On the Polyharmonic Operator with a Periodic Potential

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Abstract

In this paper we obtain asymptotic formulas for eigenvalues and Bloch
functions of the polyharmonic operator $L(l, q(x)) = 3D-\Delta^l + q(x)$, = of
arbitrary dimension d with periodic, with respect to arbitrary = lattice,
potential $q(x)$, where $l \geq 1$. Then we prove that the number of gaps = in the
spectrum of the operator $L(l, q(x))$ is finite which is the = generalisation of
the Bethe -Sommerfeld conjecture for this operator. In particular, = taking
$l = 3D1$ we get the proof of the Bethe -Sommerfeld conjecture for = arbitrary
dimension and arbitrary lattice.

INTRODUCTION

In this paper we consider the operator

L(l,q)=3D-$\Delta^l + q(x), \ x \in R^d, \ d \geq 2$
(1)

with a periodic (relative to a lattice Ω) potential $q(x) \in= W_2$
$^s(F)$, where
$s \geq 3d(3^d + d + 2)$, $F \equiv R^d/\Omega$ is a fundamental domain of
Ω, and, without loss of generality, we assume that the measure $\mu(F)$
of F is 1. Let $L_t(l, q)$ be the operator generated in F by (1) = and
the quasiperiodic conditions:

u(x+ω) $= 3De^{i(t,\omega)}u(x), \ \forall \omega \in \Omega,$
(2)

where $t \in F^* \equiv R^d/\Gamma$ and Γ is the lattice = dual to
Ω, that is, Γ is the set of all vectors $\gamma \in R^d$
satisfying $(\gamma, \omega) \in 2\pi Z$ for all $\omega \in \Omega$. It is
well-known that the operator $L_t(l, q)$ has a discrete spectrum = denoted by
$\Lambda_1(t) \leq \Lambda_2(t) \leq$ ....These functions are called = band
functions of the operator L and they form the spectrum $Spec(L)$ of = the
operator L, that is, $Spec(L) = 3D\,\{\Lambda_n(t) : t \in F^*$
,n=3D1,2,.... The corresponding eigenfunctions $\Psi_{1,t}$
(x), $\Psi_{2,t}(x)$, ..., of L_t are known as Block functions. In = the case
$q(x) = 3D0$ these eigenvalues and eigenfunctions are = $\mid \gamma + t \mid^{2l}$ and
$e^{i(\gamma+t,x)}$ respectively, for $\gamma \in \Gamma$.
The paper consists of three sections.
In section 1 the eigenvalues $|\gamma + t|^{2l}$, for = big
$\gamma \in \Gamma$, are divided into two groups: non-resonance ones and
resonance ones and for the perturbations of each group various = asymptotic
formulae are obtained. To describe the non-resonance and resonance eigenvalues
$|\gamma + t|^{2l}$ of the order of ρ^{2l} (= written
as $|\gamma + t|^{2l} \sim \rho^{2l}$ which means that
$c_1\rho < |\gamma + t| < c_2\rho$,where by c_i = for

$i = 3D1,2,...$ we denote the positive, independent of ρ, constants, $=$ whose

exact values are inessential) for big paramater ρ we write the potential $q(x) \in W_2^s$ in the form

q(x)=3D$\sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma=}$
$_1, x) + O(\rho^{-p\alpha}),$
(3)

where $\Gamma(\rho^\alpha) = 3D\{\gamma \in \Gamma : 0 < \mid \gamma \mid$
¡$\rho^\alpha)\}$, $p = 3Ds - d$, $q_0 = 3D0$, $\alpha = 3D\frac{1}{q}$,
$q = 3D3^d + d + 2$. Note that $q(x) \in W_2^s$ means that
$\sum_\gamma \mid q_\gamma \mid^2 (1+ \mid \gamma \mid^{2s}) < \infty$. If
$s \geq d$ then

$\sum_\gamma \mid q_\gamma \mid < c_3, \ \sup \mid \sum_\gamma$
$\notin \Gamma(\rho^\alpha) q_\gamma e^{i(\gamma,x)} \mid \leq \sum_{\mid}$
$\gamma \mid \geq \rho^\alpha \mid q_\gamma \mid = 3DO(\rho^{-p\alpha}),$
(4)

i.e., (3) holds. Define $\alpha_k = 3D3^k\alpha$, for $= k = 3D1, 2, ..., d-1$, and introduce the sets
$V_{\gamma_1}^l(\rho^{\alpha_1}) \equiv \{x \in R^d :\mid\mid x \mid$
$^{2l} - \mid x + \gamma_1 \mid^{2l} \mid < \rho^{\alpha_1}\},$

$\mathrm{E}_1^l(\rho^{\alpha_1}, p) \equiv \bigcup_{\gamma_1 \in \Gamma(p\rho}$
$^{\alpha)} V_{\gamma_1}^l(\rho^{\alpha_1}), \ U^l(\rho$
$^{\alpha_1}, p) \equiv R^d \backslash E_1^l(\rho^{\alpha_1}, p)$

$\mathrm{E}_k^l(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2}$
$,...,\gamma_k \in \Gamma(p\rho^{\alpha})(\cap_{i=3D1}^{k} V_{\gamma_i}$
${}^{l}(\rho^{\alpha_k})),$

where the intersection $\cap_{i=3D1}^{k} V_{\gamma_i}^{l}$ in the = definition of
E_k^l is taken over $\gamma_1, \gamma_2, ..., \gamma_k$ which = are
linearly independent and the length of γ_i is not greater than = the
length of the other vectors in $\Gamma \cap \gamma_i R$. The set U^l
(ρ^{α_1}, p) is said to be a non-resonance domain and the eigenvalue
$\mid \gamma + t \mid^{2l}$ is called a non-resonance eigenvalue if = $\gamma + t \in$
$\mathrm{U}^l(\rho^{\alpha_1}, p)$. The domains $V_{\gamma_1}^{l}(\rho^{\alpha_1}$
$) for \gamma_1 \in \Gamma(p\rho^{\alpha})$ are called resonance = domains and
the eigenvalue $\mid \gamma + t \mid^{2l}$ is called a resonance eigenvalue = if
$\gamma + t \in V_{\gamma_1}^{l}(\rho^{\alpha_1})$. In Section 1 we = prove
that for every $\gamma + t \in U^l(\rho^{\alpha_1}, p)$ there exists an
eigenvalue Λ_N of the operator $L_t(l, q)$ satisfying the following
formulae

$$\Lambda_N = 3D \mid \gamma + t \mid^{2l} + F_{k-1}(\gamma + t) + O(\rho^{-3k\alpha}) =, \tag{5}$$

for $k = 3D1, 2, ..., [\frac{1}{3}(p - \frac{1}{2}q(d-1))]$, where F_{k-1} explicitly
expresed by the potential and eigenvalues of $L_t(l, 0)$ and $[a]$ = denotes
the integer part of a. Moreover we renumber the eigenvalues $\Lambda_N(t)$
by $\Lambda_{N(\gamma+t)}(t)$ or by $\Lambda(\gamma + t)$ if

$$\mid b(N, \gamma) \mid = 3D \max_{\gamma \in \Gamma} \mid b(N, \gamma) \mid, \tag{6}$$

where the max is taken under condition (15) (in the case when the = max

is attained for several γ, then the selection from them is arbitrary), $b(N,\gamma) = 3D(\Psi_{N,t}, e^{i(\gamma+t,x)})$, $\Psi_{N,t}(x)$ is a normalized eigenfunction of $L_t(l,q)$ corresponding to $\Lambda_N(t)$, and = prove:

(a) if $\gamma + t$ is in non-resonance domain then $\Lambda(\gamma + t)$ satisfies (5).

(b) if $\gamma + t \in E_s^l \backslash E_{s+1}^l$, where = $s = 3D1, 2, ..., d-1$, then

$$\Lambda(\gamma + t) = 3D\lambda_j(\gamma + t) + O(\rho^{-k\alpha}), \tag{7}$$

where λ_j is an eigenvalue of the matrix $C(\gamma + t)$ (more precisely about this see Theorem 1(c)). Every big eigenvalue of the = operator

$L_t(l,q)$ for all values of quasimomenta t satisfies one of these formulae.

In section 2 we construct the subset B of $U^l(\rho^{\alpha_1}, p)$ with the following properties:

Pr.1. If $\gamma + t \in B$ then there exists a unique eigenvalue $\Lambda_{N(\gamma+t)}(t) \equiv \Lambda(\gamma + t)$ of operator $L_t(l,q)$ satisfying (5). This is a simple eigenvalue of $L_t(l,q)$. Therefore we call the = set B the simple set of quasimomenta.

Pr.2. The eigenfunction $\Psi_{N(\gamma+t)}(x) \equiv \Psi_{\gamma+t}(x)$ corresponding to the eigenvalue $\Lambda(\ \gamma + t)$, for $\gamma + t \in= B$, is close to $e^{i\langle\gamma+t,x\rangle}$, namely

$$\Psi_{\gamma+t}(x) = 3De^{i(\gamma+t,x)} + \Phi_{k-1}(x) + O(\rho^{-k\alpha_1=}\text{),k=3D1,2,... ,} \tag{8}$$

where $\Phi_0(x) = 3D0$, and $\Phi_{k-1}(x)$, for $k > 1$, is a linear conbination of $e^{i(\gamma+t+\gamma',x)}$ for $\gamma' \in$

$\Gamma((k-1)\rho^{\alpha})\cup\{0\}$ with explicitly expressed = coefficients.
In section 3 we prove that the set B has asymptotically ful measure in R^d and contains the intervals $\{a+sb : s\in[-1,1]\}$ such that $\Lambda(a-b)<\rho^{2l}$, $\Lambda(a+b)>\rho^{2l}$ and = $\Lambda(\gamma+t)$ is continuous on this intervals. Hence there exists $\gamma+t$ for = which $\Lambda(\gamma+t)=3D\rho^{2l}$. It implies the validity of = Bethe-Sommerfeld conjecture for $L(l,q)$. Construction of the set B consists of two = steps.
Step 1. We prove that all eigenvalues $\Lambda(\gamma+t)$, for = $\mid\gamma\mid\sim\rho$, of the operator $L_t(l,q)$, lie in the = ε_1 =3D$\rho^{-d-2\alpha}$, neighborhood of a numbers = $F(\gamma+t)=3D\mid\gamma$ +t$|^{2l}$ $+F_{k_1-1}(\gamma+t)$, $\lambda_j(\gamma+t)$ (see (5), = (7)), where $k_1=3D[\frac{d}{3\alpha}]+2$. We call these number as the known = part of the eigenvalues.
Step 2. By eliminating the set of quasimomenta $\gamma+t$, for which the known parts $F(\gamma+t)$ of $\Lambda(\gamma+t)$ are situated from the known = parts $F(\gamma^{'}+t)$, $\lambda_j(\gamma^{'}+t)$ $(\gamma^{'}\neq\gamma)$ of other eigenvalues at a distance less than $2\varepsilon_1$ we constructed the set B such that if $\gamma+t\in= B$ then the following conditions (called simplicity conditions for = eigenvalue $\Lambda(\gamma+t)\equiv\Lambda_{N(\gamma+t)}(t)$) hold

$$\mid F(\gamma+t)-F(\gamma^{'}+t)\mid\geq 2\varepsilon_1, \; = \tag{9}$$

for $\gamma^{'}\in K\backslash\{\gamma\}$, = $\gamma^{'}+t\in$ U$^1(\rho^{\alpha_1},p)$ and

$$|= F(\gamma+t)-\lambda_j(\gamma^{'}+t)\mid\geq 2\varepsilon_1, \tag{10}$$

for $\gamma^{'}\in K, \gamma^{'}+t\in E_k^1\backslash$

$\mathrm{E}^1_{k+1}, j = 3D1, 2, ...$, where K is the set of $\gamma^{'} \in \Gamma$
satisfying $|= F(\gamma + t) - \mid \gamma^{'} + t \mid^{2l} |< \frac{1}{}$
$3\rho^{\alpha_1}$. Thus B is the set of quasimomenta $\gamma + t \in$
$\mathrm{U}^1(\rho^{\alpha_1}, p)$ satisfying the simplicity conditions (9), = (10). By
these conditions the eigenvalue $\Lambda(\gamma + t)$ can not coincide = with
other eigenvalues.

The listed all results (formulas (5), (7), construction and = investigations of
the simply set B, the proof of asymptotic formulas (8) for Bloch = function
and implication the proof of the Bethe-Sommerfeld conjecture for = arbitraty
dimension and arbitrary lattices from this formulas) for the first = time were
proved in papers [12-15] for the Schrodinger operator $L(1, q)$. The main
difficults and the crucial point of papers [12-14] were the construction = and
investigations of the simple set B of quasimomenta in neighborhood of = the
surface $\{\gamma + t \in= U^1(\rho^{\alpha_1}, p) : F(\gamma + t) = 3D\rho^2\}$. If
$d = 3D2, 3$ then $F(\gamma + t) = 3D \mid \gamma + t \mid^2$, and the matrix $C(\gamma + t)$
corresponds to the Schrodinger operator with directional potential
$q_{\gamma_1}(x) = 3D \sum_{n \in Z} q_{n\gamma_1} e^{i(n\gamma_1, x)}$ (= see
[13]). So for construction of the simple set B of quasimomenta we eliminate
the vicinities of the diffraction planes ant the the sets connected with
directional potential (see (9), (10)). The simple sets B of = quasimomenta
for the first time is constructed and investigated in [13] for $d = 3D3 =$ and in
[12] for $d \geq 2$ (if $d = 3D2$ then for nonsmooth potential $q(x) \in= L_2(F)$,
if $d > 2$ for smooth potential). Then Yu.E. Karpeshina proved (see
[6],[7],[8]) the convergense of pertubation series for a set, that is similar
to B, of quasimomenta in the cases 1. $2l > n$; 2. $4l > n + 1$, $(2l \leq= n)$; 3.
$n = 3D3, l = 3D1$, and using it she proved the validity of the = Bethe-Sommerfeld

conjecture in these cases. In papers [2,3] asymptotic formulas for eigen- values

and Bloch function of the two and three dimensional operator $= L_t(1, q)$ were

obtained. In [4] asymptotic formulae for nonresonance eigenvalues of $L_0(1, q)$ were obtained.

For the first time M.M. Skriganov [10,11] proved the validity of the Bethe-Sommerfeld conjecture for the Scrodinger operator for dimension $d = 3D2, 3$

for arbitrary lattice, for dimension $d > 3$ for rational lattice and for = the

operator $L(l, q)$, for $2l > d$. The Skriganov's method is based on the = detail

investigation of the arifmetical and geometrical properties of the = lattice.

B.E.J.Dahlberg and E.Trubowits [1] using an asymptotic of Bessel = function,

gave the simpler proof of this conjecture for the two dimensional = Scrodinger

operator. B. Helffer and A. Mohamed [5] by investigations the integrated density of states proved the validity of the Bethe-Sommerfeld conjecture = for

the Scrodinger operator for $d \leq 4$, for arbitrary lattice. Recendly Parnovski and Sobelev [9] proved this conjecture for the operator $= L(l, q)$,

for $8l > d + 3$.

The method of this paper and papers [12-15] is a first and uniqie for = the

present by which the validity of the Bethe-Sommerfeld conjecture for arbitrary

lattice, and for arbitrary dimension is proved. For the operator $= L(l, q)$, in

order to avoid eclipsing the essence by technical details, we assume = that

$l \geq 1$. It can be replaced by $l > n_{d,s}$, where $n_{d,s} < 1$ and = depends on dimension d of space and smoothness s of potential $q(x) \in W_2^s(F)$.

In this paper for the different types of the measures of the subset A = of R^d we use the same notation $\mu(A)$, because it will be clear = which

measure we mean. By $\mid A \mid$ we denote the number of elements of the =
set
A and use the following obvious fact. If $a \sim \rho$ then the number = of
elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $\mid$
$\mid \gamma + t \mid -a \mid < 1$ is less than $c_\Gamma \rho^{d-1}$, where
c_Γ is a constant depending on Γ. Therefore the number = of
eigenvalues of $L_t(l, q)$ lying in $(a^{2l} - \rho^{2l-1}, a^{2l} + \rho^{2l-1})$
is less than $c_\Gamma \rho^{d-1}$. Besides we use the inequalities:
1. $\alpha_1 + d\alpha < 1 - \alpha$; 2. $d\alpha < \frac{1}{2}\alpha_d$; = 3.
$k_1 \leq \frac{1}{3}(p - \frac{1}{2}(q(d-1))$; 4. = $p_1\alpha_1 \geq$
pα
5. $3k_1\alpha > d + 2\alpha$; 6. $\alpha_k + (k-1)\alpha < 1$; 7.
$\alpha_{k+1} > 2(\alpha_k + (k-1))\alpha$,
for $k = 3D1, 2, ..., d$, that can be easily verified by using the = definitions
$p = 3Ds - d$, $\alpha_k = 3D3^k\alpha$, $\alpha = 3D\frac{1}{q}$, = $q = 3D3^d + d + 2$,
$k_1 = 3D[\frac{d}{3\alpha}] + 2$, $p_1 \equiv [\frac{p}{3}] + 1$
of the numbers $p, q, \alpha_k, \alpha, k_1, p_1$.

1 Asymptotic Formulae for Eigenvalues

In this section we obtain the asymptotic formulae for nonresonance eigenval-
ues
by iteration of the well-known formula

$(\Lambda_N - \mid \gamma + t \mid^{2l})b(N, \gamma) = 3D(\Psi_{N,t}(x)q(x), e^{i(=a}$
+t,x)).
(11)

Putting the decomposition (3) of $q(x)$ into (11), we get

$(\Lambda_N - \mid \gamma + t \mid^{2l})b(N, \gamma) = 3D\sum_{\gamma_1 \in =a}$
$(\rho^\alpha)q_{\gamma_1}b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha})$.
(12)

¿From the relations (11), (12) it follows that

b(N,$\gamma^{'}$) $= 3D$
)$\Lambda_N - \mid \gamma^{'} + t \mid^{2l} = 3D$
$$\sum_{\gamma_1 \in \Gamma(\rho^{\alpha})}$$
-$\mid \gamma^{'} + t \mid^{2l}$ $+O(\rho^{-p\alpha})$
(13)

for every vector $\gamma^{'} \in \Gamma$ satisfying the inequality

$\mid \Lambda_N - \mid \gamma^{'} + t \mid^{2l} \mid > \frac{1}{2}\rho$
$^{\alpha_1}$,
(14)

which is called the iterability condition. If

$\mid \Lambda_N - \mid \gamma + t \mid^{2l} \mid < \frac{1}{2}\rho^{\alpha_1}$,
(15)

and $\mid \gamma + t \mid^{2l}$ is a non-resonance eigenvalue, i.e., $\gamma + t \in$ U$^{l}(\rho^{\alpha_1}, p)$ then

$\mid\mid \gamma + t \mid^{2l} - \mid \gamma - \gamma_1 + t \mid^{2l} \mid > \rho$
$^{\alpha_1}$, $\mid \Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2l} \mid >$ $\underline{=1}$

$2\rho^{\alpha_1}$,
(16)

for all $\gamma_1 \in \Gamma(p\rho^{\alpha})$. Hence the vector γ
-γ_1 satisfies the iterability condition (14). Therefore, in = (13) one can replace $\gamma^{'}$ by $\gamma - \gamma_1$ and write

b(N,$\gamma - \gamma_1) = 3D$
$$\sum_{\gamma_2 \in \Gamma(\rho^{\alpha})}$$
$\gamma - \gamma_1 + t \mid^{2l} + O(\rho^{-p\alpha})$.

Putting it into right side of (12) and isolating the terms with the coefficient $b(N, \gamma)$, i.e., isolating the case $\gamma_1 + \gamma_2 = 3D0$
we get

$(\Lambda_N - \mid \gamma + t \mid^{2l}) b(N, \gamma) = 3D$
$$\sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^{\alpha})}$$
$\Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2l} + O(\rho^{-p\alpha}) = 3D$

$$\sum_{\gamma_1 \in \Gamma(\rho^{\alpha})}$$
-$\gamma_1 + t \mid^{2l} +$
$$\sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^{\alpha})}$$
$\Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2l} + O(\rho^{-p\alpha})$.

The last sum is taken under the additional condition $\gamma_1 + \gamma$
$_2 \neq 0$. Repeating this process $p_1 \equiv [\frac{p}{3}] + 1 =$ times,
i.e., in last formula replacing $b(N, \gamma^{'})$ for
$\gamma^{'} = 3D\gamma - \gamma_1 - \gamma_2$, and then for $= \gamma$
$^{'} = 3D\gamma - \gamma_1 - \gamma_2 - \gamma_3, ...,$ by its = expression
from (13) and isolating each time the terms with coefficient $= b(N, \gamma)$ we obtain

$(\Lambda_N - \mid \gamma + t \mid^{2l}) b(N, \gamma) = 3D A_{p_1}(\Lambda_N$
$, \gamma + t) b(N, \gamma) + C_{p_1} + O(\rho^{-p\alpha}),$
(17)

where $= A_{p_1}(\Lambda_N, \gamma + t) = 3D \sum_{k=3D1}^{p_1} S_k(\Lambda$
$_N, \gamma + t)$,

$\mathrm{S}_k(\Lambda_N, \gamma + t) = 3D$
$\sum_{\gamma_1, ..., \gamma_k \in \Gamma(\rho^\alpha)}$
$_2 - ... - \gamma_k \prod_{j=3D1}^{k} (\Lambda_N - \mid \gamma + t - \sum_{i=3D=1}$
$^j \gamma_i \mid^{2l}),$

$\mathrm{C}_{p_1} = 3D \sum_{\gamma_1, ..., \gamma_{p_1+1} \in \Gamma(\rho^\alpha) =}$
-$\gamma_1 - \gamma_2 - ... - \gamma_{p_1+1}) \prod_{j=3D1}^{p_1} (= da$
$_N - \mid \gamma + t - \sum_{i=3D1}^{j} \gamma_i \mid^{2l}).$

Here the sums in the expressions of S_k and C_{p_1} are taken = under the additional conditions $\gamma_1+\gamma_2+...+\gamma_s\neq 0=$ for $s=3D1,2,...,k$ and for $s=3D1,2,...,p_1$ respectively. These = conditions and the inclusion $\gamma_i\in\Gamma(\rho^\alpha)$, for $i=3D1,2,...,p_1=$ imply the relation $\sum_{i=3D1}^{j}\gamma_i\in\Gamma(p\rho^\alpha)$ for = the sum in the denominators of the fractions in expressions of S_k and C_{p_1} $.Therefore from the relation (16) it follows that the absolute values = of the$ denominators of the fractions in expressions of S_k and C_{p_1} = more than $(\frac{1}{2}\rho^{\alpha_1})^k$ and = $(\frac{1}{2}\rho^{\alpha_1}$ $)^{p_1}$ respectively. Hence the first inequality in (4) and p_1 $\alpha_1\geq p\alpha$ (see the inequality 4. in the end of = introduction) give

$$C_{p_1}=3DO(\rho^{-p_1\alpha_1})=3DO(\rho^{-p\alpha}),\ =S_k(\Lambda_N,\gamma+t)=3DO(\rho^{-k\alpha_1}),\forall=k=3D1,2,...,p_1. \quad (18)$$

Taking into account that for Λ_N we only used condition (15) = we obtain

$$S_k(a,\gamma+t)=3DO(\rho^{-k\alpha_1}),\forall k=3D1,2,...,p_1 \quad (19)$$

for all a satisfying $\mid a-\mid\gamma+t\mid^{2l}\mid<\frac{1}{2}$ ρ^{α_1}. Thus finding the number N such that $\Lambda_N=$ is close to $\mid\gamma+t\mid^{2l}$ and $b(N,\gamma)$ is not very small = then dividing both sides of (17) by $b(N,\gamma)$ we can get the asymptotic formulas for Λ_N (see Theorem 1(a)). These formulas show that = in the nonresonance case the eigenvalue of the perturbed operator = $L_t(l,q(x))$ is close to the eigenvalue of the unperturbed operator $L_t(l,0)$. = However in

Theorem 1(c) we prove that if $\mid \gamma+t \mid^{2l}$ is resonance eigenvalue, say, if $= \gamma+t \in (\cap_{i=3D1}^{k} V_{\gamma_i}^{l}(\rho^{\alpha_k}))$ for $k \geq 1$, where $\gamma_1,\gamma_2,...,\gamma_k$ are linearly independent vectors of $\Gamma(p\rho^{\alpha})$ and the length of γ_i is not more than the length of other vectors in $\Gamma \cap \gamma_i R$, then the corresponding eigenvalue of $L_t(l,q(x))$ is close to the = eigenvalue of the matrix constructed as follows. Introduse the sets

$\mathrm{B}_k(\gamma_1,\gamma_2,...,\gamma_k) == 3D\{b : b = 3D\sum_{i=3D1}^{k}$
$\mathrm{n}_i\gamma_i, n_i \in Z, \mid b \mid < \frac{1}{2}\rho^{\frac{1}{2}\alpha}$
$_{k+1}\}$,

$\mathrm{B}_k(\gamma+t) = 3D\gamma+t+B_k \equiv \{\gamma+t+b : b \in B_k,\}$,

$\mathrm{B}_k(\gamma+t,p_1) = 3DB_k(\gamma+t)+\Gamma(p_1\rho^{\alpha})$.

Denote by h_i+t for $i = 3D1,2,...,b_k$ the vectors of $B_k(\gamma$ +t,p$_1$), where $b_k \equiv b_k(\gamma_1,\gamma_2,...,\gamma_k)$ is the number = of the vectors of $B_k(\gamma+t,p_1)$. Define the matrix $C(\gamma+t,\gamma$ $_1,\gamma_2,...,\gamma_k) = 3D(c_{i,j})$ by the formula
$c_{i,j} = 3Dq_{h_i-h_j}, \forall i \neq j, c_{i,i} = 3D \mid= h_i+t \mid^{2l}$,
$i,j = 3D1,2,...,b_k$.
We use essentially the following obvious statement:
if $x \in R^d, \mid x \mid\sim \rho, \gamma_1 \in \Gamma, \mid x+\gamma_1$ $\mid\sim \rho$ then

$\mid x \mid^{2l} - \mid x+\gamma_1 \mid^{2l} = 3Da^{2(l-1)}(\mid= x \mid^2 - \mid$
x+$\gamma_1 \mid^2)$
(20)

where $a \sim \rho$. Therefore $= V^l_{\gamma_1}(\rho^{\alpha_1}) \subset$
$\mathrm{V}^1_{\gamma_1}(\rho^{\alpha_1})$,

$(\cap^k_{i=3D1} V^l_{\gamma_i}(\rho^{\alpha_k})) \subset \cap_{i=3D1=}$
${}^k V^1_{\gamma_i}(\rho^{\alpha_k}))$,
(21)

$\mathrm{U}^1(\rho^{\alpha_1}, p) \subset U^l(\rho^{\alpha_1}, p)$
(22)

for $l \geq 1$ and $k = 3D1, 2, ...$Taking into account these inclusions we consider the resonance eigenvalue $\mid \gamma + t \mid^{2l}$ for γ
+t$\in$ $(\cap^k_{i=3D1} V^1_{\gamma_i}(\rho^{\alpha_k}))$.

Theorem 1 (*a*) *Suppose* $\gamma + t \in U^l(\rho^{\alpha_1}, p)$, $\mid \gamma \mid$
$\sim \rho$. *If (15) and*

$\mid b(N, \gamma) \mid > c_4 \rho^{-c\alpha}$
(23)

hold then Λ_N *satisfies formulas (5), for* $= k = 3D1, 2, ..., [\frac{1}{}$
3(p-c)],where

$F_s = 3DO(\rho^{-\alpha_1}), \forall s = 3D0, 1, ...,$

(24)

and $F_0 = 3D0, = F_s = 3DA_s(\mid \gamma + t \mid^{2l} + F_{s-1}, \gamma + t)$, *for* $s = 3D1, 2, ...$
(b) *For* $\gamma + t \in U^l(\rho^{\alpha_1}, p), = \mid \gamma \mid \sim \rho$
there exists an eigenvalue Λ_N *of* $L_t(q(x))$ *satisfying = (5).*
(c)*Suppose* $\mid \gamma \mid \sim \rho$, $\gamma + t \in (\cap_{i=3D1}^{k}$
$V_{\gamma_i}^1(\rho^{\alpha_k}))\backslash E_{k+1}^1$, *where* $= 1 \leq$
$k \leq d-1$. *If (15) and (23) hold then there is an indeks* j *such that*

$$\Lambda_N = 3D\lambda_j(\gamma + t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)a}),$$
(25)

where $\lambda_1(\gamma + t) \leq \lambda_2(\gamma + t) \leq ... \leq \lambda$
$_{b_k}(\gamma + t)$ *are the eigenvalues of the matrix* $C(\gamma + t, \gamma$
$_1, \gamma_2, ..., \gamma_k)$.
(d) *Every eigenvalue* $\Lambda(\gamma + t) \sim \rho^{2l}$ *(see (6) for = the definition of* $\Lambda(\gamma + t)$*) of the operator* $L_t(l, q(x))$ *= satisfies either (5) or (25) for* $c = 3D\frac{q(d-1)}{2}$, *namely =* $\Lambda(\gamma + t)$
satisfies (5) for $\gamma + t \in U^l(\rho^{\alpha_1}, p)$ *and satisfies (25)*
for $\gamma + t \in E_k^1 \backslash E_{k+1}^1$, *where =* $k = 3D1, 2, ..., d-1$.

(a) To prove (5) in case $k = 3D1$ we divide both side of (17) by $= b(N, \gamma)$ and use the estimations (18). Then we obtain

$$\Lambda_N - \mid \gamma + t \mid^{2l} = 3D \mid \gamma + t \mid^{2l} + O(\rho^{-\alpha=_1})$$
(26)

This and $\alpha_1 = 3D3\alpha$ (see the end of the introduction) imply = that formula (5) for $k = 3D1$ holds and $F_0 = 3D0$. Hence (24) for $s = 3D0$ is = also

proved. Moreover by (19) we have
$S_k(\mid \gamma+t\mid^{2l} +O(\rho^{-\alpha_1}),\gamma+t)=3DO(\rho$
$^{-\alpha_1})$ for $k=3D1,2,....$ Therefore (24) for arbitrary $s=$ follows
from the definition of F_s by induction. Now we prove (5) for all $=k$ by
induction. If (5) is true for $k=3Dj$, then putting the expression from =
(5) (for
$k=3Dj$), for Λ_N into $=A_{p_1}(\Lambda_N,\gamma+t)$ in
(17) and dividing both sides of (17) by $b(N,\gamma)$ we get

$\Lambda_N=3D\mid \gamma+t\mid^{2l} +A_{p_1}(\mid \gamma+t\mid^{2l}$
$+\mathrm{F}_{j-1}+O(\rho^{-j\alpha_1}),\gamma+t)+O(\rho^{-(p-c)\alpha})=3D$

$\{\mathrm{A}_{p_1}(\mid \gamma+t\mid^{2l} +F_{j-1}+O(\rho^{-j\alpha_1}$
$),\gamma+t)-A_{p_1}(\mid \gamma+t\mid^{2l} +F_{j-1},\gamma+t)\}+$

$\mathrm{A}_{p_1}(=\mid \gamma+t\mid^{2l} +F_{j-1},\gamma+t)+O(\rho^{-(p-c)\alpha})$

To prove (a) for $k=3Dj+1$ we need to show that the expression in =
curly
brackets is equal to $O(\rho^{-2(j+1)\alpha_1})$. It can be cheked by using
(4), (16), (24) and the obvious relation

$1\overline{\prod_{j=3D1}^{s}(\mid\gamma+t\mid^{2l}+F_{j-1}+O(\rho^{-j\alpha=1}}$
)-$\mid \gamma+t-\sum_{i=3D1}^{s}\gamma_i\mid^{2l})-$

$1\overline{\prod_{j=3D1}^{s}(\mid\gamma+t\mid^{2l}+F_{j-1}-\mid\gamma}$
+t-$\sum_{i=3D1}^{s}\gamma_i\mid^{2l})$

=3D1$\overline{\prod_{j=3D1}^{s}(\mid\gamma+t\mid^{2l}+F_{j-1}-\mid\gamma}$
+t-$\sum_{i=3D1}^{s}\gamma_i\mid^{2l})(\frac{1}{1-O(\rho^{-(j+1)\alpha=1}}$
)-1)

$= 3DO(\rho^{-(j+1)\alpha_1})$, $\forall s = 3D1, 2, ..., p_1$.
(b) Let A be the set of indices N satisfying (15). Using (11) and Bessel inequality, we obtain

$\sum_{N\notin A} \mid b(N,\gamma) \mid^2 = 3D \sum_{N\notin =A} \mid$
${}_N(x), (q(x)e^{i(\gamma+t,x)})\Lambda_N - \mid \gamma + t \mid^{2l} \mid$
${}^2 = 3DO(\rho^{-2\alpha_1})$

Hence by the Parseval equality we have $\sum_{N\in A} \mid= b(N,\gamma) \mid$
${}^2 = 3D1 - O(\rho^{-2\alpha_1})$. This together with $\mid= A \mid< c_\Gamma$
$\rho^{d-1} = 3Dc_\Gamma \rho^{(d-1)q\alpha}$ (see the end of the = introduction)
implies that there exsists a number N satisfying $\mid b(N,\gamma) \mid$
¿1$\frac{}{2(c_\Gamma)^{-1}\rho^{-\frac{(d-1)q}{2}\alpha}}$, that is, = (23)
holds, for $c = 3D\frac{(d-1)q}{2}$. Thus Λ_N satisfies (5) = according to (a).
(c)Writing the equation (12) for all $h_i + t \in= B_k(\gamma + t, p_1)$, we obtain

$(\Lambda_N - \mid= h_i + t \mid^{2l})b(N, h_i) = 3D \sum_{\gamma'}$
$\in \Gamma(\rho^\alpha)q_{\gamma'} b(N, h_i - \gamma'^{=}$
$)$+O$(\rho^{-p\alpha})$,
(27)

for $i = 3D1, 2, ..., b_k$. First we show that if $= (h_i - \gamma'$
+t)$\notin B_k(\gamma + t, p_1)$ then

b(N,h$_i - \gamma') = 3D$

$\sum_{\gamma_1,...,\gamma_{p_1-1}\in\Gamma(\rho^\alpha)}$
$q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1}}b(N,h_i-\gamma^{'}$ $^{=ime}$
-$\sum_{i=3D1}^{p_1}\gamma_i)\prod_{j=3D0}^{p_1-1}(\Lambda_N-=id$
$h_i-\gamma^{'}+t-\sum_{i=3D1}^{j}\gamma_i\mid^{2l})+$
(28)

$+O(\rho^{-p\alpha})=3DO(\rho^{-p\alpha})$.

For this, we apply the formula (13) p_1 times and use the inequality

$\mid\Lambda_N-\mid h_i-\gamma^{'}-\gamma_1-\gamma_2$
-...-$\gamma_s+t\mid^{2l}\mid>\frac{1}{6}\rho^{\alpha_{k+1}}$,
(29)

for $h_i+t\in B_k(\gamma+t,p_1),=(h_i-\gamma^{'}+t)\notin$
$B_k(\gamma+t,p_1)$, and $\gamma_j\in\Gamma(\rho^\alpha)$, where
$j=3D1,2,...,s$ and $s=3D0,1,...,p_1-1$. So now we prove (29). The =
relations
$p>2p_1$ (see the end of the introduction), $h_i+t\in B_k$
$(\gamma+t,p_1)$, $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$ and
$\mid\gamma^{'}\mid,\mid\gamma_1\mid,...,\mid\gamma_{p_1-1}$
$\mid<\rho^\alpha$ imply that
$a_s\equiv h_i-\gamma^{'}-\gamma_1-\gamma_2-...-\gamma$
$_s+t\in B_k(\gamma+t,p)\backslash B_k(\gamma+t)$ for all

$s = 3D0, 1, ..., p_1 - 1$. Since $B_k(\gamma + t, p) = 3D\gamma + t + B_k + \Gamma$ $(p\rho^\alpha)$ we have a decomposition $a_s = 3D\gamma + t + b + a$, where $= b \in$ $B_k, a \in \Gamma(p\rho^\alpha)$, that is,

$$\mid b \mid < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}, \mid a \mid < p\rho^\alpha$$
.
(30)

Using this notation we see that (29) follows from (15) and

$$|| \gamma + t + a + b |^{2l} - | \gamma + t |^{2l}| > \frac{1}{5}$$
$\rho^{\alpha_{k+1}}$
(31)

It follows from (20) that, to verify (31) it is enough to prove it for $l = 3D1$. To prove (31) for $l = 3D1$ we consider two cases:
Case 1. $a \in P \equiv Span\{\gamma_1, \gamma_2, ..., \gamma_k\}$. $=$ Then the relations $a + b \in P$, $\gamma + t + a + b \notin \gamma + t + B_k$ imply that $= a + b \in$ $P \backslash B_k$,i.e., $\mid a + b \mid \geq \frac{1}{2}\ \rho^{\frac{1}{}}$
$2\alpha_{k+1}$. Consider orthogonal decomposition $\gamma + t = 3Dx + v$ of $\gamma + t$, where $v \in P$ and $x \perp P$. It is not hard to show that $=$ (see equality (40) in Remark 1)

$$\mid v \mid = 3DO(\rho^{(k-1)\alpha + \alpha_k}).$$
(32)

It together with above inequality for $\mid a + b \mid$ and the relations $\alpha_{k+1} > 2(\alpha_k + (k-1)\alpha)$ (see the inequality 7. in the $=$ end of the introduction), $\gamma + t = 3Dx + v$, $(x, v) = 3D(x, a) = 3D(x, b) = 3D0$,

$| \gamma + t + a + b |^2 - | \gamma + t |^2 = 3D | a + b + v |^2 - |$
v$|^2$
(33)

give us the estimation (31) for $l = 3D1$.
Case 2. $a \notin P$. First we show that

$|| \gamma + t + a |^2 - | \gamma + t |^2 | \geq \rho^{\alpha_{k+1}=}$.
(34)

Suppose that this inequality does not hold. Then $\gamma + t \in V_a^1$ $(\rho^{\alpha_{k+1}})$. On the other hand $\gamma + t \in \cap_{i=3D1}^k$ V$_{\gamma_i}^1(\rho^{\alpha_{k+1}})$. Therefore we have $\gamma + t \in$ E$_{k+1}^1$ which contradicts to the conditions of the Theorem 1(c). = The difference $| \gamma + t + a + b |^2 - | \gamma + t |^2$ can be = written as the sum of the differences $d_1 \equiv | \gamma + t + a + b |^2 - |$ $\gamma + t + b |^2$ and $d_2 \equiv | \gamma + t + b |^2 - | \gamma$ +t$|^2$. Since $d_1 = 3D | \gamma + t + a |^2 - | \gamma + t |^2 + 2\langle$ a,b$\rangle$, it follows from the inequalities (34), (30) that $|= d_1$ $|> \frac{2}{3}\ \rho^{\alpha_{k+1}}$. Taking $a = 3D0$ in (33) we have $d_2 = 3D | b + v |^2 - | v |^2$. Therefore (32) and (30) imply = that $| d_2 | < \frac{1}{3}\ \rho^{\alpha_{k+1}}$, $| d_1 | - |$ d$_2$ $|> \frac{1}{3}\rho^{\alpha_{k+1}}$, that is, (31) holds.
Thus in all cases (31) and hence (29) is proved. Therefore relation (28) follows from (4) and (29), since $p_1\alpha_{k+1} > p_1\alpha_1 \geq$ pα (see the inequality 4. in the end of introduction). Hence (27) = has form

$(\Lambda_N - | h_i + t |^2) b(N, h_i) = 3D \sum_{\gamma'}$

$q_{\gamma'} b(N, h_i - \gamma') + O(\rho^{-p\alpha}$
),i=3D1,2,...,b_k,

where the sum is taken under the conditions $\gamma' \in \Gamma$
(ρ^{α}) and
$h_i - \gamma' + t \in B_k(\gamma + t, p_2)$. It can be written = as

(C-$\Lambda_N I)(b(N, h_1), b(N, h_2), ...b(N, h_{b_k})) = 3DO(\rho^{-p=pha})$,

where the rigth side of this system is a vector having the norm
$\| O(\rho^{-p\alpha}) \| = 3DO(\sqrt{b_k}\rho^{-p\alpha})$. = Now by
(23) we have

$c_4\rho^{-c\alpha} < (\sum_{h_i} |= b(N, h_i) |^2)^{\frac{1}{2}}$
$\leq \| (C - \Lambda_N I)^{-1} \| \sqrt{b_k} c_5 \rho^{-p=ha}$
,

$\max_{i=3D1,2,...,b_k} =| \Lambda_N - \lambda_i |^{-1} = 3D \|$
(C-$\Lambda_N I)^{-1} \| > c_4 c_5^{-1} b_k^{-\frac{1}{2}}$
$\rho^{-c\alpha+p\alpha}$.

Using that $| B_k | = 3DO(\rho^{\frac{k}{2}\alpha_{k+1}})$, $|$
$\Gamma(p_1\rho^{\alpha}) | = 3DO(\rho^{d\alpha})$ and $= d\alpha < \frac{1}{}$
$2\alpha_d$ (see the end of introduction) we get

$b_k = 3DO(\rho^{d\alpha + \frac{k}{2}\alpha_{k+1}}) = 3DO(\rho^{\frac{d}{2}\bar{l}pha_d}$
)=3DO($\rho^{\frac{d}{2}3^d\alpha}), \forall k = 3D1, 2, ..., d-1$
(35)

Thus the formula (25) follows from (36) and (37).
(d) Let $\Lambda_N \equiv \Lambda(\gamma+t)$ be any eigenvalue = of order ρ^{2l} of the operator $L_t(l,q(x))$. Denote by D the set = of all vectors $\gamma\in\Gamma$ satisfying (15). From (11), arguing as in = the proof of (b), we obtain $\sum_{\gamma\in D} \mid b(N,\gamma)\mid^2$ =3D1-O($\rho^{-2\alpha_1}$). Since $\mid D \mid = 3DO(\rho^{d-1})$ (see = the end of the introduction), from definition of $\Lambda(\gamma+t)$ we get $\mid b(N,\gamma)\mid > c_6\rho^{-\frac{(d-1)}{2}} = 3Dc_6\rho^{-}$ $\overline{(d-1)q2\alpha}$, that is, condition (23) for $c = 3D\frac{(d-1)q}{2}$ = holds.
Now the proof follows from (a) and (c) since either $\gamma+t\in$ U$^1(\rho^{\alpha_1},p)$ or $\gamma+t\in E_k^1\backslash E_{k+1}^1$ for $k=3D1,2,...,d-1$ (see (41) in Remark 1). The Theorem is proved.$\diamondsuit$

Remark 1 *Here we note that the nonresonance domain* $= U^l(c_7\rho^{\alpha_1},p)$ *has an asymptotically full measure on* R^d *in the sence that* $\overline{\mu(U\cap B(\rho))\mu(B(\rho))}$ tends to 1 as ρ tends to infinity, where $B(\rho)=3D\{x\in R^d :\mid x\mid = 3D\rho\}$. By (22) it is enough = to prove this for $l=3D1$. It is clear and well known that $B(\rho)\cap = V_b^1$ (c$_7\rho^{\alpha_1}$) is the part of sphere $B(\rho)$ which is = contained between two parallel hyperplanes $\{x :\mid x\mid^2 - \mid x+b\mid^2 = 3D - c_7\rho^{\alpha_1}\}$and = $\{x :\mid$ x$\mid^2 - \mid x+b\mid^2 = 3Dc_7\rho^{\alpha_1}\}$. The distance of = these hyperplanes from origin are $\frac{c_7\rho^{\alpha_1}}{\mid b\mid}$. This, the relation $\mid \Gamma(p\rho^{\alpha})\mid = 3DO(\rho^{d\alpha})$, and $\alpha_1+d\alpha<1-\alpha$ (see the inequality 1. in the end of the introduction) imply

$\mu(B(\rho)\cap V_b^1(c_7\rho^{\alpha_1}))\sim\rho$
α_1+d-2 $\overline{\mid b\mid,\ \mu(E_1^1\cap = B(\rho)) = 3DO(\rho}$
$d-1-\alpha)$

$\mu(U^1(c_7\rho^{\alpha_1},p)\cap B(\rho)) == 3D(1+O(\rho^{-\alpha}$

$))\mu(B(\rho))$.

Now we consider $= E \equiv (\cap_{i=3D1}^{k} V_{\gamma_i}^{1}(\rho^{\alpha_k}$ $))\cap B(\rho)$. We turn the coordinate axis so that $= Span\{\gamma_{1,}$ $\gamma_2, ..., \gamma_k\}$ coincides with the span of the vectors $e_1 = 3D(1,0,0,...,0)$, $e_2 = 3D(0,1,0,...,0), ...,$ e_k. Then $= \gamma$ $_s = 3D\sum_{i=3D1}^{k}\gamma_{s,i}e_i$ for $s = 3D1,2,...,k$. = Therefore the relation $x \in \cap_{i=3D1}^{k} V_{\gamma_i}^{1}(\rho^{\alpha_k})$ = implies that

$\sum_{i=3D1}^{k}\gamma_{s,i}x_i = 3DO(\rho^{\alpha_k}), s = 3D1,2,...,k; \ddot{=}ext$
$\mathrm{x}_n = 3D\frac{\det(b_{j,i}^{n})}{\det(\gamma_{j,i})} = ,n = 3D1,2,...,k,$

where $x = 3D(x_1, x_2, ..., x_d), \gamma_j = 3D(\gamma_{j,1}, \gamma$ $_{j,2}, ..., \gamma_{j,k}, 0,0,...,0)$, $b_{j,i}^{n} = 3D\gamma_{j,i}$ for $= n \neq j$ and $b_{j,i}^{n} = 3DO(\rho^{\alpha_k})$ for $n = 3Dj$. Taking into = account that the determinant $\det(\gamma_{j,i})$ of the system is a volume of the parallelepiped $\{\sum_{i=3D1}^{k} b_i\gamma_i : b_i \in [0,1], i = 3D1,2,...,k\} =$, hence greater than some constant c and using that $\mid \gamma_{j,i} \mid$ ¡pρ^{α} (since $\gamma_j \in \Gamma(p\rho^{\alpha})$) we get = the estimations

$\mathrm{x}_n = 3DO(\rho^{\alpha_k+(k-1)\alpha})$ $,\forall n = 3D1,2,...,k$
(36)

¿From this for projection v of $x \in \cap_{i=3D1}^{k} V_{\gamma_i}(\rho$ $^{\alpha_k})$ onto $Span\{\gamma_1, \gamma_2, ..., \gamma_k\}$ we obtain estimation (32). Moreover, note that if $k = 3Dd$, then $v = 3Dx$ and by = (40) we have
$\mid x \mid = 3DO(\rho^{\alpha_d+(d-1)\alpha})$ which is imposible, = becouse

$\alpha_d + (d-1)\alpha < 1$ (see the inequality 6. in the end of the introduction) and $x \in B(\rho)$. It means that the intersection of d resonance domains lying in $\{\mid x \mid > \rho_0\}$ for $\rho_0 \gg 1 =$ is empty set. Therefore we have

$$\mathrm{R}^d \cap \{\mid x \mid > \rho_0\} = 3D(U^1 \cup (\cup_{s=3D1}^{d-1}(E_s$$
$$^1 \backslash E_{s+1}^1))) \cap \{\mid x \mid > \rho_0\}.$$
(37)

Remark 2 *Here we note some properties of the known part* $\mid \gamma + t \mid^{2l} + F_k(\gamma + t)$ *(for* $k = 3D1, 2, ...$ *see Theorem = 1) of the nonresonance eigenvalues of* $L_t(l, q(x))$*. Denoting* $\gamma + t$ *by =* x *, where* $\mid \gamma + t \mid \sim \rho$, $\gamma + t \in= U^1(\rho^{\alpha_1}, p)$, *we prove*

$$\partial F_k(x)\overline{_{\partial = x_i = 3DO(\rho^{2-2l-2\alpha_1+\alpha}}}$$
),$\forall i = 3D1, 2, ..., d; \forall k = 3D1, 2, ...$
(38)

We prove (42) by induction with respect to k. Using (20) one can = easily verify that

$$\mid x \mid^{2l} - \mid x - \gamma_1 \mid^{2l} \sim \rho^{2l-2}(\mid x \mid$$
$$^2 - \mid x - \gamma_1 \mid^2),$$

$$\mid x \mid^{2l-2} - \mid x - \gamma_1 \mid^{2l-2} \sim \rho^{2l-4}(\mid = x \mid$$
$$^2 - \mid x - \gamma_1 \mid^2),$$

where $\mid\mid x \mid^2 - \mid = x - \gamma_1 \mid^2 \mid > \rho^{\alpha_1}$, *since* $x \notin V^1_{\gamma_1}(\rho^{\alpha_1})$. *Therefore*

$\partial_{\overline{\partial x_i(}}$
$_1 |^{2l}) = 3D$

$x|^{2l} - | x - \gamma_1 |^{2l})^2) +$
(39)

$x\text{-}\gamma_1 |^{2l})^2 = 3DO(\rho^{2-2l-2\alpha_1+\alpha})$,

where $\gamma_1(i)$ *is the* i*-th component of the vector* γ_1
$\in \Gamma(p\rho^\alpha)$ *hence is equal to* $O(\rho^\alpha)$. *Now* $=$ *(42) for*
$k = 3D1$ *follows from (4) and (43). Suppose that (42) holds for* $k = 3Ds$.
$=$ *Using this*
and (24), replacing $| x |^{2l}$ *by* $| x |^{2l} + F_s(x)$ *in* $=$ *(43)*
and evaluating as above we obtain

$\partial_{\overline{\partial x_i(}}$
$x\text{-}\gamma_1 |^{2l}) = 3D$

$O(\rho^{2l-2-2\alpha_1+\alpha}) +$

$x_i(\mid x\mid^{2l} + F_s - \mid x-\gamma_1\mid^{2l})^2 = 3D$

$O(\rho^{2-2l-2\alpha_1+\alpha}) + O(\rho^{2-2l-4\alpha_1+\alpha}$
$)$=3D$O(\rho^{2-2l-2\alpha_1+\alpha})$

This formula with the definition of F_k implies (42) for $k = 3Ds + 1$.

2 Asymptotic Formulas for Bloch Functions

In this section using the asymptotic formulas for eigenvalues (Theorem = 1) and
simplisity conditions (9), (10) we prove asymptotic formulas for Bloch
functions with a quasimomenta of the simple set B. To use the = simplicity
conditions we write the asymptotic formulas in the following form. By Theorem
1(b) for $\gamma + t \in B \subset U^1(\rho^{\alpha_1}, p)$ there exists = an
eigenvalue Λ_N satisfying (5). Taking $k = 3Dk_1$ in (5), = where
$k_1 = 3D[\frac{d}{3\alpha}] + 2,$ (we can take $k = 3Dk_1$, since = k_1
$\leq \frac{1}{3}(p - \frac{1}{2}q(d-1))$ (see the inequality 3. in the end = of
introduction)) using the relation $3k_1\alpha > d + 2\alpha$ (see the
inequality 5. in the end of the introduction) and notations $F(\gamma$
$+$t)=3D$\mid \gamma + t\mid^{2l} + F_{k_1-1}(\gamma+t)$, ε_1
=3D$\rho^{-d-2\alpha}$ (see introduction) we obtain

$\Lambda_N - F(\gamma+t) = 3Do(\varepsilon_1)$
(40)

Theorem 2 *If $\gamma+t\in B$ and $\mid\gamma+t\mid\sim\rho$ then there exists a = unique eigenvalue Λ_N satisfying (5) for = $k=3D1,2,...,n=3D[\frac{p}{3}]$. This is a simple eigenvalue and the corresponding eigenfunction $\Psi_N(x)$ satisfies*

$$\Psi_N(x)=3De^{i(\gamma+t,x)}+O(\rho^{-\alpha_1}). \tag{41}$$

This eigenvalue is the $\Lambda(\gamma+t)$ (see (6) for the definition = of $\Lambda(\gamma+t)$).

We noted that (see above) there exists an eigenvalue Λ_N satisfying (44). Let Ψ_N be any normalized eigenfunction corresponding to Λ_N. The formula (45) is equivalent to = $b(N,\gamma)=3D1+O(\rho$ $^{-\alpha_1})$, where $b(N,\gamma)=3D(\Psi_N,e^{i(\gamma+t,x)})$. = Since the normalized eigenfunction is defined up to constant of modulas 1 we can suppose that $\arg b(N,\gamma)=3D0$. Hence to prove (45) it suffices to = prove the equality $\mid b(N,\gamma)\mid=3D1+O(\rho^{-\alpha_1})$ which = equavalent to

$$\sum_{\gamma'\in\Gamma\backslash\{\gamma\}}\mid b(N,\gamma')\mid^2=3DO(\rho^{-\alpha_1}) \tag{42}$$

Thus we need to show that (46) holds. First consider the case $\gamma'\notin K$ (for the definition of K see introduction). = Using (44), definition of K , and then (11), we get

$$\mid\Lambda_N-\mid\gamma'+t\mid^{2l}\mid>\frac{1}{4}\rho^{\alpha_1},\gamma'\notin K,$$

$\sum_{\gamma' \notin K} \mid b(N,\gamma^{'}) \mid$
$^{2} = 3D \parallel \Psi_N(x)q(x) \parallel^2 O(\rho^{-2\alpha_1}) = 3DO(\stackrel{\circ}{=}ho$
$^{-2\alpha_1}).$

If $\gamma^{'} \in K$, then clearly

$\mid \Lambda_N - \mid \gamma^{'} + t \mid^{2l} \mid < \frac{1}{2}\rho$
$^{\alpha_1},$
(43)

Now we prove that the simplicity conditions (9), (10) imply

$\mid b(N,\gamma^{'}) \mid \leq c_4 \rho^{-c\alpha}, \forall$
$\gamma^{'} \in K\backslash\{\gamma\},$
(44)

for $c = 3D2dq - \frac{1}{4}d3^d - d - 3 = 3D\frac{7}{4}d3^d + 2d^2 + 3d - 3$. If $=$ for $\gamma^{'} + t \in U^1(\rho^{\alpha_1}, p)$ and $= \gamma^{'}$
$\in K\backslash\{\gamma\}$ the inequality in (49) is not true then by Theorem 1(a) (see Theorem 1(a), and inequalities (48), (49)) we have

$\Lambda_N = 3D \mid \gamma^{'} + t \mid^{2l} + F_{k-1}(\gamma^{'=}$
+t)+O($\rho^{-3k\alpha}$),
(45)

for $k = 3D1, 2, ..., [\frac{1}{3}(p-c)]$. Now taking $= k = 3Dk_1 = 3D[\frac{d}{3\alpha}]$ (we can take $k = 3Dk_1$, since one can easily verify that $= \frac{d}{3\alpha}$

$+2\leq \frac{1}{3}(p-c)$), and arguing as in the prove of (44) we get

$$\Lambda_N - F(\gamma^{'}+t) = 3Do(\varepsilon_1) \ ,$$

because $3k_1\alpha > d+2\alpha$ (see the inequality 5. in the end of introduction). This with (44) contradicts the simplicity condition = (9). Similarly if, for $\gamma^{'}+t \in (E_k^1\backslash = E_{k+1}^1)$ and $\gamma^{'}\in K$, the inequality in (49) does not hold then = by Theorem 1(c)

$$\Lambda_N = 3D\lambda_j(\gamma^{'}+t) + O(\rho^{-(p-c-\frac{1}{4}=}$$
$$d3^d)\alpha).$$
(46)

It is not hard to verify that $(p-c-\frac{1}{4}d3^d)\alpha > d+2\alpha =$. Therefore we have

$$\Lambda_N - \lambda_j(\gamma^{'}+t) = 3Do(\varepsilon_1)$$

This with (44) contradicts the simplicity condition (10). Hence the inequality in (49) holds. Therefore using $\mid K \mid = 3DO(\rho^{q(d-1)\alpha})$ we have

$$\sum_{\gamma^{'}\in K\backslash\{\gamma\}} \mid b(N,\gamma^{'}$$
$$)|^2 = 3DO(\rho^{-(2c-q(d-1))\alpha}),$$
(47)

for $c = 3D2dq - \frac{1}{4}d3^d - d - 3$. Since $= c - q(d-1) > 3, \alpha_1 = 3D3\alpha$ the equality (52) and the equality in (47) imply (46). Thus we proved that = for any

normalized eigenfunction Ψ_N corresponding to any eigenvalue

Λ_N satisfying (5) the equality (45) holds. If there exist = two

different eigenvalues or multiple eigenvalue satisfying (5) then there = exist

two orthogonal normalized eigenfunction satisfying (45) which is = imposible.

Therefore Λ_N is a simple eigenvalue. It follows from Theorem 1(a)

that Λ_N satisfies (5) for $k = 3D1, 2, ..., n = 3D[\frac{p}{3}]$, = because the

inequality (23) holds for $c = 3D0$ (see (45)). Since (45) holds it = follows from

the definition of $\Lambda(\gamma + t)$ (see (6)) that if $\gamma + t \in B$ then

the eigenvalue Λ_N satisfying (5) is $\Lambda(\gamma + t)$. The

theorem is proved■

Now we prove the asymptotic formulas of arbitrary order for Bloch = functions.

Theorem 3 *If $\gamma + t \in B$ and $\mid \gamma + t \mid \sim \rho$ then the eigenfunction*

$\Psi_{\gamma+t}(x) \equiv \Psi_{N(\gamma+t)}(x)$ corresponding to the

eigenvalue $\Lambda_N \equiv \Lambda(\gamma + t)$ satisfies formulas (8) = for

$k = 3D1, 2, ..., n = 3D[\frac{1}{3}(c - \frac{1}{2}q(d-1))]$, where = $c = 3D2dq - \frac{1}{}$

$4d3^d - d - 3$, $\Phi_0(x) = 3D0$, $\Phi_1(x) = 3D$

$$\sum_{\gamma_1 \in \Gamma(\rho^\alpha)}$$

-$\mid \gamma + \gamma_1 + t \mid^{2l}$), and $\Phi_{k-1}(x)$ is a linear

conbination of $e^{i(\gamma+t+\gamma',x)}$ for γ'

$\in \Gamma((k-1)\rho^\alpha) \cup \{0\}$ with explicitly expressed

coefficients (58), (59).

By Theorem 2 formula (8) for $k = 3D1$ is proved. To prove formula (8) for

arbitrary $k \leq n$ we prove the following equivalent relations

$\sum_{\gamma' \in \Gamma^c(k-1)} \mid b(N, \gamma + \gamma'$

$)|^2 = 3DO(\rho^{-2k\alpha_1}),$
(48)

$\Psi_N = 3Db(N,\gamma)e^{i(\gamma+t,x)} + \sum_{\gamma' \in}$
$\Gamma((k-1)\rho^\alpha)b(N,\gamma+\gamma')e^{i(\gamma}$
$+t+\gamma', x) + H_k(x),$
(49)

where $= \Gamma^c(m) \equiv \Gamma\backslash(\Gamma(m\rho^\alpha) \cup \{0\})$
and $\| H_k(x) \| = 3DO(\rho^{-k\alpha_1})$. The case $= k = 3D1$ is
proved accoding to (46). Assume that (53) is true for $k = 3Dm$. Then
by = (54) for
$k = 3Dm$, and by (3) we have $= \Psi_N(x)(q(x)) = 3DH(x) + O(\rho^{-m\alpha_1})$,
where $H(x)$ is linear conbination of $= e^{i(\gamma+t+\gamma',x)}$
for $\gamma' \in \Gamma(m\rho^\alpha) \cup \{0\}$. Hence
$(H(x), e^{i(\gamma+t+\gamma',x)}) = 3D0$ for $= \gamma'$
$\in \Gamma^c(m)$. Therefore using (11) and the inequality in (47), we = get

$\sum_{\gamma'} \mid b(N,\gamma+\gamma') \mid^2$
$=3D\sum_{\gamma'} \mid$
$+t+\gamma', x))\Lambda_N - \mid \gamma+\gamma'$
$+t|^{2l}|^2 = 3DO(\rho^{-2(m+1)\alpha_1}),$
(50)

where the sum is taken under conditions $= \gamma' \in \Gamma^c(m)$
and $\gamma+\gamma' \notin K$. On the other hand, using $= \alpha$
$_1 = 3D3\alpha$ and definition of n, from (52) we obtain

$\sum_{\gamma' \in K\backslash\{\gamma\}} \mid b(N,\gamma'$
$)|^2 = 3DO(\rho^{-2n\alpha_1}),$

This with (55) implies (53) for $k = 3Dm+1$. Thus formula (54) is also = proved.

Here $b(N,\gamma)$ and $b(N,\gamma+\gamma^{'})$ for $\gamma^{'}$
$\in \Gamma((n-1)\rho^{\alpha})$ can be calculated as follows. In (12) replase
γ by $\gamma+\gamma^{'}$. Iterate it n times and every times
isolate the terms with maltiplicant $b(N,\gamma)$. In other word apply = (13)
for $b(N,\gamma+\gamma^{'})$, where $\gamma^{'} \in$
$\Gamma((n-1)\rho^{\alpha})$, and hence $\gamma^{'} \neq 0$. Then = apply
(13) for $b(N,\gamma+\gamma^{'}-\sum_{i=3D1}^{j}\gamma_i)$ = when
$\gamma^{'}-\sum_{i=3D1}^{j}\gamma_i \neq 0$ where = $\gamma_i \in$
$\Gamma(\rho^{\alpha})$, $j = 3D1, 2, ..., n-1$. Then using (4) and the = relation
$\mid \Lambda_N - \mid \gamma+t+\gamma^{'}-\sum_{i=3D1}^{j}\gamma_{=i}$
$|^{2l}|> \frac{1}{2}\rho^{\alpha_1}$ (see (16) and use the fact = that
$\gamma^{'}-\sum_{i=3D1}^{j}\gamma_i \in \Gamma(p\rho^{\alpha})$ = since
$p > 2n$),
$\Lambda_N = 3DP(\gamma+t)+O(\rho^{-n\alpha_1})$, where $P(\gamma$
+t)=3D$\mid \gamma+t \mid^{2l} + F_{[\frac{p}{3}]}(\gamma+t)$ (see Theorem = 2), we obtain

b(N,$\gamma+\gamma^{'}) = 3D\sum_{k=3D1}^{n-1} A_k(\gamma^{'=}$
)b(N,$\gamma) + O(\rho^{-n\alpha_1}),$
(51)

where
$A_1(\gamma^{'}) \equiv$
+t)-$\mid \gamma+\gamma^{'}+t \mid^{2l} = 3D$
$\mid \gamma+t \mid^{2l} - \mid \gamma+\gamma^{'}+t \mid^{2l}$
+O$(\rho^{-3\alpha_1})$
$A_k(\gamma^{'}) = 3D$
$$\sum_{\gamma_1,\dots,\gamma_{k-1}}$$

-$\gamma_1-\gamma_2-...-\gamma_{k-1}\prod_{j=3D0}^{k-1}(P(\gamma$
+t)-$|\gamma+t+\gamma^{'}-\sum_{i=3D1}^{j}\gamma_i|^{2l})=$
=3DO$(\rho^{-k\alpha_1})$,

$\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})}|=A_1(\gamma^*$
$)|^2=3DO(\rho^{-2\alpha_1}),\sum_{\gamma^*\in\Gamma((n-1)\rho=}$
$^{\alpha})\mid A_k(\gamma^*)\mid=3DO(\rho^{-k\alpha_1}),$
(52)

for $k>1$. Now from (54) for $k=3Dn$ and (56), we obtain

$\Psi_N=3Db(N,\gamma)e^{i(\gamma+t,x)}+\sum_{\gamma^*\in\Gamma}$
((n-1)$\rho^{\alpha})\sum_{k=3D1}^{n-1}(A_k(\gamma^*)b(N,\gamma$
)+O$(\rho^{-n\alpha_1}))e^{i(\gamma+t+\gamma^*,x)})+H_n$

Since $\parallel\Psi_N\parallel=3D1,\arg b(N,\gamma)=3D0,$ the = functions
$e^{i(\gamma+t,x)}$, $H_n(x)$, $e^{i(\gamma+t+\gamma^*,x)}$,
$(\gamma^*\in\Gamma((n-1)\rho^{\alpha}))$ are orthogonal and = $\parallel$
H$_n$ $\parallel=3DO(\rho^{-n\alpha_1})$ we have
$1=3D\mid b(N,\gamma)\mid^2+\sum_{k=3D1}^{n-1}(\sum_{\gamma^*\in}$
$\Gamma((n-1)\rho^{\alpha})\mid A_k(\gamma^*)b(N,\gamma)\mid^2$
+O$(\rho^{-n\alpha_1}))$,

b(N,γ) $=3D(1+\sum_{k=3D1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)=o}$
$^{\alpha})\mid A_k(\gamma^*)\mid^2))^{-\frac{1}{2}}+O(\rho$
$^{-n\alpha_1}))$.
(53)

(See the second equality in (57).) Now from (56) we obtain

b(N,$\gamma+\gamma^{'}) = 3D(\sum_{k=3D1}^{n-1} A_k(\gamma^{=e}$
$))(1+\sum_{k=3D1}^{n-1}\sum_{\gamma^*} \mid= A_k(\gamma^*)\mid$
$^2)^{-\frac{1}{2}} + O(\rho^{-n\alpha_1})$
(54)

Consider the case $n = 3D2$. By (58), (57), (59) we have $b(N,\gamma$
)=3D1+O($\rho^{-2\alpha_1}$),
$b(N,\gamma+\gamma^{'}) = 3DA_1(\gamma^{'}) + O(\rho^{-2=pha}$
$_1) = 3D$
$+\gamma^{'} + t \mid^{2l} + O(\rho^{-2\alpha_1})$ for all
$\gamma^{'} \in \Gamma(\rho^{\alpha})$. These and (54) for $k = 3D2 =$ imply
the formula for $\Phi_1\diamondsuit$

3 Simple Sets and Bethe-Sommerfeld conjecture

In this section we construct a part of the simple set in neighbourhood = of the
surface $S_\rho = 3D\{x \in= U^1(2\rho^{\alpha_1},p) : F(x) = 3D\rho^{2l}\}$. Due
to (5) (replace $\gamma+t$ by x in $F(\gamma+t)$) it is natural to = call
S_ρ the approximated isoenergetic surfaces in the nonresonance domain.
As we noted in introduction in order that the nonresonance eigenvalue
$\Lambda(\gamma+t)$ can not coincide with any other nonresonance = eigenvalue
$\Lambda(\gamma+t+b)$ we choose the $\gamma+t$ such that $\mid F(\gamma$
+t)-F$(\gamma+t+b) \mid > 2\varepsilon_1$ for $\gamma+t+b \in U^1(\rho$
$^{\alpha_1},p)$ and $b \in \Gamma\backslash\{0\}$ (see (9)). Therefore = we
eliminate

P$_b = 3D\{x : x, x+b \in U^1(\rho^{\alpha_1},p), \mid F(x) - F(x+b) \mid$
¡$3\varepsilon_1\}$

(55)

for $b \in \Gamma\backslash\{0\}$ from S_ρ. Denote the remaining = part of S_ρ by $S_\rho^{'}$. Then we consider the ε neighbourhood $U_\varepsilon(S_\rho^{'}) = 3D\cup_{a\in=S_\rho}$ $^{'}U_\varepsilon(a)\}$ of $S_\rho^{'}$, where $\varepsilon = 3D\frac{\varepsilon_1}{7l\rho^{2l-1}}$, U_ε (a)=3D{x$\in R^d :\mid x-a\mid<\varepsilon\}$. In this set the first simplicity condition (9) holds (see Lemma 4(a)). Denote by $Tr(E) = 3D\{\gamma + x \in U_\varepsilon(S_\rho^{'}) : \gamma \in \Gamma, x \in E\}$ and $Tr_{F^\star}(E) \equiv \{\gamma + x \in F^\star : \gamma \in \Gamma, x \in= E\}$ the translations of $E \subset R^d$ into $U_\varepsilon(S_\rho^{'})$ and $F^\star$ respectively. In order that the second simplicity = condition (10) holds, we discart from $U_\varepsilon(S_\rho^{'})$ the translation $Tr(A(\rho))$ of

A$(\rho) = 3D\cup_{k=3D1}^{d-1}(\cup_{\gamma_1,\gamma_2,\dots,\gamma_k{}^1=n}$ $\Gamma(p\rho^\alpha)(\cup_{i=3D1}^{b_k}A_{k,i}(\gamma_1,\gamma$ $_2,\dots,\gamma_k))),$

(56)

where $A_{k,i}(\gamma_1,\dots,\gamma_k) = 3D\{x \in (\cap_{i=3D1}^{k}V_{\gamma_{i=}}$ $^1(\rho^{\alpha_k})\backslash E_{k+1}^1) \cap K_\rho : \lambda_i$ (x)$\in (\rho^{2l} - 3\varepsilon_1, \rho^{2l} + 3\varepsilon_1)\}$, and

K$_\rho = 3D\{x \in R^d :\mid\mid= x\mid^{2l} - \rho^{2l}\mid<\rho^{\alpha_1}\}.$

(57)

As a result we construct the part $U_\varepsilon(S'_\rho$
$)\backslash Tr(A(\rho))$ of the simple set B (see Theorem 5(a)) which contains the intervals $\{a+sb : s\in[-1,1]\}$ such that Λ
(a-b)¡ρ^{2l}, $\Lambda(a+b)>\rho^{2l}$ and $\Lambda(\gamma+t)$ is
continuous on this intervals. Hence there exists $\gamma+t$ for which $\Lambda(\gamma+t)=3D\rho^{2l}$. It implies the validity of = Bethe-Sommerfeld conjecture for $L(l,q)$. For this we need the following lemma.

Lemma 4 *(a) If $x\in U_\varepsilon(S'_\rho)$ and $x+b\in U^1$
(ρ^{α_1},p), where $b\in\Gamma$, then*

$$| F(x)-F(x+b) |> 2\varepsilon_1 \tag{58}$$

Hence for $\gamma+t\in U_\varepsilon(S'_\rho)$ the first simplicity condition (9) holds.
*(b) If $x\in U_\varepsilon(S'_\rho)$ then $x+b\notin$
$U_\varepsilon(S'_\rho)$ for all $b\in\Gamma$.*
(c)If $E\subset R^d$ is bounded set then $\mu(Tr(E))\le\mu(E)$.
*(d) If $E\subset U_\varepsilon(S'_\rho)$ then μ
$(Tr_{F^\star}(E))=3D\mu(E)$*

(a) If $x\in U_\varepsilon(S'_\rho)$ then there is a = point
a in S'_ρ such that $x\in U_\varepsilon(a)$. Since
S'_ρ does not contain the set P_b (see (60)) we = have

$$| F(a)-F(a+b) |\ge 3\varepsilon_1 \tag{59}$$

On the other hand using (42) and the relations
$| x |<\rho+1$, $| x-a |<\varepsilon$, $| x+b-a-b |$
¡ε we obtain

$| F(x) - F(a) |< 3l\rho^{2l-1}\varepsilon,$

$| F(x+b) - F(a+b) |< 3l\rho^{2l-1}\varepsilon.$

These inequalities together with (64) give (63), since $6l\rho^{2l-1}$
$\varepsilon < \varepsilon_1$.
(b) This is a simple consequence of (a). Indeed if x and $x+b$ lie = in
$U_\varepsilon(S_\rho^{'})$ then there are points a and = c in
$S_\rho^{'}$ such that $x \in U_\varepsilon(a)$ and $x+b \in$
$U_\varepsilon(c)$. Repeating the proof of (65) we get $| F(c) - F(x+b) |$
¡3l$\rho^{2l-1}\varepsilon$. This, the first ineguality in (65), and the
relations $F(a) = 3D\rho^2, F(c) = 3D\rho^2$ (see the definition of = S_ρ)
give $| F(x) - F(x+b) |< \varepsilon_1$ which contradicts (63).
(c) Clearly for any bounded set E there are only finite number of = the
vectors $\gamma_1, \gamma_2, ..., \gamma_s$ such that $E(k) \equiv$
(E+$\gamma_k) \cap U_\varepsilon(S_\rho^{'}) \neq \emptyset$
,k=3D1,2,...,s*and*Tr(E)*istheunionofthesets*E(k).*For* =E(k)-γ
$_k$ we have the relations = $\mu(E(k) - \gamma_k) = 3D\mu(E(k)), E(k) - \gamma$
$_k \subset E$. Moreover by (b)
$(E(k) - \gamma_k) \cap (E(j) - \gamma_j) = 3D\emptyset$ for $k \neq j$. = Therefore
(c) is true.
(d) Now let $E \subset U_\varepsilon(S_\rho^{'})$. Then by (b)
the set E can by devided into finite number of the pairwise disjoint = sets
$E_1, E_2, ..., E_n$ such that there are the vectors = γ_1, γ
$_2, ..., \gamma_n$ satisfying $(E_k + \gamma_k) \subset F^\star$
,(E$_k + \gamma_k) \cap (E_j + \gamma_j) \neq \emptyset$,for $k \neq j$ = and
$k, j = 3D1, 2, ..., n$. Using $\mu(E_k + \gamma_k) = 3D\mu(E_k)$ we get the = proof of
(d), becouse $Tr_{F^\star}(E)$ and E are union of piarwise = disjoint sets
$E_k + \gamma_k$ and E_k for $k = 3D1, 2, ..., n$ = respectively$\diamondsuit$

Theorem 5 (a) *The set* $U_\varepsilon(S_\rho^{'}) \backslash = Tr(A(\rho))$ *is*
a subset of B. *Hence if* $\gamma + t$ *lies in this subset then Theorem 2 holds.*
For every connected open subset E *of* = $U_\varepsilon(S_\rho^{'}$

$)\backslash Tr(A(\rho)$ there exists a unique index N such that
$\Lambda(\gamma+t)=3D\Lambda_N(t)$ for all $\gamma+t\in E$.
(b) For the part $V_\rho\equiv S_\rho^{'}\backslash$
$U_\varepsilon(Tr(A(\rho)))$ of the approximated isoenergetic surface = the following holds

$$\mu(V_\rho)>(1-c_8\rho^{-\alpha}))\mu(B(\rho)).$$
(60)

Moreover $U_\varepsilon(V_\rho)$ lies in the subset U_ε
$(S_\rho^{'})\backslash Tr(A(\rho))$ of the simple set B.
(c) The number ρ^{2l} for $\rho\gg 1$ lies in the spectrum of $L(l,q)$, that is, the number of the gaps in the spectrum of $L(l,q)$ is finite.

(a) Suppose $\gamma+t\equiv x\in U_\varepsilon(S_\rho^{'}$
$)\backslash Tr(A(\rho))$, that is , there is $a\in= S_\rho^{'}$ such
that $x\in U_\varepsilon(a)$. Then, by lemma 4(a), the simplisity condition
(9) holds. Now we prove that (10) holds too. The first inequality in =
(65) and
the relation $F(a)=3D\rho^{2l}$ give

$$\mathrm{F}(\gamma+t)\in(\rho^{2l}-3l\rho^{2(l-1)}\varepsilon,\rho^{2l}+3l\rho^{2(l-1)}\varepsilon)\subset(\rho^{2l}-\varepsilon_1,\rho^{2l}+\varepsilon_1)$$
(61)

for $\gamma+t\in U_\varepsilon(S_\rho^{'})$. On the other = hand
$\gamma+t\notin Tr(A(\rho))$. It means that for any $\gamma^{'}$
$\in\Gamma$, we have $\gamma^{'}+t\notin A(\rho)$. If γ
$^{'}\in K$ and $\gamma^{'}+t\in E_k^1\backslash$
E_{k+1}^1 then by definition of K (see introduction) the = inequality
$\mid F(\gamma+t)-\mid\gamma^{'}+t\mid^{2l}\mid<\frac{1}{3}$

ρ^{α_1} holds. This (24) and (67) imply that $= \gamma^{'}$
$+$t$\in (E_k^1 \backslash E_{k+1}^1) \cap K_\rho$ (see (62) for the
definition of K_ρ). Sinse $\gamma^{'} + t \notin A(\rho)$ we have
$\lambda_i(\gamma + t) \notin (\rho^{2l} - 3\varepsilon_1, \rho^{2l}$
$+3\varepsilon_1)$ for $\gamma^{'} \in K$ and $\gamma^{'} + t \in$
E$_k \backslash E_{k+1}$. Therefore (10) follows from (67). Moreover it = is
clear that the inclusion $S_\rho^{'} \subset U(2\rho^{\alpha_1}, p)$
(see definition of S_ρ and $S_\rho^{'}$) implies that
$U_\varepsilon(S_\rho^{'}) \subset U(\rho^{\alpha_1}, p).$ = Thus
$U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)) \subset B.$
Now let E be a connected open subset of $U_\varepsilon(S_\rho^{'}$
$) \backslash Tr(A(\rho) \subset B$. By Theorem 2 for $a \in E \subset$
U$_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)$ the = eigenvalue
$\Lambda(a)$ is simple and the corresponding eigenfunction $\Psi_a(x)$
satisfies $\mid (\Psi_a(x), e^{i(a,x)}) \mid^2 > \frac{1}{2}$. So for $= y \in E$
there exists a unique $N(y)$ such that $\Lambda(y) = 3D\Lambda_{N(y)}(y)$
$, \Psi_y(x) = 3D\Psi_{N(y),y}(x),$

$$\mid (\Psi_{N(y),y}(x), e^{i(y,x)}) \mid^2 > \tfrac{1}{2}. \qquad (62)$$

On the other hand for fixed N the functions $\Lambda_N(t)$ and
$(\Psi_{N,t}(x), e^{i(t,x)})$ are continuous in neighborhood of t if
$\Lambda_N(t)$ is a simple eigenvalue. Therefore for $a \in E$ there = exists a
neighborhood $U(a) \subset E$ of a such that $\mid (\Psi_{N(a),y}$
(x),e$^{i(y,x)}) \mid^2 > \frac{1}{2}$, for every $y \in U(a)$. This and = (68)
yield that $N(y) = 3DN(a)$ for $y \in U(a)$, since there is a unique = integer
N
satisfying (68). Hence we proved the Proposition 1 which is $\forall = a \in$
E,$\exists U(a) \subset E : N(y) = 3DN(a), \forall y \in U(a)$. Now let $a_1 =$ and
a_2 be two points of E and $C \subset E$ be the arc which joins = these
points. (Note that the open connected subset of R^d is arcwise connected).
Let $U(y_1), U(y_2), ..., U(y_k)$ be finite subcover of the open = cover
$\cup_{a \in C} U(a)$ of the compact C, where $U(a)$ is a neighborhood = of a

satisfying Proposition 1. By Proposition 1 $N(y) = 3DN(y_i) = 3DN_i$ for $= y \in$
U(y_i). Clearly if $U(y_i) \cap U(y_j) \neq \emptyset$ then $N_i = 3DN_j$
since for $a \in U(y_i) \cap U(y_j)$ there is a unique $N(a)$. Thus
$N_1 = 3DN_2 = 3D... = 3DN_k$ and $N(a_1) = 3DN(a_2)$.
(b) First we note that if $a \in V_\rho$ then $U_\varepsilon(a) \in$
U$_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$, because = the
relation $a \in S_\rho^{'}$ implies that $U_\varepsilon(a) \in$
U$_\varepsilon(S_\rho^{'})$ and the relation $a \notin$
U$_\varepsilon(Tr(A(\rho)))$ implies that $U_\varepsilon(a) \cap$
Tr(A(ρ)) $= 3D\emptyset$. To prove (66) first we estimate the measure of
$S_\rho, S_\rho^{'}, U_{2\varepsilon}(A(\rho))$, namely we = prove

$$\mu(S_\rho) > (1 - c_9\rho^{-\alpha})\mu(B(\rho)),$$

$$\mu(S_\rho^{'}) = 3D(1 - c_{10}\rho^{-\alpha})\mu(B(\rho)),$$

$$\mu(U_{2\varepsilon}(A(\rho))) == 3DO(\rho^{-\alpha})\mu(B(\rho))\varepsilon.$$

(See below, Estimations 1, 2, 3). The estimation (64) of the measure of = set
V_ρ is done in Estimation 4 by using Estimations 1, 2, 3.
(c) Since $F(a) = 3D\rho^{2l}$ for $= a = 3D(a_1, a_2, ..., a_d) = 3D\sum_{i=3D1}$
$^d a_i e_i \in V_\rho \subset S_\rho$ it follows from (24) that
$\rho - 1 <\mid a \mid< \rho + 1$, and there is an indeks i such that $\mid$
a$_i$ $\mid> \frac{1}{d}\rho$. Without loss of generality assume that
$a_i > 0$. Then (42) and (44) imply that $F(a - \varepsilon e_i) < \rho$
$^{2l} - c_{11}\varepsilon_1$, $F(a + \varepsilon e_i) > \rho^{2l} + c_{11}$
ε_1 and $\Lambda(a - \varepsilon e_i) < \rho^{2l}$, Λ
(a+εe_i) $> \rho^{2l}$. Since $\Lambda(\gamma + t)$ is = continuous
there exists a point $y(a) \in [a - \varepsilon e_i, a + \varepsilon e_i]$
such that $\Lambda(y(a,i)) = 3D\rho^{2l}$.The Theorem is proved = $\diamondsuit$
In Estimations 1-4 we use the notations: $G(+i, a) = 3D\{x \in G, x_i > a\}$,
$G(-i, a) = 3D\{x \in G, x_i < -a\}$, where $= x = 3D(x_1, x_2, ..., x_d), a > 0$.
It is

not hard to verify that for any subset G of $U_\varepsilon(S_\rho$
$') \cup U_{2\varepsilon}(A(\rho))$, hence for all considered sets
G in these estimations, and for any $x \in G$ the followings hold

$$\rho - 1 <| x |< \rho + 1, G \subset (\cup_{i=3D1}^{d}(G(+i, \rho d^{-1})\cup$$
G(-i,ρd^{-1}))
(63)

Indeed if $x \in S_\rho^{'}$ then $F(x) = 3D\rho^{2l}$ and by = (24) we
have $| x |= 3D\rho + O(\rho^{-1-\alpha_1})$. Hence the inequalities = in (72)
hold for $x \in U_\varepsilon(S_\rho^{'})$. If $x \in = A(\rho)$
then by definition of $A(\rho)$ we have $x \in K_\rho$, and hence |
x$|= 3D\rho + O(\rho^{-1+\alpha_1})$. Therefore the inequalities in = (72) hold
for $x \in U_{2\varepsilon}(A(\rho))$ too. The inclusion in (72) follows from
these inequalities.
If $G \subset S_\rho$ then for $x \in G(+k, \rho^{-\alpha})$ by (42) = we have
$\frac{\partial F(x)}{\partial x_k} > 0,$. Therefore to calculate the measure
of $G(+k, a)$ for $a \geq \rho^{-\alpha}$ we use the formula

$$\mu(G(+k, a)) = 3D \int\limits_{\Pr_k(G(+k,a))} (\frac{\partial F}{\partial x_k}$$
$)^{-1} \mid grad(F) \mid dx_1...dx_{k-1}dx_{k+1}...dx_d,$
(64)

where $\Pr_k(G) \equiv \{(x_1, x_2, ..., x_{k-1}, x_{k+1}, x_{k+2}$
,...,x$_d) : x \in G\}$ is the projection of G on the hyperplane = $x_k = 3D0$.
Instead of $\Pr_k(G)$ we write $\Pr(G)$ if k is unambiguous. If $D =$ is
$m-$dimensional subset of R^m then we use

$$\mu(D) = 3D \int\limits_{\Pr_k(D)} \mu(D(x_1, ...x_{k-1}, x_{k+1}, ..., x_m =$$

$))\mathrm{dx}_1...dx_{k-1}dx_{k+1}...dx_m,$
(65)

where $D(x_1,...x_{k-1},x_{k+1},...,x_m)=3D\{x_k:(x_1,x_2$
$,...,\mathrm{x}_m)\in D\}$.

ESTIMATION 1. Here we prove (69) by using (73). During this estimation = the set
S_ρ is redenoted by G. If $x\in G$ then $x\notin V_b^1$
$(\rho^{\alpha_1}))$, for all $b\in\Gamma(p\rho^\alpha)$. Since the
rotation does not change the measure we choose the coordinate axis so = that the
direction of some $b\in\Gamma(p\rho^\alpha)$ coinsides with the = direction of
$(1,0,0,...,0)$, that is, $b=3D(b_1,0,0,...,0),b_1>0$. Then the = relation
$x\notin V_b^1(\rho^{\alpha_1}))$ implies that $\mid x_1\mid>a$, where
$a=3D(\rho^{\alpha_1}-b_1^2)(2b_1)^{-1}>\rho^\alpha$ because = |
b$|<p\rho^\alpha,\alpha_1=3D3\alpha$. Therefore $G=3DG(+1,a)\cup=G(-1,a)$.

Now we estimate $\mu(G(+1,a))$ by using (73) and following relations

$\partial F(x)\overline{{}_{\partial x_1=3Dl|x|^{2(l-1)}(2x_1}}$
$+\mathrm{O}(\rho^{-2\alpha}))$
(66)

$\partial F\overline{{}_{\partial x_1>\rho^\alpha,\ (\partial}}$
$\mathrm{F}\overline{{}_{\partial x_1)^{-1}|grad(F)|=3D\frac{|x|}{x_1}}}$
$+\mathrm{O}(\rho^{-2\alpha}),$
(67)

$\Pr(G(+1,a))\supset\Pr(A(+1,2a))$,
(68)

where $x\in G(+1,a)$, $A=3DB(\rho)\cap U^{1}(3\rho^{\alpha_1},p)$. = Here (75) follows from (42) and (76) follows from (75). Now we prove the inclusion = in
(77). If $(x_2,...,x_d)\in\Pr(A(+1,2a))$ then by definition of $A(+1,2a)$ there exists x_1 such that

$x_1>2a>2\rho^{\alpha},\ x_1^2+x_2^2+...+x_d^2=3D\rho$
$^2,\mid\sum_{i\geq1}(2x_ib_i-b_i^2)\mid\geq 3\rho^{\alpha_1}$
(69)

for all $(b_1,b_2,...,b_d)\in\Gamma(p\rho^{\alpha})$. Therefore = for
$h=3D\rho^{-\alpha}$ we have
$(x_1+h)^2+x_2^2+...+x_q^2>\rho^2+\rho^{-\alpha},(x_1$
-h$)^2+x_2^2+...+x_q^2<\rho^2-\rho^{-\alpha}$. This and (24) = give
$F(x_1+h,x_2,...,x_d)>\rho^{2l}$, $F(x_1-h,x_2,...,x_d)<\rho^{2l}$
$.Since$F$is a continuous function there is$ =$y_1\in(x_1-h,x_1+h)$
such that (see (78))

$y_1>a,F(y_1,x_2,...,x_d)=3D\rho^2,\mid 2y_1b_1-b_1^2$
$+\sum_{i\geq2}(2x_ib_i-b_i^2)\mid>2\rho^{\alpha_1}$,
(70)

because the expression under the absolute value in (79) differ from the expression under the absolute value in (78) by $2(y_1-x_1)b_1$, = whose absolute value is less than ρ^{α_1}. The relations in (79) = means that $(x_2,...,x_d)\in\Pr G(+1,a)$. Hence the inclusion in (77) is proved. Now it follows from (73), (76) and the obvious relation μ(Pr G(+1,a))=3DO(ρ^{d-1}) (see the inequality in (72))

$\mu(G(+1,a)) = 3D \int\limits_{\Pr(G(+1,a))} \frac{|x|}{x_1} dx_2$
$dx_3...dx_d + O(\rho^{-\alpha})\mu(B(\rho)) \geq$

$\int\limits_{\Pr(A(+1,2a))} \frac{|=x|}{x_1} dx_2 dx_3...dx_d$
$-c_{10}\rho^{-\alpha}\mu(B(\rho)) = 3D$

$\mu(A(+1,2a)) - c_{12}\rho^{-\alpha}\mu(B(\rho))$.

Similarly $= \mu(G(+1,a)) \geq \mu(A(-1,2a)) - c_{13}\rho^{-\alpha}\mu(B(\rho))$.
Hence $\mu(G) > \mu(A) - c_{14}\rho^{-\alpha}\mu(B(\rho))$. This inequality
implies (69) since $\mu(A)) = 3D(1 + O(\rho^{-\alpha}))\mu(B(\rho))$ (see = (39)).
ESTIMATION 2 Here we prove (70). For this we estimate the measure of the = set
$S_\rho \cap P_b$ (see (60)) by using (73). During this estimation = the set
$S_\rho \cap P_b$ is redenoted by G. We choose the coordinate axis = so
that the direction of b coincides with the direction of $= (1,0,0,...,0)$,
i.e., $b = 3D(b_1,0,0,...,0)$ and $b_1 > 0$. Clearly if $(x_1, x_2$
$,...,x_d) \in G$ then (see definition of S_ρ, G and $F(x)$)

$(x_1^2 + x_2^2 + ... + x_d^2)^l + F_{k_1-1}(x) = 3D\rho^{2l}$,

$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2)^l + F_{k_1-1}(x + b =)$
$=3D\rho^{2l} + h$

where $h \in (-3\varepsilon_1, 3\varepsilon_1)$. It follows from (80), (81), and (24) that

$$(\mathrm{x}_1^2 + x_2^2 + x_3^2 + ... + x_d^2) = 3D\rho^2 + O(\rho^{-\alpha_=1})$$
(71)

$$((\mathrm{x}_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2) = 3D\rho^2 + O(\rho$$
$$^{-\alpha_1})$$
(72)

Subtrracting (82) from (83) we get

$$(2\mathrm{x}_1 - b_1)b_1 = 3DO(\rho^{-\alpha_1}),$$
(73)

This with inequalities in (72) implies

$$\mid b_1 \mid < 2\rho + 3, \; = x_1 = 3D\tfrac{b_1}{2} + O(\rho^{-2\alpha_1}$$
$$\mathrm{b}_1^{-1}), \mid = x_1^2 - (\tfrac{b_1}{2})^2 \mid = 3DO(\rho^{-2\alpha_1})$$
(74)

Conside two cases. Case 1: $b \in \Gamma_1 \equiv \{b \in \Gamma :\mid \rho$ $^2 - (\frac{b_1}{2})^2 \mid < 3d\rho^{-2\alpha}\}$. In this case from $=$ (85), and (82) we obtain

$x_1^2 = 3D\rho^2 + O(\rho^{-2\alpha}), |= x_1 |= 3D\rho + O(\rho^{-2\alpha}$
$-1), x_2^2 + x_3^2 + ... + x_d^2 = 3DO(\rho^{-2\alpha}).$
(75)

Therefore $G \subset G(+1, a) \cup G(-1, a)$, where $a = 3D\rho - \rho^{-1}. =$ Using (73), the obvious relation $= \mu(\Pr_1(G(+1, a)) = 3DO(\rho^{-(d-1)\alpha})$ (see (82)) and the fact that the expression under the integral (73) for $= k = 3D1$ is
equal to $1 + O(\rho^{-\alpha})$ (see (76) and (86)) and doing the same $=$ for $G(-1, a)$ we get $\mu(G) = 3DO(\rho^{-(d-1)\alpha})$. Since $= | \Gamma_1$
$|= 3DO(\rho^{d-1})$ we have

$\mu(\cup_{b\in\Gamma_1}(S_\rho \cap = P_b) = 3DO(\rho^{-(d-1)\alpha}$
$+d-1)=3DO(\rho^{-\alpha})\mu(B(\rho)).$
(76)

Case 2: $b \notin \Gamma_1$, i.e., $= | \rho^2 - (\frac{b_1}{2})^2 |$
$\geq 3d\rho^{-2\alpha}$. By (85) and (82) we have

$| x_1^2 - \rho^2 |> 2d\rho^{-2\alpha}, x_2^2 + x_3^2$
$+...+x_d^2 > d\rho^{-2\alpha}, | x_k |> \rho^{-\alpha},$
(77)

for some $k = 3D2, 3, ..., d$. Therefore $= G \subset \cup_{k\geq 2}(G(+k, \rho^{-\alpha}$
$)\cup G(-k, \rho^{-\alpha}))$. We calculate $\mu(G(+d, \rho^{-\alpha})) =$ by
(73). Redenote by D the set $\Pr_d G(+d, \rho^{-\alpha})$. If $x \in$
$G(+d,\rho^{-\alpha})$ then by (82) and (42) the under integral $=$ expression in

(73) for $k = 3Dd$ is $O(\rho^{1+\alpha})$. Therefore the first equality = in

$\mu(D) = 3DO(\varepsilon_1 \mid b \mid^{-1} \rho^{d-2}),\ = \mu(G(+d, \rho$
$^{-\alpha})) = 3DO(\rho^{d-1+\alpha}\varepsilon_1 \mid b \mid^{-1})$
(78)

implies the second equality in (89). To prove the first equality of (89) = we
use (74) for $m = 3Dd - 1$; $k = 3D1$ and the relations $= \mu(\Pr_1 D) = 3DO(\rho^{d-2})$,

$\mu(D(x_2, x_3, ..., x_{d-1})) < 6\varepsilon_1 \mid b \mid^{-1}$,
$\forall(x_2, x_3, ..., x_{d-1}) \in \Pr D$.
(79)

First relation follows from the fact that $\mid x \mid< \rho + 1$ for $x \in$ G(+d,$\rho^{-\alpha}$) (see (72)). So we need to prove (90). If $x_1 \in$ D(x$_2, x_3, ..., x_{d-1}$) then (80) and (81) holds. Subtructing (80) = from (81), we get

$((\mathrm{x}_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2)^l - ((x_1^2$
$+\mathrm{x}_2^2 + x_3^2 + ... + x_d^2)^l -$

$\mathrm{F}_{k_1-1}(x - b) - F_{k_1-1}(x) = 3Dh$
(80)

where $x_2, x_3, ..., x_{d-1}$ are fixed . Hence we have two equations = (80) and (91) with respect two unknown x_1 and x_d. Using (42), the implicit function theorem, and the inequalities $\mid x_d \mid > \rho^{-\alpha}$,$\alpha_1 > 2\alpha$ from (80), we obtain

$$x_d = 3Df(x_1), \frac{df}{dx_1} = 3D2x_1 + O(\rho^{-2\alpha_1+\bar{l}pha})\overline{{}_{2x_d+O(\rho^{-2\alpha_1+\alpha})=3D\frac{x_1}{x_d}+O(\rho}} {}^{-\alpha_1})$$

(81)

Substituting this in (91), we get

$$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^l - ((x_1^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^l +$$

$$F_{k_1-1}(x_1 - b_1, x_2, ..., x_{d-1}, f(x_1)) - F_{k_1-1}(x_1, ..., x_{d-1}, f) = 3Dh$$

(82)

Using (42), (92), the first equality in (85) and $\mid x_d \mid > \rho^{-\alpha}$ we see the derivative (w.r.t. x_1) of the rigth side $a_l(x)$ of (93) is

$$\partial a_l(x)\overline{{}_{\partial x_1=3Dl|=x-b|^{2(l-1)}(2(x_1}} - b_1) + 2f(x_1)f^{'}(x_1))$$

(83)

$-l \mid x \mid^{2(l-1)} (2x_1 + 2f(x_1)f'(x_1)) + O(\rho$
$^{-2\alpha_1+\alpha})(1+\frac{x_1}{x_d}) + O(\rho^{-3\alpha_1+a})$.
If $l = 3D1$ then using the first equality in (85) and $\mid x_d \mid$
¿$\rho^{-\alpha}$ we have

$\mid \frac{\partial a_l(x)}{\partial x_1} \mid > b_1$.
(84)

If $l > 1$ then it follows from (80), (81), and (24) that

$\mid x \mid^{2l} = 3D\rho^{2l}(1+O(\rho^{-2l-\alpha_1}))$, |
x-b$|^{2l} = 3D\rho^{2l}(1+O(\rho^{-2l-\alpha_1}))$

$|= x \mid^{2(l-1)} = 3D\rho^{2(l-1)}(1+O(\rho^{-2l-\alpha_1}))^{\frac{l-1}{}}$
l=3D$\rho^{2(l-1)} + O(\rho^{-2-\alpha_1})$

$|= x - b \mid^{2(l-1)} = 3D\rho^{2(l-1)}(1+O(\rho^{-2l-\alpha_1}))$
$\overline{l-1l=3D\rho^{2(l-1)}+O(\rho^{-2-\alpha_1})}$

Using these in (94) and arguing as in proof of (95), for $l = 3D1$, we get
= the
proof of (95) for $l > 1$ Thus in any case (95) holds. Therefore from (93)
= by
implicit function theorem, we get $= \mid \frac{dx_1}{dh} \mid < \frac{1}{|}$
b| . This inequality and relation $= h \in (-3\varepsilon_1, 3\varepsilon$

$_1$) imply (90). Hence (89) is proved. In the same way we get the same estimation for $G(+k, \rho^{-\alpha})$ and $G(-k, \rho^{-\alpha})$ for $k \geq 2$.
Thus $\mu(S_\rho \cap P_b) = 3DO(\rho^{d-1+\alpha}\varepsilon_1 \mid$
b$|^{-1})$, for $b \notin \Gamma_1$. Since $\mid b \mid< 2\rho + 3$ (see (85))
and $\varepsilon_1 = 3D\rho^{-d-2\alpha}$, we infer $= \mu(\cup_{b\notin\Gamma}$
$_1(S_\rho \cap = P_b)) = 3DO(\rho^{2d-1+\alpha}\varepsilon_1) = 3DO(\rho$
$^{-\alpha})\mu(B(\rho))$. This, (87) and (69) give the proof of (70).
ESTIMATION 3. Here we prove (71). Let $G \equiv U_{2\varepsilon}(A_{k,j}$
$(\gamma_1,\gamma_2, ..., \gamma_k))$, where γ_1,γ
$_2, ..., \gamma_k \in \Gamma(p\rho^\alpha), k = 3D \leq d - 1$. We turn the coordinate axis so that
$Span\{\gamma_1,\gamma_2, ..., \gamma_k\} = 3D\{x = 3D(x_1, x_2, ..., x_=k$
,0,0,...,0):x$_1, x_2, ..., x_k \in R\}$. Then by (40) we have x_n
=3DO$(\rho^{\alpha_k+(k-1)\alpha})$, for $n \leq k$, $x \in G$. This, = (72), and
$\alpha_k + (k - 1)\alpha < 1$ (see the end of the introduction) give
$G \subset (\cup_{i>k}(G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1}))$,
$\mu(\Pr_i(G(+i, \rho d^{-1}))) = 3DO(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)})$,
for $i > k$. Now using this and (74) for $m = 3Dd$ we prove

$\mu(G(+i, \rho d^{-1})) = 3DO(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha}$
)+(d-1-k)),$\forall i > k$.
(85)

For this we redenote by D the set $G(+i, \rho d^{-1})$ and prove

$\mu((D(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \leq (42d^2 + 4) = n$
,$\forall i > k$,
(86)

for all $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(D)$. To = prove
(97) it is sufficient to show that if both $x = 3D(x_1, x_2, ..., x_i$
,...x$_d)$ and $x^{'} = 3D(x_1, x_2, ..., x_i^{'}, ..., x_d)$

are in D then $| x_i - x_i^{'} |\leq (42d^2 + 4)\varepsilon$.
Assume the converse. Then $| x_i - x_i^{'} |> (42d^2$
$+4)\varepsilon$. We may assume that $\rho = d^{-1} < x_i < x_i^{'}$ (
see definition of D). Since x and $x^{'}$ lie in the
2ε neighborhood of $A_{k,j}$ there exist points a and
$a^{'}$ in $A_{k,j}$ such that $| x - a |< 2\varepsilon$ and $= |$
$x^{'} - a^{'} |< 2\varepsilon$. Then we have the $=$ inequalities
$\rho d^{-1} - 2\varepsilon < a_i < a_i^{'}$, $a_i^{'}$
$-a_i > 42d^2\varepsilon$, $(a_i^{'})^2 - (a_i)^2 > 2(\rho$
$d^{-1} - 2\varepsilon)(a_i^{'} - a_i)$,
$|| a_s | - | a_s^{'} ||< 4\varepsilon, = ||$
$a_s |^2 - | a_s^{'} |^2|< 12\rho\varepsilon$ for $s \neq$
$i, and hence$
$\sum_{s\neq i} || a_s |^2 - | a_s^{'} |^2$
$|< 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i - a_i^{'})$,

$$|| a |^2 - | a^{'} |^2|> \frac{3}{2}\rho = d^{-1} | a_i^{'} - a_i |, | a_i^{'} - a_i |> 42d^2 \varepsilon \qquad (87)$$

Moreover using mean value theorem and the relations $| a |= 3D\rho+O(1)$, $|$
$a^{'} |= 3D\rho + O(1)$, we get

$$| a |^{2l} - |= a^{'} |^{2l}= 3Dl(\rho + O(1))^{2(l-1)})(| a|^2 - | a^{'} |^2) \qquad (88)$$

Let $r_i(x) = 3D\lambda_i(x)- | x |^{2l}$. Hence $r_1(x) \leq$
$r_2(x) \leq ... \leq r_{b_k}(x)$ be the eigenvalue of the matrix $C(x)- |$
$x|^{2l}$ $I \equiv C^{'}(x)$, where $C(x)$ is defined in Section $= 1$ (see
Theorem 1(c)). by definition of $C^{'}(x)$ only diagonal $=$ elements of

the matrix $C^{'}(x)$ depend on x and they are $\mid x-b_i$
$|^{2l} - \mid x\mid^{2l}$, where $b_i = 3Dh_i + t - \gamma - t \in B_k$
$+\Gamma(p_1\rho^{\alpha})$. Hence $\mid b_i \mid < \rho^{\frac{1}{2}}$
α_d . Clearly

$\mid r_j(a^{'}) - r_j(a) \mid \leq \parallel C^{'}(a^{'}$
)-C$^{'}(a) \parallel = 3D\max_i \mid a_{i,i} \mid$.
(89)

where $C^{'}(a^{'}) - C^{'}(a) = 3D(a_{i,j})$, $a_{i,i} = 3D \mid$
a$|^{2l} - \mid a - b_i \mid^{2l} - \mid a^{'} \mid^{2l} + \mid a^{'}$
-b$_i$ $|^{2l}$, $a_{i,j} = 3D0$ for $i \neq j$, that is $C^{'}$
(x)-C$^{'}(x^{'})$ is diagonal matrix. To estimate $\mid$
a$_{i,i}$ $\mid$ using mean value theorem and the relations $\mid a \mid$
=3D$\rho + O(1), \mid a^{'} \mid = 3D\rho + O(1)$, $\mid = a - b_i \mid = 3D\rho$
+O$(\rho^{\frac{1}{2}\alpha_d}), \mid a^{'} - b_i \mid = 3D\rho$
+O$(\rho^{\frac{1}{2}\alpha_d})$, we obtain

a$_{i,i} = 3D(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid = a\mid^2 - \mid$
a$^{'}$ $|^2)-$

$(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid = a - b_i \mid^2 - \mid$
a$^{'} - b_i \mid^2) = 3D$

$(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(l-1)}(\mid a \mid^2 - \mid$

a-b$_i$ $|^2 - | a^{'} |^2 + | a^{'} - b_i |^2)+$

O$(\rho^{\frac{1}{2}\alpha_d+2l-3})(| a |^2 - |= a^{'} |$
$^2)$.

Since $| a |^2 - | a - b_i |^2 - |= a^{'} |^2 + |$
a$^{'} - b_i |^2 = 3D2(a - a^{'}, b_i)$ we have (see = (100))
$| r_j(a) - r_j(a^{'}) |\leq 3\rho^{\frac{1}{2}\alpha_d}$
+2l-2$| a - a^{'} | -c_{15}\rho^{\frac{1}{2}\alpha_d+2l-3} ||$
a$|^2 - | a^{'} |^2|$ Therefore using (99), (100) = and (98)
we obtain
$| \lambda_j(a) - \lambda_j(a^{'}) |\geq|| a |^{2l} - |$
a$^{'} |^{2l}| - | r_j(a) - r_j(a^{'}) |> l\rho$
$^{2l-1}d^{-1} | a_i^{'} - a_i |>$
$l\rho^{2l-1}42d\varepsilon > 6\varepsilon_1$ which controdicts the fact that
both $\lambda_j(a)$ and $\lambda_j(a^{'})$ lie in $(\rho$
$^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$, becouse a and
$a^{'}$ lie in $A_{k,j}$ (see definition $A_{k,j}$). Thus (97), hence
(96) is proved. In the same way we get the same formula for $= G(-i, \rho$
$\overline{d).}$ So $= \mu(U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma$
$_k))) = 3DO(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+d-1-k})$, where
$j = 3D1, 2, ..., b_k(\gamma_1, \gamma_2, ..., \gamma_k)$, and γ
$_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^{\alpha})$. From this = using that
$b_k = 3DO(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$ and the number of = the
vectors $(\gamma_1, \gamma_2, ..., \gamma_k)$ for γ_1, γ
$_2, ..., \gamma_k \in \Gamma(p\rho^{\alpha})$ is $O(\rho^{dk\alpha})$, = we
obtain (71) if

d$\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) + d - 1 - k\bar{l}eq$
d-1-α

(90)

for $1 \leq k \leq d-1$. Using $\alpha_k = 3D3^k\alpha, = \alpha = 3D\frac{1}{q}$,
$q = 3D3^d + d + 2$ we see that (101) is equivalent to
$\frac{d+1}{k} + \frac{3^{k+1}}{2} + 3^k + k - 1 \leq 3^d + 2$
The right-hand side of this inequality gets its maximum value at $= k = 3Dd - 1$.
Therefore we need to show that $\frac{d+1}{d-1} + \frac{5}{6}3^d + d \leq 3^d + 4$
which follows from $\frac{d+1}{d-1} \leq 3, d < \frac{1}{6}3^d + 1$ for $d \geq 2$.
ESTIMATION 4. Here we prove (66). During this estimation we denote by $= G$ the
set $S^{'}_{\rho} \cap U_\varepsilon(Tr(A(\rho))$. Since $= V_\rho$
$\equiv S^{'}_{\rho} \backslash G$, by (70) and (72), it is = enough to
prove $\mu(G(+i, \rho d^{-1})) = 3DO(\rho^{-\alpha})\mu(B(\rho))$, for
$i = 3D1, 2, ..., d$. (The same estimation for $G(-i, \rho d^{-1})$ can be = proved in
the same way.) We prove this by using (73). By (42) for $x \in G(+i, \rho$
d^{-1}) the under integral expression in (73), for $k = 3Di; a = 3D\rho = d^{-1}$, is
less than $d+1$. Therefore it is sufficient to prove

$$\mu(\Pr(G(+i, \rho d^{-1})) = 3DO(\rho^{-\alpha})\mu(B(\rho))$$
(91)

Clearly, if $= (x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(G(+i, \rho$
d^{-1})) then
$\mu(U_\varepsilon(G)(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \geq$
2ε and by (74),

$$\mu(U_\varepsilon(G)) \geq 2\varepsilon\mu(\Pr(G(+i, \rho d^{-1})).$$
(92)

Hence to prove (102) we need to estimate $\mu(U_\varepsilon(G))$. For = this we prove that

$$\mathrm{U}_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'}), U_\varepsilon (\mathrm{G}) \subset U_{2\varepsilon}(Tr(A(\rho))), U_\varepsilon(G) \subset \mathrm{Tr}(\mathrm{U}_{2\varepsilon}(A(\rho))). \quad (93)$$

The first and second inclusions follow from $G \subset= S_\rho^{'}$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see definition = of G). Now we prove the third inclusion in (104). If $x \in U_\varepsilon(G)$ then by the second inclusion of (104) there exists b such that $b \in$ Tr(A(ρ)), $\mid x - b \mid < 2\varepsilon$. Then by the definition of $Tr(A(\rho))$ there are $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = 3D\gamma + c$. Therefore $\mid x - \gamma - c \mid = 3D \mid = x - b \mid < 2\varepsilon$, $x - \gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho))$. This together with $x \in U_\varepsilon(G) \subset U_\varepsilon(S_\rho{'})$ (see the first inclusion of (104)) give $x \in$ Tr(U$_{2\varepsilon}(A(\rho))$) , i.e., the third inclusion in (104) is proved. The third inclusion, Lemma 4 (c) and (71) imply that $\mu(U_\varepsilon$ (G))=3DO($\rho^{-\alpha}$)$\mu(B(\rho))\varepsilon$. This and (103) give = the proof of (102)$\diamondsuit$

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