly such consequences of these implications:*

Let c be a real number such that $c > 0$. Then

$$\begin{array}{llccl}
S_+(c^{-1}U) \neq \{0\} & \Rightarrow \exists M' > 0 \forall N \geq 0 & |u_N| & \leq & M'(c+1)^N \\
S_+(c^{-1}U^{*-1}) \neq \{0\} & \Rightarrow \exists M' > 0 \forall N \geq 0 & |u_N|^{-1} & \leq & M'(c+1)^N \\
S_+(c^{-1}U^{-1}) \neq \{0\} & \Rightarrow \exists M' > 0 \forall N \geq 0 & |u_{-N}| & \leq & M'(c+1)^N \\
S_+(c^{-1}U^{*}) \neq \{0\} & \Rightarrow \exists M' > 0 \forall N \geq 0 & |u_{-N}|^{-1} & \leq & M'(c+1)^N
\end{array}$$

□

Lemma L2-1 *Let c be a real number such that $c > 0$.*

Set

$$u_n := (c+2)^{|n| \sin(\frac{\pi}{2} \log_2(1+|n|))} \qquad (n = ..., -1, 0, 1, ...)$$

Then the associated shift U is bounded with its inverse and

$$S_+(c^{-1}U) = S_+(c^{-1}U^{*-1}) = S_+(c^{-1}U^{-1}) = S_+(c^{-1}U^*) = \{0\}\,.$$

[8]We have here meant the format

$$\exists f \in H_0 \setminus \{0\}, M > 0, a > 0 \forall N \geq 0 \quad \cdots \quad \Rightarrow \quad \exists M' > 0 \forall N \geq 0 \quad \cdots \quad .$$

PROOF The derivative of the real-valued function

$$x \mapsto |x| \sin(\frac{\pi}{2} \log_2(1+|x|))$$

is equal to

$$(\sin(\frac{\pi}{2} \log_2(1+|x|)) + \frac{\pi}{2\ln 2} \frac{|x|}{1+|x|} \cos(\frac{\pi}{2} \log_2(1+|x|)))\mathrm{sgn}\, x$$

and its absolute value does not exceed the value of $\alpha := 1 + \pi/(2\ln 2)$. By the Mean Value Theorem (Lagrange),

$$(c+2)^{-\alpha} \le |u_{n+1}/u_n| \le (c+2)^{\alpha}.$$

Hence U and U^{-1} are bounded.

Now choose two sequences of integers defining them by

$$n_k := 2^{1+4k} - 1; \quad m_k := 2^{3+4k} - 1 \qquad (k = 1, 2, ...).$$

Then n_k, $m_k \in \mathbf{N}$, $n_k \to +\infty$,$m_k \to +\infty$ (as $k \to +\infty$), and simultaneously

$$u_{n_k} = u_{-n_k} = (c+2)^{n_k}; \quad u_{m_k}^{-1} = u_{-m_k}^{-1} = (c+2)^{m_k}.$$

We see that no estimation of the form

$$|u_N| \le M'(c+1)^N, \quad |u_{-N}| \le M'(c+1)^N, \quad |u_N|^{-1} \le M'(c+1)^N, \quad |u_{-N}|^{-1} \le M'(c+1)^N$$

(for $N = 0, 1, \cdots$) is possible. On looking at the Observation **O2-2**, we see that

$$S_+(c^{-1}U) = \{0\}, \, S_+(c^{-1}U^{-1}) = \{0\}, \, S_+(c^{-1}U^{*-1}) = \{0\}, \, S_+(c^{-1}U^*) = \{0\}.$$

This is just what was to be proven.

□

The proof of Lemma **L2-1** is completed, so is the proof of Theorem **Th2-1**.

□

Remark R2-1 *Actually, we have taken a number $c > 0$ quite arbitrarily, and then we have constructed a system, such that* **both** *of the Ljapunov indices $\lambda_\pm$ of* **every** *non-zero displacement $\ge \ln(2+c)$. And at the same time, there are displacements such that their* **lower** *index of growth $= -\ln(2+c)$.*

3 Another Examples of J-unitary Operators

In this section, we construct two more operators which properties looks something strange. The elements of the constructions are the same as that which we have introduced in the previous sections, namely:

H_0, it stands for any separable Hilbert space; $\{b_n\}_n$, it stands for a orthonormal basis of H_0, the elements of that basis will be indexed by $n = ..., -1, 0, 1, ...$. In addition,

$$\hat{H}_0 := H_0 \oplus H_0\,, \quad J(x \oplus y) := y \oplus x\,, \quad \mathcal{J}(x \oplus y) := -y \oplus x\,, \quad (x, y \in H_0)$$

and given a linear operator $T : H_0 \to H_0$, we put $\hat{T} := T \oplus T^{*-1}$, whenever T^{*-1} exists.

We construct two bilateral sequences of numbers $\{v_n\}_n$, $\{w_n\}_n$, $n = ..., -1, 0, 1, ...$, so that the associated shifts, V and W, and the corresponding J-unitary and symplectic automorphisms, $\hat{V}$ and $\hat{W}$, have especial properties.

Definition D3-1 *Let c be a real number such that $c \geq 1$. Let $v_n := (2c)^{-|n|}$ for any integer n. Let $V : H_0 \to H_0$ denote the associated shift, defined by*

$$V\ :\ b_n \mapsto \frac{v_{n+1}}{v_n} b_{n+1}\,.$$

With other words, let

$$Vb_n := \frac{1}{2c} b_{n+1} \ \textit{for}\ n = 0, 1, 2, ... \quad Vb_n := 2cb_{n+1} \ \textit{for} \quad n = ..., -2, -1.$$

Remark R3-1 *The just now defined V is bounded and invertible and its inverse is bounded as well. Using the definition one can show that*

(1) $\|V^N b_n\| = (2c)^{-|n+N|+|n|}$ *for all integers* n, N;

(2)
$r(V) = r(V^{-1}) = 2c$;

(3) $\overline{S_0\left(\frac{3c}{2}V\right)} = H_0$, $S_0\left(\frac{2}{3c}V^{*-1}\right) = \{0\}$, $\overline{S_0\left(\frac{3c}{2}V^{-1}\right)} = H_0$, $S_0\left(\frac{2}{3c}V^{*}\right) = \{0\}$.

Lemma L3-1 *Let L, M be (linear closed) subspaces of $\hat{H}_0$ such that*

$$\hat{V}L = L, \quad |\textit{spectrum}\,\hat{V}|L| \,\leq\, c\,, \hat{V}^{-1}M = M, \quad |\textit{spectrum}\,\hat{V}^{-1}|M| \,\leq\, c\,.$$

Then:

(a) $L = L_1 \oplus \{0\}$, $\quad M = M_1 \oplus \{0\}$, *for some* $L_1 \subset H_0\,,\, M_1 \subset H_0$;

(b) $VL_1 = L_1$, $|\textit{spectrum}\,V|L_1| \,\leq\, c$;
$V^{-1}M_1 = M_1$, $|\textit{spectrum}\,V^{-1}|M_1| \,\leq\, c$;

(c) $L_1 \neq H_0$, $\quad M_1 \neq H_0$.

PROOF

Proof of (a) : Follow from

$$\begin{aligned} L \,\subset\, S_0\left(\frac{2}{3c}\hat{V}\right) &= S_0\left(\frac{2}{3c}V \oplus \frac{2}{3c}V^{*-1}\right) \\ &= S_0\left(\frac{2}{3c}V\right) \oplus S_0\left(\frac{2}{3c}V^{*-1}\right) = S_0\left(\frac{2}{3c}V\right) \oplus \{0\}; \\ M \,\subset\, S_0\left(\frac{2}{3c}\hat{V}^{-1}\right) &= S_0\left(\frac{2}{3c}V^{-1} \oplus \frac{2}{3c}V^{*}\right) \\ &= S_0\left(\frac{2}{3c}V^{-1}\right) \oplus S_0\left(\frac{2}{3c}V^{*}\right) = S_0\left(\frac{2}{3c}V^{-1}\right) \oplus \{0\}; \end{aligned}$$

Proof of (b): After (a) is proven, we can state:

$$L = L_1 \oplus \{0\},\ M = M_1 \oplus \{0\},\ \hat{V} = V \oplus V^{*-1}.$$

Hence

$$\mathit{spectrum}\,\hat{V}|L = \mathit{spectrum}\,V|L_1, \quad \mathit{spectrum}\,\hat{V}^{-1}|M = \mathit{spectrum}\,V^{-1}|M_1.$$

Therefore

$$|\mathit{spectrum}\,V|L_1| = |\mathit{spectrum}\,\hat{V}|L| \leq c,$$
$$|\mathit{spectrum}\,V^{-1}|M_1| = |\mathit{spectrum}\,\hat{V}^{-1}|M| \leq c.$$

Proof of (c) : We have $|\mathit{spectrum}\,V|L_1| \leq c$, and $r(V) = 2c$. Hence $L_1 \neq H_0$. Similarly, $|\mathit{spectrum}\,V^{-1}|M_1| \leq c$, and $r(V^{-1}) = 2c$. Hence $M_1 \neq H_0$.

□

Now recall that $H \oplus \{0\}$ is J-neutral subspace of $\hat{H}_0$ (see [Krein65]). In particular, $H_0 \oplus \{0\}$ is a semidefinite subspace.

So, we now come to

Theorem Th3-1 *Let L be a semidefinite subspace of $\hat{H}_0$ such that*

$$\hat{V}L = L, \qquad |\mathit{spectrum}\,\hat{V}|L| \leq c.$$

Then L is not maximal.

Let M be a semidefinite subspace of $\hat{H}_0$ such that

$$\hat{V}^{-1}M = M, \qquad |\mathit{spectrum}\,\hat{V}^{-1}|M| \leq c.$$

Then M is not maximal.

Remark R3-2

V has an interesting property:

$b_n \in S_0(V) \cap S_0(V^{-1})$ for every integer n; as a result, $S_0(V) \cap S_0(V^{-1})$ is dense in H_0 . Moreover, let L_k denote the closed linaer span of $\{V^s b_k |\, s \geq k\}$. Then

$$VL_k \subset L_k\ , r(V|L_k) \leq 1/2c \quad \text{and } H_0 = \overline{\cup\{L_k\,|k = ... - 1, 0, 1...\}}\ .$$

In spite of that $r(V) = 2c$.

V^{-1} has the similar property. But there L_k is to be replaced by the closed linaer span of $\{V^{-s} b_k |\, s \geq k\}$.

What we now want to know is what kind of growth of $\|V^N f\|^2$ is possible, as $N \to \pm\infty$. Is there an $f \in H_0$ such that $\|V^N f\|^2 \to \infty$, as $N \to \pm\infty$?

Let $f := \sum_{n\neq 0} |n|^{-1} b_n$. Then $f \in H_0$ and

$$\begin{aligned}
\|V^N f\|^2 &= \sum_n |(b_n, f)|^2 \left|\frac{v_{n+N}}{v_n}\right|^2 \\
&= \sum_{n>0} \frac{1}{|n|^2} \frac{(2c)^{-2|n+N|}}{(2c)^{-2|n|}} + \sum_{n<0} \frac{1}{|n|^2} \frac{(2c)^{-2|n+N|}}{(2c)^{-2|n|}} \\
&= \sum_{n>0} \frac{1}{|n|^2} \frac{(2c)^{-2|n+N|}}{(2c)^{-2|n|}} + \sum_{n>0} \frac{1}{|n|^2} \frac{(2c)^{-2|n-N|}}{(2c)^{-2|n|}}
\end{aligned}$$

Let us estimate the sequence $\|V^N f\|^2$ **from below** .
Let $N \geq 0$. *Then*

$$\begin{aligned}
\|V^N f\|^2 \quad &\geq \quad + \sum_{n>N} \frac{1}{|n|^2} \frac{(2c)^{-2|n-N|}}{(2c)^{-2|n|}} \\
&= \sum_{n>N} \frac{1}{|n|^2} \frac{(2c)^{-2n+2N}}{(2c)^{-2n}} \\
&= \sum_{n>N} \frac{1}{|n|^2} (2c)^{2N} \\
&\geq \frac{1}{N+1} (2c)^{2N} \geq \frac{1}{2^N} (2c)^{2N} = 2^N c^{2N} = 2^{|N|} c^{2|N|} .
\end{aligned}$$

Besides we observe that for the current V *and* f *the quantity* $\|V^N f\|^2$ *depends on* N *so that* $\|V^N f\|^2 = \|V^{-N} f\|^2$ *for all* N.
Thus we have seen that

$$\|V^N f\|^2 \geq \frac{1}{|N|+1} (2c)^{2N} \geq 2^{|N|} c^{2|N|} \textit{ for all } N .$$

A very rapid growth of $\|V^N f\|^2$, *as* $N \to \pm\infty$ *!!*

The example of J-unitary and symplectic automorphism we are now describing shows that two mathematecally very natural formulations of the phrase " ... is stable with respect to the action of ... " can in the real situation appear as "orthogonal" to one another.

Definition D3-4 *Let* $w_n := 2^{-|n|} = 2^n$ *for* $n \leq 0$ *and* $w_n := 1/(n+1)$ *for* $n > 0$. *Let* $W : H_0 \to H_0$ *denote the associated shift generated by*

$$W \,:\, b_n \mapsto \frac{w_{n+1}}{w_n} b_{n+1} .$$

Remark R3-3 *Since* $1/2 \leq w_{n+1}/w_n \leq 2$ *for all integers* n, W *is bounded invertible and* W^{-1} *is bounded as well.*

Lemma L3-2 *That just now defined* W *has the properties:*

$$\overline{S_0(W)} = H_0, \quad 1 \leq |\textit{spectrum}\, W| \leq 2, \quad r(W) = 2,$$
$$S(W^{*-1}) = \{0\}, \quad \frac{1}{2} \leq |\textit{spectrum}\, W^{*-1}| \leq 1, \quad r(W^{*-1}) = 1 .$$

Proof. The proof is founded on the well-known formula for spectral radius, on Remark **R3-3** and on the formulae in Observation **O2-1**. We have:

$$\|W^N\| = sup\{\frac{w_{n+N}}{w_n} \,|\, n = \ldots -1, 0, 1, \ldots\},$$

$$\|W^{*-N}\| = sup\{\frac{w_n}{w_{n+N}} \,|\, n = \ldots -1, 0, 1, \ldots\}.$$

Take $N > 0$ arbitrarily, and analyse the w_n/w_{n+N} in details. We observe:

a) $w_n/w_{n+N} = 1/2^N$ for $n + N \leq 0$;

b) $w_n/w_{n+N} = (1+n+N)/(1+n) = N/(n+1) + 1 \le N+1$ for $0 < n$;

c) $w_n/w_{n+N} = 2^n(1+n+N) \le N+1$ for $n \le 0 < N+n$

Therefore $\|W^{*-N}\| \le N+1$ (for $N > 0$) . Note that $w_0/w_N = 1+N$. Hence $\|W^{*-N}\| = N+1$ (for $N > 0$) . Therefore $r(W^{*-1}) = 1$ and $r(W^{-1}) = 1$. Quite similarly we can analyse $r(W)$ and $r(W^*)$: Note $\|W^N\| = 2^N$ (for $N > 0$). Therefore $r(W) = 2$ and $r(W^*) = 2$.

Finally, if $n+N > 0$, then $W^N b_n = {w_n}^{-1}(1+N+n)^{-1}b_{n+N}$. Therefore $b_n \in S_0(W)$ for all integers n. Hence $\overline{S_0(W)} = H$ and $S(W^{*-1}) = \{0\}$.

□

Theorem Th3-2
There exists a J-unitary operator $\hat{W}$ and a maximal semidefinite subspace L such that:

(a) $\hat{W}L^\perp = L^\perp$, $\quad 1 \le |spectrum \hat{W}|L^\perp| \le 2$, $\quad r(\hat{W}|L^\perp) = 2$

but in spite of that, $L^\perp = \overline{S_0(\hat{W})} = \overline{S(\hat{W})}$.

(b) $\hat{W}L = L$, $\quad |spectrum \hat{W}|L| \le 1$,

although $L \cap \overline{S(\hat{W})} = \{0\}$.

Proof Set

$$L := \{0\} \oplus H_0 \, , \; M := H_0 \oplus \{0\} \equiv L^\perp \, ,$$

and apply Lemma **L3-2** to the formulae for $S_0(\hat{W})$ and $S(\hat{W})$ (see Introduction):

$$S_0(\hat{W}) = S_0(W) \oplus S_0(W^{*-1}) = S_0(W) \oplus \{0\} \subset M$$

$$S(\hat{W}) = S(W) \oplus S(W^{*-1}) = S(W) \oplus \{0\} \subset M$$

But $S_0(W)$ and $S(W)$, both of them are dense in H_0. Hence the closures of $S_0(\hat{W})$ and $S(\hat{W})$ coincide with M. To complete the proof, note that $\hat{W}|M$ is unitarily equivalent to W, $\hat{W}|L$ is unitarily equivalent to W^{*-1}, and then again apply Lemma **L3-2**.

□

4 Coming to Models of Dynamics in Continuous Time

A quite traditional way to obtain a model of dynamics in continuous time from a given model of dynamics in discrete time consists in rewriting the relations of the latter replacing, in appropriate positions, symbols of sequences (functions of a discrete time) by symbols of functions of a continuous time [9] , symbols of discrete-valued (integer-valued) variables representing time, by symbols of continuum-valued (real-valued) variables, that remain to call "time", provided by a suitable redefining such notions as "sum", and all that.

So, in this way, the definition of the shift given in a previous section is being transformed as follows:

$$V^N : b_n \mapsto \frac{v_{n+N}}{v_n} b_{n+N}$$

$$V^N : \sum_n f(n) b_n \mapsto \sum_n \frac{v_{n+N}}{v_n} f(n) b_{n+N} = \sum_n \frac{v_n}{v_{n-N}} f(n-N) b_n \, ;$$

$$V^N : f(n) \mapsto \frac{v_n}{v_{n-N}} f(n-N)$$

$$V(t) : f(x) \mapsto \frac{v(x)}{v(x-t)} f(x-t)$$

Notice

$$\begin{aligned} V(t)V(\tau)^{-1} \ : & \\ f(x) \quad & \stackrel{V(\tau)^{-1}}{\longmapsto} \frac{v(x)}{v(x+\tau)} f(x+\tau) \stackrel{V(t)}{\longmapsto} \quad \frac{v(x)}{v(x-t)} \frac{v(x-t)}{v(x-t+\tau)} f(x-t+\tau) \\ & = \left(V(t-\tau) f\right)(x) \end{aligned}$$

With other words the dynamics generated by V is time-autonomous and its formal generator is:

$$\begin{aligned} (Hf)(x) &= \left(V(t)f\right)'_{t=0}(x) \\ &= \frac{\partial}{\partial t} \left[\frac{v(x)}{v(x-t)} f(x-t) \right]_{t=0} \\ &= -\frac{\partial f(x)}{\partial x} + \frac{v'(x)}{v(x)} f(x) = -v(x) \frac{\partial}{\partial x} \Big(\frac{1}{v(x)} f(x) \Big) \, . \end{aligned}$$

If we apply the conversion method presented above especially to the discrete systems, which we described in the previous sections, we will see that the corresponding continuous systems may be described as:

a) $v(x) = e^{|x| sin\left(ln(1+|x|)\right)}$

$$\begin{aligned} &(Hf)(x) \\ &= -\frac{\partial f(x)}{\partial x} + \Big(sin(ln(1+|x|)) + \frac{|x|}{1+|x|} cos\big(ln(1+|x|)\big) \Big) sgn(x) f(x) \end{aligned}$$

b) $v(x) = e^{-|x|}$

$$(Hf)(x) = -\frac{\partial f(x)}{\partial x} - (sgn\, x) f(x)$$

[9] symbols that suggest that "this object is a function defined on a set having a discrete structure" by symbols that suggest that "this object is a function defined on a set having a continuous structure"

c)

$$v(x) = \left\{ \begin{array}{lcr} e^x & , & x < 0 \\ \dfrac{1}{x+1} & , & x > 0 \end{array} \right\}$$

$$(Hf)(x) = -\frac{\partial f(x)}{\partial x} + \left\{ \begin{array}{c} 1, x < 0 \\ -\dfrac{1}{x+1}, x > 0 \end{array} \right\} f(x)$$

Of course, the behaviour of these systems is irregular likewise the the behaviour of their prototypes; but we will not here discuss it.

5 Appendix A.

A Possible Definition of Abstract Linear Dynamical System

Definition of abstract linear dynamical system.
Let L be a linear space, $\mathcal{T}_{\mathcal{A}}$ be a set (abstract time), $\geq$ be a transitive relation on $\mathcal{T}_{\mathcal{A}}$.
Let a two-parameter family of linear operators on L,

$$\{V_{t,s}\}_{t,s} \qquad (t \geq s \quad t, s \in \mathcal{T}_{\mathcal{A}}) \, ,$$

be such that

$$V_{t,r}V_{r,s} = V_{t,s} \, , \qquad \text{if } t \geq r \geq s \qquad \textbf{(consistency relation)}$$

In this case the structure

$$L \quad , \quad \mathcal{T}_{\mathcal{A}} \quad , \quad \geq \quad , \quad \{V_{t,s}\}_{t,s}$$

is said to be an **abstract linear dynamical system** *and $\{V_{t,s}\}_{t,s}$ is called a* **propagator**, *alias* **evolution operator**. [10] .

Given a $t_0 \in \mathcal{T}_{\mathcal{A}}$ and a $x_0 \in L$, we say that the one-parameter family, $\{x(t)\}_{t \geq t_0}$, defined by

$$x(t) = V_{t,t_0} x_0 \qquad (t \geq t_0) \, ,$$

is a (future or forward) **trajectory**. *In such a case we say that t_0, x_0 are* **initial data**.

If the propagator is such that each $V_{t,s}$ is invertible, then we say that the dynamics is **invertible**. *In this case we put*

$$V_{s,t} := V_{t,s}^{-1} \qquad (t \geq s) \, .$$

Finally, if $\mathcal{T}_{\mathcal{A}}$ is equipped with the discrete topology, we say that the dynamics is **discrete**.

[10] we often write $V_{t,s}$ instead of $\{V_{t,s}\}_{t,s}$

References

[Arn] V.I. ARNOL'D, *Mathematical methods of classical mechanics.*
(Matematičeskije metody klassičeskoj mehaniki)
(Russian) Moskva:Nauka, 1974.

[Ber] F.A. BEREZIN, *The Method of Second Quantization*, Academic Press, New York, 1966.

F.A. BEREZIN, *Methode der zweiten Quantelung. Zweite, neubearbeitete Auflage.* (Russian),
(Metod vtoričnogo kvantovanija, 2-e izd.) M.: Nauka, 1986,

[Bogn] J. BOGNÁR, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin Heidelberg New York, 1974.

[Bog] N.N. BOGOLIUBOV, *Ausgewaehlte Werke in 3 Baenden. Band 2.*
(Izbrannyje Trudy v 3 tomah. Tom 2.)
(Russian) Kiev: Verlag "Naukova Dumka", 1970.
N.N. BOGOLIUBOV, *Ausgewaehlte Werke in 3 Baenden. Band 3.*
(Izbrannyje Trudy v 3 tomah. Tom 3.)
(Russian) Kiev: Verlag "Naukova Dumka", 1971.

[BraRob] O. BRATTELI AND D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*, Vol. II, Springer-Verlag, New York, Heidelberg and Berlin, 1981.

[DadKul] L.A. DADAŠEV, V.JU. KULIEV, Diagonalization of bilinear Bose Hamiltonians and asymptotic behavior of corresponding Heisenberg fields. (Russian) *Teoret. Mat. Fiz.* **39**(1979), no3, 330–346. **MR 80e:81105**
(Diagonalizacija bilinejnyh boze-gamil'tonianov i asimptotičeskoe povedenie poroždaemyh imi gejzenbergovyh polej)
//TMF.1979.**T.39**,N3,330-346.

[DalKre] JU. L. DALETSKIJ, M. G. KREIN, *The Stability of the Solutions of Differental Equations in a Banach Space*, Moscow.: Nauka, 1970 (Russian)

[DR] MICHAEL A. DRITSCHEL AND JAMES ROVNYAK, Operators on Indefinite Inner Product Spaces,

in *Lectures on operator theory and its applications (Waterloo, ON, 1994)*
, Fields Institute Monographs, vol. 3, Amer. Math. Soc., Providence, RI, 1996, pp. 141–232.

This document is available via the web in two forms:

`http://faraday.clas.virginia.edu/~jlr5m/papers/fields/fieldslectures.ps`
postscript version (900K)

`http://faraday.clas.virginia.edu/~jlr5m/papers/fields/dvi_version.html`
dvi version (450K)

It has 91 pages, including bibliography and index. Supplementary materials and errata may be foand at

`http://faraday.clas.virginia.edu/~jlr5m/papers/fields/Supplement.ps`
postscript version

`http://faraday.clas.virginia.edu/~jlr5m/papers/fields/Supplement.dvi`
dvi version

The Abstract is available via the web in form:
`http://www.math.purdue.edu/~mad/pubs/abs10.html`

[Emch] G. G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley-Interscience, New York, 1972.

[Fey] R.P. Feynman, *Statistical Mechanics. A Set of Lectures*, W. A. Benjamin, Inc. Advanced Book Program Reading, Massachusetts 1972.

[Ikr89] Kh.D. Ikramov, The Theorem on the Diagonalization of One Kind of Hamiltonians from the Point of View of the Theory of Linear Operators in Indefinite Scalar Product
// Žurnal Vyčislitel'noj Matematiki i Matematičeskoj Fiziki, 1989, **v. 29**, N 1, 3-14.

[IK] I. S. Iokhvidov and M. G. Kreĭn, *Spectral theory of operators in spaces with indefinite metric. II*, Trudy Moskov. Mat. Obšč. **8** (1959), 413–496, English transl.: Amer. Math. Soc. Transl. (2) **34** (1963), 283–373.
(**MR21:6543**)

[IKL] I. S. Iokhvidov, M. G. Kreĭn, and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Mathematical Research, vol. 9, Akademie-Verlag, Berlin, 1982.

[Kul] Kuliev,V.Ju. On the general theory of diagonalization of bilinear Hamiltonians. (Russian) **MR 82f:82014**
Dokl. Akad. Nauk SSSR **253**(1980), no. 4, 860–863.

[KL1] M. G. Kreĭn and H. Langer, *Über die verallgemeinerten Resolventen and die charakteristische Funktion eines isometrischen Operators im Raume* Π_κ, Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), North-Holland, Amsterdam, 1972, pp. 353–399. Colloq. Math. Soc. János Bolyai, 5.
(**MR54:11103**)

[KL2] M. G. Kreĭn and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume* Π_κ *zusammenhängen. I. Einige Funktionenklassen and ihre Darstellungen*, Math. Nachr. **77** (1977), 187–236.
MR57:1173

[Kre64] M.G. Krein, A New Application of the Fixed-Point Principle in the Theory of Operators in a Space with Indefinite Metric. //DAN SSSR. 1964. **154**, N 5, 1023 –1026.(russisch)

[Kre65] M.G. Krein, *Introduction to the geometry of idefinite J-spaces and to the theory of operators in those spaces.* In: Second mathematical summer school, Part 1, pp 15-92, Kiev.: Naukova dumka, 1965 (Russian)

[Maj] W.A. Majewski, Does quantum chaos exist? (A quantum Lyapunov exponents approach.) // LANL E-Print, Paper: quant-ph/9805068 (http://arXiv.org/abs/quant-ph/9805068)

[MV] J. Manuceau, A. Verbeure, Quasi-free states of the CCR-algebra and Bogoliubov transformations, *Commun. Math. Phys.*, **9**, (1968), 293–302.

[Oks] A.I. Oksak, Non-Fock linear boson systems and their applications in two dimensional models. (Russian) *Teoret. Mat. Fiz.* **48** (1981),no. 3, 297-318.
(**MR84i:81079**)
Nefokovskie linejnye bozonnye sistemy i ih primenenija v dvumernyh modeljah
//TMF.1981.**T.48**,N3,297-318.

[RS1] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol 1 Functional analysis*, - N.Y.: Academic Press, 1972.

[RS2] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol 2, Fourier analysis, Self-Adjointness*, - N.Y.: Academic Press, 1975.

[RS3] M. Reed, B. Simon, *Scattering Theory*, Methods of Modern Mathematical Physics, vol 3, - N.Y.: Academic Press, 1979.

[RS4] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol 4, Analysis of Operators*, - N.Y.: Academic Press, 1978.

[Shk99] A.A. Shkalikov, On the Existence of Invariant Subspaces of Dissipative Operators in Space with Indefinite Metric.
// Fundamental'naja i prikladnaja matematika, vol.5(1999), N5, pp.625–637.

[Rob] D.W. Robinson, The ground state of the Bose gas,
// Commun. Math. Phys., **1** , (1965), 159–171.

[Will36] J. Williamson, On the algebraic problem concerning the normal forms of linear dynamical systems.
// Amer. J. of Math. 1936, **V. 58**, 141-163.

[Will37] J. Williamson, On the normal forms of linear canonical transformations in dynamics.
// Amer. J. of Math. 1937, **V. 59**, 599-617.

[Ch81] S.A. Chorošavin, *On Krein spaces and *-algebras.*
O svjazi ponjatij teorii prostranstv Krejna i *-algebr.
// VINITI 27.04.81, Nr.1916–81 (Russian)

[Ch83] S.A. Chorošavin, *On quadratic states on Weyl *-algebra.*
O kvadratičnyh sostojanijah na *-algebre Vejlja .
// VINITI 30.08.83, Nr.4823–83 (Russian)

[Ch84] S.A. Chorošavin, *Quadratic majorants of sesquilinar forms and *-representations.*
Kvadratičnyje mažoranty polutoralinejnyh form i *-predstavlenija.
// VINITI 09.04.84, Nr.2135–84 (Russian) (Russian)

[Ch84D] S.A. Chorošavin, *Linear Operators in Indefinite Inner Product Spaces and Quadratic Hamiltonians* (Russian) Ph.D. thesis, Voronezh state university, 1984

[Ch89-1] S.A. Chorošavin, *Some theorems of non-trivial neutral invariant subspaces existence. Krein approximations terms.* Nekotorye teoremy suśestvovanija netrivial'nyh invariantnyh mažorant v terminah approksimacij Krejna

//Kur. gos. ped. in-t. Kursk,1989.- 17 s. Bibliogr.:5 nazv.-
//VINITI 21.03.89, Nr.1765 - V89 RŽMAT 1989 ,7B931 DEP (Russian)

[Ch89-2] S.A. Chorošavin, *A case of non-trivial neutral invariant subspaces existence.* Odin priznak suśestvovanija nejtral'nogo invariantnogo podprostranstva

//VINITI 06.07.89, Nr.4495 - V89 RŽMAT 1989 11B799 DEP (Russian)

[Ch96T] Chorošavin S. A. *On convergence of angle operators for Krein approximations of J-unitary operator.*
O shodimosti uglovyh operatorov, sootvetstvujuśih approksimacijam Krejna J-unitarnogo operatora

// Voronež. vesen. mat. šk. "Sovrem. metody v teorii kraev. zadač "Pontrjag. čtenija-7", 17-23 apr., 1996: Tez.dokl.-Voronež, 1996.- S.181. - Rus. RŽMAT 1996 11B824. (Russian)

[Ch97T] S.A. Chorošavin, *A decomposition of linear bounded operators on Hilbert spaces.*
Odno razloženie linejnogo ograničennogo obratimogo operatora, dejstvujuśego v gil'bertovom prostranstve /
//"Pontrjag. čtenija-8" na Voronež. ves. mat. šk. "Sovrem. metody v teorii kraev. zadač", Voronež, 4-9maja, 1997 : Tez.dokl.-Voronež, 1997.- S.159. - Rus. RŽMAT 1997 10B706. (Russian)

[Ch97] S.A. Chorošavin, On one M. G. Krein problem.
//TRANSACTIONS of RANS, series MMMIC, 1997, **v.1**, N.2, 95-101. (Russian)

[Ch98] S.A. Chorošavin, An Example of J-Unitary U wich Has no Nonzero Invariant Subspace L such that $r(U|L) \leq 1$. //TRANSACTIONS of RANS, series MMMIC. 1998. **v.2**, N 2, 97–103 (Russian)

[Ch00] S.A. Chorošavin, A Nonlinear Approximation of Operator Equation $V^*QV = Q$: Nonspectral Decomposition of Nonnormal Operator and Theory of Stability // arXiv:math.DS/0312016. (http://arXiv.org/abs/math/0005117)
see also // mp_arc, Paper: 00-221 (http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=00-221, http://mpej.unige.ch/mp_arc-bin/mpa?yn=00-221, http://www.maia.ub.es/mp_arc-bin/mpa?yn=00-221)