STRONG SOLUTIONS FOR DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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Abstract. Let $(E, \mathcal{F})$ be a locally convex space. We denote the bounded elements of E by $E_b := \{x \in E : \sup_{\rho \in \mathcal{F}} \rho(x) < \infty\}$. In this paper we prove that if B_{E_b} is relatively compact with respect to the $\mathcal{F}$ topology and $f : I \times E_b \to E_b$ is a measurable family of $\mathcal{F}$-continuous maps then for each $x_0 \in E_b$ there exists a norm-differentiable local solution to the Initial Valued Problem $u_t(t) = f(t, u(t))$, $u(t_0) = x_0$. Our final goal is to study the Lipschitz stability of a differential equation involving the Hardy-Littlewood maximal operator.

1. Introduction

Differential equations modelled in Banach spaces have attracted the attention of many researchers throughout the last century. Most of the efforts are concentrated in the study of the classical Cauchy problem, also called the initial value problem and denoted by IVP

$$\begin{cases} u_t(t) &= f(t, u(t)) \text{ in } (a, b) \\ u(a) &= u_0 \end{cases} \tag{1.1}$$

The map f is a 1-parameter family of fields between a Banach space, i.e., $f: [a, b] \times E \to E$. The theory of differential equations in Banach spaces has shown to be a clever and useful strategy to study many problems that appear in the applied as well as the abstract mathematics. Its most common applications concern partial differential equations on the euclidian spaces which arise from physical systems.

Let X be a Banach space and $F: [a, b] \times X \to X$ be continuous. It is well known that if either $\dim X < \infty$ or if F is Lipschitz, then for each pair $(t_0, x_0) \in [a, b] \times X$, there exists a C^1-curve $x : (t_0 - \delta, t_0 + \delta) \to X$ such that $x(t_0) = x_0$ and $x_t(t) = F(t, x(t))$. J. Dieudonné in [9] provided the first example of a continuous map from an infinitely dimensional Banach space for which there is no solution to the related IVP. In his simple and insightful example, $X = c_0$ and $F(x_1, x_2, ...) := (|x_n|^{1/2} + 1/n)$. He noticed that there is no solution for the IVP $x(0) = 0$, $x_t(t) = F(x(t))$. J. A. Yorke [35] gave an example of the same phenomena in a Hilbert space. Afterwards, Godunov in [14] proved that for every infinity dimensional Banach space, there exists a continuous field such that there is no solution to the related Initial Valued Problem. It turned out then that continuity was not the right assumption on the field F. Many celebrated works have been developed since the 70's in order to obtain suitable extensions for the continuity notion on finitely dimensional spaces. Basically two branch were born on this journey: Uniformly continuity and continuity in the weak topology. The former came from the observation that if $R_0 := [a, b] \times \overline{B}_X(x_0, r)$ and $F: R_0 \to X$ is continuous then, if $\dim X < \infty$, due to the compactness of R_0, F is automatically uniformly continuous. For reference in this type of research direction, i.e., strong topology assumptions,

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we cite, for instance, [20] and [23]. The latter came from one of the most fruitful ideas in functional analysis. Weak topology appeared as a prototype to grapple with the lack of local compactness in infinitely dimensional Banach spaces. At least if the Banach space X is reflexive we recover locally compactness by endowing it with the weak topology. We observe that weak topology coincides with strong topology in a Banach space X if, and only if, $\dim X < \infty$.

The first paper related to the existence of weak solutions for differential equations in Banach spaces relative to the weak topology was [32]. Its main result is

Theorem 1.1 (Szep). *Let E be a reflexive Banach space and f be a weak-weak continuous function on $P = \{t_0 \leq t \leq t_0 + a, \|x - x_0\| \leq b\}$. Let $\|f(t,x)\| \leq M$ on P. Then the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$ has at least one weak solution defined on $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/M)$.*

We also cite the work of Chow and Schuur [5], where they treat the case where E is separable and reflexive, $f\colon (0,1) \times E \to E$ is a weak continuous function with bounded range. The next step was given by S. Kató in [16]. In this paper he observed that if $f\colon [0,T] \times \overline{B}_E(u_0, r) \to E$ is weakly continuous, then all we needed to assure the existence of solutions to the related IVP is the relatively weak compactness of $f([0,T] \times \overline{B}_E(u_0, r))$. Afterwards Szufla in [31] proved that, under the assumptions of theorem 1.1, the set of all weak solutions of $x' = f(t,x)$, $x(t_0) = x_0$ defined on a compact subinterval J of $[0,a]$ is a continuum in the space $C_w(J, E)$.

One of the ideas that appeared toward the generalization of those previous results for nonreflexive Banach spaces was the so called measure of weak noncompactness. Probably the first work in this direction is [8]. Roughly speaking, the idea behind this technic is to impose some condition on f involving the measure of weak noncompactness to, somehow, recover the locally compactness lost by the fact that the Banach space we are working on is no longer reflexive. Since [8], many researchers have improved and generalized results involving assumptions on the measure of weak noncompactness. Some of the recent progress in this direction are [4], [6], [7] and [15]. The only disadvantage of this theory is that, when E is not reflexive, it is really hard or even impossible to check the measure of weak noncompactness assumptions. We should mention though that Astala in [3], proved that a Banach space E is reflexive if and only if the IVP (1.1) admits a local solution for every weakly continuous field. Thus there is no hope to extend Peano's theorem in the weak topology setting to nonreflexive spaces.

In this paper we explore another line of generalization to the theory of differential equations in Banach spaces. The idea of this paper is based on the study of differential equations in locally convex spaces. The theory of differential equations in general locally convex spaces differs brutally from the theory in Banach spaces, even in the linear case. For instance, it is well known that every linear ordinary differential equation $u_t = Au$, $u(x) = u_0$ in a Banach space is globally and uniquely solved. Its solution is given by the following convergent series: $\sum_{k=0}^{\infty} \frac{t^n}{n!} A^n u_0$. In some non-normable locally convex spaces, this series diverges for all $t \neq 0$. Let us also cite that Lobanov in [22] proved that for each non-normable Fréchet space E one can find a continuous mapping $f\colon E \to E$ and a closed infinite-dimensional subspace L such that the Cauchy problem $u_t = f(u)$, $u(0) = u_0$ has no solutions for all $u_0 \in L$. On the other hand, a good theory of differential equations in locally convex spaces might be used as a powerful technic to study several important problems that arise in various parts of nonlinear functional analysis and evolution differential equations. The interpretation of some partial differential equations as an ordinary differential equation in Banach spaces may face the problem that the field f in (1.1) is not continuous, even in very natural circumstances. The freedom of choosing a more convenient notion of topology rather than normable topologies can be used in order to grapple with such a difficulty. This is precisely the case of the differential equation studied in section 4 of this paper.

Astala in [3] considered the IVP (1.1) in sequentially complete locally convex spaces that contain a compact barrel. The existence result provided there asserts that if E is a sequentially complete locally convex space and B is a compact barrel, then for every $f: I \times E \to E$ continuous there exists a local solution to the IVP (1.1). The derivative in (1.1) is understood in the sense of differentiation in locally convex spaces. For instance if we are dealing with the weak topology, the derivative in (1.1) is understood as the weak derivative, (see the conclusion of theorem 1.1). The main goal of our paper is to extend the results in [3] in two directions. The first direction is from the quantitative point of view. We shall consider measurable family of continuous maps between a locally convex space rather than considering continuous family. The second extension is from the qualitative point of view. To explain the latter, we first of all argue that in practical situations we often have that the range of f lies in a suitable subspace of the locally convex space the problem is modelled in: the bounded elements E_b. Such a subspace admits a norm and it happens to be a Banach space provided the locally convex space is sequentially complete. We now can talk about the norm derivative of the solution curve of (1.1). We prove that if $(E, \mathcal{F})$ is locally convex space such that B_{E_b} is relatively compact w.r.t. the $\mathcal{F}$-topology and $f: I \times E_b \to E_b$ is a measurable family of continuous maps then there exists a strong, i.e. norm-differentiable solution, to the IVP (1.1).

Our paper is organized as follows: In the section 2 we gather together all the facts we shall use in the proof of the main existence result. We suggest a locally convex topology in $L_\infty(I, E_b)$: The T-topology. It seems to be the right calibration between continuity and compactness. In the next section state and prove the existence theorem for differential equations in abstract spaces. In the last section we study in details a nonlinear differential equation involving the remarkable Hardy-Littlewood maximal operator. The main information given here is a sort of smoothness of the solution. This type of results might be useful in the regularization theory for differential equations involving averages over fixed domains.

2. Preliminaries Results

In this section we shall present the main tools that will be used in the proof of our existence result for differential equations in locally convex spaces. For convenience of the reader we shall state some of the classical results we will make use and afterwards, we will develop some new technics that will be needed to properly approach the problem.

A topological vector space E is called a locally convex space if E has a local base consisting of convex sets. Typical examples are normed spaces, Banach spaces endowed with the weak topology and dual spaces endowed with the weak-* topology. We shall assume by definition that all locally convex spaces are Hausdorff. A seminorm on a real vector space V is a map $\rho: V \to [0, \infty)$ obeying:

(1) $\rho(x + y) \leq \rho(x) + \rho(y)$
(2) $\rho(\alpha x) = |\alpha| \rho(x)$.

A family of seminorms $\{\rho_\lambda\}_{\alpha \in A}$ is said to separate points if $\rho_\alpha(x) = 0 \ \forall \alpha \in A$ implies $x = 0$. It is well known that every (Hausdorff) locally convex space admits a family of seminorms separating points which generates its topology. Thus from now on we shall consider locally convex spaces endowed with a family of seminorms. We will denote by $(E, \mathcal{F})$, where $\mathcal{F}$ stands for the family of seminorms that generates the locally convex topology on E.

For the theory of differentiation in locally convex spaces, we refer the readers to [34] chapter II. The next two results have a wide influence in almost all branch of the mathematical analysis.

Theorem 2.1 (Schauder-Tychonoff fixed point theorem). *Let K be a closed convex set in a locally convex Hausdorff space E. Suppose $f\colon K \to K$ is continuous and $Im(f)$ is relatively compact. Then f has a fixed point in K.*

The second classical theorem we shall need is the general version of the Ascoli-Arzelá theorem found in [17].

Theorem 2.2 (Generalized version of Ascoli-Azerlá theorem). *Let K be a compact set and E be a Hausdorff linear topological space. Then a subset $\mathcal{G} \subset C(K, E)$ is relatively compact with respect to the topology of uniform convergence if and only if G is equicontinuous and $\overline{\{g(t) : g \in \mathcal{G}\}}$ is compact for each $t \in K$.*

Let $(E, \mathcal{F})$ be a locally convex space. We denote by E_b the following set:

$$E_b := \{x \in E : \sup_{\rho \in \mathcal{F}} \rho(x) < \infty\}. \tag{2.1}$$

The elements in E_b is called the bounded elements of E. The subspace E_b will be the base space in the theory of differential equations we shall develop, in the sense that actually our solution will lie in this subspace. To this end we will consider maps $f : I \times E_b \to E_b$. It is important to mention that the assumption that the image of f lies in E_b is essentially a necessary condition for the existence of solutions to the IVP. Indeed, when $u_t(t) = f(t, u(t))$ holds, by the definition of derivative in locally convex spaces, the left hand side belongs to $\mathcal{L}(\mathbb{R}, E) = E_b$. For further details, see [34]. At this point we should also mention that in order to have a good theory for differential equation in locally convex spaces one must impose the sequential completeness of the spaces we shall work on (see for instance the comment in [3] pg. 215). The connection of this fact with E_b is clear.

Proposition 2.3. *Let $(E, \mathcal{F})$ be a sequentially complete locally convex space. Then $\|x\|_{\mathcal{F}} := \sup\limits_{\rho \in \mathcal{F}} \rho(x)$ is a norm in E_b for which $(E_b, \|\cdot\|_{\mathcal{F}})$ is a Banach space.*

Let us turn our attention now for the measure theory that will support our existence result. Let $(\Omega, \mathcal{B}, \mu)$ be a complete and σ-finite measure space, and let X be a Banach space. A simple function $f = \sum_{i=1}^m x_i \chi_{A_i}$ with $x_i \in X$ is called measurable if $A_i \in \mathcal{B}$ for every i. In general a function $f\colon \Omega \to X$ is called measurable if there is a sequence $\{f_n\}$ of measurable simple functions which converges a.e. to f as $n \to \infty$.

Definition 2.4. Let $(E, \mathcal{F})$ be a sequentially complete locally convex space. We will say $f\colon \Omega \to E_b$ is $\mathcal{F}$ measurable, if for each $\rho \in \mathcal{F}$ the real function $\rho(f)\colon \Omega \to \mathbb{R}$ is measurable in the classical sense.

An important result we shall need to develop is the following general version of Pitt's theorem.

Theorem 2.5. *Let $(E, \mathcal{F})$ be a sequentially complete locally convex space. A function $f\colon \Omega \to E_b$ is measurable if and only if it is $\mathcal{F}$ measurable and μ-almost separably-valued.*

Proof. It is clear that if f is measurable, then it is $\mathcal{F}$ measurable and μ-almost separably-valued. Conversely, suppose $f\colon \Omega \to E_b$ is $\mathcal{F}$ measurable and μ-almost separably-valued.

We may suppose then E_b is separable.
Claim: There exists a countable subset $\overline{\mathcal{F}}$ of $\mathcal{F}$, for which,

$$\|x\|_{\mathcal{F}} = \sup_{\rho\in\overline{\mathcal{F}}} \rho(x).$$

Indeed, let $(x_j)_{j=1}^{\infty} \subset E_b$ be a dense subset. For each j fixed, let $(\rho_i^j)_{i=1}^{\infty}$ be a sequence in $\mathcal{F}$ such that

$$\|x_j\|_{\mathcal{F}} = \lim_{i\to\infty} \rho_i^j(x_j).$$

Define $\overline{\mathcal{F}} := \bigcup_{i,j} \rho_i^j$. Let $x \in E_b$ be fixed and $\varepsilon > 0$ be arbitrary. By density, there exists a x_{j_0} such that

$$\|x - x_{j_0}\|_{\mathcal{F}} < \varepsilon.$$

We also have by triangular inequality that

$$\rho_i^{j_0}(x) \geq \rho_i^{j_0}(x_{j_0}) - \rho_i^{j_0}(x - x_{j_0}).$$

Thus,

$$\begin{aligned} \sup_{\rho\in\overline{\mathcal{F}}} \rho(x) &\geq \liminf_i \rho_i^{j_0}(x) \\ &\geq \liminf_i \rho_i^{j_0}(x_{j_0}) - \rho_i^{j_0}(x - x_{j_0}) \;\; lll \\ &\geq \|x_{j_0}\|_{\mathcal{F}} - \varepsilon \\ &\geq \|x\|_{\mathcal{F}} - 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was taken arbitrarily, we conclude $\sup_{\rho\in\overline{\mathcal{F}}} \rho(x) \geq \|x\|_{\mathcal{F}}$. This proves the claim.

By reenumerating we may write $\overline{\mathcal{F}} = (\rho_i)_{i\in\mathbb{N}}$. For any real number a put $A := \{s \in \Omega : \|f(s)\|_{\mathcal{F}} \leq a\}$ and $A_i := \{s : \rho_i(f(s)) \leq a\}$. Clearly $A \subseteq \bigcap_{i=1}^{\infty} A_i$. For a fixed $s \in \Omega$, there exists a subsequence $\{\rho_i\}_{i\in\mathbb{N}_s}$ such that $\rho(f(s)) \to \|f(s)\|_{\mathcal{F}}$ as $i \in \mathbb{N}_s$ goes to infinity. We have shown $A = \bigcap_{i=1}^{\infty} A_i$. Since, by hypothesis, each A_i is measurable, so is A. This proves the real function $s \mapsto \|f(s)\|_{\mathcal{F}}$ is measurable. Now it is easy to conclude f is measurable. Indeed, since $f(\Omega)$ is separable, for any $n \in \mathbb{N}$, we can find balls

$$B_{\frac{1}{n}}(x_{j,n}) \text{ such that } f(\Omega) \subset \bigcup_{j=1}^{\infty} B_{\frac{1}{n}}(x_{j,n}).$$

We have already proven the map $s \mapsto \|f(s) - x_j\|_{\mathcal{F}}$ is measurable. Thus the sets $\Omega_{j,n} := \{s \in \Omega : f(s) \in \Omega_{j,n}\}$ are measurable and for each $n \in \mathbb{N}$ fixed $\Omega = \bigcup_j \Omega_{j,n}$. We finally define $\widetilde{\Omega}_{k,n} := \Omega_{k,n} \setminus \bigcup_{j=1}^{k-1} \Omega_{j,n}$ and

$$f_n(s) = \sum_{k=1}^{\infty} \chi_{\widetilde{\Omega}_{k,n}} x_{k,n}.$$

Since $\Omega = \sum_{k=1}^{\infty} \chi_{\widetilde{\Omega}_{k,n}}$, we have $\|f(s) - f_n(s)\|_{\mathcal{F}} < 1/n$ for every $s \in \Omega$. □

If $f = \sum_{i=1}^{m} x_i\chi_{A_i}$ is a simple measurable function with $\mu(A_i) < \infty$ for all i, naturally we define $\int_\Omega f d\mu := \sum_{i=1}^{m} x_i\mu(A_i)$. A measurable function f is called Bochner integrable if there exists a sequence of measurable simple functions $\{f_n\}$ converging a.e. to f so that $\int_\Omega \|f_n - f_m\| d\mu \to 0$. The integral $\int_\Omega f d\mu$ is then defined as $\lim \int_\Omega f_n d\mu$. We recall that a function $f : \Omega \to X$ is Bochner integrable if and only if it is measurable and $\int_\Omega \|f\| d\mu <$

∞. Finally we define $L_p(\Omega, X) := \{f\colon \Omega \to X : f \text{ is measurable and } \int_\Omega \|f\|^p d\mu < \infty\}$. $L_p(\Omega, X)$ endowed with its natural norm is a Banach space. In addition, simple functions are dense in $L_p(\Omega, X)$ for $1 \le p < \infty$. For $p = \infty$, the symbol $L_\infty(\Omega, X)$ stands for the space of all equivalence classes of X-valued measurable functions defined on Ω that are essentially bounded, i.e., such that $\|f\|_\infty := \text{ess sup}\{\|f(s)\| : s \in \Omega\} < \infty$. This is also a Banach space under the norm $\|\cdot\|_\infty$.

The base space in our analysis will be $L_\infty(I, E_b)$ where $I = [0, T]$ and $(E, \mathcal{F})$ is a sequentially complete locally convex space. Our first step is to suggest a new locally convex topology to $L_\infty(I, E_b)$. This new topology seems to be a harmonic calibration of to important topological concepts: continuity and compactness. Before defining such a topology, let us justify the above claim. Let us suppose for the moment that we are dealing with a dual space endowed with the weak-* topology. Thus $E_b = E$. Assume moreover that E has the RNP. It follows therefore that $L_\infty(I, E)$ is the dual space of $L_1(I, E_*)$, where E_* is the predual of E, i.e. $E_*^* = E$. Thus from the Banach-Alaoglu theorem, $B_{L_\infty(I,E)}$ is compact in the weak-* topology. However weak-* convergence in $L_\infty(I, E)$ gives us very few information. For instance it is easy to cook up examples of sequences $\{u_n\}$ which converge weak-* in $L_\infty(I, E)$, such that there is no subsequence converging weak-* a.e. in E. Hence, even in the simplest case, $E = \mathbb{R}$, naïve nonlinear maps such as $f \mapsto (f)^+$ fails to be weak-* continuous in $L_\infty(I)$. On the other hand, if a sequence $\{u_n\}$ converges weakly in $L_\infty(I, E)$ to u, then for a.e. $t \in I$, $u_n(t) \rightharpoonup u(t)$ in E. This fact allows weak continuity results for nonlinear operators acting on vector-valued Lebesgue spaces (see [33] for these facts). The problem in this case is that $B_{L_\infty(I,E)}$ is far from being compact when endowed with the weak topology, (see [30]). The next definition try to remedy these difficulties.

Definition 2.6. Let $(E, \mathcal{F})$ be a sequentially complete locally convex space. For each $\rho \in \mathcal{F}$, we define the following seminorm in $L_\infty(I, E_b)$:

$$\Phi_\rho(f) := \sup_{s \in I} \rho(f(s)).$$

We then define the T-topology in $L_\infty(I, E_b)$ to be the locally convex topology obtained by these seminorms.

A local base around 0 for the T-topology is:

$$N(\varepsilon, i_1, i_2, ..., i_n) := \left\{ u \in L_\infty(I, E_b) : \sup_{s \in I} \rho_{i_j}(u(s)) < \varepsilon \ \forall\ j = 1, 2, ...n \right\}.$$

The next result shows a first advantage of the T-topology.

Theorem 2.7. *Let $(u_\alpha)_{\alpha \in \mathcal{A}}$ be a net in $L_\infty(I, E_b)$ which converges to u in the T-topology. Then, for a.e. $t \in I$, $u_\alpha(t) \to u(t)$ in E_b with respect to the $\mathcal{F}$-topology. Furthermore, $(L_\infty(I, E_b), T)$ is locally metrizable provided $(E_b, \mathcal{F})$ is locally metrizable.*

The last theorem we shall prove in this section refers to a generalization of one of the deepest results in measure theory.

Theorem 2.8 (Vector-Valued version of the Lebesgue Differentiation Theorem)**.** *Let (Ω, μ) be a Radon measure space and X be an arbitrary Banach space. Let $f \in L^1_{loc}(\Omega, X)$, then*

$$f(x) = \lim_{r \to 0} \frac{1}{B_r(x)} \int_{B_r(x)} f(\xi) d\xi,$$

for almost every $u \in \Omega$.

Proof. It follows from Pitt's theorem (or even theorem 2.5) that, after discarding a negligible set, we might suppose X is separable. Let $\{\zeta_k\}_{k=1}^{\infty}$ be a dense set in X. For each k we consider the real function $z_k\colon \Omega \to \mathbb{R}$ defined by:

$$z_k(x) := \|f(x) - \zeta_k\|.$$

For such a function we may employ the classical Lebesgue Differentiation Theorem and conclude there exists a negligible set A_k for which

$$z_k(x) = \lim_{r\to 0} \frac{1}{B_r(x)} \int_{B_r(x)} z_k(\xi) d\xi,$$

for all $x \in \Omega \setminus A_k$. Let $A := \bigcup_{k=1}^{\infty} A_k$. In this way, $\mu(A) = 0$ and for any $x \in \Omega \setminus A$ and any ζ_k there holds

$$\begin{aligned} \limsup_{r\to 0} \frac{1}{B_r(x)} \int_{B_r(x)} \|f(\xi) - f(x)\| d\xi &\leq \limsup_{r\to 0} \frac{1}{B_r(x)} \int_{B_r(x)} \|f(\xi) - \zeta_k\| + \|f(x) - \zeta_k\| d\xi \\ &= 2\|f(x) - \zeta_k\|. \end{aligned}$$

Finally, from the fact that $\{\zeta_k\}$ is a dense subset of X, letting ζ_{k_j} goes to $f(x)$, we conclude

$$\limsup_{r\to 0} \frac{1}{B_r(x)} \int_{B_r(x)} \|f(\xi) - f(x)\| d\xi = 0,$$

which in particular implies the theorem. □

It is worthwhile to point out that a priori theorem 2.8 is surprising. It is well known that Lipschitz maps from an interval of $\mathbb{R}$ into a Banach space X are almost differentiable if and only if X has the Radon-Nikodym property. The whole point here is that functions given by the Bochner integral of L^1_{loc} functions are a bit better than generic absolute continuous functions.

3. Existence theory for differential equations in locally convex spaces

In this section we shall prove our existence result for differential equations in abstract spaces. One of the most important advantage of our approach is the fact that we provide strong solutions rather than "weak" solutions. Let us explain what we mean by that.

Let (E, τ) be a topological vector spaces and let $u\colon \mathbb{R} \to E$ be a curve. We say u is differentiable according to the topology τ at $t_0 \in \mathbb{R}$ provided

$$\lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0}$$

converges to a certain $u'(t_0) \in E$ in the τ topology.

Let $f \in \mathbb{R} \times E \to E$. The problem we are considering is

$$\left\{ \begin{array}{rcl} u_t(t) &=& f(t, u(t)) \\ u(t_0) &=& u_0 \end{array} \right. \tag{3.1}$$

An E-valued function u, defined on some open interval I containing t_0 is a solution to the problem (3.1) if

(1) u is τ-differentiable for any $t \in I$

(2) $u_t(t) = f(t, u(t))$ for any $t \in I$

(3) $u(t_0) = u_0$.

Suppose now E has a Banach space structure as well. It means that besides the τ topology in E we also have a norm in E that induces a complete metric on E. We then have a notion of a norm solution of the problem (3.1), i.e. a curve defined on some open interval I containing t_0 such that $u(t_0) = u_0$ and items (1) and (2) above hold in the norm topology. This is the case when one has a sequentially complete locally convex space $(E, \mathcal{F})$ and considers E_b with the topology it inherits from E and with the norm defined in proposition 2.3. In general norm derivative is a much strong notion of differentiability.

Definition 3.1. Let (E, τ) be a topological space endowed with a complete norm $\|\cdot\|$. Let u be an E-valued curve defined on some open interval containing t_0. We will say that u is a "weak" solution to the problem (3.1) if it is a τ-differentiable function satisfying $u_t(t) = f(t, u(t))$ for any $t \in I$ and $u(t_0) = u_0$. We will say that u is a strong solution to the problem (3.1) if $u(t_0) = u_0$ and it is almost everywhere differentiable with respect to the norm topology and for almost every $t \in I$, $u_t(t) = f(t, u(t))$ in the norm topology sense.

Another advantage of the existence theorem we shall present in this section is the wide class of maps it can be applied to. We recall that most of the existence theorems to problem (3.1) developed so far deal with continuous family of continuous maps, i.e., deal with maps $f: \mathbb{R} \times (E, \tau) \to (E, \tau)$ that is continuous from $(\mathbb{R} \times E, |\cdot| \times \tau)$ to (E, τ). Instead we shall allow measurable family of continuous maps. The precise definition is as follows.

Definition 3.2. Let $(E, \mathcal{F})$ be a sequentially complete locally convex space. We will say a map $f: I \times E_b \to E_b$ is an $\mathcal{F}$-Carathéodory map if:

(1) For each $u \in E_b$ fixed, the map $f(\cdot, u): I \to E_b$ is measurable.

(2) For almost every $s \in I$ the map $f(s, \cdot): E_b \to E_b$ is $\mathcal{F}$-continuous.

We are ready to show the main theorem of this section.

Theorem 3.3. *Let $(E, \mathcal{F})$ be a sequentially complete locally convex space such that $(E_b, \mathcal{F})$ is locally metrizable and $f: I \times E_b \to E_b$ be a $\mathcal{F}$-Carathéodory map satisfying*

$$\|f(t, u)\|_{\mathcal{F}} \leq \Psi\big(t, \|u\|_{\mathcal{F}}\big), \tag{3.2}$$

where for each $t \in I$ fixed, the map $\Psi(t, \cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and for every $a > 0$ the map $\Psi(\cdot, a) \in L_1(I)$. Suppose furthermore that

$$B_{E_b} \text{ is relatively compact with respect to the } \mathcal{F} \text{ topology.} \tag{3.3}$$

Then for each $x_0 \in E_b$ there exists a $\delta = \delta(\|x_0\|_{\mathcal{F}})$ such that the IVP (3.1) admits a strong solution defined on $[0, \delta)$.

Proof. We start by fixing an $M > \|x_0\|$. Define $\lambda = \lambda(M)$ to be

$$\lambda := \frac{\|\Psi(\cdot, M)\|_{L_1}}{M - \|x_0\|} \tag{3.4}$$

Let us then define $F_\lambda: L_\infty(I, E_b) \to L_\infty(I, E_b)$ to be

$$F_\lambda(u)(t) := x_0 + \frac{1}{\lambda} \int_0^t f(s, u(s))ds \tag{3.5}$$

We estimate:

$$
\begin{aligned}
\|F_\lambda(u)(t)\| &\leq \|x_0\| + \frac{1}{\lambda}\int_0^t \|f(s,u(s))\|ds \\
&\leq \|x_0\| + \frac{1}{\lambda}\int_0^t \Psi\big(s,\|u(s)\|\big)ds \\
&\leq \|x_0\| + \frac{1}{\lambda}\int_0^t \Psi\big(s,\|u\|_\infty\big)ds \\
&= \|x_0\| + \frac{\|\Psi\big(\cdot,\|u\|_\infty\big)\|_{L_1}}{\lambda}
\end{aligned}
\tag{3.6}
$$

Let $X := \Big(B_{L_\infty(I,E_b)}(M), T\Big)$, i.e. the ball in $L_\infty(I,E_b)$ with radius M, endowed with the T-topology. It follows from (3.6) that if $u \in X$

$$
\begin{aligned}
\|F_\lambda(u)\|_\infty &\leq \|x_0\| + \frac{\|\Psi(\cdot,M)\|_{L_1}}{\lambda} \\
&= M,
\end{aligned}
\tag{3.7}
$$

due to the suitable choice of λ in (3.4). We have verified F_λ maps X into itself. Our next step is to show that F_λ is actually a continuous map from X into itself. To this end, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X which converges to u in the T-topology. Let $\rho \in \mathcal{F}$ be fixed. We have

$$
\begin{aligned}
\Phi_\rho\big(F_\lambda(u_n) - F_\lambda(u)\big) &= \sup_{t\in I} \rho\left(\int_0^t f(s,u_n(s))ds - \int_0^t f(s,u(s))ds\right) \\
&\leq \sup_{t\in I}\int_0^t \rho\Big(f(s,u_n(s)) - f(s,u(s))\Big)ds \\
&\leq \int_0^T \rho\Big(f(s,u_n(s)) - f(s,u(s))\Big)ds,
\end{aligned}
\tag{3.8}
$$

where in the second inequality we have used Jensen's inequality. It follows from theorem 2.7 that for a.e. $s \in I$, $u_n(s) \to u(s)$ in E_b with respect to the $\mathcal{F}$-topology. ¿From the fact that the field f is $\mathcal{F}$-Carathéodory, for a.e. $s \in I$, the map $f(s,\cdot)\colon E_b \to E_b$ is $\mathcal{F}$-continuous. It implies that

$$\rho\Big(f(s,u_n(s)) - f(s,u(s))\Big) \to 0,$$

for a.e. $s \in I$. Invoking the Lebesgue dominated convergence theorem, we conclude

$$\int_0^T \rho\Big(f(s,u_n(s)) - f(s,u(s))\Big)ds \to 0.$$

The above combined with (3.8) implies $F_\lambda(u_n) \to \mathcal{F}_\lambda(u)$ in the T-topology, i.e., we have proven $F_\lambda : X \to X$ is a continuous map.

Our next step is to study the relatively T-compactness of $F_\lambda(X)$. Let $0 \leq t_1 \leq t_2 \leq T$. We have, for all $u \in X$,

$$
\begin{aligned}
\|F_\lambda(u)(t_1) - F_\lambda(u)(t_2)\| &\leq \int_{t_1}^{t_2} \|f(s,u(s))\|ds \\
&\leq \int_{t_1}^{t_2} \Psi(s,M)ds.
\end{aligned}
\tag{3.9}
$$

Since $\Psi(\cdot,M) \in L_1(I)$, we obtain $F_\lambda(X)$ is strongly equicontinuous. Furthermore, inequality (3.6) implies for each $t \in I$ fixed, the set $F_\lambda(X)(t)$ is bounded. It follows therefore from theorem 2.2 that for any sequence $(u_n)_{n\in\mathbb{N}} \subset X$, up to a subsequence, there exists a $\mathcal{F}$-continuous map $\xi\colon I \to E_b$ such that $F(u_n)$ converges $\mathcal{F}$ uniformly to ξ, as $n \to \infty$. This implies

$$F_\lambda(u_n) \to \xi,$$

as $n \to \infty$ with respect to the T-topology. Clearly $\|\xi\|_\infty \le M$ and moreover, theorem 2.5 implies ξ is a measurable map. We have proven $F_\lambda(X)$ is relatively compact with respect to the T-topology.

It follows now from theorem 2.1 the existence of a fixed point to F_λ. Let us denote by u^λ such a fixed point. Easily one verifies that u^λ is absolutely continuous with respect to the strong topology in E_b. Furthermore, it is almost everywhere differentiable by theorem 2.8. In this way, u^λ is a strong solution for the following IVP

$$(P_\lambda) \qquad \begin{cases} u_t^\lambda(t) &= \dfrac{1}{\lambda} f(t, u^\lambda(t)) \text{ in } I \\ u^\lambda(0) &= u_0 \end{cases}$$

The next step is to pass from problem (P_λ) to problem (P_1) which is precisely problem (3.1). To this end let us define

$$\widetilde{f}(t,x) := \begin{cases} f\left(\frac{t}{\lambda}, x\right) & \text{if } 0 \le t \le \frac{T}{\lambda} \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\widetilde{f}$ satisfies the same hypothesis as f does. Hence, applying the result we have established so far, we obtain a map $\widetilde{u}^\lambda\colon I \to E_b$, which solves problem (P_λ) with the field $\widetilde{f}$. Finally we set $u\colon [0, \frac{T}{\lambda}] \to E_b$ to be $u(t) := \widetilde{u}^\lambda(\lambda t)$. Clearly $u(0) = u_0$ and

$$u_t(t) = \lambda \widetilde{u}_t^\lambda(\lambda t) = \widetilde{f}(\lambda t, \widetilde{u}^\lambda(\lambda t)) = f(t, u(t)).$$

□

Remark 3.4. (i) It is worthwhile to point out that we do not need the sequential completness of $(E, \mathcal{F})$ in theorem 3.3. All we need is $(E_b, \|\cdot\|_{\mathcal{F}})$ to be a Banach space.
(ii) The hypothesis that $(E_b, \mathcal{F})$ if locally metrizable is not crucial. In a general case we can argue as in [2]. However in most of the practical applications this hypothesis is easily verified.
(iii) One could replace hypothesis (3.3) by the following weaker hypothesis:

$$\exists M > \|x_0\| \text{ and } \delta > 0 \text{ such that } f([0,\delta] \times B_M) \text{ is relatively compact w.r.t. } \mathcal{F}-\text{topology}.$$

Corollary 3.5. *Let E be a reflexive Banach space and $f\colon I \times E \to E$ a measurable family of sequentially weakly continuous map, satisfying the growth condition (3.2). Then there exists a $\delta = \delta(\|x_0\|)$ such that the IVP admits a strong solution defined on $[0, \delta)$.*

Regarding global solution, we would like to state the following result for completeness.

Theorem 3.6 ([20] pg. 145)**.** *Let X be a Banach space. Assume the growth condition (3.2), where $\Psi \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\Psi(t, \cdot)$ is nondecreasing for each $t \in I$ and the maximal solution $x(t, 0, x_0)$ of the scalar differential equation*

$$\begin{cases} x_t(t) &= \Psi(t, x(t)) \text{ in } I \\ x(t_0) &= x_0 \end{cases}$$

exists on I. Suppose f is smooth enough to assure local existence of solutions of 3.1 *for any $(t_0, u_0) \in I \times X$. Then the largest interval of existence of any solution $u(t, t_0, u_0)$ of (1.1) such that $\|u_0\|_E \le x_0$ is $[t_0, \infty)$.*

We finish this section by characterizing condition (3.3) in theorem 3.3. The next theorem can be thought as the converse of Banach-Alaoglu theorem.

Theorem 3.7. *Let E be a Banach space. Suppose E admits a locally convex Hausdorff topology $\mathcal{F}$ such that B_E is compact with respect to the $\mathcal{F}$. Then there exists a norm-closed linear subspace E_* of E^* such that*

(1) $E = (E_*)^*$,
(2) *On B_E, the weak-* topology $\sigma(X, X_*)$ coincides with $\mathcal{F}$.*

Proof. Let us define

$$E_* := \{f : E \to \mathbb{R} \mid f \text{ is linear and } \mathcal{F}-\text{continuos on } B_E\} \subseteq E^*.$$

Clearly E_* is a linear subspace of E^*. To see that it is norm-closed, let (f_n) be a sequence in E_* which converges to $f \in E^*$ in norm. Let $(x_i)_{i\in\mathcal{I}}$ be a net in B_E which converges to $x \in B_E$ in the $\mathcal{F}$ topology. We have to show $\lim_{i\in\mathcal{I}} f(x_i) = f(x)$. Let $\varepsilon > 0$ be given. There exits an $n \in \mathcal{N}$ such that $\|f - f_n\| < \varepsilon/2$. Thus

$$\begin{aligned} \lim_{i\in\mathcal{I}} |f(x_i) - f(x)| &\leq |f(x) - f_n(x)| + \lim_{i\in\mathcal{I}} |f_n(x) - f_n(x_i)| + \lim_{i\in\mathcal{I}} |f_n(x_i) - f(x_i)| \\ &\leq 2\|f_n - f\| \\ &< \varepsilon. \end{aligned}$$

Next we proof that

$$\|x\| = \sup\{|f(x)| : f \in E_*,\ \|f\| \leq 1\}.$$

Clearly $\sup\{|f(x)| : f \in E_*,\ \|f\| \leq 1\} \leq \|x\|$. Now suppose $\|x\| > 1$. Since B_E is convex and $\mathcal{F}$-closed, by the geometric version of the Hahn-Banach theorem for locally convex spaces, there exists a $\mathcal{F}$-continuous functional $g : E \to \mathcal{R}$ such that

$$|g(x)| > \alpha > |g(y)|\ \forall y \in B_E.$$

This in particular means

$$|g(x)| > \sup\{|g(y)| \mid y \in B_E\} = \|g\|.$$

Once $g \in E_*$, we conclude

$$\sup\{|f(x)| \mid f \in E_*, \|f\| \leq 1\} \geq \left|\frac{g(x)}{\|g\|}\right| > 1,$$

which by a rescaling argument drives us to

$$\|x\| \leq \sup\{|f(x)| \mid f \in E_*, \|f\| \leq 1\}.$$

Now for each $x \in E$ we define $I(x) \in (E_*)^*$ to be

$$I(x)(f) := f(x).$$

We have already proven I is a linear isometry. Moreover it is clearly continuous from B_E endowed with the $\mathcal{F}$ topology into $B_{(E_*)^*}$ endowed with the weak-* topology. Thus $I(B_E)$ is weak-* compact. It is just remaining to prove I is onto. Let us suppose by contradiction there exists $\phi B_{(E_*)^*}$ but $\phi \notin I(B_E)$. Then there would exist an $f \in E_*$ such that

$$|\phi(f)| > \sup\{|I(x)(f) : x \in B_E\} = \sup\{|f(x) : x \in B_E\} = \|f\|,$$

a contradiction. Hence $I(B_E) = B_{(E_*)^*}$. It follows therefore that I is an isometric isomorphism from E onto $(E_*)^*$. Moreover I is a homeomorphism between B_E with respect to the $\mathcal{F}$ topology and $B_{(E_*)^*}$ with respect to the weak-* topology. □

4. Lipschitz-stability for a nonlinear differential equation involving the Hardy-Littlewood maximal operator

In this section we will study a nonlinear differential equation involving the celebrated Hardy-Littlewood maximal operator. Such operator plays an important hole in many parts of the Applied Mathematics such as: Harmonic Analysis, Singular Integrals, Partial Differential Equations, among others. Its precise definition is as follows.

Definition 4.1. Let Ω be an open set in $\mathbb{R}^N$. For a locally integrable function $u\colon \Omega \to [-\infty,+\infty]$, we define the (local) Hardy-Littlewood maximal function, $M(u)\colon \Omega \to [0,+\infty]$ as

$$M(u)(x) := \sup_{0<r<d(x,\partial\Omega)} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y)|dy.$$

We start by mentioning a remarkable theorem due to Hardy, Littlewood and Wiener:

Theorem 4.2 (Hardy-Littlewood-Wiener)**.** *Let $1 < p \le \infty$ and $u \in L^p(\Omega)$. Then $M(u) \in L^p(\Omega)$ and*

$$\|M(u)\|_{L^p(\Omega)} \le A_p \|u\|_{L^p(\Omega)}.$$

At this point is worthwhile to point out that the Hardy-Littlewood maximal operator is usually used to estimate the absolute size and hence questions about differentiability related to its image are, in general, much more delicate. Nevertheless, it was shown in [18] the following result:

Theorem 4.3 (Kinnunen-Lindquist)**.** *Let $1 < p \le \infty$. If $u \in W^{1,p}(\Omega)$, then $M(u) \in W^{1,p}(\Omega)$ and*

$$|D(M(u))(x)| \le 2M(|Du|)(x).$$

In particular, this theorem together with theorem 4.2 yields

$$\|M(u)\|_{W^{1,p}(\Omega)} \le 2A_p \|u\|_{W^{1,p}(\Omega)}. \tag{4.1}$$

Moreover it was also proven in [18] that the local maximal operator preserves zero boundary values. More precisely, for every $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, the function $M(u)$ lies in $W_0^{1,p}(\Omega)$. In this section $W_0^{1,\infty}(\Omega)$ stands for the space of all Lipschitz maps defined on Ω that vanishes at the boundary $\partial\Omega$. We endow this space with the following norm:

$$\|u\|_{W_0^{1,p}} := \|Du\|_\infty.$$

Using the fact that the local maximal operator preserves zero boundary values for $1 < p < \infty$, it is easy to justify that for every $u \in W_0^{1,\infty}(\Omega)$, $M(u)$ also lies in $W_0^{1,\infty}(\Omega)$. Indeed, if $u \in W_0^{1,\infty}(\Omega)$, $u \in W_0^{1,p}(\Omega)$ for any $p \ge 1$. If $p > n$, $u \in C^\varepsilon(\Omega)$ for some $\varepsilon \in (0,1)$, thus its trace value agrees with its value on the boundary.

After these comments, let us turn our attention to the problem we shall work on. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\xi\colon I \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a map satisfying

1. For each $z \in \Omega \times \mathbb{R}_+$, the map $\xi(\cdot, z)\colon I \to \mathbb{R}$ is measurable.
2. For almost every $t \in I$, the map $\xi(t,\cdot)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is Lipschitz and $\sup\limits_{t\in I} \|\xi\|_{\mathrm{Lip}(\Omega\times\mathbb{R}_+)} := L < \infty$.
3. $\xi(t,x,0) = 0$, for all $(t,x) \in I \times \Omega$.

Given $\varphi\colon \Omega \to \mathbb{R}$ with $\varphi\big|_{\partial\Omega} \equiv 0$, we are interested in finding a map $u\colon I \times \Omega \to \mathbb{R}$ which solves

$$\begin{cases} u_t(t,x) &= \xi(t,x,M(u(t,x))) & \text{in } I\times\Omega \\ u(t,x) &= 0 & \text{on } \partial\Omega \\ u(0,x) &= \varphi(x) & \text{in } \Omega \end{cases} \tag{4.2}$$

Let us mention that our motivation to this problem is related to regularity results to partial differential equations involving averages in fixed domains under Lipschitz nonlinearities. Moreover we should also mention that once we are interested in $W^{1,p}$ stability to problem (4.2), hypothesis (1) (2) and (3) above are necessary hypothesis. We need a simple lemma.

Lemma 4.4. *Let E be a reflexive Banach space and $f\colon E \to E$ a bounded map. Suppose E is compacted embedded into F and $f\colon F \to F$ is continuous. Then f is sequentially weakly continuous in E.*

Proof. Let $u_n \rightharpoonup u$ in E. Hence, the sequence $\{u_n\} \subset E$ is bounded and by hypothesis, so is $\{f(u_n)\} \subset E$. Once E is reflexive, we can assume, up to a subsequence, that $f(u_n)$ converges weakly to some v in E. Using now the fact that E is compacted embedded into F, we get that

$$u_n \to u \text{ in } F \quad \text{ and } \quad f(u_n) \to v \text{ in } F.$$

Finally, by the continuity of $f\colon F \to F$, we obtain that $f(u_n) \to f(u)$ in F, and hence, $v = f(u)$. □

Remark 4.5. All one needs in Lemma 4.4 is the demicontinuity of the map $f\colon F \to F$ to conclude the sequential weak continuity of $f\colon E \to E$. This type of lemma, although simple, has shown to be useful in some practical applications.

Theorem 4.6 ($W^{1,p}$-Stability)**.** *For each $\varphi \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, there exists a unique Lipschitz curve $\hat{u}\colon I \to W_0^{1,p}(\Omega)$, such that the map $u(t,x) := \hat{u}(t)(x)$ globally solves the differential equation (4.2).*

Proof. Let $E = W_0^{1,p}(\Omega)$, $F = L^p(\Omega)$ and $f\colon I \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ be given by

$$f(t,u)(x) := \xi(t,x,M(u)(x)).$$

Let us estimate $\|f(t,u)\|_{W_0^{1,p}(\Omega)}$:

$$\begin{aligned} \|f(t,u)\|_{W_0^{1,p}(\Omega)} &:= \left\{ \int_\Omega |\nabla_x f(t,u)(x)|^p dx \right\}^{1/p} \\ &= \left\{ \int_\Omega |\nabla_x \xi + \partial_s \xi \cdot D(M(u))|^p dx \right\}^{1/p} \\ &\leq L\big(|\Omega|^{1/p} + 2A_p \|u\|_{W_0^{1,p}(\Omega)}\big). \end{aligned} \tag{4.3}$$

This proves that $f\colon I \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ has a linear growth. Let us now fix a $t \in I$ for which the map $\xi(t,\cdot)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is Lipschitz. We shall estimate $\|f(t,u) - f(t,v)\|_{L^p(\Omega)}$:

$$\begin{aligned} \|f(t,u) - f(t,v)\|_{L^p(\Omega)} &:= \left\{ \int_\Omega |\xi(t,x,M(u)(x)) - \xi(t,x,M(v)(x))|^p dx \right\}^{1/p} \\ &\leq L \left\{ \int_\Omega |M(u)(x) - M(v)(x)|^p dx \right\}^{1/p} \\ &\leq L \left\{ \int_\Omega |M(u-v)(x)|^p dx \right\}^{1/p} \\ &\leq A_p L \|u - v\|_{L^p(\Omega)}. \end{aligned} \tag{4.4}$$

The above calculation shows in particular that for a.e. $t \in I$, the map $f(t,\cdot)\colon L^p(\Omega) \to L^p(\Omega)$ is continuous and then, by lemma 4.4, $f\colon I \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is a weak-Carathéodory map. We have verified all the hypothesis of corollary 3.5, which assures the existence of a

Lipschitz curve $\hat{u}\colon I \to W_0^{1,p}(\Omega)$, which globally solves (due to estimate (4.3) and theorem 3.6) the below equation

$$\left\{\begin{array}{lcl} \hat{u}_t(t) & = & f(t, \hat{u}(t)) \\ \hat{u}(0) & = & \varphi \end{array}\right. \tag{4.5}$$

Let us turn our attention to uniqueness. Suppose u and v are two solutions to problem (4.5). Let $g\colon L^p(\Omega) \to \mathbb{R}$ be given by

$$g(\varphi) := \int_\Omega |\varphi(x)|^p dx.$$

It is well know g is differentiable and

$$D(g)(\varphi)\cdot\psi = p\int_\Omega |\varphi(x)|^{p-2}\varphi(x)\cdot\psi(x)dx.$$

Thus, applying Hölder inequality and afterwards inequality (4.4) we obtain

$$\begin{array}{rcl} \dfrac{d}{dt}\|u(t)-v(t)\|_p^p & = & p\displaystyle\int_\Omega |u(t)-v(t)|^{p-2}\big(u(t)-v(t)\big)\cdot\big(u_t(t)-v_t(t)\big) \\ & \leq & p\|u(t)-v(t)\|_p^{p-1}\cdot\|u_t(t)-v_t(t)\|_p \\ & = & p\|u(t)-v(t)\|_p^{p-1}\cdot\|f(t,u(t))-f(t,v(t))\|_p \\ & \leq & pA_pL\|u(t)-v(t)\|_p^p. \end{array} \tag{4.6}$$

If we call $d(t) := \|u(t)-v(t)\|_p^p$, we have $d(0)=0$ and inequality (4.6) says

$$d_t(t) \leq C_p d(t).$$

This implies $d \equiv 0$ (Gronwall's inequality) and hence $u(t) = v(t)$ for all $t \in I$.

To conclude we define $u\colon I\times\Omega \to \mathbb{R}$ to be

$$u(t,x) := \hat{u}(t)(x).$$

Such a function satisfies

$$\left\{\begin{array}{lcll} u_t(t,x) & = & \xi(t,x,M(u(t,x))) & \text{in } I\times\Omega \\ u(t,x) & = & 0 & \text{on } \partial\Omega \\ u(0,x) & = & \varphi(x) & \text{in } \Omega \end{array}\right. \tag{4.7}$$

as requested.

□

Finally, let us move our attention toward the Lipschitz stability of the solution to problem (4.2). Suppose φ is Lipschitz. For each $p > 1$ we can apply the existence and uniqueness result we have proven in theorem 4.6 and conclude $\hat{u}(t) \in W_0^{1,p}(\Omega)$ for any $p > 1$. This gives $\hat{u}(t) \in C^\alpha(\Omega)$ for any $\alpha \in (0,1)$. It is a fairly good regularity but it does not imply Lipschitz regularity. This type of phenomena is quite common in regularity problems involving elliptic operators, for instance, obstacle problems or fully nonlinear elliptic equations. In this setting Harnack inequality plays, in general, a crucial whole that yields to pass from the C^α-regularity for any $\alpha \in (0,1)$ to Lipschitz regularity. In our setting we shall obtain this by a topological framework. To do it so, we shall make use of all the generality provided by theorem 3.3.

Before going through the construction of the mathematical tools we shall use in the regularity process, let us roughly explain the difficulty of dealing with Lipschitz stability in this setting. The first observation is that $W_0^{1,\infty}(\Omega)$ is not a reflexive space. Thus one cannot use the weak topology on $W_0^{1,\infty}(\Omega)$ to apply theorem 3.3. On the other hand, although not being a classical dual space, one can think $W_0^{1,\infty}(\Omega)$ inside of $\left[L^\infty(\Omega)\right]^{N+1}$. So in some sense we could endow $W_0^{1,\infty}(\Omega)$ with a sort of weak-* topology. The problem is, as we have already pointed out before, that with this topology "almost all" nonlinear operators fails to be continuous. The whole point is to find a reasonable topology in $W_0^{1,\infty}(\Omega)$ for which one can verify relative compactness of bounded subsets and continuity of the operator we are dealing with. This is the content of what follows.

Definition 4.7. Let $n > 2$ be a natural number. We put

$$\mathcal{F}_n := \Big\{\rho\colon W_0^{1,\infty}(\Omega) \to \mathbb{R}_+ \ \Big|\ \rho(f) = \rho_\phi(f) := |\phi(f)|,\ \phi \in \big[W_0^{1,n}(\Omega)\big]^*,\ \|\phi\| = 1\Big\}.$$

We then define $\mathcal{F} := \bigcup_{n\geq 2} \mathcal{F}_n$.

Notice that $\mathcal{F}$ is a family of seminorms in $W_0^{1,\infty}(\Omega)$. We shall consider the locally convex topological space $\mathbb{X} := \Big(W_0^{1,\infty}(\Omega),\ \mathcal{F}\Big)$.

Lemma 4.8. $\mathbb{X}_b = W_0^{1,\infty}(\Omega)$ *and* $\|f\|_{\mathcal{F}} = \|f\|_{W_0^{1,\infty}}$

Proof. We might suppose without losing generality that $|\Omega| = 1$. Let $f \in W_0^{1,\infty}(\Omega)$ be fixed. Let $\rho_\phi \in \mathcal{F}_n$. We compute

$$\rho_\phi(f) = |\phi(f)| \leq \|f\|_{W_0^{1,n}} \leq \|f\|_{W_0^{1,\infty}}.$$

This proves $\mathbb{X}_b = W_0^{1,\infty}(\Omega)$ and $\|f\|_{\mathcal{F}} \leq \|f\|_{W_0^{1,\infty}}$. On the other hand, if $f \in W_0^{1,\infty}(\Omega)$ and $n \geq 2$ we have

$$\|f\|_{W_0^{1,n}} = \sup_{\substack{\phi\in[W_0^{1,n}]^* \\ \|\phi\|=1}} |\phi(f)| \leq \|f\|_{\mathcal{F}}.$$

Letting n goes to infinity in the above inequality, we conclude $\|f\|_{W_0^{1,\infty}} \leq \|f\|_{\mathcal{F}}$. □

Lemma 4.9. *The ball of* $W_0^{1,\infty}(\Omega)$ *endowed with the* $\mathcal{F}$ *topology is a compact metrizable space.*

Proof. The fact that it is metrizable follows from the fact that one can find a enumerable subset of $\mathcal{F}$ that generates the $\mathcal{F}$ topology. Let us turn our attention to the compactness. We observe that a sequence converges with respect to the $\mathcal{F}$ topology if and only if it converges weakly in $W_0^{1,n}(\Omega)$ for every $n \geq 2$. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $W_0^{1,\infty}(\Omega)$ with $\|f_j\|_{W_0^{1,\infty}} \leq 1$. For each $n \geq 2$ we notice that

$$\|f_j\|_{W_0^{1,n}} \leq \|f_j\|_{W_0^{1,\infty}} \cdot |\Omega|^{1/n} \leq |\Omega|^{1/n}. \tag{4.8}$$

Thus, due to the reflexibility of $W_0^{1,n}(\Omega)$, there exists a subset of the natural numbers $\mathbb{N}_n \subseteq \mathbb{N}$ such that

$$(f_j)_{j\in\mathbb{N}_n} \rightharpoonup f_n \text{ in } W_0^{1,n}(\Omega).$$

Moreover $\|f_n\|_{W_0^{1,n}} \leq |\Omega|^{1/n}$. We now implement a diagonal argument as follows: There exists a subset $\mathbb{N}_2 \subseteq \mathbb{N}$ such that $f_j \rightharpoonup f$ in $W_0^{1,2}(\Omega)$. By repeating this argument we can find a subset $\mathbb{N}_3 \subseteq \mathbb{N}_2$, such that $f_j \rightharpoonup f \in W_0^{1,3}(\Omega)$. Carrying this process on we find a nested sequence of subsets

$$\mathbb{N}_2 \supseteq \mathbb{N}_3 \supseteq ...\mathbb{N}_n \supseteq ...$$

such that

$$(f_j)_{j\in\mathbb{N}_n} \rightharpoonup f \text{ in } W_0^{1,n}(\Omega).$$

Finally if we set $\mathbb{N}_d$ to be the diagonal subset, i.e., the nth element of $\mathbb{N}_d$ is the nth element of $\mathbb{N}_n$, we have

$$(f_j)_{j\in\mathbb{N}_d} \rightharpoonup f \text{ in } W_0^{1,n}(\Omega),\ \forall n \geq 2.$$

It remains to show that $f \in W^{1,\infty}(\Omega)$ and that $\|f\|_{W_0^{1,\infty}} \leq 1$. It follows from the weak lower semicontinuity of the norm and inequality (4.8) that

$$\|f\|_{W_0^{1,n}} \leq |\Omega|^{1/n} \ \forall n \geq 2.$$

Letting $n \to \infty$ in the above inequality we conclude the lemma. □

We now can state the final goal of the section

Theorem 4.10 (Optimal Stability). *For each $\varphi \in W_0^{1,\infty}(\Omega)$, there exists a unique Lipschitz curve $\hat{u}\colon I \to W_0^{1,\infty}(\Omega)$, such that the map $u(t,x) := \hat{u}(t)(x)$ solves globally the differential equation (4.2).*

Proof. The work is almost done. Let $\mathbb{X} := \left(W_0^{1,\infty}(\Omega),\ \mathcal{F}\right)$ as defined above and $f\colon \mathbb{X} \to \mathbb{X}$ be given by

$$f(t,u)(x) := \xi(t,x,M(u)(x)).$$

As we have seen in theorem 4.6, f has linear growth and for almost every $t \in I$ and for any $n \geq 2$, $f(t,\cdot) : W_0^{1,n}(\Omega) \to W_0^{1,n}(\Omega)$ is sequentially weakly continuous. This implies that f is an $\mathcal{F}$-Carathéodory map. Finally Lemma 4.8 and Lemma 4.9 provide the remaining hypothesis of Theorem 3.3, which asserts the existence of a strong solution to problem (4.5). Theorem 4.10 is finished. □

Remark 4.11. The main information given by theorem 4.6 and ultimately by theorem 4.10 is the fact that $\hat{u}(t) \in W_0^{1,p}(\Omega)$, provided $\varphi \in W_0^{1,p}(\Omega)$, $1 < p \leq \infty$. We remark that, the field studied in theorem 4.6 and theorem 4.10, seen as a field defined on $L^p(\Omega)$, is Lipschitz (see estimate (4.4)). Thus, for every $\varphi \in L^p(\Omega)$, the classical Cauchy-Picard theorem asserts that there exists a unique (local) solution in $L^p(\Omega)$. Therefore, theorem 4.6 and theorem 4.10 are a sort of regularization result for such type of equations.

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