ON A THEOREM OF KAC AND GILBERT

BARRY SIMON

ABSTRACT. We prove a general operator theoretic result that asserts that many multiplicity two selfadjoint operators have simple singular spectrum.

1. Introduction

In 1963, I.S. Kac [5] proved that whole-line Schrödinger operators, $-\frac{d}{dx^2} + V(x)$, for fairly general V's have simple singular spectrum. It is well known (e.g., $V = 0$) that the absolutely continuous spectrum can have multiplicity two and that under limit point hypotheses, eigenvalues are simple. But the simplicity of the singular continuous spectrum is surprisingly subtle. Some insight into the result was obtained by Gilbert [3], who found a proof using the subordinacy theory of Gilbert-Pearson [4]. The proof is elegant but depends on the substantial machinery of subordinacy. Our purpose here is to note an abstract result that relates these things to the celebrated result of Aronszajn-Donoghue [1]:

Theorem 1. *Let A be a bounded selfadjoint operator on $\mathcal{H}$ and $\varphi \in \mathcal{H}$ a cyclic vector for A. Suppose $\lambda \in \mathbb{R}\backslash\{0\}$ and*

$$B = A + \lambda\langle\varphi, \cdot\rangle\varphi \tag{1.1}$$

Then the singular spectral measures for A and B are disjoint.

We state this and the next theorem in the bounded case for simplicity; we discuss the general case later. Here's the main result of this note:

Theorem 2. *Let $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and $P : \mathcal{H} \to \mathcal{K}_1$, the canonical projection. Let $A_j \in \mathcal{L}(\mathcal{K}_j)$ for $j = 1, 2$, and $\varphi \in \mathcal{H}$ so that $\varphi_1 = P\varphi$ and $\varphi_2 = (1-P)\varphi$ are cyclic for A_1 and A_2. Let $\lambda \in \mathbb{R}\backslash\{0\}$. Then*

$$C = A_1 \oplus A_2 + \lambda(\varphi, \cdot)\varphi \tag{1.2}$$

has simple singular spectrum.

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* Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592.

Remark. If A_1 is unitarily equivalent to A_2 and has a.c. spectrum, then C has multiplicity two a.c. spectrum. So it is interesting that the singular spectrum is simple.

In Section 2, we prove Theorem 2. In Section 3, we apply it to whole-line Jacobi matrices. In Section 4, we discuss extensions of Theorem 2 to the case of unbounded selfadjoint operators and to unitary operators. In Section 5, we apply the results of Section 4 to Schrödinger operators and to extended CMV matrices.

2. Proof of Theorem 2

Let $d\mu_j$ be the spectral measure for φ_j and A_j and $B = A_1 \oplus A_2$. Thus $\mathcal{K}_j \cong L^2(\mathbb{R}, d\mu_j)$ in such a way that A_j is multiplication by x and $\varphi_j \cong 1$.

Pick disjoint sets $X, Y, Z \subset \mathbb{R}$ whose union is $\mathbb{R}$ so $d\mu_1 \restriction X$ is equivalent to $d\mu_2 \restriction X$ and $\mu_1(Y) = \mu_2(Z) = 0$. For example, if $d\mu_1 = f(d\mu_1 + d\mu_2)$, then one can take $X = \{x \mid 0 < f(x) < 1\}$, $Y = \{x \mid f(x) = 0\}$, $Z = \{x \mid f(x) = 1\}$.

Let $\mathcal{L}_1$ be the cyclic subspace generated by φ and B and $\mathcal{L}_2 = \mathcal{L}_1^\perp$. Then $\psi = (\chi_X, -\chi_X)$ is a cyclic vector for $B \restriction \mathcal{L}_2$ and its spectral measure is

$$d\mu_\psi^B = \chi_X(x)(d\mu_1 + d\mu_2) \tag{2.1}$$

In particular,

$$(d\mu_\psi^B)_\text{s} \leq (d\mu_1 + d\mu_2)_\text{s} \tag{2.2}$$

By definition of $\mathcal{L}_1$, φ is cyclic for $B \restriction \mathcal{L}_1$ and

$$C \restriction \mathcal{L}_1 = B + \lambda(\varphi, \,\cdot\,)\varphi$$

so, by Theorem 1,

$$(d\mu_\varphi^C)_\text{s} \perp (d\mu_\varphi^B)_\text{s} = (d\mu_1 + d\mu_2)_\text{s} \tag{2.3}$$

Thus, the singular parts of $d\mu_\varphi^C$ and $d\mu_\psi^C = d\mu_\psi^B$ are disjoint, which implies that the singular spectrum of C is simple. $\square$

The proof shows that the singular parts of B and C overlap in $\chi_X(d\mu_1 + d\mu_2)_\text{s}$ and, in particular,

Corollary 2.1. *B and C have mutually singular parts if and only if A_1 and A_2 have mutually singular parts.*

3. Application to Jacobi Matrices

A two-sided Jacobi matrix is defined by two two-sided sequences, $\{b_n\}_{n=-\infty}^{\infty}$ and $\{a_n\}_{n=-\infty}^{\infty}$ with $b_n \in \mathbb{R}$ and $a_n \in (0,\infty)$ and $\sup_n(|a_n| + |b_n|) < \infty$. It defines a bounded operator J on $\ell^2(\mathbb{Z})$ by

$$(Ju)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} \tag{3.1}$$

so

$$J = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.2}$$

Theorem 3.1. *The singular spectrum of J is simple.*

Proof. Let $\mathcal{K}_1 = \ell^2((-\infty,-1])$, $\mathcal{K}_2 = \ell^2([0,\infty))$, and φ the vector with components

$$\varphi_j = \begin{cases} 1 & j = -1, 0 \\ 0 & j \neq -1, 0 \end{cases} \tag{3.3}$$

so $P\varphi = \delta_{-1}$; $(1-P)\varphi = \delta_0$.

Then

$$J - a_{-1}(\varphi, \varphi) = A_1 \oplus A_2$$

where A_2 is the one-sided Jacobi matrix with

$$A_2 = \begin{pmatrix} b_0 - a_{-1} & a_0 & 0 & \dots \\ a_0 & b_1 & a_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and A_1 in $\delta_{-1}, \delta_{-2}, \dots$ basis is

$$A_1 = \begin{pmatrix} b_{-1} - a_{-1} & a_{-2} & 0 & \dots \\ a_{-2} & b_{-2} & a_{-3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Thus $P\varphi$ is cyclic for A_1 and $(1-P)\varphi$ is cyclic for A_2. Theorem 2 applies and implies the desired result. $\square$

4. Unitary and Unbounded Selfadjoint Operators

Let U_1, U_2, and W be unitary operators on $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ so $W - U_1 \oplus U_2$ is rank one, and so if $\varphi \in \ker(W - U_1 \oplus U_2)^{\perp}$, then $P\varphi$ is cyclic for U_1 and $(1-P)\varphi$ is cyclic for U_2, where P is the canonical projection of $\mathcal{H}$ to $\mathcal{K}_1$. Suppose $W - U_1 \oplus U_2 \neq 0$. Then

Theorem 4.1. *The singular spectrum of W is simple.*

Proof. We begin by proving the unitary analog of the Aronszajn-Donoghue theorem. If $W - V$ is rank one and nonzero, then for φ a unit vector in $\ker(W - V)^{\perp}$, we have $W\varphi = \lambda V\varphi$ for some $\lambda \in \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$. By a direct calculation (see, e.g., [6]),

$$\left(\varphi, \frac{W+z}{W-z}\,\varphi\right) = \frac{1+zg(z)}{1-zg(z)} \tag{4.1}$$

$$\left(\varphi, \frac{V+z}{V-z}\,\varphi\right) = \frac{1+zf(z)}{1-zf(z)} \tag{4.2}$$

and

$$g(z) = \lambda^{-1} f(z) \tag{4.3}$$

If φ cyclic for W, the singular spectrum in W is supported on those $z \in \partial\mathbb{D}$ with

$$\lim_{r\uparrow 1} rz\, g(rz) = 1$$

and similarly, the singular spectrum of V on the set of $z \in \partial\mathbb{D}$ with

$$\lim_{r\uparrow 1} rz\, f(rz) = 1$$

By (4.3), these sets are disjoint.

This proves the Aronszajn-Donoghue theorem in the unitary case, and that implies this theorem by mimicking the proof of Theorem 2. □

Next, let A_1, A_2, and C be potentially unbounded selfadjoint operators on $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Suppose $D \equiv (A_1 \oplus A_2 - i)^{-1} - (C - i)^{-1}$ is rank one with $\varphi \in (\ker D)^{\perp}$ so $P\varphi$ is cyclic for A_1 and $(1 - P)\varphi$ for A_2. Then with

$$U_j = (A_j + i)(A_j - i)^{-1} \qquad W = (C + i)(C - i)^{-1}$$

Theorem 4.1 applies, so

Theorem 4.2. *C has simple singular spectrum.*

5. Extended CMV Matrices and Schrödinger Operators

Extended CMV matrices enter in the theory of the orthogonal polynomials on the unit circle [6]. They are defined by a family of Verblunsky coefficients $\{\alpha_j\}_{j=-\infty}^{\infty}$ with $\alpha_j \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ as follows. Let $\Theta(\alpha)$ be the 2×2 matrix

$$\Theta(\alpha) = \begin{pmatrix} \bar{\alpha} & \rho \\ \rho & -\alpha \end{pmatrix} \tag{5.1}$$

where $\rho = (1-|\alpha|^2)^{1/2}$.

Think of $\ell^2(\mathbb{Z})$, first as a direct sum $\oplus_{n=-\infty}^{n} \mathbb{C}^2$ with the n-th factor spanned by $(\delta_{2n}, \delta_{2n+1})$ and let $\mathcal{M} = \oplus\Theta(\alpha_{2n})$, then as a direct sum with the n-th factor spanned by $(\delta_{2n+1}, \delta_{2n+2})$ and $\mathcal{L} = \oplus\Theta(\alpha_{2n+1})$. Then $\mathcal{E} = \mathcal{L}\mathcal{M}$ is the extended CMV matrix.

We claim

Theorem 5.1. *$\mathcal{E}$ always has simple singular spectrum.*

Proof. Let

$$x = \frac{1+\bar{\alpha}_{-1}}{1+\alpha_{-1}}$$

Then

$$\det\left(\Theta(\alpha_{-1}) - \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

by a simple calculation. It is thus rank one, so if $\tilde{\mathcal{E}}$ is defined by replacing $\Theta(\alpha_{-1})$ by $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\tilde{\mathcal{E}} - \mathcal{E}$ is rank one. On the other hand, $\tilde{\mathcal{E}}$ is a direct sum of two half-line CMV matrices and it is easy to see $P\varphi$ and $(1-P)\varphi$ are cyclic. Thus Theorem 4.1 applies. $\square$

Finally, we turn to the Schrödinger operator case that motivated us in the first place. Suppose $H = -\frac{d^2}{dx^2} + V$ where $V \in L^1_{\text{loc}}(-\infty, \infty)$ is limit point at both $+\infty$ and $-\infty$. Let H_1 (resp. H_2) be H on $L^2(0,\infty)$ (resp. $L^2(-\infty, 0)$) with $u(0) = 0$ boundary conditions. Then

$$(H-i) - (H_1 \oplus H_2 - i)^{-1}$$

is rank one by the explicit Green's function formulae [2], and its kernel is spanned by a function φ with

$$\left(-\frac{d^2}{dx^2} + V\right)\varphi = i\varphi \qquad x \neq 0 \tag{5.2}$$

with φ L^2 at $+\infty$ and $-\infty$ and $\varphi(0_+) = \varphi(0_-)$. (5.2) implies $\varphi \restriction [0,\infty)$ (resp. $\varphi \restriction (-\infty, 0]$) is cyclic for H_1 (resp. H_2). Theorem 4.2 applies and yields the Kac-Gilbert theorem:

Theorem 5.2. $-\frac{d^2}{dx^2} + V$ *has simple singular spectrum.*

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