Existence and localization of solutions of the one dimensional Ginzburg-Landau system

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Abstract. Following [12, 13], we construct subsolutions and supersolutions for the one dimensional Ginzburg-Landau system on the bounded interval $[0, d]$ in the weak-κ limit and for κd large. These constructions lead to a localization of positive solutions of the Ginzburg-Landau system satisfying $f(0) > \frac{1}{\sqrt{2}}$. Then, using a minmax method [26] and following [2] and [23], we deduce that there exist solutions which are not local minimizers of the relevant functional.

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1 Introduction

The states of a superconducting material in an exterior magnetic field are described by the Ginzburg-Landau theory which introduces a functional $\Delta G_h(f, A)$ depending in particular on a complex wave function and on the magnetic potential A. These states are characterized as global or local minima of $\Delta G_h(f, A)$. When the sample is a film and the exterior magnetic field is parallel to the surface, the Ginzburg-Landau model reduces to a one-dimensional problem where the wave function is real (and denoted by f) and where the functional is the following:

$$\begin{aligned}&\varepsilon_d(f, A; h)\\ &= \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[\frac{1}{2}(1 - f(x)^2)^2 - \frac{1}{2} + \kappa^{-2} f'(x)^2 + f(x)^2 A(x)^2 + (A'(x) - h)^2\right] dx,\end{aligned}$$

with $(f, A) \in \left(H^1([-\frac{d}{2}, \frac{d}{2}])\right)^2$. Here, d is proportional to the thickness of the film, h is proportional to the exterior magnetic field and κ is the Ginzburg-Landau parameter characterizing the properties of the material.

We restrict us to the research of *symmetric solutions*. By symmetric solutions, we mean solutions (f, A) such that f is even and A is odd. Hence, we reduce the study of ε_d to the interval $]-d, 0[$ and then to $]0, d[$ by a translation (the edge of the film is then at 0). A reduced GL functional is then defined by

$$\varepsilon_d(f, A; h) = \int_0^d [\frac{1}{2}f^4 - f^2 + \kappa^{-2}(f')^2 + f^2A^2 + (A' - h)^2]dx, \tag{1.1}$$

for the pairs (f, A) of $H^1(]0, d[)^2$ such that $A(d) = 0$.

The corresponding Ginzburg-Landau equations expressing the necessary conditions for minima are then

$$(GL)^s_d \quad \begin{cases} (a) \ -\kappa^{-2} f'' - f + f^3 + f\ A^2 & = 0 \ \text{ on } \ ]0, d[, \\ (b) \ -A'' + A f^2 & = 0 \ \text{ on } \ ]0, d[, \\ (c) \ f'(0) = 0, \ \ f'(d) = 0, \ \ A'(0) = h, \ \ A(d) = 0. \end{cases} \tag{1.2}$$

When the width d of the film is large (in the sense that κd is large), a slightly different modelization is considered, which was first introduced by Ginzburg and which is usually called the superconducting half-space. To get formally the limiting problem on the interval $[0, +\infty[$, we put $d = +\infty$ in the definition of the GL functional (1.1) after a renormalization obtained by adding the term $(h^2 - \frac{1}{2})d$. We then get

$$\varepsilon_\infty(f, A; h) = \int_0^{+\infty} \left[\frac{1}{2}(1 - f(x)^2)^2 + \kappa^{-2}\, f'(x)^2 + f(x)^2\, A(x)^2 + A'(x)^2 \right] dx \ + 2\, h\, A(0) \ ,$$

defined for (f, A) such that $(1 - f, A) \in (H^1(]0, +\infty[))^2$. The corresponding Ginzburg-Landau equations expressing the necessary conditions for minima are then

$$(GL)_\infty \quad \begin{cases} -\kappa^{-2} f'' - f + f^3 + f\ A^2 & = 0, \\ -A'' + A f^2 & = 0, \\ f'(0) = 0, \quad A'(0) = h, \end{cases} \tag{1.3}$$

with $(1 - f, A) \in (H^2(]0, +\infty[))^2$. The problem $(GL)_\infty$ is called the half-space model and was studied in [21] and [25] where computational solutions are given.
We introduce the superheating field of the problem $(GL)_\infty$ defined as (see [4])

$$h^{sh}(\kappa, +\infty) = \sup\{h > 0;\ (f, A; h) \text{ is a solution of } (GL)_\infty\}.$$

Every solution $(f, A; h)$ of $(GL)_\infty$ has the property that f is strictly positive (see [10]). But, it is no more the case when the interval is bounded(see [7], for solutions with no constant sign, in the limiting case $h = 0$). In the case of a bounded interval, the superheating field is defined in restriction to solutions $(f, A; h)$ with $f > 0$ as

$$h^{sh,+}(\kappa, d) = \sup\{h > 0;\ (f, A; h) \text{ is a solution of } (GL)^s_d\ \}.$$

In [19], in the weak-κ limit, De Gennes obtained heuristically for the half-space model an approximation of the set of the pairs (f_0, h) for which there exists a solution of (1.3) such that $f(0) = f_0$ with $f_0 \in]0, 1[$ (see also [15] and [20]). It is given by

$$\kappa h^2 \sim \sqrt{2} f_0^2 (1 - f_0^2). \tag{1.4}$$

On the other hand, let us recall that C. Bolley and B. Helffer [13] have proved that for $\rho \in]0, 1[$, there exist L_ρ, κ_ρ, C_ρ and C'_ρ such that for $\kappa \le \kappa_\rho$ and $\kappa d \ge L_\rho$, and for any positive solution $(f, A; h)$ of $(GL)^s_d$ such that $f(0) \ge \rho$, then

$$|\kappa A'(0)^2 - \sqrt{2} f(0)^2 (1 - f(0)^2)| \le C_\rho \kappa^{\frac{1}{2}} + C'_\rho \exp(-\frac{f(0)}{2} \kappa d).$$

In [13] (Lemma A.1. p. 182), in the case κ small and κd large, these authors have constructed solutions of $(GL)^s_d$ at the superheating field (which corresponds approximately to $f(0) = \frac{1}{\sqrt{2}}$ in (1.4)). Here, constructing subsolutions and supersolutions of $(GL)^s_d$, we deduce the existence of solutions of (1.2) for pairs (h, f_0) satisfying (1.4) with $f_0 > \frac{1}{\sqrt{2}}$. We are guided in the computations by the structure of the asymptotic matched solution of $(GL)_\infty$ constructed in [15] and [20]. We prove the following theorem:

Theorem 1.1 *Let $(\delta_0,\delta_1)\in]\frac{1}{\sqrt{2}},1[^2$ such that $\delta_0<\delta_1$, $\epsilon\in]0,\frac{1}{4}[$ and $\alpha_0\in]0,\frac{\sqrt{2}}{2}[$. There exist $\kappa_0>0$ and d_0 such that, for (κ,d) satisfying $\kappa\le\kappa_0$ and $\kappa d\ge d_0$, for all $f_0\in[\delta_0,\delta_1]$ and for $h>0$ satisfying*

$$\kappa h^2=2^{\frac{1}{2}}f_0^2(1-f_0^2), \tag{1.5}$$

there exists (f,A) solution of $(GL)^s_d$ satisfying the estimate for some $C>0$

$$|f(0)-f_0|\le C\left(\kappa^{1-\epsilon}+\exp(-\alpha_0\kappa d)\right). \tag{1.6}$$

Let us remark that Estimate (1.5) suggests that for h given, there will be two solutions of $(GL)^s_d$ corresponding values of f_0.
Then, constructing accurate supersolutions (in [13], C. Bolley and B. Helffer use the function $f\equiv 1$), and using a maximum principle, we get approximate solutions of $(GL)^s_d$ in terms of known functions. From Theorem 1.1, we deduce easily the following theorem:

Theorem 1.2 *Let $(\delta_0,\delta_1)\in]\frac{1}{\sqrt{2}},1[^2$such that $\delta_0<\delta_1$ and $\epsilon\in]0,\frac{1}{4}[$. There exist $\tilde{C}$ and $\kappa_0>0$ such that, for $\kappa\le\kappa_0$, for $\kappa d\ge\kappa^{-\epsilon}$, for all $f_0\in[\delta_0,\delta_1]$, for all $h>0$ satisfying*

$$\kappa h^2=2^{\frac{1}{2}}f_0^2(1-f_0^2), \tag{1.7}$$

the solutions (f,A) of $(GL)^s_d$, defined in Theorem 1.1 satisfy the estimates

$$|f-\bar{f}_0|\le\tilde{C}\kappa^{1-2\epsilon}\exp(-\frac{1}{\sqrt{2}}\kappa x),\ \ \forall x\in[0,d], \tag{1.8}$$

$$|A-\bar{A}_0|\le\tilde{C}\kappa^{\frac{1}{2}-2\epsilon}\exp(-\frac{\delta_0}{2}x),\ \ \forall x\in[0,d], \tag{1.9}$$

where $\bar{f}_0(x):=\tanh(\frac{\kappa x}{\sqrt{2}}+\bar{x}_0)$, $\tanh(\bar{x}_0)=f_0$ and $\bar{A}_0$ is the unique solution in $H^2([0,d])$ of

$$\begin{cases}-\bar{A}_0''+\bar{f}_0^2\bar{A}_0=0,\\ \bar{A}_0'(0)=h,\quad \bar{A}_0(d)=0.\end{cases} \tag{1.10}$$

In [16] (Theorem 6.2), we have obtained similar estimates as the one exposed in (1.8) and (1.9) in the case of the half-space model, but at the superheating field (which corresponds approximately to $f(0)=\frac{1}{\sqrt{2}}$ when κ tends to 0). Theorem 1.2 lead to a certain boundary layer for the solution, which is seen by plotting this approximation to the solution.
In the last part, we study the existence of non-minimizing solutions of $(GL)^s_d$. Let us recall that C. Bolley and B. Helffer have proved in [5] the unstability of bifurcating solutions when κd is large (see also [24]). We developp here an other approach to prove their existence. In [23] (Theorem 1 and 2), S. Hastings, M.K. Kwong and W.C. Troy have studied the existence of symmetric solutions. They have shown that for an appropriate range of the parameter values, the relevant boundary value has at least two positive symmetric solutions. With the help of these results, we deduce the existence of solutions of $(GL)^s_d$ which not minimize the Ginzburg-Landau functional. We establish the following theorem:

Theorem 1.3 *For all $\kappa>0$, there exists d_0 such that, for all d satisfying $\kappa d\ge d_0$, there are values of h for which the problem $(GL)^s_d$ admits a non-minimizing solution with $A'(0)=h$.*

To prove Theorem 1.3, we follow S. Alama, L. Bronsard and T. Giorgi in [2]. We consider the restriction of the functional ε_d to a closed convex in the form of

$$C_\alpha:=\{(f,A)\in H^1(]0,d[)\times\tilde{H}^1(]0,d[),\ \text{ such that }\ 0<\alpha\le f\le 1\}, \tag{1.11}$$

where $\tilde{H}^1([0,d]):=\{A\in H^1([0,d]),\ \ A(d)=0\}$. Then, we prove that ε_d satisfies the Palais-Smale condition on C_α. In conclusion, we apply the following minmax theorem (see [26], p. 166) with $V=\{(f,A)\in H^1(]0,d[)\times\tilde{H}^1(]0,d[)\}$, $C=C_\alpha$ and $\varepsilon=\varepsilon_d$:

Theorem 1.4 *Let C a closed convex subset of a Hilbert space V, $\varepsilon \in C^1(V)$ satisfying $(PS)_C$ on C, and let us assume that ε admits two distinct relative minimas u_1 and u_2 in C. We set*

$$\beta := \inf_{p \in P} \sup_{t \in [0,1]} \varepsilon(p(t)),$$

where

$$P := \{p \in C^0([0,1], C) : \; p(0) = u_1, \; p(1) = u_2\}.$$

Then, either $\varepsilon(u_1) = \varepsilon(u_2) = \beta$ and u_1 and u_2 can be connected in any neighbourhood of the set of the relative minima $u \in C$ of ε in C with $\varepsilon(u) = \beta$,
or
there exists a critical point $\bar{u}$ of ε in C, which is not a relative minimizer of ε.

On the other hand, from an analogous method to the previous one and from Theorems 1.1 and 1.4, we deduce the existence of non-minimizing solutions of $(GL)^s_d$ for h near to the superheating field. More precisely, we establish the following theorem:

Theorem 1.5 *Let $(\delta_0, \delta_1) \in]\frac{1}{\sqrt{2}}, 1[^2$ such that $\delta_0 < \delta_1$. There exist $\kappa_0 > 0$ and d_0 such that, for (κ, d) satisfying $\kappa \leq \kappa_0$ and $\kappa d \geq d_0$, and for $\kappa h^2 \in [2^{\frac{1}{2}}\delta_1^2(1-\delta_1^2), 2^{\frac{1}{2}}\delta_0^2(1-\delta_0^2)]$, there exist solutions (f, A) of $(GL)^s_d$ which are not local minimizers of the functional ε_d.*

The plan of the paper is the following. In Section 2, we recall some properties of the solutions of $(GL)^s_d$. In Section 3, we construct subsolutions and supersolutions of $(GL)^s_d$. In Section 4, we get solutions of the Ginzburg-Landau system. In Section 5, we give a localization of these solutions. In Section 6, we prove that ε_d satisfies the Palais-Smale condition and we obtain non-minimizing solutions of $(GL)^s_d$.

2 General properties for the solutions of $(GL)^s_d$

We first recall some properties of the solutions of $(GL)^s_d$ (see [6] and [12] for proofs).

Proposition 2.1 *Let $(\kappa, d) \in (]0, +\infty[)^2$. For all solutions (f, A, h) of $(GL)^s_d$, we have*

$$|f| \leq 1, \quad \forall x \in [0, d]. \tag{2.1}$$

Moreover, if we suppose that $f > 0$ on $[0, d]$,

$$\text{then } f \text{ is increasing on } [0, d], \tag{2.2}$$

and the function A satisfies the property :

$$h\frac{\sinh(d-x)}{\cosh d} \leq -A(x) \leq \frac{h}{f(0)}\frac{\sinh(f(0)(d-x))}{\cosh(f(0)d)}, \quad \forall x \in [0, d]. \tag{2.3}$$

In the next sections, we use one useful version of the maximum principle (cf. [12]).

Lemma 2.2 *Let $d \in \mathbb{R}^+$. Let C be a bounded function on $]0, d[$ such that*

$$C(x) \geq 0, \quad \forall \, x \in]0, d[,$$

and let $u \in C^2([0, d])$ be a function such that

$$\begin{cases} -u'' + Cu \leq 0 \;\; \text{on} \;\;]0, d[, \\ u'(0) \geq 0, \quad u(d) \leq 0. \end{cases} \tag{2.4}$$

Then,

$$u \leq 0 \;\; \text{on} \;\;]0, d[.$$

3 Construction of subsolutions and supersolutions of $(GL)^s_d$

3.1 Construction of subsolutions

Let us recall the definition of a subsolution of $(GL)^s_d$.

Definition 3.1 *A triplet $(f, A; h)$ is a subsolution of $(GL)^s_d$ if it satisfies the conditions*

$$\begin{cases} -\kappa^{-2} f'' + f(-1 + f^2 + A^2) \leq 0, \quad \forall\, x \in [0, d], \\ f'(0) = 0, \; f'(d) = 0, \end{cases} \tag{3.1}$$

where A is the unique solution in $H^2([0, d])$ of

$$\begin{cases} -A'' + f^2 A = 0, \\ A'(0) = h, \; A(d) = 0. \end{cases} \tag{3.2}$$

We say more briefly that f is a subsolution of $(GL)^s_d$, if the triplet $(f, A; h)$ with A solution of (3.2) is a subsolution.
We say that $(f, A; h)$ is a supersolution of $(GL)^s_d$ if $(f, A; h)$ satisfies the opposite inequality in (3.1).

In [13] (Lemma A.1), to construct a subsolution of $(GL)^s_d$ for the critical value $f_0 = \frac{1}{\sqrt{2}}$, C. Bolley and B. Helffer use the function

$$\begin{aligned} &f(x) = \tanh\left(\tfrac{\kappa}{\sqrt{2}}(1 + C_1\kappa^{1-\epsilon} + C_1' g(\kappa d))x + x_{\kappa,h}\right) \\ &+ \frac{\kappa^2 h^2}{\sqrt{2}} \exp(-\sqrt{2}x) + \lambda_{d,\kappa,h} \exp(-\sqrt{2}\kappa(d - x))E(\frac{x}{d}), \end{aligned} \tag{3.3}$$

where E is an increasing C^∞ cutoff function and C_1 and C_1' are parameters. In [15], we have constructed an asymptotic matched solution $(f^{vd,(n)}, A^{vd,(n)})$ of $(GL)_\infty$. Let us recall that

$$f^{vd,(1)}(x) = \tanh(\frac{\kappa x}{\sqrt{2}} + x_0) + \kappa \frac{\sqrt{2}(1 - A_0^2)}{4A_0} \exp(-A_0 x),$$

and

$$\kappa^{\frac{1}{2}} A^{vd,(0)}(x) = -2^{\frac{1}{4}}(1 - A_0^2)^{\frac{1}{2}} \exp(-A_0 x),$$

where $A_0 = \tanh(x_0)$ with $A_0 \in]0, 1[$. This suggests to replace the term $\frac{\kappa^2 h^2}{\sqrt{2}} \exp(-\sqrt{2}x)$ which appears in (3.3) with the term $\frac{\kappa^2 h^2}{4 f_0^3} \exp(-2 f_0 x)$. Then, we can show the following proposition:

Proposition 3.2 *Let $\epsilon \in]0, 1[$, $\alpha_0 \in]0, \frac{\sqrt{2}}{2}[$ and $(\delta_0, \delta_1) \in]0, 1[^2$. Let E an increasing C^∞ cutoff function such that*

$$E(y) = \begin{cases} 0 & \text{if } y \leq \frac{1}{2} \\ 1 & \text{if } y \geq \frac{3}{4}. \end{cases} \tag{3.4}$$

There exist $\kappa_0 > 0$, C_0, C_0' and d_0 such that, for (κ, d) satisfying $\kappa \leq \kappa_0$ and $\kappa d \geq d_0$, for all $f_0 \in [\delta_0, \delta_1]$ and for h such that

$$\kappa h^2 = 2^{\frac{1}{2}} f_0^2 (1 - f_0^2) + C_0 \kappa^{1-\epsilon} + C_0' g(\kappa d), \tag{3.5}$$

where

$$g(y) = \exp(-\alpha_0 y),$$

then, the problem $(GL)^s_d$ admits a positive subsolution. This subsolution is given, for suitable constants C_1 C_1', by

$$\begin{array}{l} f(x) = \tanh\left(\frac{\kappa}{\sqrt{2}}(1 + C_1\kappa^{1-\epsilon} + C_1' g(\kappa d))x + x_{\kappa,h}\right) \\ +\dfrac{\kappa^2 h^2}{4 f_0^3}\exp(-2f_0 x) + \lambda_{d,\kappa,h}\exp(-\sqrt{2}\kappa(d-x))E(\dfrac{x}{d}), \end{array} \tag{3.6}$$

where the constants $x_{\kappa,h}$ and $\lambda_{d,\kappa,h}$ are determined by the conditions :

$$f'(0) = 0, \quad f'(d) = 0. \tag{3.7}$$

Proof of Proposition 3.2 Let us introduce,

$$\nu := 1 - \epsilon, \tag{3.8}$$

$$\tilde{f}_1(x) := \tanh(\frac{\kappa}{\sqrt{2}}(1 + C_1\kappa^\nu + C_1' g(\kappa d))x + x_{\kappa,h}), \tag{3.9}$$

$$\tilde{f}_2 := f - \tilde{f}_1, \tag{3.10}$$

$$z_{\kappa,h} := \exp(-2x_{\kappa,h}), \tag{3.11}$$

$$f_1(x) := \tanh(\frac{\kappa}{\sqrt{2}}(1 + C_1\kappa^\nu + C_1' g(\kappa d))x + x_{\kappa,h}) + \kappa^2\frac{h^2}{4f_0^3}\exp(-2f_0 x), \tag{3.12}$$

and

$$f_2(x) := \lambda_{d,\kappa,h}\exp(-\sqrt{2}\kappa(d-x))E(\frac{x}{d}). \tag{3.13}$$

Step 1: Estimates for $x_{\kappa,h}$, $z_{\kappa,h}$ and $\lambda_{d,\kappa,h}$ The constant $x_{\kappa,h}$ is such that $f'(0) = 0$, so it satisfies

$$(1 - \tanh^2(x_{\kappa,h}))(1 + C_1\kappa^\nu + C_1' g(\kappa d)) - \frac{\sqrt{2}\kappa h^2}{2f_0^2} = 0.$$

As h satisfies (3.5), we get

$$(1 - \tanh^2(x_{\kappa,h}))(1 + C_1\kappa^\nu + C_1' g(\kappa d)) = 1 - f_0^2 + \frac{C_0\kappa^\nu}{\sqrt{2}f_0^2} + \frac{C_0' g(\kappa d)}{\sqrt{2}f_0^2},$$

then

$$\begin{array}{l} \tanh^2(x_{\kappa,h}) \\ = f_0^2 + \left((1-f_0^2)C_1 - \frac{C_0}{2\sqrt{2}f_0^2}\right)\kappa^\nu + \left((1-f_0^2)C_1' - \frac{C_0'}{2\sqrt{2}f_0^2}\right)g(\kappa d) + \mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2). \end{array}$$

Thus, we get

$$\begin{array}{l} \tanh(x_{\kappa,h}) \\ = f_0 + \left(\frac{1-f_0^2}{2f_0}C_1 - \frac{C_0}{2\sqrt{2}f_0^3}\right)\kappa^\nu + \left(\frac{1-f_0^2}{2f_0}C_1' - \frac{C_0'}{2\sqrt{2}f_0^3}\right)g(\kappa d) + \mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2). \end{array} \tag{3.14}$$

Therefore, for κ and $g(\kappa d)$ small, we get

$$z_{\kappa,h} = \frac{1 - \tanh(x_{\kappa,h})}{1 + \tanh(x_{\kappa,h})} = \frac{1-f_0}{1+f_0} + \mathcal{O}(\kappa^\nu) + \mathcal{O}(g(\kappa d)). \tag{3.15}$$

Let us remark that, from (3.15), there exists κ_0 such that, for $\kappa \leq \kappa_0$, we have, for all $f_0 \in [\delta_0, \delta_1]$,

$$\frac{1-\delta_1}{1+\delta_1} + \mathcal{O}(\kappa^\nu) + \mathcal{O}(g(\kappa d)) \leq z_{\kappa,h} \leq 1. \tag{3.16}$$

The constant $\lambda_{d,\kappa,h}$ is such that $f'(d)=0$, so that $\lambda_{d,\kappa,h} = -\frac{f_1'(d)}{\sqrt{2}\kappa}$. Therefore, we get

$$\begin{aligned}\lambda_{d,\kappa,h} &= \frac{\kappa h^2}{2\sqrt{2}f_0^2}\exp(-2f_0 d)\\ &-2(1+C_1\kappa^\nu + C_1'g(\kappa d))\frac{z_{\kappa,h}\exp(-\sqrt{2}\kappa d(1+C_1\kappa^\nu + C_1'g(\kappa d)))}{(1+z_{\kappa,h}\exp(-\sqrt{2}\kappa d(1+C_1\kappa^\nu + C_1'g(\kappa d)))^2}.\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we deduce the existence of κ_0 and d_0 such that, for all $f_0 \in [\delta_0, \delta_1]$, for $\kappa \le \kappa_0$ and $\kappa d \ge d_0$, we get

$$\begin{aligned}\lambda_{d,\kappa,h} &= -2(1+C_1\kappa^\nu + C_1'g(\kappa d))z_{\kappa,h}.\exp(-\sqrt{2}(1+C_1\kappa^\nu + C_1'g(\kappa d))\kappa d)\\ &\times\left(1+\mathcal{O}(\exp(-\sqrt{2}\kappa d(1+C_1\kappa^\nu)))\right).\end{aligned} \tag{3.18}$$

It results that, for $\kappa \le \kappa_0$ and $\kappa d \ge d_0$

$$\lambda_{d,\kappa,h} = \mathcal{O}\left(\exp(-\sqrt{2}(1-\mathcal{O}(\kappa^\nu))\kappa d)\right), \quad \text{with } \lambda_{d,\kappa,h} < 0. \tag{3.19}$$

Step 2: Estimates for f' and f From (3.6), we get :

$$\begin{aligned}f'(x) &= \frac{\kappa(1+C_1\kappa^\nu + C_1'g(\kappa d))}{\sqrt{2}(\cosh(2^{-1/2}\kappa(+C_1\kappa^\nu + C_1'g(\kappa d))x + x_{\kappa,h}))^2}\\ &-\frac{\kappa^2 h^2}{2f_0^2}\exp(-2f_0 x)\\ &+\kappa\lambda_{d,\kappa,h}\exp(-\sqrt{2}\kappa(d-x))\left(\sqrt{2}E(\frac{x}{d}) + \frac{1}{\kappa d}E'(\frac{x}{d})\right).\end{aligned}$$

We assume that

$$C_1 < 0, \quad C_1' < 0. \tag{3.20}$$

From (3.20), and as $\lambda_{d,\kappa,h} < 0$ (see (3.19)), $E \ge 0$ and $E' \ge 0$, it results that

$$f'(x) \le \frac{\kappa(1+C_1\kappa^\nu + C_1'g(\kappa d))}{\sqrt{2}(\cosh(2^{-1/2}\kappa(+C_1\kappa^\nu + C_1'g(\kappa d))x + x_{\kappa,h}))^2}. \tag{3.21}$$

According to (3.14) and (3.15), for all $f_0 \in [\delta_0, \delta_1]$, for all $x \in [0,d]$, we get, for some positive C

$$-C\kappa \le f'(x) \le 2^{-\frac{1}{2}}\kappa. \tag{3.22}$$

Upper bound and lower bound for f From Lemma 8.5 p. 75 in [8], the function f_1 is increasing. As $\lambda_{d,\kappa,h} < 0$, we have $\forall\, x \in [0,d],\;\; f(x) \le f_1(x) \le f_1(d)$. Moreover, using

$$\tanh(y) = 1 - 2\exp(-2y) + \mathcal{O}(\exp(-4y)), \tag{3.23}$$

when $y \to +\infty$, we get, for κd large and κ small and for $x \in [0,d]$:

$$f(x) \le 1 - 2z_{\kappa,h}\exp(-\sqrt{2}\kappa d(1+C_1\kappa^\nu + C_1'g(\kappa d)))(1+\mathcal{O}(\exp(-\sqrt{2}\kappa d(1+C_1\kappa^\nu)))). \tag{3.24}$$

For $x \in [0, \dfrac{d}{2}]$, as $f_2(x) = 0$ and f_1 is increasing, we have

$$f(x) \ge f_1(0). \tag{3.25}$$

From (3.5) and (3.14), we get

$$\begin{aligned} f_1(0) &= \tanh(x_{\kappa,h}) + \kappa^2 \tfrac{h^2}{4f_0^3} \\ &= f_0 + \left(\tfrac{1-f_0^2}{2f_0} C_1 - \tfrac{C_0}{2\sqrt{2}f_0^3}\right)\kappa^\nu + \left(\tfrac{1-f_0^2}{2f_0} C_1' - \tfrac{C_0'}{2\sqrt{2}f_0^3}\right) g(\kappa d) \\ &+\mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2) + \mathcal{O}(\kappa). \end{aligned} \tag{3.26}$$

On the interval $[\frac{d}{2}, d]$, we remark, using the fact that f_1 is increasing and f_2 is decreasing that

$$\begin{aligned} f(x) &\geq f_1(d/2) + f_2(d) \\ &= \lambda_{d,\kappa,h} + \tanh(\tfrac{\kappa d}{2\sqrt{2}}(1 + C_1\kappa^\nu + C_1' g(\kappa d)) + x_{\kappa,h}) + \kappa^2 \frac{h^2}{4f_0^3}\exp(-f_0 d). \end{aligned} \tag{3.27}$$

Using (3.23), we get the estimate

$$\begin{aligned} &\tanh(\tfrac{\kappa d}{2\sqrt{2}}(1 + C_1\kappa^\nu + C_1' g(\kappa d)) + x_{\kappa,h}) \\ &= 1 - 2z_{\kappa,h}\exp(-2^{-1/2}\kappa d(1 + C_1\kappa^\nu + C_1' g(\kappa d))) + \mathcal{O}(\exp(-2^{1/2}\kappa d(1 + C_1\kappa^\nu + C_1' g(\kappa d))). \end{aligned}$$

According to (3.18) and (3.27), we deduce the estimate, for κd large and for κ small, for $x \in [\frac{d}{2}, d]$

$$f(x) \geq 1 - 2z_{\kappa,h}\exp\left(-2^{-1/2}\kappa d(1 + C_1\kappa^\nu + C_1' g(\kappa d))\right).(1 + \mathcal{O}(\exp(-2\kappa d(1 + C_1\kappa^\nu)))). \tag{3.28}$$

From (3.5), (3.25), (3.26) and (3.28), it results that there exists $\kappa_0 > 0$ such that, for all $\kappa \leq \kappa_0$, for κd large, for all $f_0 \in [\delta_0, \delta_1]$ and for $x \in [0, d]$, we get

$$\begin{aligned} f(x) &\geq \tanh(x_{\kappa,h}) + \kappa^2 \tfrac{h^2}{4f_0^3} \\ &= f_0 + \left(\tfrac{1-f_0^2}{2f_0} C_1 - \tfrac{C_0}{2\sqrt{2}f_0^3}\right)\kappa^\nu + \left(\tfrac{1-f_0^2}{2f_0} C_1' - \tfrac{C_0'}{2\sqrt{2}f_0^3}\right) g(\kappa d) \\ &+\mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2) + \mathcal{O}(\kappa). \end{aligned} \tag{3.29}$$

Step 3: Upper bound for $A^2 f$ We shall need estimates for $A^2 f$. For this purpose, as in [13, 14], we split the interval $[0, d]$ into two intervals $[0, \kappa^{-\epsilon}]$ and $[\kappa^{-\epsilon}, d]$. Let us recall that (see [13] Lemma A.1. p. 185)

$$A^2(x)f(x) \leq \frac{h^2}{f(0)^2}\exp(-2f(0)x)f(x), \quad \forall\, x \in [0, d]. \tag{3.30}$$

For (κ, d) such that $\kappa d \geq 2$ and $d \geq 2$, we have $(\kappa d)^\epsilon d^{1-\epsilon} \geq 2$, so that $k^{-\epsilon} \leq \frac{d}{2}$. Moreover, from (3.22), we get $0 \leq f(x) \leq f(0) + 2^{-1/2}\kappa, \quad \forall x \in [0, \kappa^{-\epsilon}]$. Using (3.14), then (3.30), we get for $x \in [0, \kappa^{-\epsilon}]$,

$$\begin{aligned} &A^2 f \leq \frac{h^2}{f_0}\exp(-2f(0)x) \\ &\times(1 + \left(-\tfrac{1-f_0^2}{2f_0^2} C_1 + \tfrac{C_0}{2\sqrt{2}f_0^4} + \tfrac{1}{\sqrt{2}f_0}\right)\kappa^\nu + \left(-\tfrac{1-f_0^2}{2f_0^2} C_1' + \tfrac{C_0'}{2\sqrt{2}f_0^4}\right) g(\kappa d) \\ &+\mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2) + \mathcal{O}(\kappa)). \end{aligned} \tag{3.31}$$

For $x \in [\kappa^{-\epsilon}, d]$, according to (3.14), (3.24) and (3.30), we get, for κ and $g(\kappa d)$ small and for κd large enough the other upper bound

$$A^2(x)f(x) \leq \frac{h^2}{f_0^2}\exp(-2f(0)x)\left(1 + \mathcal{O}(\kappa^\nu) + \mathcal{O}(g(\kappa d)) + \mathcal{O}(\exp(-\sqrt{2}\kappa d(1 + C_1\kappa^\nu)))\right). \tag{3.32}$$

Step 4: The function f is a subsolution Using the splitting of f, $f = f_1 + f_2$ (see (3.6), (3.12) and (3.35)), we get

$$
\begin{aligned}
&-\kappa^{-2}f'' + (-1 + f^2 + A^2)f \\
&= (2C_1\kappa^{\nu} + 2C_1'g(\kappa d) + \mathcal{O}(\kappa^{2\nu} + g(\kappa d))^2)\tilde{f}_1(1-\tilde{f}_1^2) \\
&-\tfrac{h^2}{f_0}\exp(-2f_0x) \\
&-\lambda_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x)).\left[2E(\tfrac{x}{d}) + \frac{2\sqrt{2}}{\kappa d}E'(\frac{x}{d}) + \frac{1}{(\kappa d)^2}E''(\frac{x}{d})\right] \\
&+\tilde{f}_2.(-1 + f^2 + f\,\tilde{f}_1 + \tilde{f}_1^2) + A^2 f.
\end{aligned}
\tag{3.33}
$$

We have to prove that the right-hand-side of (3.33) is negative when κ and $g(\kappa d)$ are small, κd is large, $C_1 < 0$ and $C_1' < 0$. We again distinguish the two intervals $[0, \kappa^{-\epsilon}]$ and $[\kappa^{-\epsilon}, d]$.
Step 4.1: Study on the interval $[0, \kappa^{-\epsilon}]$ From (3.4), we remark that

$$E(\frac{x}{d}) = 0 \ \text{ on } \ [0, \kappa^{-\epsilon}]. \tag{3.34}$$

According to (3.6), (3.9) and (3.10), it results that

$$\tilde{f}_2.(-1 + f^2 + f\,\tilde{f}_1 + \tilde{f}_1^2) = \mathcal{O}(\tilde{f}_2) = \mathcal{O}(\kappa^2 h^2).\exp(-2f_0x). \tag{3.35}$$

Moreover, from(3.34), we get

$$-\lambda_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x)).[2E(\frac{x}{d}) + \frac{2\sqrt{2}}{\kappa d}E'(\frac{x}{d}) + \frac{1}{(\kappa d)^2}E''(\frac{x}{d})] = 0. \tag{3.36}$$

As $0 < \tilde{f}_1 < 1$, $C_1 < 0$ and $C_1' < 0$, the first term in the right-hand-side of (3.33) is negative. It results that the sign of the right-hand-side of (3.33) depends on the sign of $\Lambda := A^2 f - \frac{h^2}{f_0}\exp(-2f_0x)$. Using (3.14), (3.31) and expanding the term $\exp(-2f(0)x)$, we get

$$\Lambda \le \frac{h^2}{f_0}\exp(-2f_0x)(\exp(\Lambda_1 x)(1 + \frac{\Lambda_1}{2f_0} + \frac{1}{\sqrt{2}f_0}\kappa^{\nu}) - 1),$$

where

$$
\begin{aligned}
&\Lambda_1 := \\
&-2\left(\left(\tfrac{1-f_0^2}{2f_0}C_1 - \tfrac{C_0}{2\sqrt{2}f_0^3}\right)\kappa^{\nu} + \left(\tfrac{1-f_0^2}{2f_0}C_1' - \tfrac{C_0'}{2\sqrt{2}f_0^3}\right)g(\kappa d) + \mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(g(\kappa d)^2) + \mathcal{O}(\kappa)\right).
\end{aligned}
\tag{3.37}
$$

We remark that the term $\exp(\Lambda_1 x)$ inside the bracket is lower than 1 for all $f_0 \in [\delta_0, \delta_1]$, $(\delta_0, \delta_1) \in]0, 1[^2$ when κ and $g(\kappa d)$ are small, if we assume

$$\frac{C_0}{2\sqrt{2}\delta_1^3} < \frac{1-\delta_0^2}{2\delta_0}C_1, \quad \frac{C_0'}{2\sqrt{2}\delta_1^3} < \frac{1-\delta_0^2}{2\delta_0}C_1'. \tag{3.38}$$

The sum Λ will then be negative if $\frac{\Lambda_1}{2f_0} + \frac{1}{\sqrt{2}f_0}\kappa^{\nu} < 0$, hence if the two following conditions are satisfied for all $f_0 \in [\delta_0, \delta_1]$,

$$\frac{C_0}{2\sqrt{2}\delta_1^4} < \frac{1-\delta_0^2}{2\delta_0^2}C_1 - \frac{1}{\sqrt{2}\delta_0}, \quad \frac{C_0'}{2\sqrt{2}\delta_1^4} < \frac{1-\delta_0^2}{2\delta_0^2}C_1'. \tag{3.39}$$

We can choose (C_0, C_0', C_1, C_1') satisfying (3.38) and (3.39). We thus get the result in $[0, \kappa^{-\epsilon}]$.
Step 4.2: Study on the interval $[\kappa^{-\epsilon}, d]$ We use that $1 - \tanh^2 y \geq \exp(-2y)$, that $C_1 < 0$ and $C_1' < 0$. For the term $A^2 f$, we now use (3.31) and get

$$\begin{aligned}
&-\kappa^{-2} f'' + (-1 + f^2 + A^2) f \\
&\leq (2C_1\kappa^\nu + 2C_1' g(\kappa d) + \mathcal{O}((\kappa^{2\nu} + g(\kappa d))^2).z_{d,h,\kappa} \exp(-2^{1/2}\kappa(1 + C_1\kappa^\nu + C_1' g(\kappa d))x) \\
&-\tfrac{h^2}{f_0}(1 + \mathcal{O}(\kappa^\nu + g(\kappa d)) \exp(-2f_0 x) \\
&+|\lambda_{d,h,\kappa}| \exp(-\sqrt{2}\kappa(d-x)) E(\frac{x}{d}).(12 z_{\kappa,h} \exp(-\sqrt{2}\kappa d(1 + C_1\kappa^\nu + C_1' g(\kappa d)))) \qquad (3.40) \\
&\times(1 + \mathcal{O}(1 + \exp(-\sqrt{2}\kappa d)))) + |\lambda_{d,h,\kappa}| \exp(-\sqrt{2}\kappa(d-x)). \left[\frac{2\sqrt{2}}{\kappa d} E'(\frac{x}{d}) + \frac{1}{(\kappa d)^2} E''(\frac{x}{d})\right] \\
&+\frac{h^2}{f_0^2}(1 + \mathcal{O}(\kappa^\nu) + \mathcal{O}(g(\kappa d))). \exp(-2f(0)x).
\end{aligned}$$

The first term (with $C_1\kappa^\nu$) will control the last terms for κ and $g(\kappa d)$ small, as soon as $C_1 < 0$ and the second term $C_1' g(\kappa d)$ will control the terms where E, E' and E'' appear. We set

$$\Lambda_2 := 1 + C_1\kappa^\nu + C_1' g(\kappa d). \qquad (3.41)$$

From (3.14) and as $f_0 \in [\delta_0, \delta_1]$, we have indeed, for any $x \in [\kappa^{-\epsilon}, d]$

$$\begin{aligned}
\Lambda' &:= 2C_1\kappa^\nu(1 + \mathcal{O}(\kappa^\nu)) z_{\kappa,h} \exp(-\sqrt{2}\kappa x\Lambda_2) + \tfrac{h^2}{f_0^2}. \exp(-2f(0)x) \\
&\leq 2\kappa^\nu(1 + \mathcal{O}(\kappa^\nu)) \exp(-\sqrt{2}\kappa x\Lambda_2)(C_1 z_{\kappa,h} + 2^{-\frac{3}{2}}\kappa^{-1-\nu}(1 + \mathcal{O}(\kappa^\nu)) \\
&\times \exp(-(\delta_0 - \sqrt{2}\kappa)\kappa^{-\epsilon}).
\end{aligned}$$

Using the control (3.15) of $z_{\kappa,h}$, we deduce that Λ' is negative for all $f_0 \in [\delta_1, \delta_2]$, for $C_1 < 0$ and for κ and $g(\kappa d)$ small enough. Let us now consider

$$\begin{aligned}
\Lambda'' &:= 2C_1' g(\kappa d) z_{\kappa,h} \exp(-\sqrt{2}\kappa\Lambda_2 x) + |\lambda_{d,\kappa,h}| \exp(-\sqrt{2}\kappa(d-x)) E(\frac{x}{d})) 12 z_{\kappa,h} \exp(-\sqrt{2}\kappa d\Lambda_2) \\
&+|\lambda_{d,\kappa,h}| \exp(-\sqrt{2}\kappa(d-x)) \left(\frac{2\sqrt{2}}{\kappa d} E'(\frac{x}{d}) + \frac{1}{(\kappa d)^2} E''(\frac{x}{d})\right).
\end{aligned}$$

From (3.5) and (3.18), we have

$$\begin{aligned}
\Lambda'' &= 2g(\kappa d) z_{\kappa,h} \exp(-\sqrt{2}\kappa\Lambda_2 x)[C_1' z_{\kappa,h} + 12(g(\kappa d))^{-1} z_{\kappa,h}^2 \Lambda_2(\exp(-\sqrt{2}\kappa\Lambda_2 d)) \\
&\times \exp(-\sqrt{2}\kappa\Lambda_2(d-x)). \big(1 + \mathcal{O}(\exp(-\sqrt{2}\kappa d\Lambda_2'))\big) \exp(-\sqrt{2}\kappa(d-x))) E(\frac{x}{d})) \\
&+2(g(\kappa d))^{-1} z_{\kappa,h}\Lambda_2 \exp(-\sqrt{2}\kappa\Lambda_2(d-x))) \exp(-\sqrt{2}\kappa(d-x))) \\
&\big(1 + \mathcal{O}(\exp(-\sqrt{2}\kappa d\Lambda_2'))\big) (\frac{2\sqrt{2}}{\kappa d} E'(\frac{x}{d}) + \frac{1}{(\kappa d)^2} E''(\frac{x}{d}))],
\end{aligned}$$

where $\Lambda_2' = 1 + C_1\kappa^\nu$. Using now that $\operatorname{supp} E \subset [\frac{1}{2}, \frac{3}{4}]$, we get

$$\begin{aligned}
\Lambda'' &\leq 2g(\kappa d) z_{\kappa,h} \exp(-\sqrt{2}\kappa\Lambda_2 x)[C_1' z_{\kappa,h} + 12(g(\kappa d))^{-1}\Lambda_2 z_{\kappa,h}^2(\exp(-\tfrac{\sqrt{2}}{4}\kappa d(1 + 5\Lambda_2)) \\
&. \big(1 + \mathcal{O}(\exp(-\sqrt{2}\kappa d\Lambda_2'))\big) E(\frac{x}{d})) + \Lambda_2.(g(\kappa d))^{-1} z_{\kappa,h} \exp(-\frac{\sqrt{2}}{4}\kappa d(1 + \Lambda_2)) \\
&\big(1 + \mathcal{O}(\exp(-\sqrt{2}\kappa d\Lambda_2'))\big) (\frac{2\sqrt{2}}{\kappa d}\|E'\|_\infty + \frac{1}{(\kappa d)^2}\|E''\|_\infty)].
\end{aligned}$$

If $C_1' < 0$, the term between the last square brackets, and then quantity Λ'' will be negative, for κ small enough and κd large enough if

$$g(y)^{-1} \exp\left(-\frac{\sqrt{2}}{4} y(2 + C_1\kappa^\nu + C_1' g(y))\right)$$

tends to 0 when y tends to $+\infty$. We deduce the result when g satisfies

$$g(y) = \exp(-\alpha_0 y) \text{ with } 0 < \alpha_0 < \frac{\sqrt{2}}{2}.$$

The proof of Proposition 3.2 follows.

3.2 Construction of supersolutions of $(GL)_d^s$

In [13], C. Bolley and B. Helffer remark that $f \equiv 1$ is a supersolution of $(GL)_d^s$. In order to get approximate solutions, we should construct more accurate supersolutions. This is the goal of the following proposition:

Proposition 3.3 *Let $\epsilon \in]0,1[$, $\alpha_0 \in]0, \frac{\sqrt{2}}{2}[$, $(\delta_0, \delta_1) \in]0,1[^2$ and E be the function defined in (3.4). There exist $\kappa_0 > 0$, $\bar{C}_0$, $\bar{C}_0'$ and d_0 such that, for (κ, d) satisfying $\kappa \leq \kappa_0$ and $\kappa d \geq d_0$, for all $f_0 \in [\delta_0, \delta_1]$ and for h such that*

$$\kappa h^2 = 2^{\frac{1}{2}} f_0^2 (1 - f_0^2) + \bar{C}_0 \kappa^{1-\epsilon} + \bar{C}_0' \bar{g}(\kappa d), \tag{3.42}$$

where

$$\bar{g}(y) = \exp(-\alpha_0\, y),$$

then, $(GL)_d^s$ admits a positive supersolution.
This supersolution is given, for suitable constants $\bar{C}_1$ $\bar{C}_1'$, by

$$\begin{aligned} f(x) &= \tanh\left(\tfrac{\kappa}{\sqrt{2}}(1 + \bar{C}_1 \kappa^{1-\epsilon} + \bar{C}_1' \bar{g}(\kappa d))x + \bar{x}_{\kappa,h}\right) \\ &+ \kappa^2 \frac{h^2}{4 f_0^3} \exp(-2 f_0 x) + \bar{\lambda}_{d,\kappa,h} \exp(-\sqrt{2}\kappa(d - x)) E(\frac{x}{d}), \end{aligned} \tag{3.43}$$

where the constants $\bar{x}_{\kappa,h}$ and $\bar{\lambda}_{d,\kappa,h}$ are determined by the conditions:

$$f'(0) = 0, \quad f'(d) = 0. \tag{3.44}$$

Remark 3.4 *Let us remark that $\bar{x}_{\kappa,h}$ and $\bar{\lambda}_{d,\kappa,h}$ are determined by (3.14) and (3.18) replacing C_0, C_1, C_0' and C_1' with $\bar{C}_0$, $\bar{C}_1$, $\bar{C}_0'$ and $\bar{C}_1'$ in these estimates.*

To construct the supersolution, we should get a lower bound for $A^2 f$ in the interval $[0, \kappa^{-\epsilon}]$. In the interval $[\kappa^{-\epsilon}, d]$, we will show that we can neglect this term. Following the method used in [17], we improve Inequality (2.3).

Lemma 3.5 *Let $(\kappa, d) \in (]0, +\infty[)^2$ and $\epsilon \in]0,1[$. Let A be the solution of (3.2) where f is defined in (3.43). There exist κ_0, d_0 and C such that for all $\kappa \leq \kappa_0$, for $\kappa d \geq d_0$, we have*

$$A(x) \leq -\frac{h}{\Lambda(\kappa)}(1 - C\kappa^2) \exp(-\Lambda(\kappa) x), \quad \forall\, x \in [0, \kappa^{-\epsilon}], \tag{3.45}$$

where

$$\Lambda(\kappa) := f(0) + \tfrac{1 - f_0^2}{\sqrt{2}} \kappa^\nu + \mathcal{O}(\kappa^{2\nu}) + \mathcal{O}(\kappa^\nu \bar{g}(\kappa d)). \tag{3.46}$$

Proof From (3.14) and (3.43), as f is increasing on $[0,\kappa^{-\epsilon}]$, we remark that, for all $x\in[0,\kappa^{-\epsilon}]$, we have for κ small and κd large

$$f(x)\le\Lambda(\kappa):=f(\kappa^{-\epsilon})=\tanh(\bar{x}_{\kappa,h})+\frac{1-f_0^2}{\sqrt{2}}\kappa^{\nu}+\mathcal{O}(\kappa^{2\nu})+\mathcal{O}(\kappa^{\nu}\bar{g}(\kappa d)). \tag{3.47}$$

As $f(0)=\tanh(\bar{x}_{\kappa,h})+\mathcal{O}(\kappa)$, we get the estimate

$$\Lambda(\kappa)=f(0)+\tfrac{1-f_0^2}{\sqrt{2}}\kappa^{\nu}+\mathcal{O}(\kappa)+\mathcal{O}(\kappa^{2\nu})+\mathcal{O}(\kappa^{\nu}\bar{g}(\kappa d)). \tag{3.48}$$

To get an upper bound for A, we compare this function with the unique solution in $H^2([0,d])$ of

$$\begin{cases} a.\ -y''+\Lambda(\kappa)^2y=0,\\ b.\ y'(0)=h,\ \ y(\kappa^{-\epsilon})=0.\end{cases} \tag{3.49}$$

Applying Lemma 2.2 with $u=A-y$ and $C(x)=\Lambda(\kappa)^2$, from (3.47), it derives that $A\le y$ on $[0,\kappa^{-\epsilon}]$. The solution of $(3.49)_a$ is given by

$$y(x)=C_1(\kappa)\exp(-\Lambda(\kappa)x)+C_2(\kappa)\exp(\Lambda(\kappa)x), \tag{3.50}$$

and from $(3.49)_b$, it results that

$$C_1(\kappa)=-\frac{h}{\Lambda(\kappa)}+C_2(\kappa),\quad\text{where}\quad C_2(\kappa)=\frac{h\exp(-\Lambda(\kappa)\kappa^{-\epsilon})}{2\Lambda(\kappa)\cosh(\Lambda(\kappa)\kappa^{-\epsilon})}. \tag{3.51}$$

From (3.48) and (3.51), we have for κ small and for all $n\in\mathbb{N}$, $C_2(\kappa)=\mathcal{O}(\kappa^n)$. From (3.50), it results that there exist κ_0 and d_0 such that for $\kappa\le\kappa_0$ and $\kappa d\ge d_0$, we have on the interval $[0,\frac{\kappa^{-\epsilon}}{2}]$ the estimate

$$y(x)=\left(-\frac{h}{\Lambda(\kappa)}+\mathcal{O}(\kappa^n)\right)\exp(-\Lambda(\kappa)x).$$

Finally, for $n=2$ and for some C, we get

$$y\le-\frac{h}{\Lambda(\kappa)}(1-C\kappa^2)\exp(-\Lambda(\kappa)x),\quad\forall\,x\in[0,\frac{\kappa^{-\epsilon}}{2}].$$

If we make κ_0 smaller, we get (3.45) on $[0,\kappa^{-\epsilon}]$. The proof of Lemma 3.5 follows.

Proof of Proposition 3.3

Step 1: Lower bound for A^2f From Lemma 3.5, and as $f(x)\ge f(0)$, on $[0,\kappa^{-\epsilon}]$, we get

$$A^2f\ge f(0)\frac{h^2}{\Lambda(\kappa)^2}(1-C\kappa^2)\exp(-2\Lambda(\kappa)x). \tag{3.52}$$

According to (3.14) and (3.46), we get

$$\begin{aligned}\Lambda(\kappa)&=f_0+\left(\tfrac{1-f_0^2}{2f_0}\bar{C}_1-\tfrac{\bar{C}_0}{2\sqrt{2}f_0^3}+\tfrac{1-f_0^2}{\sqrt{2}}\right)\kappa^{\nu}\\&+\left(\tfrac{1-f_0^2}{2f_0}\bar{C}_1'-\tfrac{\bar{C}_0'}{2\sqrt{2}f_0^3}\right)\bar{g}(\kappa d)+\mathcal{O}(\kappa)+\mathcal{O}(\kappa^{2\nu})+\mathcal{O}(\bar{g}(\kappa d)^2)+\mathcal{O}(\kappa^{\nu}\bar{g}(\kappa d)).\end{aligned} \tag{3.53}$$

From (3.52) and (3.53), we get

$$\begin{aligned}A^2f\ge\frac{h^2}{f_0}\exp(-2\Lambda(\kappa)x)(1&+\left(-\frac{1-f_0^2}{2f_0^2}\bar{C}_1+\frac{\bar{C}_0}{2\sqrt{2}f_0^4}-\frac{\sqrt{2}(1-f_0^2)}{f_0}\right)\kappa^{\nu}\\+\left(-\tfrac{1-f_0^2}{2f_0^2}\bar{C}_1'+\frac{\bar{C}_0'}{2\sqrt{2}f_0^4}\right)&\bar{g}(\kappa d)+\mathcal{O}(\kappa)+\mathcal{O}(\kappa^{2\nu})+\mathcal{O}(\kappa^{\nu}\bar{g}(\kappa d))+\mathcal{O}(\bar{g}(\kappa d)^2).\end{aligned} \tag{3.54}$$

Step 2: The function f is a supersolution

Step 2.1: Lower bound on $[0,\kappa^{-\epsilon}]$ We start from (3.33). We assume that $\bar{C}_1>0$ and $\bar{C}_1'>0$, so we can also eliminate the first term in the right-hand-side of (3.33). As in the proof of Proposition 3.2, the sign of the right-hand-side of (3.33) depends on the sign of $A^2f-\frac{h^2}{f_0}\exp(-2f_0x)$. From (3.54) and following the Step 3.1. in the proof of Proposition 3.2, we get

$$A^2f-\tfrac{h^2}{f_0}\exp(-2f_0x)\geq\tfrac{h^2}{f_0}\exp(-2f_0x)\left(\exp(\Lambda_1(\kappa)x)(1+\tfrac{\Lambda_1(\kappa)}{2f_0}-\tfrac{\sqrt{2}(1-f_0^2)}{f_0}\kappa^\nu)-1\right),$$

where $\Lambda_1(\kappa)$ is given by $-2(\Lambda(\kappa)-f_0)$. We remark that the term $\exp(\Lambda_1(\kappa)x)$ inside the bracket is larger than 1 for all $f_0\in[\delta_0,\delta_1]$, when κ and $g(\kappa d)$ are small, under the two conditions

$$\frac{1-\delta_0^2}{2\delta_0}\bar{C}_1+\frac{1-\delta_0^2}{\sqrt{2}}<\frac{\bar{C}_0}{2\sqrt{2}\delta_1^3},\quad\frac{1-\delta_0^2}{2\delta_0}\bar{C}_1'<\frac{\bar{C}_0'}{2\sqrt{2}\delta_1^3}.\tag{3.55}$$

The sum $\Lambda(\kappa)$ defined in (3.46) is then positive for all f_0 when the two following conditions are satisfied

$$\frac{\bar{C}_0}{2\sqrt{2}\delta_1^4}>\frac{1-\delta_0^2}{2\delta_0^2}\bar{C}_1+\frac{\sqrt{2}(1-\delta_0^2)}{\delta_0},\quad\frac{\bar{C}_0'}{2\sqrt{2}\delta_1^4}>\frac{1-\delta_0^2}{2\delta_0^2}\bar{C}_1'.\tag{3.56}$$

We can choose $(\bar{C}_0,\bar{C}_0',\bar{C}_1,\bar{C}_1')$ satisfying (3.55) and (3.56). We thus get the result in $[0,\kappa^{-\epsilon}]$.

Step 2.2: Lower bound on $[\kappa^{-\epsilon},d]$ On the interval $[\kappa^{-\epsilon},d]$, we can neglect the positive term A^2f. From (3.33), we get

$$\begin{aligned}&-\kappa^{-2}f''+(-1+f^2+A^2)f\geq\\&(2\bar{C}_1\kappa^\nu+2\bar{C}_1'\bar{g}(\kappa d)+\mathcal{O}(\kappa^{2\nu}+\bar{g}(\kappa d))^2)\tilde{f}_1(0)\bar{z}_{\kappa,h}\exp(-\sqrt{2}\kappa(1+\bar{C}_1\kappa^\nu+\bar{C}_1'\bar{g}(\kappa d))x)\\&-\frac{h^2}{f_0}(1+\mathcal{O}(\kappa^2))\exp(-2f_0x)\\&-\bar{\lambda}_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x)).\left[3E(\frac{x}{d})+\frac{2\sqrt{2}}{\kappa d}E'(\frac{x}{d})+\frac{1}{(\kappa d)^2}E''(\frac{x}{d})\right]\\&+3\tilde{f}_1^2\bar{\lambda}_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x))E(\frac{x}{d})+\bar{\lambda}_{d,h,\kappa}^3\exp(-3\sqrt{2}\kappa(d-x))E(\frac{x}{d})^3.\end{aligned}$$

As $E'>0$, $\bar{\lambda}_{d,h,\kappa}<0$ and $1-\tilde{f}_1^2>0$, we get

$$\begin{aligned}&-\kappa^{-2}f''+(-1+f^2+A^2)f\geq\\&(2\bar{C}_1\kappa^\nu+2\bar{C}_1'\bar{g}(\kappa d)+\mathcal{O}(\kappa^{2\nu}+\bar{g}(\kappa d))^2)\tilde{f}_1(0)\bar{z}_{\kappa,h}\exp(-\sqrt{2}\kappa(1+\bar{C}_1\kappa^\nu+\bar{C}_1'\bar{g}(\kappa d))x)\\&-\frac{h^2}{f_0}(1+\mathcal{O}(\kappa^2))\exp(-2f_0x)-\bar{\lambda}_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x)).\frac{1}{(\kappa d)^2}E''(\frac{x}{d})\\&+\bar{\lambda}_{d,h,\kappa}^3\exp(-3\sqrt{2}\kappa(d-x))E(\frac{x}{d})^3.\end{aligned}$$

From (3.41), for all $f_0\in[\delta_0,\delta_1]$ and κ small enough, we have on $[\kappa^{-\epsilon},d]$

$$\begin{aligned}\Lambda'&:=2\bar{C}_1\kappa^\nu\tilde{f}_1(0)\bar{z}_{\kappa,h}\exp(-\sqrt{2}\kappa\Lambda_2x)-\frac{h^2}{f_0}(1+\mathcal{O}(\kappa^2))\exp(-2f_0x)\\&\geq\kappa^\nu\exp(-\sqrt{2}\kappa\Lambda_2x)\left(2\bar{C}_1\tilde{f}_1(0)\bar{z}_{\kappa,h}-\sqrt{2}\kappa^{-1-\nu}(1+\mathcal{O}(\kappa^2))\exp(-(2\delta_0-\sqrt{2}\kappa\Lambda_2)d)\right).\end{aligned}$$

For $\bar{C}_1>0$, Λ' is positive for κ small and κd large enough. Let us now consider the term

$$\begin{aligned}\Lambda''&:=2\bar{C}_1'\bar{g}(\kappa d)\tilde{f}_1(0)\bar{z}_{\kappa,h}\exp(-\sqrt{2}\kappa\Lambda_2x)-\bar{\lambda}_{d,h,\kappa}\exp(-\sqrt{2}\kappa(d-x)).\frac{1}{(\kappa d)^2}E''(\frac{x}{d})\\&+|\bar{\lambda}_{d,h,\kappa}|^3\exp(-3\sqrt{2}\kappa(d-x))E(\frac{x}{d})^3.\end{aligned}$$

If we assume $\bar{C}_1'>0$, we can conclude as in Proposition 3.2, taking $\bar{g}(y)=\exp(-\alpha_0y)$ with $\alpha_0\in]0,\frac{\sqrt{2}}{4}[$. The proof of Proposition 3.3 follows.

4 Existence of solutions of $(GL)^s_d$

Following the proof of Proposition 6.5 given by C. Bolley and B. Helffer in [8], p. 66, it is sufficient to construct a subsolution f_{sub} and a supersolution f_{sup} ($f_{sub} \leq f_{sup}$) in order to derive a solution (f, A) of $(GL)^s_d$.
In the same spirit of [10] and [16], from Propositions 3.2 and 3.3, we deduce the proof of Theorem 1.1.
Proof of Theorem 1.1 According Propositions 3.2 and 3.3, we can choose $g = \bar{g}$, taking $\alpha_0 \in]0, \frac{\sqrt{2}}{4}[$.
Step 1: Construction of subsolutions and supersolutions For $\frac{1}{\sqrt{2}} < \delta_0 < \delta_1 < 1$ fixed, let us consider $f_0 \in [\delta_0, \delta_1]$ and the solutions in $]0, 1[$ of

$$\sqrt{2}y^2(1-y^2) + C_0\kappa^{1-\epsilon} + C_0' g(\kappa d) = \sqrt{2} f_0^2(1-f_0^2), \tag{4.1}$$

where C_0, C_0' and g are given by Proposition 3.2. For κ small enough and κd large enough, there exist two solutions of (4.1) denoted by $y_1(f_0, \kappa)$ and $y_2(f_0, \kappa)$. We can assume $y_1(f_0, \kappa) < \delta_0 < y_2(f_0, \kappa)$.
From (4.1), as $0 < f_0 < 1$ and $y_2(f_0, \kappa) \in]\frac{1}{\sqrt{2}} 1[$, it results that there exists $\bar{C}$ such that for all $y \in]0, 1[$, for κ small enough and κd large enough, we have

$$|y_2(f_0, \kappa) - f_0| \leq \bar{C}\left(\kappa^{1-\epsilon} + g(\kappa d)\right). \tag{4.2}$$

From (3.14) and Proposition 3.2, there exist two subsolutions of the Ginzburg-Landau system, denoted by $f^{(1)}_{sub}$ and $f^{(2)}_{sub}$ satisfying

$$f^{(1)}_{sub}(0) = \tanh(x^{(1)}_{\kappa,h}) + \mathcal{O}(\kappa), \quad \text{with} \quad \tanh(x^{(1)}_{\kappa,h}) = y_1(f_0, \kappa) + \mathcal{O}\left(\kappa^\nu + g(\kappa d)\right), \tag{4.3}$$

and

$$f^{(2)}_{sub}(0) = \tanh(x^{(2)}_{\kappa,h}) + \mathcal{O}(\kappa) \quad \text{with} \quad \tanh(x^{(2)}_{\kappa,h}) = y_2(f_0, \kappa) + \mathcal{O}\left(\kappa^\nu + g(\kappa d)\right). \tag{4.4}$$

Now, for κ small and κd large, let us consider $\bar{y}_1(f_0, \kappa)$ and $\bar{y}_2(f_0, \kappa)$ the solutions in $]0, 1[$ of

$$\sqrt{2}y^2(1-y^2) + \bar{C}_0\kappa^{1-\epsilon} + \bar{C}_0' g(\kappa d) = \sqrt{2} f_0^2(1-f_0^2),$$

where $\bar{C}_0$ and $\bar{C}_0'$ are given by Proposition 3.3. We can assume $\bar{y}_1(f_0, \kappa) < \bar{y}_2(f_0, \kappa)$. Similarly, from Proposition 3.3, there exist two supersolutions of the Ginzburg-Landau system, $f^{(1)}_{sup}$ and $f^{(2)}_{sup}$ such that:

$$\tanh(\bar{x}^{(2)}_{\kappa,h}) = \bar{y}_1(f_0, \kappa) + \mathcal{O}\left(\kappa^\nu + g(\kappa d)\right) \quad \text{and} \quad \tanh(\bar{x}^{(2)}_{\kappa,h}) = \bar{y}_2(f_0, \kappa) + \mathcal{O}\left(\kappa^\nu + g(\kappa d)\right). \tag{4.5}$$

Making similar computations to the previous one, we can show that there exists $\hat{C}$ such that for all $f_0 \in]\frac{1}{\sqrt{2}}, 1[$, for κ small and κd large

$$0 < \bar{y}_2(f_0, \kappa) - f_0 \leq \hat{C}\left(\kappa^{1-\epsilon} + g(\kappa d)\right). \tag{4.6}$$

Furthermore, we have the inequalities:

$$\bar{y}_1(f_0, \kappa) < y_1(f_0, \kappa) < \delta_0 < y_2(f_0, \kappa) < \bar{y}_2(f_0, \kappa).$$

Step 2: Comparison of $f^{(2)}_{sub}$ and $f^{(2)}_{sup}$ From (4.2) and (4.6), we deduce the estimate, for κ small and κd large enough

$$\bar{y}_2(f_0, \kappa) - y_2(f_0, \kappa) = \mathcal{O}(\kappa^\nu + g(\kappa d)), \quad \forall f_0 \in [\delta_0, \delta_1]. \tag{4.7}$$

From the inequality $y_2(f_0,\kappa) < \bar{y}_2(f_0,\kappa)$, and from (4.5) and (4.7), we deduce that there exists $\kappa_0 > 0$ such that for all $\kappa \le \kappa_0$, for all $f_0 \in [\delta_0, \delta_1]$, we have

$$x^{(2)}_{\kappa,h} < \bar{x}^{(2)}_{\kappa,h} \quad\text{and}\quad \bar{x}^{(2)}_{\kappa,h} - x^{(2)}_{\kappa,h} = \mathcal{O}(\kappa^\nu + g(\kappa d)). \tag{4.8}$$

From the equality $\tanh a - \tanh b = \frac{\sinh(a-b)}{\cosh a \cosh b}$, we deduce that

$$f^{(2)}_{sup} - f^{(2)}_{sub} = H_{\kappa,d} + J_\kappa + G_{\kappa,d}, \tag{4.9}$$

where

$$\begin{aligned}
H_{\kappa,d}(x) &:= \frac{\sinh\left(\frac{\kappa}{\sqrt{2}}((\bar{C}_1 - C_1)\kappa^\nu + (\bar{C}'_1 - C'_1)g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h} - x^{(2)}_{\kappa,h}\right)}{\cosh\left(\frac{\kappa}{\sqrt{2}}(1+\bar{C}_1\kappa^\nu + g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h}\right)\cosh\left(\frac{\kappa}{\sqrt{2}}(1+C_1\kappa^\nu + g(\kappa d))x + x^{(2)}_{\kappa,h}\right)},\\
J_\kappa(x) &:= \tfrac{\kappa^2 h^2}{4\bar{y}_2(f_0,\kappa)^3}\exp(-2\bar{y}_2(f_0,\kappa)x) - \tfrac{\kappa^2 h^2}{4y_2(f_0,\kappa)^3}\exp(-2y_2(f_0,\kappa)x),\\
G_{\kappa,d}(x) &:= \left(\bar{\lambda}_{d,\kappa,h} - \lambda_{d,\kappa,h}\right)\exp(-\sqrt{2}\kappa(d-x))E(\tfrac{x}{d}).
\end{aligned} \tag{4.10}$$

First, as $C_1 < 0$, $C'_1 < 0$, $\bar{C}_1 > 0$, $\bar{C}'_1 > 0$, $E > 0$, from (3.18) and Remark 3.4, we get $\bar{\lambda}_{d,\kappa,h} - \lambda_{d,\kappa,h} > 0$. Thus

$$G_{\kappa,d}(x) > 0, \quad \forall\, x \in [0,d]. \tag{4.11}$$

Let us compare $H_{\kappa,d}$ and J_κ. There exist some positive constants β, γ and σ such that for $x \in [0,d]$ and for all $f_0 \in [\delta_0,\delta_1]$ we have the estimates

$$H_{\kappa,d}(x) \ge \gamma(\kappa^\nu + g(\kappa d))\exp(-2\sqrt{2}\kappa x), \quad |J_\kappa(x)| \le \sigma\kappa\exp(-\beta x).$$

Thus for κ small and for all $x \in [0,d]$, we get

$$|J_\kappa(x)| \le H_{\kappa,d}(x). \tag{4.12}$$

According to (4.9), (4.11) and (4.12), we deduce that

$$f^{(2)}_{sup}(x) - f^{(2)}_{sub}(x) \ge 0, \quad \forall\, x \in [0,d].$$

We deduce that there exists a pair (f,A) solution of $(GL)^s_d$ such that

$$f^{(2)}_{sub}(x) \le f(x) \le f^{(2)}_{sup}(x), \quad \forall\, x \in [0,d]. \tag{4.13}$$

According to (4.2), (4.4)-(4.6) and (4.13), it results that there exists C such that for κ small and κd large enough, we get

$$|f(0) - f_0| \le C\left(\kappa^{1-\epsilon} + g(\kappa d)\right).$$

The proof of Theorem 1.1 follows.

In the next section, we denote simply $f^{(2)}_{sub}$ and $f^{(2)}_{sup}$ by f_{sub} and f_{sup}.

5 Localization of solutions of $(GL)^s_d$

In this section, we give a localization of the solutions constructed in Theorem 1.1. Let us consider the pair $(\bar{f}_0, \bar{A}_0)$ where

$$\bar{f}_0(x) = \tanh(\frac{\kappa x}{\sqrt{2}} + \bar{x}_0), \tag{5.1}$$

and where $\bar{A}_0$ is defined in (1.10). The parameter $\bar{x}_0$ is determined by the equality $\bar{f}_0(0) = f_0,\ f_0 \in]\frac{1}{\sqrt{2}}, 1[$.
For technical reasons, we assume that

$$\kappa d \geq \kappa^{-\epsilon}, \quad \epsilon > 0. \tag{5.2}$$

Proof of Theorem 1.2 Let (f, A) be the solution defined in Theorem 1.1. By construction, we have shown in the proof of Theorem 1.1 that f satisfies $f_{sub} \leq f \leq f_{sup}$. So, we have $f_{sub} - \bar{f}_0 \leq f - \bar{f}_0 \leq f_{sup} - \bar{f}_0$.
Step 1: Estimate for $|f - \bar{f}_0|$ We have

$$\begin{aligned} & f_{sup}(x) - \bar{f}_0(x) \\ &= L_{\kappa,d}(x) + \tfrac{\kappa^2 h^2}{4\bar{y}_2(f_0,\kappa)^3} \exp(-2\bar{y}_2(f_0,\kappa)x) + \bar{\lambda}_{d,\kappa,h} \exp(-\sqrt{2}\kappa(d-x))E(\tfrac{x}{d}), \end{aligned} \tag{5.3}$$

where

$$L_{\kappa,d}(x) := \frac{\sinh\left(\frac{\kappa}{\sqrt{2}}(\bar{C}_1\kappa^\nu + \bar{C}_1' g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h} - \bar{x}_0\right)}{\cosh(\frac{\kappa}{\sqrt{2}}(1 + \bar{C}_1\kappa^\nu + \bar{C}_1' g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h}) \cosh(\frac{\kappa x}{\sqrt{2}} + \bar{x}_0)}. \tag{5.4}$$

From (4.6) and as $\kappa d \geq \kappa^{-\epsilon}$, we have

$$\bar{x}^{(2)}_{\kappa,h} - \bar{x}_0 = \mathcal{O}(\kappa^\nu). \tag{5.5}$$

Using (3.8) and (5.5), we deduce that there exists $C_1 > 0$ such that on the interval $[0, \kappa^{-1-\epsilon}]$, we have for κ small enough

$$\sinh\left(\frac{\kappa}{\sqrt{2}}(\bar{C}_1\kappa^\nu + \bar{C}_1' g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h} - \bar{x}_0\right) \leq C_1\kappa^{1-2\epsilon}.$$

As $\frac{1}{\cosh(x)} \leq 2\exp(-x)$ for all $x \in \mathbb{R}^+$, and because $\bar{C}_1 > 0$ and $\bar{C}_1' > 0$, we get for all $f_0 \in [\delta_0, \delta_1]$ and all $, x \in [0, d]$

$$\frac{1}{\cosh\left(\frac{\kappa}{\sqrt{2}}(1 + \bar{C}_1\kappa^\nu + \bar{C}_1' g(\kappa d))x + \bar{x}^{(2)}_{\kappa,h}\right)} \frac{1}{\cosh\left(\frac{\kappa x}{\sqrt{2}} + \bar{x}_0\right)} \leq C_2 \exp(-\sqrt{2}\kappa x).$$

As $\bar{\lambda}_{d,\kappa,h} < 0$, we deduce that there exists $C > 0$ such that for κ small enough, we have

$$f_{sup} - \bar{f}_0 \leq C\kappa^{1-2\epsilon} \exp(-\sqrt{2}\kappa x), \quad \forall x \in [0, \kappa^{-1-\epsilon}]. \tag{5.6}$$

Moreover, there exists $C_3 > 0$ such that for all $f_0 \in [\delta_0, \delta_1]$, for all $x \in [0, d]$, we have

$$L_{\kappa,d}(x) \leq C_3 \exp(-\tfrac{2\sqrt{2}\kappa x}{3}).$$

From (5.3), it derives that there exists $C_4 > 0$, such that, on the interval $[-c\kappa^{-1}\ln\kappa, d[$, we get

$$f_{sup} - \bar{f}_0 \leq C_4\kappa \exp(-\frac{\sqrt{2}}{2}\kappa x). \tag{5.7}$$

For κ small enough $I_1 \cap I_2 \neq \emptyset$. From (5.6) and (5.7), we deduce that there exists $\hat{C} > 0$ such that

$$f_{sup} - \bar{f}_0 \leq \hat{C}\kappa^{1-2\epsilon} \exp(-\frac{\sqrt{2}}{2}\kappa x), \ \ \forall\, x \in [0, d]. \tag{5.8}$$

Similarly, one can check that there exists a constant $C_5 > 0$ such that we have

$$f_{sub}(x) - \bar{f}_0(x) \geq -C_5\kappa^{1-\epsilon}\exp(-\frac{\sqrt{2}}{2}\kappa x), \quad \forall x \in [0,d]. \tag{5.9}$$

From (5.8) and (5.9), we deduce the existence of κ_0 and $\bar{C} > 0$ such that for $\kappa \leq \kappa_0$, for $x \in [0,d]$, we have

$$|f - \bar{f}_0| \leq \bar{C}\kappa^{1-2\epsilon}\exp(-\frac{\sqrt{2}}{2}\kappa x).$$

Inequality (1.8) follows.

Step 2: Estimate for $|A - \bar{A}_0|$ We follow the proof of Proposition 4.5 in [10]. We set $\rho := 2\epsilon$, and

$$f_{max} = \bar{f}_0 + \bar{C}\kappa^{1-\rho}\exp(-\frac{\sqrt{2}}{2}\kappa x). \tag{5.10}$$

Let A_{max} be the unique solution in $H^2([0,d])$ of the problem

$$\begin{cases} -A''_{max} + A_{max}f^2_{max} = 0, \\ A'_{max}(0) = h, \quad A_{max}(d) = 0. \end{cases}$$

From Lemma 2.2, as $f \leq f_{max}$ and $\bar{f}_0 \leq f_{max}$, it results that, on $[0,d]$

$$A \leq A_{max} < 0 \quad \text{and} \quad \bar{A}_0 \leq A_{max} < 0. \tag{5.11}$$

We introduce $Z := A_{max} - \bar{A}_0$. The function Z is positive and $Z \in H^2([0,d])$. Moreover, Z is the unique solution of

$$\begin{cases} -Z'' + f^2_{max}Z = (f^2_{max} - \bar{f}_0^2)(-\bar{A}_0), \\ Z'(0) = 0, \quad Z(d) = 0. \end{cases} \tag{5.12}$$

We set $M_{max} = \min_{x\in[0,d]} f_{max}(x)$. Let us recall that the function $\bar{A}_0$ satisfies $-\bar{A}_0 \leq \frac{h}{f_0}\exp(-f_0 x)$. From (5.10), it results that the right-hand-side of (5.12) admits as upper bound $C_6\frac{h}{f_0}\kappa^{1-\rho}\exp\left(-(\delta_0 + \frac{1}{\sqrt{2}}\kappa)x\right)$.
Let us consider W, the unique solution in $H^2([0,d])$ of

$$\begin{cases} -W'' + M^2_{max}W = C_6\kappa^{\frac{1}{2}-\rho}\exp(-\frac{\delta_0}{2}x), \\ W'(0) = 0, \quad W(d) = 0. \end{cases}$$

The solution is given by

$$W(x) = \frac{C_6\kappa^{\frac{1}{2}-\rho}}{M^2_{max} - \frac{\delta_0^2}{4}}\exp(-\frac{\delta_0}{2}x) + C_1(\kappa)\exp(-M_{max}x) + C_2(\kappa)\exp(M_{max}x), \tag{5.13}$$

where

$$C_1(\kappa) = C_2(\kappa) - \frac{C_6\kappa^{\frac{1}{2}-\rho}\frac{\delta_0}{2}}{M_{max}(M^2_{max} - \frac{\delta_0^2}{4})}, \tag{5.14}$$

with

$$C_2(\kappa) = \frac{2}{\cosh(M_{max}d)}\left(\frac{C_6\kappa^{\frac{1}{2}-\rho}\exp(-\frac{\delta_0}{2}d)}{M^2_{max} - \frac{\delta_0^2}{4}} - \frac{C_6\kappa^{\frac{1}{2}-\rho}\frac{\delta_0}{2}\exp(-M_{max}d)}{M_{max}(M^2_{max} - \frac{\delta_0^2}{4})}\right). \tag{5.15}$$

Applying Lemma 2.2 with $u = Z - W$ and $C = M_{max}^2$, we get for κ small enough

$$Z \le W, \quad \forall x \in [0, d].$$

From (5.14) and (5.15), we remark that $C_1(\kappa) < 0$ for κ small. Then, from (5.11), and using that $M_{max}^2 - \frac{\delta_0^2}{4} \ge \frac{\delta_0^2}{4}$ and $|C_2(\kappa)| \exp\left((M_{max} + \frac{\delta_0}{2})x\right) = \mathcal{O}(\kappa^{\frac{1}{2}-\rho})$, we get for κ small enough and for some constant $C_7 > 0$

$$A - \bar{A}_0 \le C_7 \kappa^{\frac{1}{2}-\rho} \exp(-\frac{\delta_0}{2}x), \quad \forall x \in [0, d]. \tag{5.16}$$

Similarly, one can check that there exists C_8 such that $A - \bar{A}_0 \ge -C_8 \kappa^{\frac{1}{2}-\rho} \exp(-\frac{\delta_0}{2}x)$. The proof of Theorem 1.2 follows.

6 Application of a min-max theorem

6.1 The functional ε_d satisfies the Palais-Smale condition on C_α

In order to use a min-max theorem, we introduce the following definitions.
Let C be a closed convex subset of a Hilbert space V, $\varepsilon : C \to \mathbb{R}$ possesses an extension $\varepsilon \in C^1(V; \mathbb{R})$ to V. For $u \in C$, we define

$$g(u) = \sup_{\substack{v \in C \\ ||u - v|| < 1}} \langle D\varepsilon(u), u - v \rangle, \tag{6.1}$$

as a measure for the slope of ε in C ($D\varepsilon$ is the differential of ε). If $C = V$, we have $g(u) = \|DE(u)\|$.

Definition 6.1 *We say that $u \in C$ is a critical point of ε if $g(u) = 0$.*

Definition 6.2 *The functional ε satisfies the Palais-Smale condition $(PS)_C$ on C if the following condition is satisfied:*
Any sequence $(u_n)_{n \in \mathbb{N}}$ such that $|\varepsilon(u_n)| \le c$ uniformly while $g(u_n) \to 0$ ($n \to \infty$) is relatively compact in V.

In the following, we take $V = \{(f, A) \in H^1(]0, d[) \times \tilde{H}^1(]0, d[)\}$, $\varepsilon = \varepsilon_d$, $C = C_\alpha$ (see (1.11)). For $(f, A) \in H^1(]0, d[)^2$ and $(u, v) \in H^1(]0, d[)^2$, we have

$$\langle D\varepsilon_d(f, A), (u, v) \rangle = \int_0^d \left(\kappa^{-2} f' u' - fu + f^3 u + A^2 fu + A' v' + A f^2 v\right) dx. \tag{6.2}$$

Proposition 6.3 *The functional ε_d satisfies the Palais-Smale condition on C_α.*

Proof Let $(f_n, A_n)_{n \in \mathbb{N}}$ be a sequence in C_α such that there exists $D \in \mathbb{R}$ satisfying

$$\epsilon_d(f_n, A_n) \le D, \;\; \forall\, n \in \mathbb{N}, \tag{6.3}$$

and such that

$$g(f_n, A_n) \to 0, \quad \text{when } n \to +\infty. \tag{6.4}$$

First, we show that these sequences are uniformly bounded in $H^1(]0, d[)$. We have (taking account that $A_n(d) = 0$)

$$A_n^2(0) = -2 \int_0^d A_n(t) A_n'(t) dt \le 2 \|A_n\|_{L^2(]0,d[)} \|A_n'\|_{L^2(]0,d[)}.$$

From the definition of ϵ_d (see (1.1)), we deduce that

$$\epsilon_d(f_n, A_n) \geq \|A_n'\|^2_{L^2(]0,d[)} + \|A_n f_n\|^2_{L^2(]0,d[)} - 2\sqrt{2}h\|A_n'\|^{\frac{1}{2}}_{L^2(]0,d[)}\|A_n\|^{\frac{1}{2}}_{L^2(]0,d[)} + (h^2 - \frac{1}{2})d.$$

As $f_n \geq \alpha > 0$, we get

$$\begin{aligned} &\epsilon_d(f_n, A_n) \\ &\geq \|A_n'\|^2_{L^2(]0,d[)} + \alpha^2\|A_n\|^2_{L^2(]0,d[)} - 2\sqrt{2}h\|A_n'\|^{\frac{1}{2}}_{L^2(]0,d[)}\|A_n\|^{\frac{1}{2}}_{L^2(]0,d[)} + (h^2 - \tfrac{1}{2})d. \end{aligned} \tag{6.5}$$

We set $B_n = \frac{A_n}{h}$. From (6.5), we can write

$$\begin{aligned} &\epsilon_d(f_n, A_n) \\ &\geq h^2\left(\|B_n'\|^2_{L^2(]0,d[)} + \alpha^2\|B_n\|^2_{L^2(]0,d[)} - 2\sqrt{2}\|B_n'\|^{\frac{1}{2}}_{L^2(]0,d[)}\|B_n\|^{\frac{1}{2}}_{L^2(]0,d[)}\right) + (h^2 - \tfrac{1}{2})d. \end{aligned} \tag{6.6}$$

For $a > 0$, $b > 0$ and $\alpha > 0$, we have

$$\alpha^2 a^2 + b^2 - 2\sqrt{2}a^{\frac{1}{2}}b^{\frac{1}{2}} \geq \frac{\alpha}{2}\frac{a^2 + b^2}{2} - \frac{2}{\alpha}. \tag{6.7}$$

According to (6.6) and (6.7), we get

$$\epsilon_d(f_n, A_n) \geq \frac{\alpha}{2}\frac{\|A_n'\|^2_{L^2(]0,d[)} + \|A_n\|^2_{L^2(]0,d[)}}{2} - \frac{2h^2}{\alpha} + (h^2 - \frac{1}{2})d. \tag{6.8}$$

From (6.3), we deduce that there exists $E > 0$ such that

$$\|A_n\|_{H^1(]0,d[)} \leq E. \tag{6.9}$$

On the other hand, we have

$$\kappa^{-2}\|f_n'\|^2_{L^2(0,d)} + \|1 - f_n\|^2_{L^2(0,d)} - 2\sqrt{2}h\|A_n'\|^{\frac{1}{2}}_{L^2(0,d)}\|A_n\|^{\frac{1}{2}}_{L^2(0,d)} \leq \varepsilon_d(f_n, A_n) \leq D.$$

From (6.9), it results that there exists $\bar{E} > 0$ such that

$$\|1 - f_n\|_{H^1(]0,d[)} \leq \bar{E}. \tag{6.10}$$

According to (6.9) and (6.10) and the weak compactness of the unit ball, we deduce that we can extract subsequences, also denoted by f_n and A_n such that

$$f_n \rightharpoonup f \text{ in } H^1(]0,d[) \text{ and } A_n \rightharpoonup A \text{ in } \tilde{H}^1(]0,d[), \tag{6.11}$$

and

$$f_n \to f \text{ in } L^2(]0,d[) \text{ and } A_n \to A \text{ in } L^2(]0,d[). \tag{6.12}$$

In order to use the fact that $g(f_n, A_n)$ tends to 0, we set

$$t_{m,n} = \begin{cases} 2||f_n - f_m||_{H^1(]0,d[)}, \text{ if } ||f_n - f_m||_{H^1(]0,d[)} \geq \frac{1}{2}, \\ 1 \text{ if } ||f_n - f_m||_{H^1(]0,d[)} < \frac{1}{2}. \end{cases}$$

From (6.10), we deduce that

$$1 \leq t_{m,n} \leq 4\bar{E}. \tag{6.13}$$

Now, we choose

$$\bar{f}_{m,n} = f_n + \frac{1}{t_{m,n}}(f_m - f_n).$$

As C_α is convex, $(\bar{f}_{m,n}, A_n) \in C_\alpha$, and

$$||(f_n, A_n) - (\bar{f}_{m,n}, A_n)||_{H^1(]0,d[)^2} = ||f_n - \bar{f}_{m,n}||_{H^1(]0,d[)} < 1$$

taking account the choice of $t_{m,n}$. From (6.4) and (6.13), $t_{m,n}\, g(f_n, A_n) \to 0$ when $n \to +\infty$. Coming to (6.1), it results that for all $\epsilon > 0$, there exists $A > 0$ such that for all pairs $(m,n) \in \mathbb{N}^2$ such that $n \geq A$, we get

$$\langle D\varepsilon_d(f_n, A_n)(f_n - f_m, 0)\rangle = t_{m,n}\, \langle D\varepsilon_d(f_n, A_n)(f_n - \bar{f}_{m,n}, 0)\rangle \leq t_{m,n}\, g(f_n, A_n) \leq \epsilon. \tag{6.14}$$

We may reverse the roles of m and n. On the other hand, we get

$$\begin{aligned} &\langle D\varepsilon_d(f_n, A_n)(f_n - f_m, 0)\rangle - \langle D\varepsilon_d(f_m, A_m)(f_n - f_m, 0)\rangle \\ &= \int_0^d \{\kappa^{-2}(f_n' - f_m')^2 + (f_n - f_m)(A_n^2 f_n - A_m^2 f_m) \\ &-[(1 - f_n^2)f_n - (1 - f_m^2)f_m](f_n - f_m)\}dx. \end{aligned} \tag{6.15}$$

According to (6.14) and (6.15), we deduce

$$\langle D\varepsilon_d(f_n, A_n)(f_n - f_m, 0)\rangle - \langle D\varepsilon_d(f_m, A_m)(f_n - f_m, 0)\rangle \leq 2\epsilon. \tag{6.16}$$

From (6.9) and (6.10), and as $H^1(]0,d[) \hookrightarrow C^0([0,d])$, we deduce that f_n and A_n are bounded in $C^0([0,d])$ and $f_n \to f$ in $C^0([0,d])$ when $n \to +\infty$. We deduce that for all $\epsilon > 0$, there exists $\tilde{A} > 0$ such that for all pairs $(m,n) \in \mathbb{N}^2$ such that $m \geq \tilde{A}$ et $n \geq \tilde{A}$, we have

$$|\int_0^d \{(f_n - f_m)(A_n^2 f_n - A_m^2 f_m) - [(1 - f_n^2)f_n - (1 - f_m^2)f_m](f_n - f_m)\}dx| \leq \epsilon.$$

According to (6.15) and (6.16), it results that, for all $\epsilon > 0$, there exists $B > 0$ such that for all pairs $(m,n) \in \mathbb{N}^2$ such that $n \geq B$, $m \geq B$

$$\int_0^d \kappa^{-2}(f_n' - f_m')^2 dx \leq \epsilon. \tag{6.17}$$

From (6.12) and (6.17), we deduce that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy's sequence in $H^1(]0,d[)$. It derives that $f \in H^1(]0,d[)$ and $f_n \to f$, in $H^1(]0,d[)$.
Similarly, evaluating $\langle D\varepsilon_d(f_n, A_n)(0, A_n - A_m)\rangle - \langle D\varepsilon_d(f_m, A_m)(0, A_n - A_m)\rangle$, one can check that $(A_n)_{n\in\mathbb{N}}$ is Cauchy's sequence in $H^1(]0,d[)$. Consequently, there exists $A \in H^1(]0,d[)$ such that $A_n \to A$, in $H^1(]0,d[)$. The proof of Proposition 6.3 follows.

Now, we show that a critical point of ε_d in the sense of the definition 6.1 is a solution of $(GL)_d^s$. We follow a proof due to T. Giorgi, S. Alama, L. Bronsard (see Lemma 4.1 in [2]).

Proposition 6.4 *Let us assume that $(f, A) \in C_\alpha$ with $g(f, A) = 0$. Then (f, A) is a solution of $(GL)_d^s$.*

Proof Let $\phi, \psi \in C_0^\infty(]0,d[)$ and $\epsilon > 0$. From (6.2) and the definition of g (see (6.1)), we have $\langle D\varepsilon_d(f,A),(0,\frac{\psi}{||\psi||})\rangle \leq g(f,A)$. Thus as $g(f,A)=0$, we get

$$\langle D\varepsilon_d(f,A),(0,\psi)\rangle \leq 0.$$

Replacing ψ with $-\psi$ in this previous inequality, we get

$$\langle D\varepsilon_d(f,A),(0,\psi)\rangle = 0, \ \ \forall\, \psi \in C_0^\infty([0,d]). \tag{6.18}$$

We deduce that

$$-A'' + Af^2 = 0, \ \text{ in } \ D'(]0,d[). \tag{6.19}$$

We set $\phi_\epsilon = -(f+\epsilon\phi-\alpha)_- \geq 0, \quad \phi^\epsilon = (f+\epsilon\phi-1)_+ \geq 0$, and define w_ϵ to be the truncation of $f+\epsilon\phi$

$$w_\epsilon = f + \epsilon\phi + \phi_\epsilon - \phi^\epsilon.$$

Then $(w_\epsilon, A) \in C_\alpha$. For $\epsilon > 0$, it results that

$$\langle D\varepsilon_d(f,A),(f-w_\epsilon,0)\rangle \leq 0. \tag{6.20}$$

We set

$$\Omega_\epsilon := \{x : f+\epsilon\phi \leq \alpha\}, \ \ \Omega^\epsilon := \{x : f+\epsilon\phi \geq 1\}.$$

From (6.20), we deduce that

$$0 \leq \epsilon\langle D\varepsilon_d(f,A),(\phi,0)\rangle - \langle D\varepsilon_d(f,A),(\phi^\epsilon,0)\rangle + \langle D\varepsilon_d(f,A),(\phi_\epsilon,0)\rangle,$$

thus

$$\langle D\varepsilon_d(f,A),(\phi,0)\rangle \geq \frac{1}{\epsilon}[\langle D\varepsilon_d(f,A),(\phi^\epsilon,0)\rangle - \langle D\varepsilon_d(f,A),(\phi_\epsilon,0)\rangle].$$

We have $\langle D\varepsilon_d(1,A),(\phi^\epsilon,0)\rangle = \int_{\Omega^\epsilon} A^2\phi^\epsilon dx$. It results that

$$\langle D\varepsilon_d(f,A),(\phi^\epsilon,0)\rangle \ \geq \ \langle D\varepsilon_d(f,A) - D\varepsilon_d(1,A),(\phi^\epsilon,0)\rangle.$$

Consequently,

$$\begin{aligned}
&\langle D\varepsilon_d(f,A),(\phi^\epsilon,0)\rangle \geq \int_{\Omega^\epsilon} (\kappa^{-2}f'(\phi^\epsilon)' + f(-1+f^2)\phi^\epsilon + A^2(f-1)\phi^\epsilon dx \\
&\geq \int_{\Omega^\epsilon} [\kappa^{-2}(f-1)'(f-1+\epsilon\phi)' \\
&+A^2(f-1)(f-1+\epsilon\phi) + f(-1+f^2)(f-1+\epsilon\phi)]dx \\
&\geq \epsilon \int_{\Omega^\epsilon} (\kappa^{-2}f'\phi' + f(-1+f^2)\phi + A^2(f-1)\phi dx.
\end{aligned}$$

As $mes(\Omega^\epsilon) \to 0$ when $\epsilon \to 0^+$, we get

$$\langle D\varepsilon_d(f,A),(\phi^\epsilon,0)\rangle \geq o(\epsilon). \tag{6.21}$$

Similarly, we get

$$\begin{aligned}
&\langle D\varepsilon_d(f,A),(\phi_\epsilon,0)\rangle = \int_{\Omega_\epsilon} (\kappa^{-2}f'{\phi_\epsilon}' + f(-1+f^2+A^2)\phi_\epsilon dx \\
&\leq -\epsilon \int_{\Omega_\epsilon} \kappa^{-2}f'\phi' + A^2 f\phi dx = o(\epsilon).
\end{aligned} \tag{6.22}$$

According to (6.21) and (6.22), we get $\langle D\varepsilon_d(f,A),(\phi,0)\rangle \geq \frac{1}{\epsilon}o(\epsilon),\ \ \forall\, \epsilon>0$, thus

$$\langle D\varepsilon_d(f,A),(\phi,0)\rangle \geq 0. \tag{6.23}$$

Since we might just as well have considered $-\phi$ in the above calculation, we obtain

$$\langle D\varepsilon_d(f,A),(\phi,0)\rangle = 0,\ \ \forall\, \phi\in C_0^\infty(]0,d[), \tag{6.24}$$

thus

$$-\kappa^{-2}f''-f+f^3+A^2f=0,\ \text{ in }\ D'(]0,d[). \tag{6.25}$$

From (6.19) and (6.25), we deduce that $f\in H^2([0,d])$ and $A\in H^2([0,d])$, then $f\in\cap_{m\in\mathbb{N}}H^m([0,d])$ and $A\in\cap_{m\in\mathbb{N}}H^m([0,d])$. The functions f and A are C^∞ and satisfy the Ginzburg-Landau equations in the usual sense. Let us show that (f,A) satisfies the boundary conditions at zero (see (1.2)). Let $\tilde{\phi}\in C_0^1([0,d[)$. From (6.18), we get

$$\int_0^d (A'f'+Af^2\tilde{\phi})dx+h\tilde{\phi}(0)=0. \tag{6.26}$$

On the other hand

$$\int_0^d A''\tilde{\phi}dx=[A'\tilde{\phi}]_0^d-\int_0^d A'\tilde{\phi}'dx,$$

thus, taking account (6.26), and the fact that $-A''+f^2A=0$, we deduce that

$$\tilde{\phi}(0)(A'(0)-h)=0,\ \ \forall\tilde{\phi}\in C_0^1([0,d[).$$

It derives that

$$A'(0)=h.$$

Similarly, using (6.24), we can show that $f'(0)=0$. To show that $f'(d)=0$, we make similar computations with test functions $\tilde{\psi}\in C^1(]0,d])$. The pair (f,A) satisfies the boundary conditions. The proof of Proposition 6.4 follows.

Proof of Theorem 1.3 From Theorem 1 and Theorem 2 in [24], it results that for κd large enough, there are values of h such that there exist at least two strictly positive solutions ϕ_1 and ϕ_2 of $(GL)_d^s$. Then either one of these solutions is not a local minimum, and then the theorem is true, or the two solutions are local minima. On the other hand, from Proposition 6.3, the functional ϵ_d satisfies The Palais-Smale condition on C_α. Then we can apply Theorem 1.4 and Proposition 6.4 with the choice $C=C_\alpha$ with $\alpha=\min(\min_{x\in[0,d]}\phi_1(x),\min_{x\in[0,d]}\phi_2(x))$. It follows that there exist a non-minimizing solution for the Ginzburg-Landau system and this achieves the proof of Theorem 1.3.

6.2 Proof of Theorem 1.5

In order to prove that there exist non-minimizing solutions for h near the superheating field, we will show that there exist some pairs (f,A) of $H^1(]0,d[)\times\tilde{H}^1(]0,d[)$ with very small energy, such that $f>0$. Moreover, we will prove that f can be chosen arbitrarly small. First, we give a variant of the proposition due to C. Bolley and B. Helffer (see [11], Proposition 2.1, p. 126).

Proposition 6.5 *Let $(\tilde{f}_0,d,h)\in]0,1]\times[0,\infty[^2$. We set $U(\tilde{f}_0,d):=\{(f,A)\in H^1(]0,d[)\times\tilde{H}^1(]0,d[);f=\tilde{f}_0$ on $]0,d[\}$. The restriction of the functionnal ε_d to the set $U(\tilde{f}_0,d)$ admits a global minimum and*

$$\epsilon_0(\tilde{f}_0,d):=\inf_{(f,A)\in U(\tilde{f}_0,d)}\varepsilon_d(f,A),$$

is given by

$$\epsilon_0(\tilde{f}_0,d)=\frac{d}{2}(1-\tilde{f}_0^2)^2-\frac{h^2}{\tilde{f}_0}\tanh(\tilde{f}_0\,d). \tag{6.27}$$

Proof For all $(f,A)\in U(\tilde f_0,d)$, we have the equality $\varepsilon_d(\tilde f_0,A)=\frac{d}{2}(1-\tilde f_0^2)^2+e_1(A)$, where

$$e_1(A)=\int_0^d(\tilde f_0^2A^2+(A')^2)dx+2hA(0).$$

The functional $A\to e_1(A)$ is strictly convex on $\tilde H^1(]0,d[)$. Its minimum is reached for A such that

$$-A''+\tilde f_0^2A=0 \ \text{ on } \ ]0,d[\tag{6.28}$$

with $A'(0)=h$ and $A(d)=0$. We get a unique solution given by

$$A_{\tilde f_0}(x)=-\frac{h}{\tilde f_0\cosh(\tilde f_0\,d)}\sinh(\tilde f_0(d-x)) \ \text{ for } \ x\in[0,d]. \tag{6.29}$$

The infimum of e_1 is equal to $\inf_{A\in U_1}e_1(A)=-\frac{h^2}{\tilde f_0}\tanh(\tilde f_0\,d)$. Consequently, the restriction of ε_d to $U(\tilde f_0,d)$ admits a unique minimum achieved at $(\tilde f_0,A_{\tilde f_0})$ and equal to

$$\epsilon_0(\tilde f_0,d)=\frac{d}{2}(1-\tilde f_0^2)^2-\frac{h^2}{\tilde f_0}\tanh(\tilde f_0\,d).$$

The proof of Proposition 6.5 follows.

Proposition 6.6 *There exist $\kappa_0>0$ and d_0, such that for all $\kappa\le\kappa_0$ and for all $\kappa d\ge d_0$, for all $f_0\in[\delta_0,\delta_1]$ and $h>\frac{1}{\sqrt2}$ such that*

$$\kappa h^2=2^{\frac12}f_0^2(1-f_0^2),$$

there exist a pair (f,A) solution of $(GL)_d^s$ and a pair $(\tilde f,\tilde A)\in H^1(]0,d[)\times\tilde H^1(]0,d[)$ such that

$$\varepsilon_d(\tilde f,\tilde A)<\varepsilon_d(f,A) \tag{6.30}$$

and

$$0<\tilde f(0)<f(0). \tag{6.31}$$

Proof We consider a solution (f,A) obtained in Theorem 1.1. The function f satisfies

$$f(0)>\frac{1}{\sqrt2}. \tag{6.32}$$

From $-A''+Af^2=0$ and the condition $A'(0)=h$, we get $\int_0^d((A')^2+A^2f^2)dt=-h\,A(0)$. It results that, in a critical point of ε_d, from (1.1), the energy of this solution can be expressed in the form of:

$$\varepsilon_d(f,A)=\int_0^d(\kappa^{-2}(f')^2+\frac{(1-f^2)^2}{2})dx+hA(0)+(h^2-\frac12)d. \tag{6.33}$$

From Proposition 6.5 (see (2.3)), (6.32), (6.33) and as $h>\frac{1}{\sqrt2}$, it results that

$$\varepsilon_d(f,A)\ge h.A(0)\ge-\frac{h^2}{f(0)}\ge-\sqrt2h^2. \tag{6.34}$$

For $\tilde f_0\in]0,1[$, using that $\tanh(x)\ge x-\frac{x^3}{3}$ for $x\ge0$ and (6.27), we get

$$\epsilon_0(\tilde f_0,d)\le\frac{d}{2}(1-2h^2+\frac23(\tilde f_0d)^2h^2)-\frac{d\tilde f_0}{2}(2-\tilde f_0^2). \tag{6.35}$$

For all $h > \frac{1}{\sqrt{2}}$, there exists $a > 0$ such that $1 - 2h^2 + \frac{2}{3}a^2h^2 < 0$. Let d_0 defined by $d_0 := \frac{-2\sqrt{2}h^2}{1-2h^2+\frac{2}{3}a^2h^2}$. Then, for all $d \geq d_0$ and for all $\tilde{f}_0 \leq \dfrac{a}{d}$, we have $\epsilon_0(\tilde{f}_0, d) \leq -\sqrt{2}h^2$. Let $(\tilde{f}, \tilde{A}) \in U(\tilde{f}_0, d)$ where the function $\tilde{A}$ is defined in (6.28). Then, for all $d \geq d_0$, we have

$$\varepsilon_d(\tilde{f}, \tilde{A}) \leq -\sqrt{2}h^2,$$

and

$$0 < \tilde{f}(0) < f(0).$$

We can take $d_0(\kappa) = \mathcal{O}(\kappa^{-1})$. The proof of Proposition 6.6 follows.

Proposition 6.7 *The restriction of ε_d to the convex set C_α admits a global minima.*

Proof We have shown in the proof of Proposition 6.3 that the restriction of ε_d to C_α is semi-bounded. Let (f_n, A_n) a minimizing sequence of ε_d. As f_n and A_n are bounded in $H^1([0,d])$, we can extract subsequences such that

$$f_{n_k} \rightharpoonup f, \quad A_{n_k} \rightharpoonup A \text{ in } H^1([0,d]).$$

As the functional ε_d is semi-bounded for the weak topology, we get

$$\varepsilon_d(f, A) \leq \lim \inf_{k\to\infty} \varepsilon_d(f_{n_k}, A_{n_k}).$$

This achieves the proof of Proposition 6.7.

Then, we can prove Theorem 1.5. Proposition 6.6 leads us to consider the closed convex C_α defined in (1.11) with $\alpha := \tilde{f}(0)$.

Proof of Theorem 1.5 From Proposition 6.6, there exists κ_0 such that for all $\kappa \leq \kappa_0$ and for all $\kappa d \geq d_0$, there exist a pair (f, A) solution of $(GL)^s_d$ and a pair $(\tilde{f}, \tilde{A}) \in H^1(]0,d[) \times \tilde{H}^1(]0,d[)$ such that $\varepsilon_d(\tilde{f}, \tilde{A}) < \varepsilon_d(f, A)$, and $f(0) > \tilde{f}(0) > 0$. Let us consider the restriction of ε_d to C_α, where we have chosen α equal to $\tilde{f}(0)$. Either the solution (f, A) is not a local minimum, and then the theorem is true, or it is a local minimum. In this case, from Proposition 6.7, the restriction of ε_d to C_α admits a global minimum, and consequently, ε_d admits two distinct local minimas $(\bar{f}, \bar{A})$ and (f, A) such that $\varepsilon_d(\bar{f}, \bar{A}) < \varepsilon_d(f, A)$. Then, we can conclude as in the proof of Theorem 1.3. This achieves the proof of Theorem 1.5.

7 Conclusion

The first question is the study of the stability of the solutions localized in Theorem 1.2. Numerical computations [18] lead us to express the following conjecture:

Conjecture 7.1 *Let (f_κ, A_κ) be a solution of $(GL)^s_d$ constructed in Theorem 1.1. It is stable, in the sense that at the point (f_κ, A_κ), $D^2\epsilon_d(f_\kappa, A_\kappa)$ is definite positive.*

On the other hand, let us recall that C. Bolley and P. Del Castillo have proved in [3] the existence and the uniqueness of a solution $(f, A; h)$ of $(GL)_\infty$ for all $\kappa > 0$ when the value of $f(0) \in]0,1]$ is given, but without localizate these solutions. A localization of solutions of $(GL)_\infty$ for pairs (f_0, h) satisfying the de Gennes condition can probably be obtain following the arguments developped in [16]. The study of the stability of the critical points of ϵ_∞ is an open problem. The results obtained at a formal level by S.J. Di Bartolo, A.T. Dorsey and A.J. Dolgert in [20] for the half-space model suggest that there exist κ_0 and $\theta \in C^\infty([0, \kappa_0])$ satisfying $\theta(0) = \frac{1}{\sqrt{2}}$, such that for all $\kappa \leq \kappa_0$, the unstable solutions satisfy $f(0) < \theta(\kappa)$ and the stable solutions satisfy $f(0) > \theta(\kappa)$.

Theorems 1.3 and 1.5 give the existence of non-minimizing solutions, but the approach used here does not permit to determine a localization of these solutions. This question is also an open problem.

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