Graded cluster expansion for lattice systems

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Abstract

In this paper we develop a general theory which provides a unified treatment of two apparently different problems. The weak Gibbs property of measures arising from the application of Renormalization Group maps and the mixing properties of disordered lattice systems in the Griffiths' phase. We suppose that the system satisfies a mixing condition in a subset of the lattice whose complement is sparse enough namely, large regions are widely separated. We then show how it is possible to construct a convergent multi-scale cluster expansion.

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1. Introduction

In this paper we develop a general theory which provides a unified treatment of two apparently different problems: (i) the weak Gibbs property of measures arising from the application of Renormalization Group (RG) maps to Gibbs states of lattice systems and (ii) the mixing properties of disordered lattice systems in the so called Griffiths' phase. Let us explain the main features of these issues.

1.1. Weak Gibbsianity of renormalized measures

Renormalization group is a fundamental method in modern theoretical physics. It has been originally introduced to analyze scale invariant situations that are typical of statistical mechanical systems at their critical point. However it also exhibits its power for non–critical systems that deserve to be analyzed on appropriate scales, see [6, 23]. The RG maps are defined as follows.

Consider a d–dimensional lattice spin system (object system) whose state space is $\mathcal{X} := \bigotimes_{x\in\mathcal{L}} \mathcal{X}_x$, where $\mathcal{L} := \mathbb{Z}^d$ and each $\mathcal{X}_x$ is a copy of the same finite set $\mathcal{X}_0$. We set $\mathcal{L}^{(\ell)} := \ell\mathbb{Z}^d$, with $\ell \in \mathbb{N}$, and partition $\mathcal{L}$ as the disjoint union of ℓ–boxes $\mathcal{L} = \bigcup_{i\in\mathcal{L}^{(\ell)}} Q_\ell(i)$ where $Q_\ell(i)$ is the cube of side length ℓ with i the site with smallest coordinates. Moreover, to each $i \in \mathcal{L}^{(\ell)}$ we associate a *renormalized spin* m_i taking value in a finite state space $\mathcal{M}_i^{(\ell)}$, with each $\mathcal{M}_i^{(\ell)}$ a copy of the same finite set $\mathcal{M}_0^{(\ell)}$. We assign the normalized non–negative kernel $T_\ell(\sigma_{Q_\ell(i)}, m_i)$ with $\sigma_{Q_\ell(i)} \in \bigotimes_{x\in Q_\ell(i)} \mathcal{X}_x$ and $m_i \in \mathcal{M}_i^{(\ell)}$. Given a Gibbs measure (w.r.t. an absolutely summable potential, for instance a finite range potential) μ on $\mathcal{X}$, the renormalized measure $\nu^{(\ell)}$ on the renormalized space $\mathcal{M}^{(\ell)} = \bigotimes_{i\in\mathcal{L}^{(\ell)}} \mathcal{M}_i^{(\ell)}$ is defined by its finite dimensional distributions

$$\nu^{(\ell)}(M^{(\ell)} : M_V^{(\ell)} = m_V) = \sum_{\sigma\in\mathcal{X}_\Lambda} \mu(\eta : \eta_\Lambda = \sigma) \prod_{i\in V} T_\ell(\sigma_{Q_\ell(i)}, m_i)$$

where V is a finite subset of $\mathcal{L}^{(\ell)}$, $\Lambda := \bigcup_{i\in V} Q_\ell(i)$, $\mathcal{X}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{X}_x$, and $m_V \in \mathcal{M}_V^{(\ell)} := \bigotimes_{i\in V} \mathcal{M}_i^{(\ell)}$. We shall write $\nu^{(\ell)} = T_\ell\mu$. For the usual choices of the kernel, the semigroup property holds, namely $T_\ell T_{\ell'} = T_{\ell\ell'}$.

An easy example is the *decimation* transformation where $\mathcal{M}_i^{(\ell)} = \mathcal{X}_i$, for all $i \in \mathcal{L}^{(\ell)}$, and $T_\ell^{\text{dec}}(\sigma_{Q_\ell(i)}, m_i) = \delta(\sigma_i - m_i)$; m_i, with $i \in \mathcal{L}^{(\ell)}$, are the "surviving spins." Another important example is the Block Averaging Transformation (BAT) that we discuss in the case $\mathcal{X}_0 := \{-1, +1\}$: for each $i \in \mathcal{L}^{(\ell)}$ the single renormalized spin configuration space is $\mathcal{M}_i^{(\ell)} = \{-\ell^d, -\ell^d + 2, \ldots, +\ell^d\}$ and $T_\ell^{\text{bat}}(\sigma_{Q_\ell(i)}, m_i) = \delta\Big(\sum_{x\in Q_\ell(i)} \sigma_x - m_i\Big)$.

Theoretical reasons and many applications lead us to analyze the map on the potentials induced by the map T_ℓ that was defined on (infinite volume) measures. A preliminary condition is that the renormalized measure is Gibbsian in the Dobrushin–Lanford–Ruelle sense i.e., its conditional probabilities have the Gibbs form with respect to an absolutely summable potential that we call renormalized potential [10,22]. Another, hopefully equivalent, approach consists in defining at finite volume a map acting directly on the Hamiltonians in the following way. Given a box $V \subset\subset \mathcal{L}$, let $\Lambda := \bigcup_{i\in V} Q_\ell(i)$ be the corresponding box in $\mathcal{L}$, $\mathcal{X}_\Lambda := \mathcal{X}_0^\Lambda = \{-1,+1\}^\Lambda$, and $-H_\Lambda$ the energy of the object system, we write

$$e^{+H_V^{(\ell)}(m)} = \sum_{\sigma\in\mathcal{X}_\Lambda} e^{+H_\Lambda(\sigma)} \prod_{i\in V} T_\ell(\sigma_{Q_\ell(i)}, m_i)$$

where we have included in H the inverse temperature. Now the problem is to extract the potential from the renormalized Hamiltonian $H_V^{(\ell)}$ via a procedure still having a sense in the thermodynamic limit. For this purpose a crucial role is played by the so called *constrained systems* i.e., the object system conditioned on some fixed renormalized spin configuration. Given a renormalized spin configuration $m \in \mathcal{M}_V^{(\ell)}$, the constrained measure in Λ is defined by:

$$\mu_{m,\Lambda}^{(\ell)}(\sigma) = \frac{e^{+H_\Lambda(\sigma)} \prod_{i\in V} T_\ell(\sigma_{Q_\ell(i)}, m_i)}{\sum_{\sigma\in\mathcal{X}_\Lambda} e^{+H_\Lambda(\sigma)} \prod_{i\in V} T_\ell(\sigma_{Q_\ell(i)}, m_i)}$$

Note that the configuration m plays the role of a parameter. In the case of the decimation, $\mu_{m,\Lambda}^{(\ell)}$ is nothing but the original Gibbs measure in the volume obtained by removing from Λ the set V of the surviving sites, *conditioned* to the configuration m on V.

As it has been shown in many examples, see [19], it may happen that $\nu^{(\ell)}$ is not Gibbsian. Typically this pathology manifests itself as violation of *quasi–locality*, a necessary condition for Gibbsianity. More precisely, quasi–locality is a continuity property of the conditional probabilities consisting in a weak dependence on very far conditioning spins, see [19] for more details. This violation of quasi–locality, in turn, is often a consequence of a first order phase transition of a constrained model corresponding to a particular renormalized configuration m. On the positive side, to avoid the pathology of non–Gibbsianity, we need absence of phase transitions, in a very strong sense, of the constrained model for *all* possible values of m. For instance when the object system is in the high temperature regime, the usual perturbative expansion for the constrained models is sufficient to compute the renormalized potentials, see [8,26,28]. However, in order to get close to the critical point, certainly we have to use other, more powerful, perturbative theories.

We discuss, now, these different perturbative theories in the concrete case of systems above their critical temperature T_c. Usual high temperature expansions work only for temperatures T sufficiently larger than T_c; they basically involve perturbations around a universal reference system consisting of independent spins; in other words the small parameter is the inverse temperature and all interactions are expanded treating, in this way, every lattice system in the same manner. In [36, 37] another perturbative expansion has been introduced, around a non trivial model–dependent reference system, that we call *scale–adapted expansion*. The small parameter is no more $\beta = 1/T$ but, rather, the ratio between the correlation length (at the given temperature $T > T_c$) and the length scale L at which we analyze our system. The geometrical objects (polymers) involved in the scale–adapted expansion live on scale length L whereas in the usual expansions they live on scale 1. Of course the smaller is $T - T_c$ the larger has to be taken the length L.

A similar situation occurs for low temperature Ising ferromagnets at arbitrarily small but non zero magnetic field h. Now we have another characteristic length, beyond the correlation length, the *critical length* which is of order $1/h$; it represents the minimal size of a droplet whose growth is energetically favorable and, at the same time, the minimal length required to *screen* the effect of a boundary condition opposite to the field. Thus in the part of the bulk far apart from the boundary more than the critical length, we see, uniformly in the boundary condition, the unique phase with magnetization parallel to the field. Also in this case of low temperature and not vanishing magnetic field we have to look at our system on a scale length L sufficiently larger than the critical length (at low temperature the correlation length is of order one).

The scale–adapted expansions are based on a suitable *finite size condition* saying, roughly speaking, that if we look at the Gibbs measure in a box of sufficiently large side length L, then, uniformly in the boundary conditions, the correlations between observables localized at distance of order L are smaller than $L^{-2(d-1)}$. It is proven in [36,37], that, for general short range lattice systems, assuming this finite size condition, we can construct a convergent cluster expansion implying, in particular, exponential decay of correlations for any volume Λ (finite or infinite) given as disjoint union of L–boxes, with a decay rate independent of Λ. We call this decay property *strong mixing*. Such a property implies uniqueness of the infinite volume Gibbs measure, we refer to [32, 36, 37] for more details. In some cases, like the two–dimensional standard Ising model in the uniqueness region, strong mixing has been theoretically proven, see [34, 39].

Starting from a strong mixing condition for the measure $\mu_m^{(\ell)}$, *uniform* in the renormalized configuration m, it is possible to prove, using the scale–adapted perturbative expansion, the Gibbsianity of $\nu^{(\ell)}$ by explicitly computing the renormalized potentials as convergent series, [2, 27]. In the particular case of BAT the condition on the constrained model can be deduced from a strong mixing property of the original model by using a strong form of the equivalence of ensembles [2, 9].

The philosophy behind the use of scale–adapted expansions to study RG maps is that one first fixes the thermodynamic parameters and consequently chooses the renormalization scale ℓ. It may happen that for a given scale ℓ and for particular values of the thermodynamic parameters a RG map is ill defined but, keeping fixed the thermodynamic parameters, provided one chooses a large enough renormalization scale ℓ, the pathology is removed. This is exactly the case for decimation transformation for a two–dimensional standard Ising model away from the coexistence line. In [19] it is proven that for any decimation scale ℓ there exist values of β and h such that the renormalized measure $\nu_{\beta,h}^{(\ell),\mathrm{dec}}$ is not Gibbsian; however in [33] it is shown that, given any β and h which, for some ℓ, give rise to the pathology, then, if one chooses, instead, a scale $\bar{\ell}$ sufficiently large in terms of β and h, the renormalized measure $\nu_{\beta,h}^{(\bar{\ell}),\mathrm{dec}}$ becomes Gibbsian and actually the renormalized potential converges to zero as $\bar{\ell}$ goes to infinity. In [2], adopting this philosophy, the block averaging transformation on scale ℓ has been analyzed and it has been shown, in particular, that for the two–dimensional standard Ising model at *any* $T > T_c$ and arbitrary h the renormalized measure $\nu_{\beta,h}^{(\ell),\mathrm{bat}}$ is Gibbsian for ℓ *large enough*. Moreover the renormalized potential converges, in a suitable sense, to the expected trivial fixed point as ℓ goes to infinity. In order to perturbatively study convergence properties of the iterates of the renormalization group maps, even far from criticality, the use of scale–adapted expansions on increasing scales appears therefore very natural.

We mention that there is another, stronger, notion of strong mixing originally introduced, for general short range lattice systems, by Dobrushin and Shlosman in [14, 15], before [36, 37]. It has been called by the authors *complete analyticity*: it consists in the exponential decay of correlations for *all* finite or infinite domains Λ (of arbitrary shape). This point of view contrasts the one of scale–adapted expansions. Indeed in Dobrushin–Shlosman's approach there is no minimal scale length. Scale–adapted perturbative theory gives rise to a notion that has been called *restricted complete analyticity* or *complete analyticity for regular domains*; here "regular" means "multiple" of a sufficiently large box. Dobrushin and Shlosman also developed a finite size condition that involves *all* the possible subsets of a given, sufficiently large box and not just the box itself like [32]: for some respects Dobrushin–Shlosman's approach could be still seen as a perturbative theory on scale one.

The standard Ising model for (i) $d = 2$ outside the closure of the coexistence line, (ii) $d = 3$, $h \neq 0$, and $T \ll T_c$, provides two examples where there is a diverging characteristic length as $T \to T_c$ and $h \to 0$ respectively. In both cases restricted complete analyticity has been proven whereas complete analyticity in Dobrushin and Shlosman's sense has not yet been proven in the case (i) and actually disproven in the case (ii) [32, 34, 39]. We finally note that an interesting direct proof of restricted complete analyticity, starting from a finite size condition similar to the one in [36, 37], has been established in [30] without the use of cluster expansion.

The physical interest of the above discussion in connection with RG maps lies in the possibility of well defining a renormalization map for potentials close to their critical point. Actually it is believed that even if the object system is critical it may happen that the constrained systems are in the one phase, weakly coupled regime, so that the renormalized potential is still well defined, see [1, 12, 27].

Let us go back to the discussion of the possible pathology of non–Gibbsianity. It frequently happens that the renormalized configuration m inducing non–Gibbsianity, via violation of quasi locality, is very atypical with respect to renormalized measure $\nu^{(\ell)}$. It is then natural and physically relevant to introduce a weaker notion of Gibbsianity by requiring that conditional probabilities are well behaved only $\nu^{(\ell)}$ almost surely. More precisely weak Gibbsianity of $\nu^{(\ell)}$ means the following: there exists a "good set" $\bar{\mathcal{M}}^{(\ell)} \subset \mathcal{M}^{(\ell)}$ with $\nu^{(\ell)}(\bar{\mathcal{M}}^{(\ell)}) = 1$, such that for $m \in \bar{\mathcal{M}}^{(\ell)}$ the conditional probabilities of the measure $\nu^{(\ell)}$ have the usual Gibbs form with respect to a potential $\{\Phi_I^{(\ell)}\}_{I \subset\subset \mathcal{L}^{(\ell)}}$, $\Phi_I^{(\ell)} : \mathcal{M}_I^{(\ell)} \to \mathbb{R}$, satisfying the *pointwise* absolute summability: for each $i \in \mathcal{L}^{(\ell)}$ and $m \in \bar{\mathcal{M}}^{(\ell)}$ we have $\sum_{I \ni i} |\Phi_I^{(\ell)}(m_I)| < \infty$, but not the usual *uniform* absolute summability namely, $\sup_{i \in \mathcal{L}^{(\ell)}} \sum_{I \ni i} \sup_{m_I \in \mathcal{M}_I^{(\ell)}} |\Phi_I^{(\ell)}(m_I)| < \infty$. The idea of looking at the weak Gibbs property goes back to Dobrushin [13]. It has subsequently developed in [17] and in many other papers, see for instance [7, 35]. The main point of weak Gibbsianity is the construction of the set $\bar{\mathcal{M}}^{(\ell)}$ of full $\nu^{(\ell)}$ measure. The key property is that for $m \in \bar{\mathcal{M}}^{(\ell)}$ the "bad situations," giving rise to long range correlations in $\mu_m^{(\ell)}$, are very "sparse" namely, larger and larger bad regions are farther and farther apart.

We discuss in some detail the example of the block averaging transformation. It is known, see [19], that BAT for the Ising model at low temperature is non–Gibbsian because of violation of quasi locality induced by configuration $m_i = 0$ for all $i \in \mathcal{L}^{(\ell)}$. It is clear that any constrained system and in particular the one corresponding to $m_i = 0$ for all i does not depend at all on the value of the magnetic field h. On the other hand if h is different from zero the configuration $m_i = 0$ for all i is very unlikely with respect to the renormalized measure $\nu^{(\ell)}$. Therefore, with high $\nu^{(\ell)}$ probability the regions with $m_i = 0$ are very sparse; however, with probability one there are arbitrarily large regions with bad magnetization $m_i = 0$. Inside these regions, the situation described by the constrained measure $\mu_m^{(\ell)}$ is close to a first order phase transition with long range order. This prevents a good Gibbsian behavior of conditional probabilities of $\nu^{(\ell)}$ as well as estimates of the renormalized potential, *uniform* in the renormalized conditioning configuration m. Moreover it is reasonable to expect weak Gibbsianity of the renormalized system, indeed this is proven in [5].

1.2. Disordered systems and Griffiths' singularity

The above scenario, leading to the replacement of the notion of Gibbsianity with the one of weak Gibbsianity, is very similar to what happens for disordered systems in presence of the so called Griffiths' singularity.

Let us consider the case of high temperature Ising–like spin glasses. They are described by the following formal Hamiltonian

$$H(\sigma) = -\sum_{x,y} J_{x,y}\sigma_x\sigma_y - h\sum_x \sigma_x, \tag{1.1}$$

where $\sigma_x \in \{-1,+1\}$, $J_{x,y}$, for all $x,y \in \mathcal{L} := \mathbb{Z}^d$, are i.i.d. random variables, and $h \in \mathbb{R}$. For the sake of simplicity we further specify the model by assuming $J_{x,y} = 0$ for $|x-y| \neq 1$ and $J_{x,y} \sim \mathcal{N}(0,1)$, namely $J_{x,y}$ are Gaussian independent random variables with mean zero and variance one. We denote by $\mu^{(J)}$ the Gibbs measure corresponding to the Hamiltonian (1.1). The "typical" (with respect to the disorder) interaction energy between neighboring spins is of order one so that for small inverse temperature β our random system is in the weak coupling regime. However, with probability one there are arbitrarily large regions where the random couplings $J_{x,y}$ take large positive values giving rise, inside these regions, to the behavior of a low temperature ferromagnetic Ising system with long range order. For a similar case, the one of a low temperature ferromagnetic diluted Ising system, it has been shown, see [25, 40], that the infinite volume specific free energy is infinitely differentiable but not analytical in h. This is a sort of infinite order phase transition also called "Griffiths' singularity".

For a high temperature spin glass with unbounded random couplings a similar behavior is expected. We also expect exponential decay of correlations with a non–random decay rate but with a random unbounded prefactor. More precisely, let us denote by Ω the collection of all $\underline{J} := \{J_{x,y},\ x,y \in \mathcal{L},\ |x-y| = 1\}$; in the above conditions we expect that there exist $m > 0$ and a set $\bar{\Omega} \subset \Omega$ of full measure such that for all $\underline{J} \in \bar{\Omega}$, there exists a positive real $C(\underline{J})$ such that the spin correlations have the following bound

$$\left|\mu^{(J)}(\sigma_x;\sigma_y)\right| \leq C(\underline{J})\ \exp\{-m|x-y|\} \tag{1.2}$$

There are several approaches to the study of disordered systems in the above regime; let us just quote the two papers [18, 20]. In [18] a very elegant method is presented proving, in particular, that (1.2) holds for some $C(\underline{J})$ having bounded expectations. The set of full measure where $C(\underline{J})$ are bounded is not explicitly constructed; this method does not make use of the cluster expansion but there are some restrictions to its applicability; however the case of high temperature Ising spin glass is covered. In [20], that appeared several years before [18], a more powerful and more widely applicable method is presented, involving a graded cluster expansion. The set of full measure $\bar{\Omega}$ is explicitly constructed; indeed its construction makes use of a multi–scale analysis similar to the one introduced in [21] to study the Anderson localization. It emerges from the analysis developed in [20], based on a hierarchy of "scales of badness" that, with high probability, larger and larger bad regions are farther and farther apart and the largest scale of badness seen close to the origin is finite. The theory developed in [20] gives rise directly to estimates valid with probability one and requires very mild assumptions on the probability distribution of random couplings.

However it is important to remark that in the perturbative theory developed in [20] the first length scale of the hierarchy is one. This restricts its applicability to the regime of high temperature or large magnetic field.

1.3. A graded cluster expansion

To analyze disordered systems close to criticality, we need an approach based on scale–adapted expansions. This is inspired by the approach in [20] with the first step being a scale–adapted expansion, as in [36, 37], instead of a high temperature expansion. To describe its application suppose, for example, to consider the case of small random perturbations a ferromagnetic system at a given temperature larger but arbitrarily close to the critical value. To be concrete consider a ferromagnetic two dimensional Ising system with zero magnetic field and coupling constants given by i.i.d. random variables for different bonds with distribution

$$J = \begin{cases} 1 & \text{with probability } 1-p \\ J_0 & \text{with probability } p \end{cases} \tag{1.3}$$

Fix a temperature T slightly larger than the critical value T_c corresponding to a deterministic system with coupling constant one. To our knowledge the above described situation has never been studied in the literature. We expect the following result [4]: given $T > T_c$ there exists $p > 0$ such that for any arbitrarily large $J_0 \leq +\infty$ we can construct a convergent graded cluster expansion implying, in particular, the decay property (1.2).

We also mention that, adapting the methods in [30], an effective finite size condition involving the quenched expectation of correlations can be obtained [31].

It is also clear that, when studying weak Gibbsianity of renormalized measures, in order to compute renormalized potentials as convergent series, we need a complete theory based on graded cluster expansion since the methods developed in [18], which avoid the use of cluster expansion, are not sufficient for this purpose. On the other hand it is also clear that if we want to study weak Gibbsianity only assuming strong mixing, in particular for systems close to criticality and/or to study convergence properties of the iterates of RG maps, we need to consider a graded cluster expansion whose first scale is not one but, rather, depends on the parameters. In other words we have necessarily to make use of scale–adapted cluster expansion to treat the first step in the hierarchy of badness.

In [5] we study the BAT transformation only assuming strong mixing of the object system. In the framework of a graded cluster expansion, with a sufficiently large minimal scale length, using scale–adapted expansion to treat the first step of the hierarchy, we establish the weak Gibbsianity of the renormalized measure. Moreover we show, in a suitable sense, convergence to a (trivial) fixed point of renormalized potential as the RG scale ℓ goes to infinity. Our results apply to the two–dimensional Ising model in the uniqueness region i.e., for $h \neq 0$ or $h = 0$, $T > T_c$ and, in particular below T_c where non–Gibbsianity has been proven in [19]. At the moment we are not able to cover the case $h = 0$, $T < T_c$.

In [35] as well as in [7] the authors establish weak Gibbsianity for measures arising from the application of general decimation transformations to a low temperature Ising or Pirogov–Sinai system. They have to analyze constrained systems on arbitrarily large scales but they have to choose a sufficiently low temperature and their minimal length is of order one. Therefore their methods work only very far below the critical point.

In both papers the authors first fix the scale ℓ of RG transformation and then choose a sufficiently low temperature. In particular they both have to choose lower and lower temperatures starting from larger and larger RG scales. This behavior is not in agreement with the general RG philosophy. In [38] this anomaly is fixed, as it is shown that at a given sufficiently low temperature is enough to get weak Gibbsianity for all large enough scales. This approach is still based on a low temperature expansion and it is neither suited to approach the critical point nor to study convergence properties of the iterates of RG maps.

1.4. Synopsis and table of contents

In the present paper we construct the graded cluster expansion that will be used to treat weak Gibbsianity for the block averaging transformation [5] and disordered systems in the Griffiths' phase [4]. Here there is no random disorder in the interactions; however, we suppose deterministically possible to analyze the bad interactions on suitable increasing scale lengths. We treat iteratively the regions of increasing badness and prove convergence of the expansion on the basis of suitable assumptions on the potential in the good region and sufficient "sparseness" of bad regions. In [5] we prove that, with probability one with respect to the disorder or to the renormalized spin configuration, the situation is the one deterministically assumed in the present paper.

The assumption that the system is weakly coupled on the complement of the bad region of the lattice namely, the good part, is here formalized by the following assumption. Let Δ be a finite subset of the good region and $Z_\Delta(\sigma)$ be the partition function in Δ with boundary condition σ. We assume

$$\log Z_\Delta(\sigma) = \sum_{X \cap \Delta \neq \emptyset} V_{X,\Delta}(\sigma) \tag{1.4}$$

where the effective potential $V_{X,\Delta}$ satisfies the following conditions.

(a) Given a finite subset $X \subset \mathbb{Z}^d$ the functions $V_{X,\Delta}$ are constant w.r.t. Δ for the sets Δ with a fixed intersection with X.

(b) The functions $V_{X,\Delta}$ have a suitable decay property w.r.t. X uniformly in Δ and σ.

The expression (1.4) can be obtained via cluster expansion in the weak coupling (high temperature and/or small activity) region but it also holds in the more general situation of the scale–adapted cluster expansion discussed before. In the latter case it holds provided the volume Δ is a disjoint union of cubes whose side length equals the scale L of the expansion. We also note that, whenever the good region extends to the whole lattice, (1.4) implies one of the Dobrushin–Shlosman complete analyticity conditions [16] namely, Condition IVa, see [3]. In the applications we discussed above condition (1.4) will be derived via a scale–adapted cluster expansion and therefore it will hold only for volumes Δ which are disjoint unions of cubes whose side length equals the scale L of the expansion. However, by rescaling the lattice and redefining the single spin state space we reduce to the case in which (1.4) holds for any finite subset Δ of the good region, which is the basic assumption of the present paper.

The main result concerns an expression, similar to (1.4), of the logarithm of the partition function on a generic finite subset of the whole lattice, possibly intersecting its bad region. Its characteristic feature, with respect to a usual low activity expansion, is that here polymers are geometrical objects living on arbitrarily large scale. This rules out the possibility to prove analyticity of the infinite volume free energy but would allow to prove infinite differentiability and exponential tree decay of semi–invariants [3] with an unbounded prefactor as it is typical of Griffiths' phase. The proof is achieved by using condition (1.4) to integrate over the good region and by using the multi–scale geometry of the bad regions to recursively compute the effective interaction among them, i.e. to recursively integrate over the bad spins.

This paper is organized as follows: in Section 2 we introduce the model and state our results in Theorems 2.5 and 2.6. The latter, whose proof is based on the cluster expansion of the logarithm of the partition function provided by the former, states the exponential decay of the semi–invariants for suitable local functions. The proof of Theorem 2.5 is achieved via the graded cluster expansion whose basic setup is introduced in Section 3; there we also state the related technical result in Theorem 3.2, whose proof is split into two parts: the algebraic structure of the computation is provided in Section 4, while all convergence issues are discussed in Section 5. The proof of Theorem 3.2 is completed at the end of Section 5. The Theorems 2.5 and 2.6 are finally proven in Section 6.

2. Notation and results

In this Section we introduce the general framework, define precisely the model we shall consider, and state our main results. Given $a, b \in \mathbb{R}$ we adopt the usual notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Given a set A we let $|A|$ be its cardinality.

2.1. The lattice

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ we let $|x|_1 := \sum_{k=1}^{d} |x_k|$ and $|x|_\infty := \sup_{k=1,\cdots,d} |x_k|$. The spatial structure is modeled by the d–dimensional cubic lattice $\mathbb{L} := \mathbb{Z}^d$ in which we let $e_i, i = 1, \ldots, d$ be the coordinate unit vectors. We use $X^{\mathrm{c}} := \mathbb{L} \setminus X$ to denote the complement of $X \subset \mathbb{L}$. We use $X \subset\subset \mathbb{L}$ to indicate that $|X|$ is finite. On $\mathbb{L}$ we consider the distances $\mathrm{d}_1(x, y) := |x - y|_1$ and $\mathrm{d}_\infty(x, y) := |x - y|_\infty$ for $x, y \in \mathbb{L}$. As usual for $X, Y \subset \mathbb{L}$ we set $\mathrm{d}_1(X, Y) := \inf\{\mathrm{d}_1(x, y),\ x \in X,\ y \in Y\}$, $\mathrm{d}_\infty(X, Y) := \inf\{\mathrm{d}_\infty(x, y),\ x \in X,\ y \in Y\}$, $\mathrm{diam}_1(X) := \sup\{\mathrm{d}_1(x, x'),\ x, x' \in X\}$, and $\mathrm{diam}_\infty(X) := \sup\{\mathrm{d}_\infty(x, x'),\ x, x' \in X\}$. Moreover, given $r \geq 1$ and $X \subset \mathbb{L}$ we let $\partial_r X := \{x \notin X :\ \mathrm{d}_\infty(x, X) \leq r\}$ be the *r–external boundary* of X and $\overline{X} := X \cup \partial_r X$ be the *r–closure* of X.

For $x \in \mathbb{L}$ and m a positive real we let $Q_m(x) := \{y \in \mathbb{L} :\ x_i \leq y_i \leq x_i + (m - 1),\ \forall i = 1, \ldots, d\}$ the cube of side length m with x the site with smallest coordinates and $B_m(x) := \{y \in \mathbb{L} :\ \mathrm{d}_1(y, x) \leq m\}$ the ball of side length $2m + 1$ centered at x. We shall denote $Q_m(0)$, resp. $B_m(0)$, simply by Q_m, resp. B_m. For each $X \subset\subset \mathbb{L}$ we denote by $\mathcal{Q}(X) \subset\subset \mathbb{L}$ the smallest parallelepiped, with axes parallel to the coordinate directions, containing X.

2.2. The configuration space

For some applications, for instance the block averaging transformation, we have to deal with systems in which even the single spin space is not translationally invariant. We introduce the basic notation. We suppose given a collection of strictly positive integers

S_x, $x \in \mathbb{L}$, such that $S := \sup_x S_x < +\infty$. The single spin configuration space is given by a finite set $\mathcal{S}_x$, $|\mathcal{S}_x| = S_x + 1$, where $x \in \mathbb{L}$. We identify $\mathcal{S}_x$ with $\{0, 1, \ldots, S_x\}$ which we endow with the discrete topology. The configuration space in $\Lambda \subset \mathbb{L}$ is $\mathcal{S}_\Lambda \equiv \mathcal{S}(\Lambda) := \otimes_{x \in \Lambda} \mathcal{S}_x$. Finally, the configuration space in $\mathbb{L}$ is $\mathcal{S} := \otimes_{x \in \mathbb{L}} \mathcal{S}_x$, equipped with the product topology. Elements of $\mathcal{S}$, called *configurations*, are denoted by $\sigma, \tau, \eta, \ldots$. The integer $\sigma_x \equiv \sigma(x)$ is called value of the spin at the site x. For $\Lambda \subset \mathbb{L}$ and $\sigma \in \mathcal{S}$ we denote by σ_Λ the restriction of σ to Λ. We denote by $\mathcal{F}$ the Borel σ–algebra of $\mathcal{S}$ and, for each $\Lambda \subset \mathbb{L}$, we set $\mathcal{F}_\Lambda := \sigma\{\sigma_x,\, x \in \Lambda\} \subset \mathcal{F}$.

Let m be a positive integer and $\Lambda_1, \ldots, \Lambda_m \subset \mathbb{L}$ be pairwise disjoint subsets of $\mathbb{L}$; if $\sigma_i \in \mathcal{S}_{\Lambda_i}$, $i = 1, \ldots, m$, we denote by $\sigma_1\sigma_2 \cdots \sigma_m$ the configuration in $\mathcal{S}_{\Lambda_1 \cup \cdots \cup \Lambda_m}$ given by $\sigma_1\sigma_2 \cdots \sigma_m(x) := \sum_{i=1}^m \mathbb{I}_{\{x \in \Lambda_i\}} \sigma_i(x)$, $x \in \Lambda_1 \cup \cdots \cup \Lambda_m$.

A function $f : \mathcal{S} \to \mathbb{R}$ is called a *local* function if and only if there exists $\Lambda \subset\subset \mathbb{L}$ such that $f \in \mathcal{F}_\Lambda$, namely f is $\mathcal{F}_\Lambda$–measurable for some finite set Λ. For f a local function we shall denote by $\mathrm{supp}(f)$, the so called support of f, the smallest $\Lambda \subset\subset \mathbb{L}$ such that $f \in \mathcal{F}_\Lambda$. If $f \in \mathcal{F}_\Lambda$ we shall sometimes misuse the notation by writing $f(\sigma_\Lambda)$ for $f(\sigma)$. For $f \in \mathcal{F}$ we let $\|f\|_\infty := \sup_{\sigma \in \mathcal{S}} |f(\sigma)|$ be the sup norm of f.

2.3. The Gibbs state

A *potential* U is a collection of local functions on $\mathcal{S}$ labelled by finite subsets of $\mathbb{L}$, namely $U := \{U_X \in \mathcal{F}_X,\, X \subset\subset \mathbb{L}\}$. We shall consider *finite range* potential i.e., there exists a real $R \geq 0$, called *range*, such that $U_X = 0$ if $\mathrm{diam}_1(X) > R$. We, finally, introduce the norm $\|U\| := \sup_{x \in \mathbb{L}} \sum_{X \ni x} \|U_X\|_\infty$. In the sequel we shall always understand that the real r appearing in the definition of the boundary $\partial_r X$ of $X \subset \mathbb{L}$ is chosen so that $r \geq R$. We remark that we do not require the potential to be translationally invariant.

Let $\Lambda \subset\subset \mathbb{L}$, $\sigma \in \mathcal{S}$ and consider the *Hamiltonian*

$$H_\Lambda(\sigma) := \sum_{X \cap \Lambda \neq \emptyset} U_X(\sigma) \tag{2.1}$$

In this paper we shall consider only finite volume Gibbs measures defined as follows: let $\tau \in \mathcal{S}$, the finite volume Gibbs measure μ_Λ^τ, with boundary condition τ, is the probability measure on $\mathcal{S}_\Lambda$ given by

$$\mu_\Lambda^\tau(\sigma) := \frac{1}{Z_\Lambda(\tau)} \exp\{+H_\Lambda(\sigma_\Lambda \tau_{\Lambda^c})\} \tag{2.2}$$

where $Z_\Lambda \in \mathcal{F}_{\Lambda^c}$, called the *partition function*, is the normalization constant. Note that we have defined the Gibbs measure with a sign convention opposite to the usual one.

2.4. All's well ...

Our aim is the computation of the partition function $Z_\Lambda(\tau)$ under the hypotheses that the Gibbs random field is weakly coupled only on a part of the lattice, that will be called *good* and denoted by $\mathbb{G}_0$, under suitable geometric conditions on such $\mathbb{G}_0$. We shall assume the system admits a convergent cluster expansion in $\mathbb{G}_0$ with a suitable tree decay of the *effective* potential among the spins in $\mathbb{L}\setminus\mathbb{G}_0$ and resulting from the integration in $\mathbb{G}_0$.

Let $\mathbb{E} := \{\{x,y\},\, x,y\in\mathbb{L} : \mathrm{d}_1(x,y)=1\}$ the collection of *edges* in $\mathbb{L}$. Note that, according to our definitions, the edges are parallel to the coordinate directions. We say that two edges $e,e'\in\mathbb{E}$ are connected if and only if $e\cap e'\neq\emptyset$. A subset $(V,E)\subset(\mathbb{L},\mathbb{E})$ is said to be connected iff for each pair $x,y\in V$, with $x\neq y$, there exists in E a path of connected edges joining them. We agree that if $|V|=1$ then $(V,\emptyset)$ is connected. For $X\subset\subset\mathbb{L}$ we then set

$$\mathbb{T}(X) := \inf\left\{|E|\,,\ (V,E)\subset(\mathbb{L},\mathbb{E}) \text{ is connected and } V\supset X\right\} \tag{2.3}$$

Note that $\mathbb{T}(X)=0$ if $|X|=1$. We remark that for each $x,y\in\mathbb{L}$ we have $\mathbb{T}(\{x,y\}) = \mathrm{d}_1(x,y)$.

Condition 2.1. *Given $\mathbb{G}_0\subset\mathbb{L}$, we assume that for each $\Lambda\subset\subset\mathbb{L}$ and $\sigma\in\mathcal{S}$ we have the expansion*

$$Z_{\Lambda\cap\mathbb{G}_0}(\sigma) := \sum_{\eta\in\mathcal{S}(\Lambda\cap\mathbb{G}_0)} \exp\Big\{H_{\Lambda\cap\mathbb{G}_0}(\eta\sigma_{(\Lambda\cap\mathbb{G}_0)^{\mathrm{c}}})\Big\} = \exp\Big\{\sum_{X\cap\Lambda\neq\emptyset} V_{X,\Lambda}(\sigma)\Big\} \tag{2.4}$$

for suitable local functions $V_{X,\Lambda}:\mathcal{S}\to\mathbb{R}$ satisfying the following properties

1. *given $\Lambda,\Lambda'\subset\subset\mathbb{L}$ if $X\cap\Lambda = X\cap\Lambda'$ then $V_{X,\Lambda}=V_{X,\Lambda'}$;*
2. *$V_{X,\Lambda}\in\mathcal{F}_{X\cap(\Lambda\cap\mathbb{G}_0)^{\mathrm{c}}}$;*
3. *$V_{X,\Lambda}=0$ if $X\cap\left(\overline{\Lambda}\right)^{\mathrm{c}}\neq\emptyset$.*

Moreover, the effective potential $V_\Lambda := \{V_{X,\Lambda},\ X\cap\Lambda\neq\emptyset\}$ can be bounded as follows: there are reals $\alpha>0$ and $A<\infty$ such that

$$\sup_{x\in\mathbb{L}}\sum_{X\ni x} e^{\alpha\mathbb{T}(X)} \sup_{\substack{\Lambda\subset\subset\mathbb{L}:\\ \Lambda\cap X\neq\emptyset}} \|V_{X,\Lambda}\|_\infty \le A \tag{2.5}$$

We recall that our aim is to cluster expand $\log Z_\Lambda(\tau)$ with $\Lambda\subset\subset\mathbb{L}$ and $\tau\in\mathcal{S}$. Given $\Lambda\subset\subset\mathbb{L}$, we first apply (2.4) to the configuration $\sigma_\Lambda\tau_{\Lambda^{\mathrm{c}}}$ and then we integrate on the variables $\sigma_{\Lambda\setminus(\Lambda\cap\mathbb{G}_0)}$.

2.5. ... or sparse ...

We shall not make any assumption on the behavior of the Gibbs field on the complement of the good part, namely the *bad* part of the lattice $\mathbb{B}_0 := \mathbb{L} \setminus \mathbb{G}_0$, but we shall require that the bad sites are sparse enough. We start from the partition $\mathbb{L} = \mathbb{G}_0 \cup \mathbb{B}_0$ of the lattice in *good* and *bad* sites. Although such a sharp classification seems to be the reason for the never ending popularity of most American movies, it is not sufficient to our purposes. Let us forget about the good sites and look more closely at the bad ones. Some of them are not really bad, they are just bad guys far away from all the other bad sites (only close enough bad individuals form a dangerous gang). We are not really allowed to call such a behavior bad and we say they are *gentle* (more precisely 1–*gentle*). We next forget also about the 1–*gentle* sites and look at the remaining ones, which we call 1–*bad*. Even among them some are not so bad, after all. Maybe we have just a small group of bad guys very far away from all the 1–*bad* sites; those are called 2–*gentle*. Proceeding in such a way we construct a multi–scale classification of the sites and we also suppose a happy ending: there are no ∞–*bad* guys. We formalize the above discussion in the following Definitions.

Definition 2.2. *We say that two strictly increasing sequences $\Gamma = \{\Gamma_j\}_{j\geq 0}$ and $\gamma = \{\gamma_j\}_{j\geq 0}$ are* steep scales *iff they satisfy the following conditions:*

1. $\Gamma_0 = 0$, $\gamma_0 \geq 0$, $\Gamma_1 \geq 2$, $\gamma_1 > r \geq R$, *and* $\Gamma_j < \gamma_j/2$ *for any* $j \geq 1$.
2. *For* $j \geq 0$ *set* $\vartheta_j := \sum_{i=0}^{j}(\Gamma_i + \gamma_i)$ *and* $\lambda := \inf_{j\geq 0}(\Gamma_{j+1}/\vartheta_j)$*; then* $\lambda > 7$.
3. *We have* $\displaystyle\sum_{j=0}^{\infty} \frac{\Gamma_j}{\gamma_j} \leq \frac{1}{2}$ *where we understand* $\Gamma_0/\gamma_0 = 0$ *even in the case* $\gamma_0 = 0$.

It is useful to remark that from items 2 and 3 above we get that

$$\vartheta_j \leq \gamma_j \left[1 + \left(1 + \frac{1}{\lambda}\right) \sum_{i=1}^{\infty} \frac{\Gamma_i}{\gamma_i}\right] \leq 2\gamma_j, \qquad \text{for any } j \geq 0 \tag{2.6}$$

Indeed, from item 2 it follows $\gamma_j \leq \Gamma_{j+1}/\lambda$; hence for $j \geq 1$ we have

$$\begin{aligned} \vartheta_j &= \gamma_j \left(\sum_{i=0}^{j} \frac{\Gamma_i}{\gamma_j} + 1 + \sum_{i=0}^{j-1} \frac{\gamma_i}{\gamma_j} \right) \\ &\leq \gamma_j \left(1 + \sum_{i=0}^{j} \frac{\Gamma_i}{\gamma_i} + \frac{1}{\lambda} \sum_{i=0}^{j-1} \frac{\Gamma_{i+1}}{\gamma_{i+1}}\right) \leq \gamma_j \left(1 + \frac{1}{2} + \frac{1}{2\lambda}\right) \end{aligned}$$

Remark 2.3. *We note that the Conditions 2 and 3 in the above definition, force a superexponential growth of the sequences* Γ *and* γ*. It is easy to show that, given* $\beta \geq 9 \vee (4/9)\log(8r)$*, the sequences* $\Gamma_0 = \gamma_0 := 0$*,*

$$\Gamma_k := e^{(\beta+1)(3/2)^k} \qquad \text{and} \qquad \gamma_k := \frac{1}{8} e^{\beta(3/2)^{k+1}} \qquad \text{for } k \geq 1 \tag{2.7}$$

provide an example of steep scales.

Definition 2.4. *We say that* $\mathcal{G} := \{\mathcal{G}_j\}_{j\geq 0}$*, where each* $\mathcal{G}_j$ *is a collection of finite subsets of* $\mathbb{L}$*, is a* graded disintegration of $\mathbb{L}$ *iff*

1. *for each* $g \in \bigcup_{j\geq 0} \mathcal{G}_j$ *there exists a unique* $j \geq 0$*, which is called the* grade *of* g*, such that* $g \in \mathcal{G}_j$*;*

2. *the collection* $\bigcup_{j\geq 0} \mathcal{G}_j$ *of finite subsets of* $\mathbb{L}$ *is a partition of the lattice* $\mathbb{L}$ *namely, it is a collection of not empty pairwise disjoint finite subsets of* $\mathbb{L}$ *such that*

$$\bigcup_{j\geq 0} \bigcup_{g\in\mathcal{G}_j} g = \mathbb{L}. \tag{2.8}$$

Given $\mathbb{G}_0 \subset \mathbb{L}$ *and* Γ, γ *steep scales, we say that a graded disintegration* $\mathcal{G}$ *is a* gentle disintegration *of* $\mathbb{L}$ *with respect to* $\mathbb{G}_0, \Gamma, \gamma$ *iff the following recursive conditions hold:*

3. $\mathcal{G}_0 = \big\{\{x\},\, x \in \mathbb{G}_0\big\}$*;*

4. *if* $g \in \mathcal{G}_j$ *then* $\mathrm{diam}_1(g) \leq \Gamma_j$ *for any* $j \geq 1$*;*

5. *set* $\mathbb{G}_j := \bigcup_{g\in\mathcal{G}_j} g \subset \mathbb{L}$*,* $\mathbb{B}_0 := \mathbb{L} \setminus \mathbb{G}_0$ *and* $\mathbb{B}_j := \mathbb{B}_{j-1} \setminus \mathbb{G}_j$*, then for any* $g \in \mathcal{G}_j$ *we have* $\mathrm{d}_1(g, \mathbb{B}_{j-1} \setminus g) > \gamma_j$ *for any* $j \geq 1$*;*

6. *for each* $x \in \mathbb{L}$ *we have* $k_x := \sup\big\{j \geq 1 : \exists g \in \mathcal{G}_j \text{ such that } \mathrm{d}_\infty(x, \mathcal{Q}(g)) \leq \vartheta_j\big\} < \infty$*, where we recall* $\mathcal{Q}(g)$ *has been defined at the end of Section 2.1.*

Sites in $\mathbb{G}_0$ *(resp.* $\mathbb{B}_0$*) are called* good *(resp.* bad*); similarly we call* j*–gentle (resp.* j*–bad) the sites in* $\mathbb{G}_j$ *(resp.* $\mathbb{B}_j$*). Elements of* $\mathcal{G}_j$*, with* $j \geq 1$*, are called* j*–gentle atoms. Finally, we set* $\mathcal{G}_{\geq j} := \bigcup_{i\geq j} \mathcal{G}_i$*.*

For $G \subset \mathcal{G}_{\geq 0}$ we define $\widehat{G} := \bigcup_{g\in G} g \subset \mathbb{L}$; note that $\widehat{\mathcal{G}_j} = \mathbb{G}_j$ and $\widehat{\{g\}} = g$. Given the integers $j \geq 0$, $s \geq 0$ and $G \subset\subset \mathcal{G}_{\geq j}$, such that $G \cap \mathcal{G}_j \neq \emptyset$, we define

$$Y_s(G) := \big\{x \in \mathbb{L} :\, \mathrm{d}_\infty(x, \mathcal{Q}(\widehat{G})) \leq \vartheta_j + s\big\} \tag{2.9}$$

Moreover for each $s \geq 0$ we set $y_s(G) := Y_s(G) \backslash Y_{s-1}(G)$ where we understand $Y_{-1}(G) = \emptyset$.

2.6. ... that ends well

As discussed before, our aim is to prove that, under Condition 2.1 on the good part of the lattice and the sparseness condition formulated in Definitions 2.2 and 2.4 of the bad part of the lattice, the system admits a convergent cluster expansion. We set

$$a := \frac{4d}{9(44)^{1/d}}, \quad q := \frac{1}{2^5\, 3^2}, \quad \text{and} \quad \varrho := \left\{ \frac{1}{1-q} \left(1 + \frac{1}{\alpha} \log A \right) \right\} \vee 0 \tag{2.10}$$

where we recall α and A are the parameters in Condition 2.1.

Theorem 2.5. *Suppose Condition 2.1 holds with $\alpha > 0$ and $A < +\infty$ for some $\mathbb{G}_0 \subset \mathbb{L}$. Assume also that for such $\mathbb{G}_0$ there exist steep scales γ, Γ and a gentle disintegration $\mathcal{G}$ of $\mathbb{L}$ with respect to $\mathbb{G}_0, \Gamma, \gamma$ as in Definition 2.4. Finally assume the scales Γ, γ are such that*

1. *we have $\Gamma_1 > \max\{4(1 + \log 3)/\alpha,\, (8d)^3/(2a)\}$;*
2. *we have $A(\Gamma_j + 1)^d e^{-\alpha\gamma_j/4} \leq 1$ for any $j \geq 1$;*
3. *we have $\displaystyle\sum_{j=1}^{\infty} \frac{8d}{a^{1/3}} \frac{1}{\gamma_j^{1/3}} \leq \frac{\alpha}{32}$;*
4. *for each $j \geq 1$ we have $\gamma_j \geq \left(\dfrac{j}{d}\right)^{3/2} a^{1/2}$.*

Then, for each $X, \Lambda \subset\subset \mathbb{L}$ there exist functions $\Psi_{X,\Lambda}, \Phi_{X,\Lambda} \in \mathcal{F}_{X \cap \Lambda^c}$ such that the following statements hold.

1. *For each $\Lambda \subset\subset \mathbb{L}$ we have the totally convergent expansion*

$$\log Z_\Lambda(\tau) = \sum_{X \cap \Lambda \neq \emptyset} [\Psi_{X,\Lambda}(\tau) + \Phi_{X,\Lambda}(\tau)] \tag{2.11}$$

2. *Let $\Lambda, \Lambda' \subset\subset \mathbb{L}$, for each $X \subset\subset \mathbb{L}$ such that $X \cap \Lambda = X \cap \Lambda'$ we have that $\Psi_{X,\Lambda} = \Psi_{X,\Lambda'}$ and $\Phi_{X,\Lambda} = \Phi_{X,\Lambda'}$.*

3. *Let $X, \Lambda \subset\subset \mathbb{L}$, if $\operatorname{diam}_\infty(X) > \varrho$ and there exists no $g \in \mathcal{G}_{\geq 1}$ such that $Y_0(g) = X$ then $\Psi_{X,\Lambda} = 0$. Moreover for each $x \in \mathbb{L}$, recalling the integer k_x has been introduced in item 6 of Definition 2.4,*

$$\sum_{X \ni x} \sup_{\Lambda \subset\subset \mathbb{L}} \|\Psi_{X,\Lambda}\|_\infty \leq A + k_x(\Gamma_{k_x} + 1 + 2\vartheta_{k_x})^{2d} \big[\log S + \|U\| + k_x(1 \vee A)(8^d + 1) \big] \tag{2.12}$$

4. We have

$$\sup_{x\in\mathbb{L}}\sum_{X\ni x} e^{q\alpha \mathrm{diam}_\infty(X)} \sup_{\Lambda\subset\subset\mathbb{L}} \|\Phi_{X,\Lambda}\|_\infty \le e^{-\alpha} + e^{-q\alpha\gamma_1}\Big(\frac{1+e^{-q\alpha/(2d)}}{1-e^{-q\alpha/(2d)}}\Big)^d \tag{2.13}$$

Remark. We note that for β large enough, depending on A and α, the steep scales defined in Remark 2.3 do satisfy items 1–4 in the hypotheses of Theorem 2.5.

We next discuss the exponential decay of correlations which will be a simple consequence of the expansion in Theorem 2.5. We stress that the decay of correlations cannot hold for all pairs of local functions; for instance, if their supports are contained in the same gentle atom, a possible long range order inside the atom itself could prevent such a decay. Our result essentially states the exponential decay of correlations except for such a case. In order to state this result we need few more definitions: let $\Lambda\subset\subset\mathbb{L}$, $n\ge 2$ an integer, $f_1,\dots,f_n$ local functions such that $\mathrm{supp}(f_i)\subset\Lambda$ for $i=1,\dots,n$, $t_1,\dots,t_n\in\mathbb{R}$, and $\tau\in\mathcal{S}$; we define

$$Z_\Lambda\big(\tau;t_1,\dots,t_n\big) := \mu_\Lambda^\tau\Big(\exp\Big\{\sum_{i=1}^n t_i f_i\Big\}\Big) \tag{2.14}$$

The semi–invariant of $f_1,\dots,f_n$ with respect to the finite volume Gibbs measure μ_Λ^τ is defined as

$$\mu_\Lambda^\tau\big(f_1;\cdots;f_n\big) := \frac{\partial^n \log Z_\Lambda\big(\tau;t_1,\dots,t_n\big)}{\partial t_1\cdots\partial t_n}\bigg|_{t_1=\cdots=t_n=0} \tag{2.15}$$

note that for $n=2$ we have $\mu_\Lambda^\tau\big(f_1;f_2\big)=\mu_\Lambda^\tau\big(f_1\,f_2\big)-\mu_\Lambda^\tau\big(f_1\big)\mu_\Lambda^\tau\big(f_2\big)$ namely, the covariance between f_1 and f_2. Let us denote, moreover, by $(\mathbb{V}_n,\mathbb{E}_n)$ the graph obtained from $(\mathbb{L},\mathbb{E})$ by contracting each $\mathrm{supp}(f_i)$, $i=1,\dots,n$, to a single point namely, $\mathbb{V}_n := [\mathbb{L}\setminus\bigcup_{i=1}^n \mathrm{supp}(f_i)]\cup\bigcup_{i=1}^n\{\mathrm{supp}(f_i)\}$, $\mathbb{E}_n := \{\{v,v'\},\, v,v'\in\mathbb{V}_n \,:\, \mathrm{d}_1(v,v')=1\}$, and set

$$T\big(f_1;\dots;f_n\big) := \inf\Big\{|E|,\,(V,E)\subset(\mathbb{V}_n,\mathbb{E}_n)\text{ connected and }V\supset\bigcup_{i=1}^n\{\mathrm{supp}(f_i)\}\Big\} \tag{2.16}$$

Theorem 2.6. *Suppose the hypotheses of Theorem 2.5 are satisfied. Let $n\in\mathbb{N}$ and $f_1,\dots,f_n$ local functions such that the following conditions are satisfied:*

1. *for each $i\ne j\in\{1,\dots,n\}$ we have $\mathrm{d}_1(\mathrm{supp}(f_i),\mathrm{supp}(f_j))>r\ge R$;*

2. *for each $i\ne j\in\{1,\dots,n\}$ there is no $g\in\mathcal{G}_{\ge 1}$ such that $Y_0(g)\cap\mathrm{supp}(f_i)\ne\emptyset$ and $Y_0(g)\cap\mathrm{supp}(f_j)\ne\emptyset$.*

Then, there exist a real $M = M(A, \alpha, d, n; |\operatorname{supp}(f_1)|, \dots, |\operatorname{supp}(f_n)|) < +\infty$ *such that*

$$\big|\mu_\Lambda^\tau(f_1; \dots; f_n)\big| \le M \prod_{i=1}^{n} \mu_\Lambda^\tau(|f_i|) \exp\Big\{ -\frac{q\alpha}{n-1} T(f_1; \dots; f_n) \Big\} \tag{2.17}$$

for any $\tau \in \mathcal{S}$ *and any* $\Lambda \subset\subset \mathbb{L}$ *such that* $\Lambda \supset \operatorname{supp}(f_i)$, $i = 1, \dots, n$.

3. The graded cluster expansion

In this section we introduce our main technique, the graded cluster expansion, and state the related abstract results. It will be convenient to introduce the following notion.

Definition 3.1. *Given* $X, V \subset\subset \mathbb{L}$ *and the family* $F := \{f_\Lambda : \mathcal{S} \to \mathbb{R},\ \Lambda \subset\subset \mathbb{L}\}$, *we say that* F *is* (X, V)–compatible *iff*

1. *for each* $\Lambda, \Lambda' \subset\subset \mathbb{L}$ *we have that* $X \cap \Lambda = X \cap \Lambda'$ *implies* $f_\Lambda = f_{\Lambda'}$;
2. *the function* f_Λ *is* $\mathcal{F}_{(X\cap\Lambda^c)\cup V}$*–measurable.*

In other words the family $\{f_\Lambda,\ \Lambda \subset\subset \mathbb{L}\}$ is (X, V)–compatible if and only if f_Λ does not change when Λ is varied outside X and it depends only on the configuration inside V and the part of X intersecting Λ^c.

We suppose that $\mathcal{G}$, as in Definition 2.4, is a gentle disintegration of the lattice $\mathbb{L}$ with respect to $\mathbb{G}_0, \Gamma, \gamma$. We recall that a j–gentle atom $g \in \mathcal{G}_j$ is a finite subset of $\mathbb{L}$. If $G \subset\subset \mathcal{G}_{\ge 1}$ by $|G|$ we always mean the cardinality of G as a subset of $\mathcal{G}_{\ge 1}$ i.e., the number of elements $g \in \mathcal{G}_{\ge 1}$ in G. On the other hand, if $g \in \mathcal{G}_{\ge 1}$ then $|g|$ denotes the cardinality of g as a subset of $\mathbb{L}$, but note that $|\{g\}| = 1$. The building bricks of our polymers are finite subsets of $\mathcal{G}_{\ge 1}$. From now on s will always denote a positive integer.

Given $X \subset\subset \mathbb{L}$ we let $\xi(X)$ be the collection of the gentle atoms intersecting X namely,

$$\xi(X) := \{g \in \mathcal{G}_{\ge 1} :\ g \cap X \ne \emptyset\} \subset\subset \mathcal{G}_{\ge 1} \tag{3.1}$$

At scale j the relevant notion of connectedness is the following. Given $G, G' \subset \mathcal{G}_{\ge j}$ we say they are *j–connected*, and write $G \overset{j}{\longleftrightarrow} G'$, iff $G \cap G' \cap \mathcal{G}_j \ne \emptyset$. A system $G_1, \dots, G_k$ with $G_h \subset \mathcal{G}_{\ge j}$ is said to be *j–connected* iff for each $h, h' \in \{1, \dots, k\}$ there exist $h_1, \dots, h_m \in \{1, \dots, k\}$ such that $G_h = G_{h_1} \overset{j}{\longleftrightarrow} G_{h_2} \overset{j}{\longleftrightarrow} \cdots \overset{j}{\longleftrightarrow} G_{h_m} = G_{h'}$. We are now ready to define the polymers at scale j namely, we set

$$\begin{aligned}\mathcal{R}_j := \textstyle\bigcup_{k\ge 1} \big\{ \{(G_1, s_1), \cdots, (G_k, s_k)\},\ &\text{where } G_h \subset \mathcal{G}_{\ge j},\ s_h \ge 0, \\ &\text{for } h = 1, \dots, k,\ \text{and the system } G_1, \dots, G_k \text{ is } j\text{–connected} \big\}\end{aligned} \tag{3.2}$$

Elements of $\mathcal{R}_j$ will be called *j–polymers*. Given a j–polymer $R = \{(G_1, s_1), \dots, (G_k, s_k)\}$ and $i \geq j$ we set $R\restriction_i := \bigcup_{h=1}^k G_h \cap \mathcal{G}_i \subset\subset \mathcal{G}_i$ and $R\restriction_{\geq i} := \bigcup_{i' \geq i} R\restriction_{i'} \subset\subset \mathcal{G}_{\geq i}$. We also introduce the *support* of the polymer

$$\operatorname{supp} R := \bigcup_{h=1}^{k} Y_{s_h}(G_h) \subset\subset \mathbb{L} \tag{3.3}$$

We remark that a set $G \subset \mathcal{G}_{\geq 1}$, with $|G| = n > 1$, can be viewed as an n–body link, while $G = \{g\}$, with $g \in \mathcal{G}_{\geq 1}$ corresponds to one body. A pair (G, s) has to be thought as a pair made of the link G and the parallelepiped $Y_s(G)$. The latter represents an "s–extended" support of the bond G. Thus the bricks of a polymer R namely, the pairs (G_h, s_h), can be viewed as the parallelepipeds $Y_{s_h}(G_h)$ whose connectedness properties rely only upon the links G_h. The support of the polymer R, on the other hand, whose interest will become clear in the sequel, is defined as the union of the s_h–extended supports $Y_{s_h}(G_h)$.

Given two j–polymers $R, S \in \mathcal{R}_j$ we say they are *j–compatible*, and write $R \operatorname{comp}_j S$, iff $R\restriction_j \cap S\restriction_j = \emptyset$. Conversely we say that R, S are *j–incompatible*, and write $R \operatorname{inc}_j S$ iff they are not *j–compatible*. We say that a collection $\underline{R} = \{R_1, \dots, R_k\}$, where $R_h \in \mathcal{R}_j$, for $h = 1, \dots, k$, of j–polymers forms a *cluster of j–polymers* iff it is not decomposable into two non empty subsets $\underline{R} = \underline{R}_1 \cup \underline{R}_2$ such that every pair $R_1 \in \underline{R}_1$, $R_2 \in \underline{R}_2$ is j–compatible. We denote by $\underline{\mathcal{R}}_j$ the collection of all the clusters of j–polymers. More precisely we have

$$\begin{aligned}\underline{\mathcal{R}}_j := \textstyle\bigcup_{k\geq 1} \big\{&\underline{R} = \{R_1, \dots, R_k\}\ ,\ R_h \in \mathcal{R}_j\ : \forall\, h, h' \in \{1, \dots, k\} \\ &\exists\, h_1, \dots, h_m \in \{1, \dots, k\} \text{ such that } R_h = R_{h_1} \operatorname{inc}_j R_{h_2} \cdots \operatorname{inc}_j R_{h_m} = R_{h'}\big\}\end{aligned} \tag{3.4}$$

We remark that repetitions of the same j–polymer are allowed. We also define

$$\underline{\mathcal{R}}_j^{\text{comp}} := \bigcup_{k\geq 1} \big\{\, \{R_1, \dots, R_k\}\ ,\ R_h \in \mathcal{R}_j \text{ such that } R_h \operatorname{comp}_j R_{h'},\ h \neq h' \big\} \tag{3.5}$$

Given $S \in \mathcal{R}_j$ and $\underline{R} \in \underline{\mathcal{R}}_j$ we write $\underline{R} \operatorname{inc}_j S$ iff there exists $R \in \underline{R}$ such that $R \operatorname{inc}_j S$. For $i \geq j$, $\underline{R} \in \underline{\mathcal{R}}_j$ we set $\underline{R}\restriction_i := \bigcup_{R\in\underline{R}} R\restriction_i$, $\underline{R}\restriction_{\geq i} := \bigcup_{i'\geq i} \underline{R}\restriction_{i'}$; we finally set $\operatorname{supp} \underline{R} := \bigcup_{R\in\underline{R}} \operatorname{supp} R$.

The setup introduced above is needed to develop the algebraic structure of the graded cluster expansion. In order to formulate the necessary recursive estimates, which quantify the decay of the effective interaction at scale i, we also need to take into account the couplings *below* scale i and we need to introduce some more notation. Let $G = \{g_1, \dots, g_n\} \subset\subset \mathcal{G}_{\geq 1}$, we set

$$\mathcal{T}(G) := \inf_{\substack{x_m \in g_m \\ m=1,\dots,n}} \mathbb{T}\left(\{x_1, \dots, x_n\}\right) \tag{3.6}$$

We finally introduce some combinatorial factors as follows: for each $j \ge 1$, $k \ge 1$ and $\{R_1, \dots, R_k\} \in \underline{\mathcal{R}}_j$ we set

$$\varphi_T(R_1, \dots, R_k) := \frac{1}{k!} \sum_{f \in F(R_1, \dots, R_k)} (-1)^{\#\text{ edges in } f} \tag{3.7}$$

where $F(R_1, \dots, R_k)$ is the collection of connected subgraphs with vertex set $\{1, \dots, k\}$ of the graph with vertices $\{1, \dots, k\}$ and edges $\{h, h'\}$ corresponding to pairs $R_h, R_{h'}$ such that $R_h \mathrm{inc}_j R_{h'}$. We set the sum equal to zero if F is empty and one if $k = 1$.

Theorem 3.2. *Suppose the hypotheses of Theorem 2.5 are satisfied. Then, there exist functions $Z^{(j)}_{g,\Lambda}, \zeta_{\underline{R},\Lambda} : \mathcal{S} \to \mathbb{R}$, with $j \ge 1$, $g \in \mathcal{G}_j$ and $\underline{R} \in \underline{\mathcal{R}}_j$, such that $\underline{R}\restriction_{\ge j+1} = \emptyset$, and $\Lambda \subset\subset \mathbb{L}$, such that*

1. *for each $\tau \in \mathcal{S}$ and $\Lambda \subset\subset \mathbb{L}$ the free energy $\log Z_\Lambda(\tau)$ can be written as the absolutely convergent series*

$$\log Z_\Lambda(\tau) = \sum_{\substack{X \cap \Lambda \ne \emptyset: \\ \xi(X) = \emptyset}} V_{X,\Lambda}(\tau) + \sum_{j=1}^{\varkappa} \sum_{g \in \mathcal{G}_j} \log Z^{(j)}_{g,\Lambda}(\tau) + \sum_{j=1}^{\varkappa} \sum_{\substack{\underline{R} \in \underline{\mathcal{R}}_j \\ \underline{R}\restriction_{\ge j+1} = \emptyset}} \varphi_T\left(\underline{R}\right) \zeta_{\underline{R},\Lambda}(\tau) \tag{3.8}$$

   where $\varkappa = \varkappa(\Lambda) < \infty$ is the minimal integer k such that $\overline{\Lambda} \cap \bigcup_{j \ge k+1} \mathbb{G}_j = \emptyset$, so that Λ admits the partition $\Lambda = \bigcup_{j=0}^{\varkappa} \Lambda_j$, with $\Lambda_j := \Lambda \cap \mathbb{G}_j$;

2. *for each $j \ge 1$ and $g \in \mathcal{G}_j$ the family $\{Z^{(j)}_{g,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), \emptyset)$–compatible and each function $Z^{(j)}_{g,\Lambda}$ is identically equal to one whenever $\overline{g} \subset \Lambda^{\mathrm{c}}$. For each $j \ge 1$ and $\underline{R} \in \underline{\mathcal{R}}_j$, such that $\underline{R}\restriction_{\ge j+1} = \emptyset$, the family $\{\zeta_{\underline{R},\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(\mathrm{supp}\,\underline{R}, \emptyset)$–compatible and each function $\zeta_{\underline{R},\Lambda}$ is identically zero if there exists $R \in \underline{R}$, $(G, s) \in R$ and $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$;*

3. *let $\varepsilon := \exp\{-\alpha\gamma_1/8\}$; then*

$$\sup_{\Lambda \subset\subset \mathbb{L}} \| \log Z^{(j)}_{g,\Lambda} \|_\infty \le (\Gamma_j + 1)^d \big[\|U\| + \log S \big] + j(1 \vee A)(8^d + 1)\Gamma_j^d \tag{3.9}$$

   and

$$\sup_{\Lambda \subset\subset \mathbb{L}} \|\zeta_{\underline{R},\Lambda}\|_\infty \le \prod_{R \in \underline{R}} \prod_{(G,s) \in R} \varepsilon^{|G|} \exp\Big\{ -\frac{\alpha}{16} \Big[\mathcal{T}(G) + \frac{1}{2} \mathrm{d}_1(\mathcal{Q}(\widehat{G}), y_s(G)) \Big] \Big\} \tag{3.10}$$

4. Algebra of the expansion

In this Section we introduce the algebra of the graded cluster expansion without discussing any convergence issue, which will be dealt upon in Section 5. We suppose the hypotheses of Theorem 3.2 are satisfied. Moreover, for $\Lambda \subset\subset \mathbb{L}$ we define the set

$$\Upsilon_\Lambda := \{X \subset\subset \mathbb{L} : \, X \cap \Lambda \neq \emptyset \text{ and } X \cap (\overline{\Lambda})^{\mathrm{c}} = \emptyset\} \tag{4.1}$$

by item 3 in Condition 2.1 we can rewrite (2.4) as

$$Z_{\Lambda \cap \mathbb{G}_0}(\sigma) = \exp\Big\{ \sum_{X \in \Upsilon_\Lambda} V_{X,\Lambda}(\sigma) \Big\} \tag{4.2}$$

Given $\Lambda \subset\subset \mathbb{L}$, for $G \subset\subset \mathcal{G}_{\geq 1}$ and $s \geq 0$ let us define

$$\Upsilon_\Lambda(G,s) := \{X \in \Upsilon_\Lambda : \, \xi(X) = G, \, X \subset Y_s(G), \, X \cap y_s(G) \neq \emptyset\} \tag{4.3}$$

In other words $\Upsilon_\Lambda(G,s)$ is the collection of the subsets X of $Y_s(G)$ intersecting Λ, all and only the atoms of the gentle disintegration in G, and the annulus $y_s(G)$. It is easy to show that for each $\Lambda, \Lambda' \subset\subset \mathbb{L}$ one has

$$\Lambda \cap Y_s(G) = \Lambda' \cap Y_s(G) \implies \Upsilon_\Lambda(G,s) = \Upsilon_{\Lambda'}(G,s) \tag{4.4}$$

Notice, finally, that if there exists $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$ then $\Upsilon_\Lambda(G,s) = \emptyset$.

Recalling that $V_{X,\Lambda}$ has been introduced in (2.4), for $i \geq 1$, and $g \in \mathcal{G}_i$ we define the following function

$$\Psi^{(i,0)}_{g,\Lambda} := \sum_{X \in \Upsilon_\Lambda(g,0)} V_{X,\Lambda} \tag{4.5}$$

Recalling Definition 3.1, we have that (4.4) above and the items 1 and 2 of Condition 2.1 imply that the family $\{\Psi^{(i,0)}_{g,\Lambda}, \, \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), g)$–compatible. Furthermore, if $\overline{g} \subset \Lambda^{\mathrm{c}}$ then $\Upsilon_\Lambda(g,0) = \emptyset$ implies $\Psi^{(i,0)}_{g,\Lambda} = 0$. We shall look at $\Psi^{(i,0)}_{g,\Lambda}$ as the contribution to the self interaction of the i–atom g due to the integration on scale 0. It will not be expanded, but it will contribute to the reference (product) measure relative to the expansion at step i.

For $i \geq 1$, $G \subset\subset \mathcal{G}_{\geq i}$ such that $G \cap \mathcal{G}_i \neq \emptyset$, and $s \geq 0$ we define

$$\Phi^{(i,0)}_{G,s,\Lambda} := \begin{cases} \displaystyle\sum_{X \in \Upsilon_\Lambda(G,s)} V_{X,\Lambda} & \text{if} \quad (|G|, s) \neq (1,0) \\ \qquad 0 & \text{if} \quad (|G|, s) = (1,0) \end{cases} \tag{4.6}$$

As before we get that the family $\{\Phi^{(i,0)}_{G,s,\Lambda}, \, \Lambda \subset\subset \mathbb{L}\}$ is $(Y_s(G), \widehat{G})$–compatible and that $\Phi^{(i,0)}_{G,s,\Lambda} = 0$ if there exists $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$. We shall look at $\Phi^{(i,0)}_{G,s,\Lambda}$ as the effective interaction at scale i due to the integration on scale 0; it will be expanded at step i.

By using definitions (4.5) and (4.6) we have

$$\sum_{X\in\Upsilon_\Lambda} V_{X,\Lambda} = \sum_{\substack{X\in\Upsilon_\Lambda:\\ \xi(X)=\emptyset}} V_{X,\Lambda} + \sum_{i\geq 1}\sum_{g\in\mathcal{G}_i} \Psi^{(i,0)}_{g,\Lambda} + \sum_{i\geq 1}\sum_{\substack{G\subset\subset\mathcal{G}_{\geq i}\\ G\cap\mathcal{G}_i\neq\emptyset}}\sum_{s\geq 0}\Phi^{(i,0)}_{G,s,\Lambda} \tag{4.7}$$

Note that if $\xi(X)=\emptyset$ and $X\subset\Lambda$ then $V_{X,\Lambda}\in\mathcal{F}_\emptyset$, namely the function $V_{X,\Lambda}$ is constant. Moreover, since $\Lambda\subset\subset\mathbb{L}$, all but a finite number of terms on the r.h.s. of (4.7) are vanishing.

To simplify the notation for each $g\in\mathcal{G}_{\geq 1}$ we define the bare self–interaction inside g as

$$U_{g,\Lambda} := \sum_{\substack{X\subset\subset\mathbb{L}:\, X\cap\Lambda\neq\emptyset\\ X\cap\Lambda\subset g\cap\Lambda}} U_X \tag{4.8}$$

and remark that, since the potential U has range $R\leq r$, we have that the family $\{U_{g,\Lambda},\ \Lambda\subset\subset\mathbb{L}\}$ is $(\overline{g},g)$–compatible; furthermore, $U_{g,\Lambda}=0$ if $g\cap\Lambda=\emptyset$.

Note that for $g,h\in\mathcal{G}_{\geq 1}$, $g\neq h$, we have $\mathrm{d}_1(g,h)>\gamma_1>r\geq R$. Recalling that the integer $\varkappa$ has been defined in Theorem 3.2 we have $\mathcal{S}_\Lambda=\bigotimes_{j=0}^{\varkappa}\mathcal{S}(\Lambda_j)$, where we recall in item 1 of Theorem 3.2 we have defined $\Lambda_j:=\Lambda\cap\mathbb{G}_j$ for $j\geq 0$. For $i\geq 0$ we also set

$$\Lambda_{\geq i} := \bigcup_{j\geq i}\Lambda_j \qquad\text{and}\qquad \Delta_i := \Lambda_{\geq i+1}\cup\Lambda^{\mathrm{c}}$$

Then, given $\tau\in\mathcal{S}$, recalling the abuse of notation mentioned at the end of Section 2.2, we have

$$\begin{aligned} Z_\Lambda(\tau) \;&:=\; \sum_{\eta\in\mathcal{S}:\,\eta_{\Lambda^{\mathrm{c}}}=\tau} e^{H_\Lambda(\eta)} = \sum_{\substack{\eta^\varkappa\in\mathcal{S}(\Delta_{\varkappa-1})\\ \eta^\varkappa_{\Delta_\varkappa}=\tau}}\cdots\sum_{\substack{\eta^1\in\mathcal{S}(\Delta_0)\\ \eta^1_{\Delta_1}=\eta^2}}\sum_{\substack{\eta^0\in\mathcal{S}\\ \eta^0_{\Delta_0}=\eta^1}} e^{H_\Lambda(\eta_0)}\\ &=\; \sum_{\substack{\eta^\varkappa\in\mathcal{S}(\Delta_{\varkappa-1})\\ \eta^\varkappa_{\Delta_\varkappa}=\tau}}\prod_{g\in\mathcal{G}_\varkappa} e^{U_{g,\Lambda}(\eta^\varkappa)}\cdots\sum_{\substack{\eta^1\in\mathcal{S}(\Delta_0)\\ \eta^1_{\Delta_1}=\eta^2}}\prod_{g\in\mathcal{G}_1} e^{U_{g,\Lambda}(\eta^1)}\sum_{\substack{\eta^0\in\mathcal{S}\\ \eta^0_{\Delta_0}=\eta^1}} e^{H_{\Lambda_0}(\eta_0)} \end{aligned} \tag{4.9}$$

Now, by using equations (4.2) and (4.7) we get

$$\begin{aligned} Z_\Lambda(\tau) \;=\;& \exp\Bigg\{\sum_{\substack{X\in\Upsilon_\Lambda:\\ \xi(X)=\emptyset}} V_{X,\Lambda}(\tau)\Bigg\}\\ &\times\sum_{\substack{\eta^\varkappa\in\mathcal{S}(\Delta_{\varkappa-1})\\ \eta^\varkappa_{\Delta_\varkappa}=\tau}}\prod_{g\in\mathcal{G}_\varkappa} e^{U_{g,\Lambda}(\eta^\varkappa)+\Psi^{(\varkappa,0)}_{g,\Lambda}(\eta^\varkappa)}\cdot\exp\Bigg\{\sum_{\substack{G\subset\subset\mathcal{G}_{\geq\varkappa}\\ G\cap\mathcal{G}_\varkappa\neq\emptyset}}\sum_{s\geq 0}\Phi^{(\varkappa,0)}_{G,s,\Lambda}(\eta^\varkappa)\Bigg\}\\ &\times\cdots\times\sum_{\substack{\eta^1\in\mathcal{S}(\Delta_0)\\ \eta^1_{\Delta_1}=\eta^2}}\prod_{g\in\mathcal{G}_1} e^{U_{g,\Lambda}(\eta^1)+\Psi^{(1,0)}_{g,\Lambda}(\eta^1)}\cdot\exp\Bigg\{\sum_{\substack{G\subset\subset\mathcal{G}_{\geq 1}\\ G\cap\mathcal{G}_1\neq\emptyset}}\sum_{s\geq 0}\Phi^{(1,0)}_{G,s,\Lambda}(\eta^1)\Bigg\} \end{aligned} \tag{4.10}$$

We next define by recursion on $j = 0, \dots, \varkappa$ some functions $\Phi^{(i,j)}$ and $\Psi^{(i,j)}$, $0 \le j < i \le \varkappa$. As in the case $j = 0$ we look at $\Phi^{(i,j)}$ as the effective interaction at scale i due to the integration on scale $j < i$; on the other hand we look at $\Psi^{(i,j)}$ as the effective self–interaction at scale i due to the integration on scale $j < i$.

As recursive hypotheses we assume that we have already defined the families of functions $\{\Psi^{(i,m)}_{g,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$, which is $(Y_0(g), g)$–compatible, and $\{\Phi^{(i,m)}_{G,s,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$, which is $(Y_s(G), \widehat{G})$–compatible, for any $m = 0, \dots, j-1$, any $i = m+1, \dots, \varkappa$, any $g \in \mathcal{G}_i$, any $G \subset\subset \mathcal{G}_{\ge i}$, such that $G \cap \mathcal{G}_i \neq \emptyset$, and any $s \ge 0$. Moreover we assume $\Psi^{(i,m)}_{g,\Lambda} = 0$ if $\overline{g} \subset \Lambda^{\mathrm{c}}$ and $\Phi^{(i,m)}_{G,s,\Lambda} = 0$ if $(|G|, s) = (1, 0)$ or there exists $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$. We next define, by integrating on the scale j, the potentials $\Psi^{(i,j)}_{g,\Lambda}$ and $\Phi^{(i,j)}_{G,s,\Lambda}$ for $i = j+1, \dots, \varkappa$, any $g \in \mathcal{G}_i$, any $G \subset\subset \mathcal{G}_{\ge i}$, such that $G \cap \mathcal{G}_i \neq \emptyset$, and $s \ge 0$, and show that they satisfy the compatibility properties stated above.

By the recursive assumptions and the properties of $U_{g,\Lambda}$, for each $g \in \mathcal{G}_j$ the family of functions $\{U_{g,\Lambda} + \sum_{m=0}^{j-1} \Psi^{(j,m)}_{g,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), g)$–compatible and a function of the family is identically zero if $\overline{g} \subset \Lambda^{\mathrm{c}}$. Therefore, for $\eta^{j+1} \in \mathcal{S}_{\Delta_j}$ we can set

$$Z^{(j)}_{g,\Lambda}(\eta^{j+1}) := \sum_{\substack{\eta^j \in \mathcal{S}(g \cup \Lambda_{\ge j+1} \cup \Lambda^{\mathrm{c}}) \\ \eta^j_{\Delta_j} = \eta^{j+1}}} \exp\left\{ U_{g,\Lambda}(\eta^j) + \sum_{m=0}^{j-1} \Psi^{(j,m)}_{g,\Lambda}(\eta^j) \right\} \tag{4.11}$$

We note that the family $\{Z^{(j)}_{g,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), \emptyset)$–compatible and a function of the family is identically equal to one if $\overline{g} \subset \Lambda^{\mathrm{c}}$. For each $\eta^{j+1} \in \mathcal{S}_{\Delta_j}$ we can define a probability measure $\nu^{(j)}_{g,\Lambda,\eta^{j+1}}$ on $\mathcal{S}_g$ by setting, for each $\sigma \in \mathcal{S}_g$,

$$\nu^{(j)}_{g,\Lambda,\eta^{j+1}}(\sigma) := \delta_{\eta^{j+1}}(\sigma_{g \cap \Lambda^{\mathrm{c}}}) \frac{1}{Z^{(j)}_{g,\Lambda}(\eta^{j+1})} \exp\left\{ U_{g,\Lambda}(\sigma\eta^{j+1}) + \sum_{m=0}^{j-1} \Psi^{(j,m)}_{g,\Lambda}(\sigma\eta^{j+1}) \right\} \tag{4.12}$$

For each $\sigma \in \mathcal{S}_g$ the family $\{\eta^{j+1} \mapsto \nu_{g,\Lambda,\eta^{j+1}}(\sigma),\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), \emptyset)$–compatible; moreover, $\nu_{g,\Lambda,\eta^{j+1}} = \delta_{\eta^{j+1}}$ if $g \subset \Lambda^{\mathrm{c}}$.

Given $G \subset\subset \mathcal{G}_{\ge j}$ such that $G \cap \mathcal{G}_j \neq \emptyset$, and $s \ge 0$ we set

$$\Phi^{(j)}_{G,s,\Lambda} := \sum_{m=0}^{j-1} \Phi^{(j,m)}_{G,s,\Lambda} \tag{4.13}$$

which is the (cumulated) effective interaction at scale j. By the recursive hypotheses we have that the family $\{\Phi^{(j)}_{G,s,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_s(G), \widehat{G})$–compatible; moreover $\Phi^{(j)}_{G,s,\Lambda}$ is identically zero if $(|G|, s) = (1, 0)$ or there exists $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$.

Let $\eta^{j+1} \in \mathcal{S}(\Delta_j)$ and $R = \{(G_1, s_1), \dots (G_k, s_k)\} \in \mathcal{R}_j$; we define its activity $\zeta_{R,\Lambda}(\eta^{j+1})$ as

$$\zeta_{R,\Lambda}(\eta^{j+1}) := \sum_{\substack{\eta^j \in \mathcal{S}(\widehat{R\restriction_j} \cup \Lambda_{\geq j+1} \cup \Lambda^c) \\ \eta^j_{\Delta_j} = \eta^{j+1}}} \prod_{g \in R\restriction_j} \nu^{(j)}_{g,\Lambda,\eta^{j+1}}(\eta^j_g) \prod_{h=1}^{k} \left[\exp\left\{ \Phi^{(j)}_{G_h,s_h,\Lambda}(\eta^j) \right\} - 1 \right] \tag{4.14}$$

It follows that $\{\zeta_{R,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(\operatorname{supp} R, \widehat{R\restriction_{\geq j+1}})$–compatible and an element of the family is identically zero if there exists $(G, s) \in R$ and $g \in G$ such that $\overline{g} \subset \Lambda^c$. For $\underline{R} \in \underline{\mathcal{R}}_j$, we set

$$\zeta_{\underline{R},\Lambda}(\eta^{j+1}) := \prod_{R \in \underline{R}} \zeta_{R,\Lambda}(\eta^{j+1}) \tag{4.15}$$

it follows that $\{\zeta_{\underline{R},\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(\operatorname{supp} \underline{R}, \widehat{\underline{R}\restriction_{\geq j+1}})$–compatible and an element of the family is identically zero if there exists $R \in \underline{R}$, $(G, s) \in R$, and $g \in G$ such that $\overline{g} \subset \Lambda^c$.

By standard polymerization and cluster expansion, under suitable "small activity" conditions that will be specified later on, see item 7 in Lemma 5.9 below, we have, see e.g. [24],

$$\begin{aligned} &\sum_{\substack{\eta^j \in \mathcal{S}(\Delta_{j-1}) \\ \eta^j_{\Delta_j} = \eta^{j+1}}} \prod_{g \in \mathcal{G}_j} \nu^{(j)}_{g,\Lambda,\eta^{j+1}}(\eta^j_g) \exp\left\{ \sum_{\substack{G \subset\subset \mathcal{G}_j \\ G \cap \mathcal{G}_j \neq \emptyset}} \sum_{s \geq 0} \Phi^{(j)}_{G,s,\Lambda}(\eta^j) \right\} \\ &\qquad = 1 + \sum_{\underline{R} \in \underline{\mathcal{R}}_j^{\mathrm{comp}}} \zeta_{\underline{R},\Lambda}(\eta^{j+1}) = \exp\left\{ \sum_{\underline{R} \in \underline{\mathcal{R}}_j} \varphi_T\left(\underline{R}\right) \zeta_{\underline{R},\Lambda}(\eta^{j+1}) \right\} \end{aligned} \tag{4.16}$$

with φ_T defined in (3.7).

We are now ready to define the interactions due to the integration on the scale j. Let $G \subset\subset \mathcal{G}_{\geq j+1}$ and $s \geq 0$, we define

$$\underline{\mathcal{R}}_j(G, s) := \left\{ \underline{R} \in \underline{\mathcal{R}}_j :\ \underline{R}\restriction_{\geq j+1} = G\,,\ \operatorname{supp} \underline{R} \subset Y_s(G),\ \operatorname{supp} \underline{R} \cap y_s(G) \neq \emptyset \right\} \tag{4.17}$$

For $g \in \mathcal{G}_i$, $i > j$, we let

$$\Psi^{(i,j)}_{g,\Lambda} := \sum_{\underline{R} \in \underline{\mathcal{R}}_j(g,0)} \varphi_T\left(\underline{R}\right)\, \zeta_{\underline{R},\Lambda} \tag{4.18}$$

It is easy to check that $\{\Psi^{(i,j)}_{g,\Lambda},\ \Lambda \subset\subset \mathbb{L}\}$ is $(Y_0(g), g)$–compatible and an element of the family is identically zero if $\overline{g} \subset \Lambda^c$; so we met the first recursive condition. The effective interaction at scale $i > j$ due to the integration on scale j is defined as follows; for

$G \subset\subset \mathcal{G}_{\geq i}$, $G \cap \mathcal{G}_i \neq \emptyset$ and $s \geq 0$ we set

$$\Phi^{(i,j)}_{G,s,\Lambda} := \begin{cases} \displaystyle\sum_{\underline{R} \in \underline{\mathcal{R}}_j(G,s)} \varphi_T\left(\underline{R}\right) \zeta_{\underline{R},\Lambda} & \text{if} \quad (|G|, s) \neq (1,0) \\ 0 & \text{if} \quad (|G|, s) = (1,0) \end{cases} \tag{4.19}$$

As before $\{\Phi^{(i,j)}_{G,s,\Lambda}, \Lambda \subset\subset \mathbb{L}\}$ is $(Y_s(G), \widehat{G})$–compatible and an element of the family is identically zero if there exists $g \in G$ such that $\overline{g} \subset \Lambda^{\mathrm{c}}$; so we also met the second recursive condition.

By noticing that

$$\sum_{\underline{R} \in \underline{\mathcal{R}}_j} \varphi_T(\underline{R})\zeta_{\underline{R},\Lambda} = \sum_{\substack{\underline{R} \in \underline{\mathcal{R}}_j \\ \underline{R}\restriction_{\geq j+1} = \emptyset}} \varphi_T(\underline{R})\zeta_{\underline{R},\Lambda} + \sum_{i=j+1}^{\varkappa} \left\{ \sum_{g \in \mathcal{G}_i} \Psi^{(i,j)}_{g,\Lambda} + \sum_{\substack{G \subset\subset \mathcal{G}_{\geq i} \\ G \cap \mathcal{G}_i \neq \emptyset}} \sum_{s \geq 0} \Phi^{(i,j)}_{G,s,\Lambda} \right\} \tag{4.20}$$

and using recursively (4.16) in (4.10), it is easy to check that, provided all the series converges absolutely, we have got the expansion (3.8).

5. Convergence of the graded cluster expansion

In this section we prove the convergence of the cluster expansion introduced in Section 4 above.

5.1. Geometric bounds

In this section we collect bounds which hold in our geometry of wide separated *gentle atoms*. For the reader convenience we restate [3, Lemma 3.4] in the present context.

Lemma 5.1. *Let k be a positive integer and $\Pi_0(k)$ be the set of permutations π of $\{0,1,\ldots,k\}$ such that $\pi(0) = 0$. Let $X = \{x_0, x_1, \ldots, x_k\} \subset \mathbb{L}$ and $\mathbb{T}(X)$ as in (2.3); then*

$$\mathbb{T}(X) \geq \frac{1}{2} \inf_{\pi \in \Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}_1\left(x_{\pi(l-1)}, x_{\pi(l)}\right) \tag{5.1}$$

Proof. It is easy to show that the infimum in (2.3) is attained (not necessary uniquely) for a graph $T_X = (V_X, E_X) \subset (\mathbb{L}, \mathbb{E})$ which is a tree, i.e. a connected and loop–free graph. The Lemma follows from the bound

$$|E_X| \geq \frac{1}{2} \inf_{\pi \in \Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}_1\left(x_{\pi(l-1)}, x_{\pi(l)}\right) \tag{5.2}$$

which is proven as follows. By induction on the number of edges in T_X it is easy to prove, see Fig. 1, that there exists a path $(\ell_0, \ldots, \ell_{M-1})$, with $\ell_m \in E_X$ for all $m = 0, \ldots, M-1$,

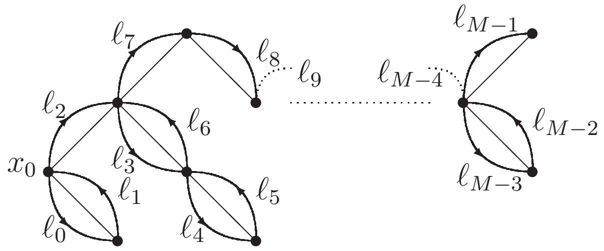

Figure 1: The path $\ell = \{\ell_0, \ldots, \ell_{M-1}\}$ introduced in the proof of Lemma 5.1.The solid circles represent the points $\{x_0, x_1, \ldots, x_k\}$.

satisfying the following properties: $\ell_{m-1} \cap \ell_m \neq \emptyset$ for all $m = 1, \ldots, M-1$, $x_0 \in \ell_0$, for each $v \in V_X$ there exists $m \in \{0, \ldots, M-1\}$ such that $\ell_m \ni v$, and each $e \in E_X$ appears in the path at most twice. The bound (5.2) then follows. $\square$

We give, now, a recursive definition that will be used to parameterize the exponential decay of the potential at different scales. Recall definitions (4.17) and (2.3), set

$$\begin{aligned}
&\mathcal{T}_0(G,s) := \inf_{x \in \mathcal{Y}_s(G)} \inf_{\substack{x_m \in g_m \\ m=1,\ldots,n}} \mathbb{T}\left(\{x, x_1, \ldots, x_n\}\right) \quad \text{for } G = \{g_1, \ldots, g_n\} \subset\subset \mathcal{G}_{\geq 1} \text{ and } s \geq 0 \\
&\mathcal{T}_j(G,s) := \inf_{\underline{R} \in \underline{\mathcal{R}}_j(G,s)} \sum_{R \in \underline{R}} \sum_{(H,u) \in R} \mathcal{T}_{j-1}(H,u) \quad \text{for } j \geq 1,\ G \subset\subset \mathcal{G}_{\geq j+1},\ \text{and } s \geq 0
\end{aligned} \tag{5.3}$$

As usual if $\underline{R}_j(G,s) = \emptyset$ we understand $\mathcal{T}_j(G,s) = +\infty$. Note that $\mathcal{T}_0(G,0) = \mathcal{T}(G)$, see (3.6). Finally for each $j \geq 0$, $G \subset\subset \mathcal{G}_{\geq j+1}$ and $s \geq 0$ we set $\widehat{\mathcal{T}}_j(G,s) := \inf_{0 \leq k \leq j} \mathcal{T}_k(G,s)$. In order to clarify the recursive definition (5.3) we consider in some detail the case $j = 1$, $G = \{g_1, g_2\} \subset \mathcal{G}_2$, and $s = 0$. Let $\underline{R}^* \in \underline{\mathcal{R}}_1(\{g_1, g_2\}, 0)$ be a minimizer for the right–hand side of (5.3). Then

$$\mathcal{T}_1(\{g_1, g_2\}, 0) = \sum_{R \in \underline{R}^*} \sum_{(H,u) \in R} \mathcal{T}_0(H,u)$$

We note that a polymer $R \in \underline{R}^*$ is built of bonds (H,u) connecting on 1–gentle atoms. Therefore, $\mathcal{T}_1(\{g_1, g_2\}, 0)$ can be strictly smaller than $\mathrm{d}_1(g_1, g_2)$ due to the presence of 1–gentle atoms between g_1 and g_2. However, by the sparseness conditions 4 and 5 of Definition 2.4, we have

$$\mathcal{T}_1(\{g_1, g_2\}, 0) \geq \frac{\gamma_1}{\Gamma_1 + \gamma_1} \mathrm{d}_1(g_1, g_2) \geq \left(1 - \frac{\Gamma_1}{\gamma_1}\right) \mathrm{d}_1(g_1, g_2)$$

Indeed, the maximum number of 1–gentle atoms that can be arranged between g_1 and g_2 is $\mathrm{d}_1(g_1, g_2)/(\Gamma_1 + \gamma_1)$. The following proposition states a similar bound for a general situation.

Proposition 5.2. *Let $j \geq 0$, $G \subset\subset \mathcal{G}_{\geq j+1}$, and $s \geq 0$. Then*

$$\mathcal{T}_j(G,s) \geq \Big(1 - \sum_{k=0}^{j} \frac{\Gamma_k}{\gamma_k}\Big)\Big\{\mathcal{T}(G) + 1\!\!1_{s\geq 1}\big[\mathrm{d}_1(\mathcal{Q}(\widehat{G}), y_s(G)) - \vartheta_j\big]\Big\} \tag{5.4}$$

where we understand $0/\gamma_0 = 0$ even if $\gamma_0 = 0$.

We remark that from the bound (5.4) above, item 3 in Definition 2.2, and (2.9) it is straightforward to deduce that

$$\widehat{\mathcal{T}}_j(G,s) \geq \frac{1}{2}\mathcal{T}(G) + \frac{1}{4}\mathrm{d}_1(\mathcal{Q}(\widehat{G}), y_s(G)) \tag{5.5}$$

To prove Proposition 5.2 one of the ingredients is a lemma about one–side projections of graphs to hyper–planes. In order to state it we need a few more definitions. Let $\widehat{n} \in \{e_i, -e_i,\, i = 1,\dots,d\}$ be a coordinate direction and $c \in \mathbb{N}$ an integer; we consider the hyper–plane $\pi \equiv \pi_{\widehat{n},c} := \{x \in \mathbb{L},\, (x - c\widehat{n}) \cdot \widehat{n} = 0\} \subset \mathbb{L}$, where $\cdot$ denotes the canonical inner product in $\mathbb{R}^d$. We then define the half–lattices $\mathbb{L}_{\pi,\leq} := \{x \in \mathbb{L},\, (x - c\widehat{n}) \cdot \widehat{n} \leq 0\}$ and $\mathbb{L}_{\pi,>} := \{x \in \mathbb{L},\, (x - c\widehat{n}) \cdot \widehat{n} > 0\}$; remark that $\mathbb{L}_{\pi,\leq} \supset \pi$.

Given a connected graph $(V,E) \subset\subset (\mathbb{L},\mathbb{E})$, recall the definition above (2.3), we define $V_{\pi,\leq} := V \cap \mathbb{L}_{\pi,\leq}$, $V_{\pi,>} := V \cap \mathbb{L}_{\pi,>}$, $E_{\pi,\leq} := \{e \in E,\, e \subset \mathbb{L}_{\pi,\leq}\}$, and $E_{\pi,>} := \{e \in E,\, e \cap \mathbb{L}_{\pi,>} \neq \emptyset\}$. We note that $V = V_{\pi,\leq} \cup V_{\pi,>}$ and $E = E_{\pi,\leq} \cup E_{\pi,>}$. We finally define $E^{\perp}_{\pi,>} := \big\{\{x,y\} \subset \pi,\, \exists k \geq 1 \text{ such that } \{x + k\widehat{n}, y + k\widehat{n}\} \in E_{\pi,>}\big\}$.

Lemma 5.3. *Let $(V,E) \subset (\mathbb{L},\mathbb{E})$ be a connected graph, $\widehat{n} \in \{e_i, -e_i,\, i = 1,\dots,d\}$ a coordinate direction, and $c \in \mathbb{N}$; consider the hyper–plane $\pi_{\widehat{n},c} \equiv \pi \subset \mathbb{L}$. With the definitions given above, if $V_{\pi,\leq} \neq \emptyset$ then*

1. *the bound*
$$|E| \geq |E_{\pi,\leq} \cup E^{\perp}_{\pi,>}| + \sup_{v \in V_{\pi,>}} \mathrm{d}_1(v,\pi) \tag{5.6}$$
holds, where we understand the second term in the right–hand side equal to zero whenever $V_{\pi,>} = \emptyset$;

2. *the pair $(V_{\pi,\leq}, E_{\pi,\leq} \cup E^{\perp}_{\pi,>})$ is a connected graph.*

We remark that this lemma depends on the use of the distance d_1 in the definition of the edge set $\mathbb{E}$. Indeed it would have been false if we had used the distance d_∞.

Proof of Lemma 5.3. Proof of the item 1. Let $E^{\parallel}_{\pi,>} := \big\{\{x,y\} \in E_{\pi,>},\, (y - x) \cdot \widehat{n} \neq 0\big\}$; it is immediate to show that

$$|E| \geq |E_{\pi,\leq} \cup E^{\perp}_{\pi,>}| + |E^{\parallel}_{\pi,>}| \tag{5.7}$$

If $V_{\pi,>} = \emptyset$ (5.6) trivially follows from (5.7). Suppose, now, $V_{\pi,>} \neq \emptyset$. Pick $v \in V_{\pi,>}$ and let $D := \mathrm{d}_1(\pi, v) = \mathrm{d}_\infty(\pi, v)$. Recalling that the graph (V,E) is connected and that by hypotheses $V_{\pi,\leq} \neq \emptyset$, we have that there exist $w \in \pi$ and a connected path $\ell_1, \ldots, \ell_h$ such that $v \in \ell_1$, $w \in \ell_h$, and $\ell_m \in E_{\pi,>}$ for all $m = 1, \ldots, h$. We have the obvious bounds

$$\big|E^{\|}_{\pi,>}\big| \geq \big|\{\{x,y\} \in \{\ell_1, \ldots, \ell_h\},\, (y-x)\cdot\widehat{n} \neq 0\}\big| \geq D \tag{5.8}$$

The inequality (5.6) follows from (5.7) and (5.8).

Proof of the item 2. The statement is trivial if $|V_{\pi,\leq}| = 1$. Suppose, now, $|V_{\pi,\leq}| \geq 2$ and pick two distinct vertexes $v, w \in V_{\pi,\leq}$. By recalling that (V,E) is a connected graph we have that there exists a connected path joining v to w namely, there exist $\ell_1, \ldots, \ell_h \in E$ such that $v \in \ell_1$, $w \in \ell_h$, and $\ell_m \cap \ell_{m+1} \neq \emptyset$ for $m = 1, \ldots, h-1$.

We let $\ell'_1, \ldots, \ell'_{h'}$ be the path obtained from $\ell_1, \ldots, \ell_h$ by removing all the edges belonging to $E^{\|}_{\pi,>}$; we remark that the path $\ell'_1, \ldots, \ell'_{h'}$ is not necessarily connected and that $1 \leq h' \leq h$. Let $\ell' = \{x', y'\}$ be an edge of such a path; ℓ' is either in $E_{\pi,\leq}$ or in $E_{\pi,>} \setminus E^{\|}_{\pi,>}$. We set $\bar{\ell}' := \ell'$ in the former case and $\bar{\ell}' := \{x' + (c - x'\cdot\widehat{n})\widehat{n}, y' + (c - y'\cdot\widehat{n})\widehat{n}\} \in E^{\perp}_{\pi,>}$ in the latter.

By construction $\bar{\ell}'_1, \ldots, \bar{\ell}'_{h'} \in E_{\pi,\leq} \cup E^{\perp}_{\pi,>}$. Moreover it is an easy task to prove that $v \in \bar{\ell}'_1$, $w \in \bar{\ell}'_{h'}$, and $\ell'_m \cap \ell'_{m+1} \neq \emptyset$ for $m = 1, \ldots, h'-1$. The proof of item 2 is completed. $\square$

Lemma 5.4. *Let $G \subset\subset \mathcal{G}_{\geq 1}$ and $s \geq 0$. Then the bound (5.4) holds true for $j = 0$.*

Proof. The statement is trivial in the case $s = 0$. Let $s \geq 1$ and label the elements of G by setting $G = \{g_1, \ldots, g_{|G|}\}$. Let $x^* \in y_s(G)$, $x^*_1 \in g_1, \ldots, x^*_{|G|} \in g_{|G|}$ be a minimizer for the infimum in the definition of $\mathcal{T}_0(G,s)$, see (5.3). Let also $V := \{x^*, x^*_1, \ldots, x^*_{|G|}\}$ and (V,E) the connected graph such that $|E| = \mathbb{T}\big(\{x^*, x^*_1, \ldots, x^*_{|G|}\}\big) = \mathcal{T}_0(G,s)$.

Let F_{x^*} the face of $y_s(G)$ such that $x^* \in F_{x^*}$ (choose anyone if it is not unique) and π the hyper–plane parallel to F_{x^*} such that $\pi \cap \mathcal{Q}(\widehat{G}) \neq \emptyset$ and $\mathrm{d}_1(\pi, F_{x^*})$ is minimal. Let also $\widehat{n}$ be the normal to π such that $(x^* - y)\cdot\widehat{n} > 0$ for any $y \in \pi$. By applying the Lemma 5.3 to the graph (V,E), the normal $\widehat{n}$, and the hyper–plane π we get $|E| \geq |E_{\pi,\leq} \cup E^{\perp}_{\pi,>}| + \mathrm{d}_1(x^*, \pi)$. Since $V_{\pi,\leq} = \{x^*_1, \ldots, x^*_{|G|}\}$, by item 2 of the Lemma 5.3 we have that $|E_{\pi,\leq} \cup E^{\perp}_{\pi,>}| \geq \mathbb{T}\big(\{x^*_1, \ldots, x^*_{|G|}\}\big) \geq \mathcal{T}(G)$. Moreover, by construction $\mathrm{d}_1(x^*, \pi) = \mathrm{d}_1(\mathcal{Q}(\widehat{G}), y_s(G))$. The thesis follows. $\square$

Proof of Proposition 5.2. We can assume $\underline{R}_j(G,s) \neq \emptyset$, otherwise $\mathcal{T}_j(G,s) = +\infty$. We prove (5.4) by induction; the step $j = 0$ has been proven in the Lemma 5.4. We suppose (5.4) holds for $j-1$ and we show it holds true for j. To bound $\mathcal{T}_j(G,s)$ we let $\underline{R}^* \in \underline{\mathcal{R}}_j(G,s)$ be a minimizer for (5.3). Note that $\underline{R}_j(G,s)$ is not a finite set because repetitions of the some bond are allowed. However a minimizer $\underline{R}^*$ does exist because without such repetitions $\underline{R}_j(G,s)$ would be finite and *repetita juvant*. We have

$$\mathcal{T}_j(G,s) = \sum_{R\in\underline{R}^*} \sum_{(H,u)\in R} \mathcal{T}_{j-1}(H,u) \tag{5.9}$$

We consider, now, the case $s=0$. Let $\mathcal{H} \equiv \mathcal{H}(\underline{R}^*) := \{H \subset \mathcal{G}_{\geq j} : \exists R \in \underline{R}^*, \exists u \geq 0 : (H,u) \in R \text{ and } |H| \geq 2\}$; we note that $\mathcal{H}$ is finite and not empty. From (5.9) and the inductive hypothesis we have

$$\mathcal{T}_j(G,0) \geq \Big(1 - \sum_{k=0}^{j-1} \frac{\Gamma_k}{\gamma_k}\Big) \sum_{H\in\mathcal{H}} \mathcal{T}(H)$$

We also remark that definitions (3.4) and (4.17) imply that the system $\mathcal{H}$ is j–connected in the sense specified just above (3.2). By adding and subtracting Γ_j/γ_j and by remarking that $|H| \geq 2$ implies $\mathcal{T}(H) \geq \gamma_j$ we get

$$\mathcal{T}_j(G,0) \geq \Big(1 - \sum_{k=0}^{j} \frac{\Gamma_k}{\gamma_k}\Big) \sum_{H\in\mathcal{H}} \mathcal{T}(H) + |\mathcal{H}|\Gamma_j \tag{5.10}$$

Let us construct a partition of the system $\mathcal{H}$: pick an element of $\mathcal{H}$, denote it by $H_{0,1}$, and set $\mathcal{H}_0 := \{H_{0,1}\}$. For any $m \geq 1$ and $H \in \mathcal{H} \setminus \bigcup_{\ell=0}^{m-1} \mathcal{H}_\ell$ we say that $H \in \mathcal{H}_m$ if and only if there exists $H' \in \mathcal{H}_{m-1}$ such that H and H' are j–connected namely, $H \cap H' \cap \mathcal{G}_j \neq \emptyset$. Recalling $\mathcal{H}$ is j–connected we have that there exists a maximal value of m that we call t; in other words there exists $t \geq 0$ such that $\mathcal{H}_m \neq \emptyset$ for all $m \leq t$ and $\mathcal{H}_m = \emptyset$ for all $m > t$. The collection $\mathcal{H}_0, \dots, \mathcal{H}_t$ is a partition of $\mathcal{H}$.

For each $m = 1, \dots, t$ we denote by $H_{m,1}, \dots, H_{m,|\mathcal{H}_m|}$ the elements of $\mathcal{H}_m$; for each $m = 0, \dots, t$ and $\ell = 1, \dots, |\mathcal{H}_m|$ we let $(V_{m,\ell}, E_{m,\ell}) \subset (\mathbb{L}, \mathbb{E})$ be a connected graph such that

$$\mathcal{T}(H_{m,\ell}) = |E_{m,\ell}| \tag{5.11}$$

and for each $h \in H_{m,\ell}$ we have that $V_{m,\ell} \cap h \neq \emptyset$. We define, now, an algorithm that constructs a graph $(V,E) \subset (\mathbb{L}, \mathbb{E})$ such that $|E| \geq \mathcal{T}(G)$ and $|E|$ is bounded from above in terms of $\mathcal{T}(H)$ for $H \in \mathcal{H}$:

1. `set` $m=0$ `and` $(V,E) = (V_{0,1}, E_{0,1})$`;`

2. `set` $m=m+1$ `and` $\ell=0$`, if` $m=t+1$ `goto 8;`

3. `set` $\ell=\ell+1$`, pick` $\ell' \in \{1,\dots,|\mathcal{H}_{m-1}|\}$ `such that` $H_{m-1,\ell'} \xleftrightarrow{j} H_{m,\ell}$`;`

4. `pick` $h \in H_{m-1,\ell'} \cap H_{m,\ell} \cap \mathcal{G}_j$`,` $y \in h \cap V_{m-1,\ell'}$`, and` $x \in h \cap V_{m,\ell}$`;`

5. `find a connected graph` $(W,F) \subset (\mathbb{L},\mathbb{E})$ `such that` $|F|$ `is minimal and the set of vertices` W `contains both` x `and` y`;`

6. `set` $V = V \cup V_{m,\ell} \cup W$ `and` $E = E \cup E_{m,\ell} \cup F$`;`

7. `if` $\ell < |\mathcal{H}_m|$ `goto 3 else goto 2;`

8. `exit;`

By recursion it is easy to prove that this algorithm outputs a connected graph (V,E) such that for each $H \in \mathcal{H}$ and $h \in H$ there exists $x \in h$ such that $x \in V$; in particular for each $g \in G$ there exists $x \in g$ such that $x \in V$, hence $|E| \geq \mathcal{T}(G)$. Moreover, by noticing that the graph (W,F) introduced at line 5 is such that $|F| \leq \mathrm{diam}_1(h) \leq \Gamma_j$ we have

$$|E| \leq \sum_{m=0}^{t} \sum_{\ell=1}^{|\mathcal{H}_m|} |E_{m,\ell}| + (|\mathcal{H}|-1)\Gamma_j \tag{5.12}$$

Now, by using (5.10)–(5.12) we get

$$\mathcal{T}_j(G,0) \geq \Big(1 - \sum_{k=0}^{j} \frac{\Gamma_k}{\gamma_k}\Big)\big[|E| - (|\mathcal{H}|-1)\Gamma_j\big] + |\mathcal{H}|\Gamma_j \geq \Big(1 - \sum_{k=0}^{j} \frac{\Gamma_k}{\gamma_k}\Big)\mathcal{T}(G)$$

which completes the inductive proof of (5.4) for $s = 0$.

We consider, now, the case $s \geq 1$. Recalling (5.9), there exists $R' \in \underline{R}^*$ and $(H',u') \in R'$ such that $Y_{u'}(H') \cap y_s(G) \neq \emptyset$. Let $\mathcal{H}' \equiv \mathcal{H}'(\underline{R}^*) := \{H \subset \mathcal{G}_{\geq j} : \exists R \in \underline{R}^*, \exists u \geq 0 : (H,u) \in R, (H,u) \neq (H',u') \text{ and } |H| \geq 2\}$. Note that, as in the previous case, $|H| \geq 2$ implies $\mathcal{T}(H) \geq \gamma_j$; on the other hand we note that $\mathcal{H}'$ can be empty. Set also $\mathcal{H} := \mathcal{H}' \cup \{H'\}$. By using (5.9) and the recursive hypothesis we have

$$\begin{aligned} \mathcal{T}_j(G,s) \geq \;& \mathcal{T}_{j-1}(H',u') + \sum_{H \in \mathcal{H}'} \mathcal{T}_{j-1}(H,u) \\ \geq \;& \Big(1 - \sum_{k=0}^{j-1} \frac{\Gamma_k}{\gamma_k}\Big)\Big\{ 1\!\!1_{u' \geq 1}\big[\mathrm{d}_1(\mathcal{Q}(\widehat{H'}), y_{u'}(H')) - \vartheta_{j-1}\big] + \sum_{H \in \mathcal{H}} \mathcal{T}(H)\Big\} \end{aligned} \tag{5.13}$$

We note that for each $H \in \mathcal{H}'$ we have $|H| \geq 2$, hence $\mathcal{T}(H) \geq \gamma_j$. Moreover, we claim that

$$\mathcal{T}(H') + 1\!\!1_{u' \geq 1}\big(\mathrm{d}_1(\mathcal{Q}(\widehat{H'}), y_{u'}(H')) - \vartheta_{j-1}\big) \geq \gamma_j \tag{5.14}$$

Indeed, if $u' = 0$ then $|H'| \geq 2$, so that $\mathcal{T}(H') \geq \gamma_j$. On the other end if $u' \geq 1$, then $\mathrm{d}_1(\mathcal{Q}(\widehat{H'}), y_{u'}(H')) = \vartheta_j + u'$ implies $\mathrm{d}_1(\mathcal{Q}(\widehat{H'}), y_{u'}(H')) - \vartheta_{j-1} > \vartheta_j - \vartheta_{j-1} = \Gamma_j + \gamma_j > \gamma_j$. Now, by adding and subtracting Γ_j/γ_j in (5.13) we get

$$\mathcal{T}_j(G,s) \geq \Big(1 - \sum_{k=0}^{j} \frac{\Gamma_k}{\gamma_k}\Big)\Big\{ \sum_{H \in \mathcal{H}} \mathcal{T}(H) + 1\!\!1_{u' \geq 1}\big[\mathrm{d}_1(\mathcal{Q}(\widehat{H'}), y_{u'}(H')) - \vartheta_{j-1}\big]\Big\} + |\mathcal{H}|\Gamma_j \tag{5.15}$$

Since $Y_{u'}(H')\cap y_s(G)\neq\emptyset$ there exists $h'\in H'$ such that $\mathrm{d}_1(h',y_s(G))=\vartheta_j+u'$. Label the elements of G by setting $G=\{g_1,\ldots,g_{|G|}\}$. By running the algorithm used in the case $s=0$, we construct a connected graph $(V,E)\subset(\mathbb{L},\mathbb{E})$ such that $V\supset\{x',x_1,\ldots,x_{|G|}\}$, for some $x'\in h'$, $x_1\in g_1,\ldots,x_{|G|}\in g_{|G|}$, and

$$\sum_{H\in\mathcal{H}}\mathcal{T}(H)\geq|E|-(|\mathcal{H}|-1)\Gamma_j \tag{5.16}$$

Let F' be the face of $y_s(G)$ such that $\mathrm{d}_1(h',F')=\vartheta_j+u'$ (choose anyone if it is not unique) and π the hyper–plane parallel to F' such that $\pi\cap\mathcal{Q}(\widehat{G})\neq\emptyset$ and $\mathrm{d}_1(\pi,F')$ is minimal. Let also $\widehat{n}$ be the normal to π such that $(y'-y)\cdot\widehat{n}>0$ for any $y'\in F'$ and $y\in\pi$. By applying the Lemma 5.3 to the graph (V,E), the normal $\widehat{n}$, and the hyper–plane π we get

$$|E|\geq\mathcal{T}(G)+\mathrm{d}_1(\mathcal{Q}(\widehat{G}),h') \tag{5.17}$$

Finally, by plugging (5.16) and (5.17) into (5.15) we get

$$\mathcal{T}_j(G,s)\geq\Big(1-\sum_{k=0}^{j}\frac{\Gamma_k}{\gamma_k}\Big)\Big\{\mathcal{T}(G)+\mathrm{d}_1(\mathcal{Q}(\widehat{G}),h')+\Gamma_j+\mathbb{1}_{u'\geq1}\big[\mathrm{d}_1(\mathcal{Q}(\widehat{H'}),y_{u'}(H'))-\vartheta_{j-1}\big]\Big\} \tag{5.18}$$

Consider, now, the sub–case $u'=0$. In this case $\mathrm{d}_1(h',F')=\vartheta_j$, hence $h'\not\subset\mathcal{Q}(\widehat{G})$. This implies $h'\in\mathcal{G}_j$; therefore $\mathrm{diam}_1(h')\leq\Gamma_j$, see item 4 in Definition 2.4. We get

$$\mathrm{d}_1(\mathcal{Q}(\widehat{G}),h')\geq\mathrm{d}_1(\mathcal{Q}(\widehat{G}),y_s(G))-\Gamma_j-\vartheta_j \tag{5.19}$$

The bound (5.4) follows from (5.18) and (5.19).

We finally consider the sub–case $u'\geq1$. Recalling how $h'\in H'$ has been chosen, we have that

$$\mathrm{d}_1(\mathcal{Q}(\widehat{H'}),y_{u'}(H'))=\mathrm{d}_1(h',y_{u'}(H'))=\mathrm{d}_1(h',y_s(G)) \tag{5.20}$$

If $h'\in G$ then $h'\subset\mathcal{Q}(\widehat{G})$; hence $\mathrm{d}_1(\mathcal{Q}(\widehat{H'}),y_{u'}(H'))\geq\mathrm{d}_1(\mathcal{Q}(\widehat{G}),y_s(G))$. Then (5.4) follows easily from (5.18). On the other hand if $h'\in\mathcal{G}_j$, we have $\mathrm{diam}_1(h')\leq\Gamma_j$, hence by using (5.20) we have

$$\mathrm{d}_1(\mathcal{Q}(\widehat{G}),h')+\Gamma_j+\mathrm{d}_1(\mathcal{Q}(\widehat{H'}),y_{u'}(H'))\geq\mathrm{d}_1(\mathcal{Q}(\widehat{G}),y_s(G)) \tag{5.21}$$

Then (5.4) follows easily from (5.18). $\square$

Lemma 5.5. *Let $j\geq0$, $G\subset\subset\mathcal{G}_{\geq j+1}$, $s\geq0$; suppose $\underline{\mathcal{R}}_j(G,s)\neq\emptyset$, see definition (4.17). For each $g\in G$, $\underline{R}\in\underline{\mathcal{R}}_j(G,s)$ and $h\in\underline{R}\restriction_j$; we have*

$$\sum_{R\in\underline{R}}\sum_{(H,u)\in R}\widehat{\mathcal{T}}_{j-1}(H,u)\geq\frac{1}{2}\mathrm{d}_1(g,h) \tag{5.22}$$

Proof. The Lemma can be proven by using (5.4), the simple bound $1-\sum_0^\infty(\Gamma_j/\gamma_j) \geq 1/2$, and by running the algorithm introduced in the proof of Lemma 5.2. $\square$

Lemma 5.6. *Let $G \subset\subset \mathcal{G}_{\geq j+1}$, $s \geq 0$ and $\underline{\mathcal{R}}_j(G,s)$ as defined in (4.17). Then, for each $\underline{R} \in \underline{\mathcal{R}}_j(G,s)$,*

$$\sum_{R\in\underline{R}} \sum_{(H,u)\in R} |H| \geq |G| + \sum_{R\in\underline{R}} |R\restriction_j| \tag{5.23}$$

Proof. The Lemma follows directly from the definition of $\underline{\mathcal{R}}_j(G,s)$. $\square$

5.2. Preliminary lemmata

In this section we collect some technical bounds needed to prove the convergence of the multi–scale cluster expansion.

Lemma 5.7. *For $m > 0$ let*

$$K(m) := \Big(\frac{1+e^{-m/2}}{1-e^{-m/2}}\Big)^d \tag{5.24}$$

where we recall d is the dimension of the lattice $\mathbb{L}$. Let also $\gamma, L \geq 0$ be positive reals; then we have

$$\sum_{\substack{x\in\mathbb{L}:\\ \mathrm{d}_1(x,0)\geq\gamma}} e^{-m\,\mathrm{d}_1(x,B_L)} \leq K(m)\, e^{-\frac{m}{2}(\gamma-2L)} \tag{5.25}$$

where we recall B_L is the ball of radius L centered at the origin defined at the end of Section 2.1.

Proof. First of all we note that $\mathrm{d}_1(x,0) \leq L + \mathrm{d}_1(x,B_L)$. Hence

$$\sum_{\substack{x\in\mathbb{L}:\\ \mathrm{d}_1(x,0)\geq\gamma}} e^{-m\,\mathrm{d}_1(x,B_L)} \leq \sum_{\substack{x\in\mathbb{L}:\\ \mathrm{d}_1(x,0)\geq\gamma}} e^{-m\,[\mathrm{d}_1(x,0)-L]} \leq e^{mL-m\gamma/2} \sum_{\substack{x\in\mathbb{L}:\\ \mathrm{d}_1(x,0)\geq\gamma}} e^{-m\mathrm{d}_1(x,0)/2}$$

Recalling that $\mathrm{d}_1(x,0) = |x_1| + \cdots + |x_d|$, where $x = (x_1,\dots,x_d)$, and using the bound above we get

$$\sum_{\substack{x\in\mathbb{L}:\\ \mathrm{d}_1(x,0)\geq\gamma}} e^{-m\,\mathrm{d}_1(x,B_L)} \leq e^{-m(\gamma-2L)/2} \sum_{x\in\mathbb{L}} e^{-m(|x_1|+\cdots+|x_d|)/2} \leq e^{-m(\gamma-2L)/2}\Big(1+2\sum_{k=1}^\infty e^{-m\,k/2}\Big)^d$$

and the Lemma follows via elementary computations. $\square$

Lemma 5.8. *For $j \geq 1$ and $m > 0$ let*

$$q_j(m) := K(m/4)\, e^{-m\,\gamma_j/8} \tag{5.26}$$

where $K(m)$ has been defined in Lemma 5.7. Assume $q_j(m) < 1$ and set

$$K_j(m) := e^{-\frac{m}{4}\vartheta_j} \frac{e^{-m/4}}{1-e^{-m/4}} + \Big[1 + e^{-\frac{m}{4}\vartheta_j} \frac{e^{-m/4}}{1-e^{-m/4}}\Big] (\Gamma_j + 1)^d \frac{q_j(m)}{1-q_j(m)} \tag{5.27}$$

Then

$$\sup_{g\in\mathcal{G}_j} \sum_{\substack{G\subset\subset\mathcal{G}_{\geq j}:\\ G\ni g}} \sum_{s=0}^{\infty} 1\!\!1_{(|G|,s)\neq(1,0)} \exp\big\{-m\widehat{\mathcal{T}}_{j-1}(G,s)\big\} \leq K_j(m) \tag{5.28}$$

Proof. Let $g_0 \in \mathcal{G}_j$; by using (5.5), definition (3.6), and Lemma 5.1 we have

$$\begin{aligned} &\sum_{\substack{G\subset\subset\mathcal{G}_{\geq j}\\ G\ni g_0}} \sum_{s=0}^{\infty} 1\!\!1_{(|G|,s)\neq(1,0)} \exp\big\{-m\widehat{\mathcal{T}}_{j-1}(G,s)\big\} \\ &\leq \sum_{s=1}^{\infty} \exp\Big\{-\frac{m}{4}\mathrm{d}_1\left(\mathcal{Q}(g_0), y_s(g_0)\right)\Big\} \\ &\quad + \sum_{k=1}^{\infty} \sum_{\substack{G\subset\subset\mathcal{G}_{\geq j}:G\ni g_0\\ |G|=k+1}} \exp\Big\{-\frac{m}{4} \inf_{\substack{x_h\in g_h:\\ h=0,1,\ldots,k}} \inf_{\pi\in\Pi_0(k)} \sum_{l=1}^{k} \mathrm{d}_1\left(x_{\pi(l-1)}, x_{\pi(l)}\right)\Big\} \\ &\quad \times \sum_{s=0}^{\infty} \exp\Big\{-\frac{m}{4}\mathrm{d}_1\left(\mathcal{Q}(\widehat{G}), y_s(G)\right)\Big\} \end{aligned} \tag{5.29}$$

For $G \subset\subset \mathcal{G}_{\geq j}$, such that $G \cap \mathcal{G}_j \neq \emptyset$, we have $\mathrm{d}_1(\mathcal{Q}(\widehat{G}), y_s(G)) = \vartheta_j + s$; then

$$\sum_{s=1}^{\infty} e^{-\frac{m}{4}\mathrm{d}_1(\mathcal{Q}(g_0),y_s(g_0))} = e^{-\frac{m}{4}\vartheta_j} \frac{e^{-m/4}}{1-e^{-m/4}} \tag{5.30}$$

and

$$\sum_{s=0}^{\infty} e^{-\frac{m}{4}\mathrm{d}_1\left(\mathcal{Q}(\widehat{G}),y_s(G)\right)} = 1 + e^{-\frac{m}{4}\vartheta_j} \frac{e^{-m/4}}{1-e^{-m/4}} \tag{5.31}$$

On the other hand

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{\substack{G\subset\mathcal{G}_{\geq j}:\,G\ni g_0\\ |G|=k+1}}\exp\Big\{-\frac{m}{4}\inf_{\substack{x_h\in g_h\\ h=0,1,\dots,k}}\inf_{\pi\in\Pi_0(k)}\sum_{h=1}^{k}\mathrm{d}_1(x_{\pi(h-1)},x_{\pi(h)})\Big\}\\
&\quad\leq\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\substack{g_1,\dots,g_k\in\mathcal{G}_{\geq j}:\\ g_h\neq g_{h'},\,g_h\neq g_0}}\exp\Big\{-\frac{m}{4}\inf_{\substack{x_h\in g_h:\\ h=0,1,\dots,k}}\inf_{\pi\in\Pi_0(k)}\sum_{h=1}^{k}\mathrm{d}_1(x_{\pi(h-1)},x_{\pi(h)})\Big\}\\
&\quad\leq\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\substack{g_1,\dots,g_k\in\mathcal{G}_{\geq j}:\\ g_h\neq g_{h'},\,g_h\neq g_0}}\sum_{\substack{x_h\in g_h:\\ h=0,1,\dots,k}}\sum_{\pi\in\Pi_0(k)}\exp\Big\{-\frac{m}{4}\sum_{h=1}^{k}\mathrm{d}_1(x_{\pi(h-1)},x_{\pi(h)})\Big\}\\
&\quad\leq\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{x_0\in g_0}\sum_{\pi\in\Pi_0(k)}\prod_{h=1}^{k}\Big(\sum_{\substack{g_{\pi(h)}\\ g_{\pi(h)}\neq g_{\pi(h-1)}}}\sum_{x_{\pi(h)}\in g_{\pi(h)}}\exp\Big\{-\frac{m}{4}\mathrm{d}_1(x_{\pi(h-1)},x_{\pi(h)})\Big\}\Big)
\end{aligned}
\tag{5.32}
$$

We now have

$$
\begin{aligned}
\sup_{g\in\mathcal{G}_{\geq j}}\sup_{x\in g}\sum_{\substack{g'\in\mathcal{G}_{\geq j}\\ g'\neq g}}\sum_{y\in g'}\exp\Big\{-\frac{m}{4}\mathrm{d}_1(x,y)\Big\}\;&\leq\sup_{x\in\mathbb{L}}\sum_{\substack{y\in\mathbb{L}\\ \mathrm{d}_1(x,y)>\gamma_j}}\exp\Big\{-\frac{m}{4}\mathrm{d}_1(x,y)\Big\}\\
&\leq K(m/4)\exp\Big\{-\frac{m}{8}\gamma_j\Big\}=q_j(m)
\end{aligned}
\tag{5.33}
$$

where we used Lemma 5.7 and (5.26).

By plugging (5.33) into the r.h.s. of (5.32) we then get

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{\substack{G\subset\mathcal{G}_{\geq j}:\,G\ni g_0\\ |G|=k+1}}\exp\Big\{-\frac{m}{4}\inf_{\pi\in\Pi_0(k)}\inf_{\substack{x_h\in g_h:\\ h=0,1,\dots,k}}\sum_{h=1}^{k}\mathrm{d}_1(x_{\pi(h-1)},x_{\pi(h)})\Big\}\\
&\quad\leq\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{x_0\in g_0}\sum_{\pi\in\Pi_0(k)}[q_j(m)]^k\\
&\quad=|g_0|\,\frac{q_j(m)}{1-q_j(m)}\leq(\Gamma_j+1)^d\,\frac{q_j(m)}{1-q_j(m)}
\end{aligned}
\tag{5.34}
$$

The estimate (5.28) now follows collecting the bounds (5.29)–(5.31) and (5.34). □

In the sequel we shall need some elementary inequalities relating the sequences Γ, γ to the parameters α and A introduced in Condition 2.1. We show how those inequalities are implied by the hypotheses of Theorem 3.2.

Lemma 5.9. *Suppose the hypotheses of Theorem 3.2 are satisfied. We define the decreasing sequence of positive numbers*

$$\delta_k := \frac{8d}{a^{1/3}}\gamma_k^{-1/3} \tag{5.35}$$

for $k \geq 1$. Moreover we set

$$m_0 = \frac{\alpha}{4}, \qquad \textit{and} \qquad m_j := \frac{\alpha}{4} - 4\sum_{k=1}^{j}\delta_k \quad \textit{for all } j \geq 1 \tag{5.36}$$

Then

1. *for each $j \geq 1$ we have $\delta_j\gamma_j \geq 8j$;*
2. *we have $\sum_{k=1}^{\infty}\delta_k \leq \frac{\alpha}{32}$;*
3. *we have $e\varepsilon < 1/3$;*
4. *let $q_j(m)$ as defined in Lemma 5.8 and δ_j as in (2), then $q_j(\delta_j) < 1$ for all $j \geq 1$;*
5. *let $K_j(m)$ as defined in Lemma 5.8, then $K_j(\delta_j) < 1/3$ for all $j \geq 1$;*
6. *for each $j \geq 1$ we have $m_{j-1}\gamma_j \geq m_j\gamma_j \geq 32j$;*
7. *we have $K(\delta_j/2)\exp\left\{-\frac{\delta_j}{4}\gamma_j\right\} \leq 1$ for all $j \geq 1$;*
8. *we have $K((m_{j-1} - 2\delta_j)/2)\exp\left\{-\frac{m_{j-1}-2\delta_j}{4}\gamma_j\right\} \leq 1$ for all $j \geq 1$;*
9. *we have $[4^d(\Gamma_i + \gamma_j)^d + 1]\exp\left\{-\frac{\delta_j}{4}\gamma_i\right\} \leq 1$ for any $1 \leq j < i$.*

Proof. Item 1 is an immediate consequence of definition (5.35) and item 4 in the hypotheses of Theorem 2.5. By definition (5.35) item 2 is equivalent to item 3 in the hypotheses of Theorem 2.5.

Item 3 is an immediate consequence of the definition of ε in item 3 of Theorem 3.2, item 1 in the hypotheses of Theorem 2.5 and the property $\gamma_1 > 2\Gamma_1$ (see item 1 in Definition 2.2).

With simple elementary computations, one can prove that definition (5.35) implies that the inequality

$$\frac{22}{3}\left(\frac{18\gamma_j}{\delta_j}\right)^d e^{-\delta_j\gamma_j/8} \leq \frac{1}{6} \tag{5.37}$$

holds for all $j \geq 1$; such inequality will be useful in the proof of the remaining items. Indeed, by using (5.35) we get that (5.37) is equivalent to $(\gamma_j^2/a)^{2/3} \exp\{-(\gamma_j^2/a)^{1/3}\} \leq 1$, which holds trivially.

Item 4 is obvious once one has proven

$$q_j(\delta_j) \leq \frac{1}{7(\Gamma_j+1)^d+1} \tag{5.38}$$

for all $j \geq 1$. To prove (5.38) we first use (5.24), (5.26) and recall $\Gamma_j \geq 2$ for all $j \geq 1$, see Definition 2.2; we then have

$$[7(\Gamma_j+1)^d+1]q_j(\delta_j) \leq \frac{22}{3}\left(\frac{3}{2}\right)^d \Gamma_j^d q_j(\delta_j) \leq \frac{22}{3}\left(\frac{3}{2}\right)^d \Gamma_j^d \left(\frac{1+e^{-\delta_j/8}}{1-e^{-\delta_j/8}}\right)^d e^{-\delta_j\gamma_j/8} \tag{5.39}$$

We note, now, that item 1 in the hypotheses of Theorem 2.5 and definition (5.35) implies $\delta_j \leq 1$ for all $j \geq 1$. Hence, the term $(1+e^{-\delta_j/8})/(1-e^{-\delta_j/8})$ can be bounded from above by $24/\delta_j$. The inequality (5.38) finally follows from (5.37) once we recall $\gamma_j \geq 2\Gamma_j$ for all $j \geq 1$.

Item 5: first note that for $j \geq 1$

$$e^{-\delta_j\vartheta_j/4}\frac{e^{-\delta_j/4}}{1-e^{-\delta_j/4}} \leq e^{-\delta_j\gamma_j/8}\frac{1+e^{-\delta_j/4}}{1-e^{-\delta_j/4}} \leq e^{-\delta_j\gamma_j/8}\frac{12}{\delta_j} \leq \frac{1}{6} \tag{5.40}$$

where we used $\vartheta_j \geq \gamma_j$ for all $j \geq 1$, inequality (5.37), and $\delta_j \leq 1$ for all $j \geq 1$. By inserting the bounds (5.38) and (5.40) inside the expression of $K_j(\delta_j)$, see definition (5.27), we get the desired inequality.

Item 6: from (5.36) and item 2 above we have that

$$m_j = \frac{\alpha}{4} - 4\sum_{k=1}^{j}\delta_k \geq \frac{\alpha}{4} - 4\frac{\alpha}{32} = \frac{\alpha}{8} \geq 4\delta_j \tag{5.41}$$

Hence, $m_{j-1}\gamma_j \geq m_j\gamma_j \geq 4\delta_j\gamma_j \geq 32j > 4(j-1)$, where we have used item 1 above.

Item 7 is a straightforward consequence of the definition (5.24) of K and the inequality (5.37).

Item 8: by using (5.41) we have that $m_{j-1} - 2\delta_j \geq \delta_j$. So the thesis follows from item 7 once we note that $K(m)$ is a decreasing function of $m \geq 0$.

Item 9 follows easily from (5.37), using that $\Gamma_i \geq 7\gamma_j$, see item 2 in Definition 2.2, and $\delta_i \leq \delta_j$ for $i > j \geq 1$. $\square$

5.3. Recursive estimate

In this section we obtain a recursive estimate on the effective interaction due to the integration on scale j, which is the key step in the proof of Theorem 3.2. More precisely, recalling ε and m_j have been defined in item 3 of Theorem 3.2 and in (5.36), we shall prove the following bounds.

Theorem 5.10. *Let the hypotheses of Theorem 3.2 be satisfied. For $i \geq 1$ set $A_i := (1 \vee A)(8^d + 1)\Gamma_i^d$. Let also $\Psi_{g,\Lambda}^{(i,j)}$ (resp. $\Phi_{G,s,\Lambda}^{(i,j)}$) as defined in (4.5) and (4.18) (resp. in (4.6) and (4.19)). Then for each $i > j \geq 0$, we have*

$$\begin{aligned} \|\Psi_{g,\Lambda}^{(i,j)}\|_\infty &\leq A_i \qquad \forall g \in \mathcal{G}_i &(5.42)\\ \|\Phi_{G,s,\Lambda}^{(i,j)}\|_\infty &\leq \varepsilon^{|G|} e^{-m_j \widehat{\mathcal{T}}_j(G,s)} \qquad \forall G \subset\subset \mathcal{G}_{\geq i} : G \cap \mathcal{G}_i \neq \emptyset, \forall s \geq 0 &(5.43) \end{aligned}$$

for any $\Lambda \subset\subset \mathbb{L}$.

The Theorem follows by complete induction from Lemma 5.11 and Proposition 5.12 below. First of all we show that (5.42) and (5.43) hold for $j = 0$.

Lemma 5.11. *Let $\Psi_{g,\Lambda}^{(i,0)}$, resp. $\Phi_{G,s,\Lambda}^{(i,0)}$, as defined in (4.5), resp. in (4.6) and assume the hypotheses of Theorem 3.2 are satisfied. Then for any $\Lambda \subset\subset \mathbb{L}$ and any $i \geq 1$*

$$\begin{aligned} \|\Psi_{g,\Lambda}^{(i,0)}\|_\infty &\leq A_i \qquad \forall g \in \mathcal{G}_i &(5.44)\\ \|\Phi_{G,s,\Lambda}^{(i,0)}\|_\infty &\leq \varepsilon^{|G|} e^{-m_0 \widehat{\mathcal{T}}_0(G,s)} \qquad \forall G \subset\subset \mathcal{G}_{\geq i} : G \cap \mathcal{G}_i \neq \emptyset, \forall s \geq 0 &(5.45) \end{aligned}$$

Proof. We first prove (5.44). Recall (4.5), given $X \in \Upsilon_\Lambda(g, 0)$, if $\xi(X) = g$ then $X \cap g \neq 0$. Hence by using Condition 2.1,

$$\|\Psi_{g,\Lambda}^{(i,0)}\|_\infty \leq \sum_{\substack{X \cap \Lambda \neq 0: \\ \xi(X) = g}} \|V_{X,\Lambda}\|_\infty \leq \sum_{x \in g} \sum_{\substack{X \cap \Lambda \neq 0: \\ X \ni x}} \|V_{X,\Lambda}\|_\infty \leq |g| A \tag{5.46}$$

The bound (5.44) follows from $|g| \leq (\Gamma_i + 1)^d$.

To prove (5.45) we first note that for $G \subset\subset \mathcal{G}_{\geq i}$, such that $G \cap \mathcal{G}_i \neq \emptyset$ and $(|G|, s) \neq (1, 0)$, and $X \in \Upsilon_\Lambda(G, s)$ we have, recalling (2.9) and item 5 in definition 2.4, that $\mathbb{T}(X) \geq \gamma_i$. Therefore by using (5.3) we have

$$\begin{aligned} \inf_{X \in \Upsilon_\Lambda(G,s)} \mathbb{T}(X) &\geq \frac{1}{4}\gamma_i + \frac{1}{4}\widehat{\mathcal{T}}_0(G,s) + \frac{1}{2} \inf_{X \in \Upsilon_\Lambda(G,s)} \mathbb{T}(X) \\ &\geq \frac{1}{4}\gamma_i + \frac{1}{4}\widehat{\mathcal{T}}_0(G,s) + \frac{1}{4}\gamma_i\big[(|G| - 1) \vee 1\big] \end{aligned} \tag{5.47}$$

where in the last step we used Lemma 5.1 in the case $|G| \geq 2$. Now, for G and s as above, remarking that $|G| \geq 2$ implies $|G| - 1 \geq |G|/2$, we have, recalling $\gamma_i \geq \gamma_1$ and

$\varepsilon = \exp\{-\alpha\gamma_1/8\}$ as in item 3 of Theorem 3.2,

$$
\begin{aligned}
\|\Phi^{(i,0)}_{G,s,\Lambda}\|_\infty &\le \sum_{X\in\Upsilon_\Lambda(G,s)} \|V_{X,\Lambda}\|_\infty = \sum_{X\in\Upsilon_\Lambda(G,s)} e^{\alpha\mathbb{T}(X)} e^{-\alpha\mathbb{T}(X)} \|V_{X,\Lambda}\|_\infty \\
&\le e^{-\frac14\alpha\gamma_i - \frac14\alpha\widehat{\mathcal{T}}_0(G,s) - \frac18\alpha\gamma_1|G|} \sum_{X\in\Upsilon_\Lambda(G,s)} e^{\alpha\mathbb{T}(X)} \|V_{X,\Lambda}\|_\infty \\
&\le \varepsilon^{|G|} e^{-\frac14\alpha\widehat{\mathcal{T}}_0(G,s)} e^{-\frac14\alpha\gamma_i} \sup_{g\in\mathcal{G}_i} \sum_{\substack{X\subset\subset\mathbb{L}\\ \xi(X)\ni g}} e^{\alpha\mathbb{T}(X)} \|V_{X,\Lambda}\|_\infty \\
&\le \varepsilon^{|G|} e^{-\frac14\alpha\widehat{\mathcal{T}}_0(G,s)} e^{-\frac14\alpha\gamma_i} \sup_{g\in\mathcal{G}_i} \sum_{x\in g} \sum_{\substack{X\subset\subset\mathbb{L}\\ X\ni x}} e^{\alpha\mathbb{T}(X)} \|V_{X,\Lambda}\|_\infty \\
&\le \varepsilon^{|G|} e^{-\frac14\alpha\widehat{\mathcal{T}}_0(G,s)} e^{-\frac14\alpha\gamma_i} (\Gamma_i+1)^d A
\end{aligned}
\tag{5.48}
$$

where we used the same bound as in (5.46). The bound (5.45) finally follows from item 2 in the hypotheses of Theorem 2.5. □

Proposition 5.12. *Let the hypotheses of Theorem 3.2 be satisfied. Let also $\Phi^{(j,h)}_{G,s,\Lambda}$ satisfy the bound (5.43) for any $G \subset\subset \mathcal{G}_{\ge j}$ with $G\cap\mathcal{G}_j \neq \emptyset$, any $s \ge 0$, and any $h = 0,\dots,j-1$. Then, for each $\Lambda \subset\subset \mathbb{L}$, the cluster expansion in (4.16) is absolutely convergent. Moreover, $\Psi^{(i,j)}_{g,\Lambda}$ and $\Phi^{(i,j)}_{G,s,\Lambda}$, as defined in (4.18) and (4.19), satisfy the bounds (5.42) and (5.43) for any $i > j \ge 1$.*

The proof of the inductive step in Proposition 5.12 is split in a series of Lemmata in which we understate the hypotheses of Proposition 5.12 itself to be satisfied.

Lemma 5.13. *For $R \in \mathcal{R}_j$, let $\zeta_{R,\Lambda}$ be as defined in (4.14). Then we have*

$$
\|\zeta_{R,\Lambda}\|_\infty \le \prod_{(G,s)\in R} \varepsilon^{|G|}\, e^{-(m_{j-1}-\delta_j)\widehat{\mathcal{T}}_{j-1}(G,s)} \tag{5.49}
$$

for any $\Lambda \subset\subset \mathbb{L}$.

Proof. Recalling (4.13), the inductive hypotheses (5.43) implies that for each $G \subset\subset \mathcal{G}_{\ge j}$, $G \cap \mathcal{G}_j \neq \emptyset$ and $s \ge 0$ with $(|G|, s) \neq (1,0)$

$$
\|\Phi^{(j)}_{G,s,\Lambda}\|_\infty \le \sum_{h=0}^{j-1} \varepsilon^{|G|} e^{-m_h\widehat{\mathcal{T}}_h(G,s)} \le j\, \varepsilon^{|G|} e^{-m_{j-1}\widehat{\mathcal{T}}_{j-1}(G,s)} \tag{5.50}
$$

where we used that m_h, $\widehat{\mathcal{T}}_h$ are decreasing in h. Note that for $g \in \mathcal{G}_{\ge j}$ and $s \ge 1$ we have, by recalling the inequality (5.5) and definition (2.9), that

$$
\widehat{\mathcal{T}}_h(g,s) \ge \frac14 \mathrm{d}_1(\mathcal{Q}(g), y_s(g)) > \frac14\gamma_j \qquad h = 0,\dots,j-1
$$

On the other hand, for $G \subset\subset \mathcal{G}_{\geq j}$, $|G| \geq 2$, there are $g, g' \in G$ with $\mathrm{d}_1(g, g') > \gamma_j$. Hence, recalling (5.5)

$$\widehat{\mathcal{T}}_h(G, s) \geq \frac{1}{2}\mathcal{T}(G) \geq \frac{1}{2}\gamma_j \geq \frac{1}{4}\gamma_j \qquad h = 0, \ldots, j-1$$

We thus conclude that for each $G \subset\subset \mathcal{G}_{\geq j}$ such that $(|G|, s) \neq (1, 0)$, $j \geq 1$, we have

$$\widehat{\mathcal{T}}_h(G, s) \geq \frac{1}{4}\gamma_j \qquad h = 0, \ldots, j-1 \tag{5.51}$$

Since $\Phi^{(j)}_{G,s,\Lambda} = 0$ if $(|G|, s) = (1, 0)$, $m_{j-1}\gamma_j \geq 32j$ (see item 6 in Lemma 5.9), and $\varepsilon \in (0, 1)$, from (5.50) we get the bound $\|\Phi^{(j)}_{G,s,\Lambda}\|_\infty \leq 1$. Recalling definition (4.14) of the activity of a j–polymer R and using the bound $|e^x - 1| \leq e^{|x|}|x|$ and (5.50), we get

$$\begin{aligned} \|\zeta_{R,\Lambda}\|_\infty &\leq \prod_{(G,s)\in R} ej\,\varepsilon^{|G|} e^{-m_{j-1}\widehat{\mathcal{T}}_{j-1}(G,s)} \\ &\leq \prod_{(G,s)\in R} \varepsilon^{|G|}\, e^{-(m_{j-1}-\delta_j)\widehat{\mathcal{T}}_{j-1}(G,s)} \sup_{j\geq 0}\Big[ej \exp\big\{-\delta_j \frac{1}{4}\gamma_j\big\}\Big] \end{aligned}$$

where we used again (5.51). The bound (5.49) follows since $\sup_{r\geq 0}\{e\,r\,e^{-r}\} = 1$ and $\delta_j\gamma_j \geq 8j \geq 4j$, see item 1 Lemma 5.9. ∎

Lemma 5.14. *For $R \in \mathcal{R}_j$, let*

$$\widetilde{\zeta}_R := \varepsilon^{|R\upharpoonright_j|} \prod_{(G,s)\in R} \exp\big\{-\delta_j \widehat{\mathcal{T}}_{j-1}(G, s)\big\} \tag{5.52}$$

Then

$$\sup_{g\in\mathcal{G}_j} \sum_{\substack{R\in\mathcal{R}_j \\ R\upharpoonright_j \ni g}} \widetilde{\zeta}_R \exp\big\{|R\upharpoonright_j|\big\} \leq 1 \tag{5.53}$$

Proof. The above Lemma follows from the estimate in [11, Appendix B], indeed the only needed ingredient is provided by Lemma 5.8. Firstly we notice that from definition (5.52) we have

$$\widetilde{\zeta}_R\, e^{|R\upharpoonright_j|} = (e\varepsilon)^{|R\upharpoonright_j|} \prod_{(G,s)\in R} \exp\big\{-\delta_j \widehat{\mathcal{T}}_{j-1}(G, s)\big\}$$

From item 4 in Lemma 5.9 and Lemma 5.8 we get

$$\sup_{g\in\mathcal{G}_j} \sum_{\substack{G\subset\subset\mathcal{G}_{\geq j} \\ G\ni g}} \sum_{s=0}^{\infty} \mathbb{1}_{(|G|,s)\neq(1,0)} \exp\big\{-\delta_j \widehat{\mathcal{T}}_{j-1}(G, s)\big\} \leq K_j(\delta_j) =: \widetilde{K}_j \tag{5.54}$$

On the other hand from items 3 and 5 in Lemma 5.9 we easily get

$$e^{\widetilde{K}_j} \le \frac{1}{e\,\varepsilon(2-e\,\varepsilon)} \qquad \text{for all } \ j\ge 1 \tag{5.55}$$

Now, by using (5.55) and item 3 in Lemma 5.9 we can indeed perform the estimate in [11, Appendix B] to obtain

$$\sup_{g\in\mathcal{G}_j} \sum_{\substack{R\in\mathcal{R}_j\\ R\upharpoonright_j\ni g}} \widetilde{\zeta}_R\, e^{|R\upharpoonright_j|} \le e\varepsilon\widetilde{K}_j\Big[1+\frac{e^{\widetilde{K}_j}-1}{1+(e\varepsilon)^2 e^{\widetilde{K}_j}-2e\varepsilon e^{\widetilde{K}_j}}\Big] \le 1 \tag{5.56}$$

where the last inequality follows from items 3 and 5 of Lemma 5.9 by elementary computations. □

The bound (5.53) allows us to justify the cluster expansion in (4.16). We are now indeed ready to apply the abstract theory developed in [29].

Lemma 5.15. *For $R\in\mathcal{R}_j$, let $\widetilde{\zeta}_R$ as in (5.52) and, for $\underline{R}\in\underline{\mathcal{R}}_j$, set $\widetilde{\zeta}_{\underline{R}} := \prod_{R\in\underline{R}}\widetilde{\zeta}_R$. Then, recalling the incompatibility inc$_j$ has been defined below (3.5), for each $S\in\mathcal{R}_j$ we have*

$$\sum_{\substack{\underline{R}\in\underline{\mathcal{R}}_j\\ \underline{R}\,\mathrm{inc}_j\, S}} |\varphi_T(\underline{R})|\;\widetilde{\zeta}_{\underline{R}} \le |S\upharpoonright_j| \tag{5.57}$$

Remark. Since, by (5.41) $m_{j-1}-\delta_j \ge m_j-\delta_j \ge 3\delta_j \ge \delta_j$, from Lemmata 5.6, 5.13, 5.15, and (5.52) it follows for each $\Lambda\subset\subset\mathbb{L}$ the cluster expansion in (4.16) is absolutely convergent if the hypotheses of Theorem 3.2 hold. This proves the first claim in Proposition 5.12.

Proof of Lemma 5.15. For each $S\in\mathcal{R}_j$ we have the bound

$$\sum_{\substack{R\in\mathcal{R}_j\\ R\,\mathrm{inc}_j\, S}} \widetilde{\zeta}_R\, e^{|R\upharpoonright_j|} \le \sum_{g\in S\upharpoonright_j}\sum_{\substack{R\in\mathcal{R}_j\\ R\upharpoonright_j\ni g}} \widetilde{\zeta}_R\, e^{|R\upharpoonright_j|} \le |S\upharpoonright_j|$$

where we applied Lemma 5.14. The bound (5.57) now follows from the Theorem in [29] by choosing there $a(R)=|R\upharpoonright_j|$. □

We can now estimate the self interaction due the integration on scale j.

Lemma 5.16. *Let $g\in\mathcal{G}_i$ and $\Psi^{(i,j)}_{g,\Lambda}$ as defined in (4.18). Then for each $i\ge j+1$*

$$\|\Psi^{(i,j)}_{g,\Lambda}\|_\infty \le A_i \tag{5.58}$$

for any $\Lambda\subset\subset\mathbb{L}$.

Proof. Recalling (5.52), by using Lemmata 5.6 and 5.13, we get

$$\begin{aligned}\|\Psi^{(i,j)}_{g,\Lambda}\|_\infty &\le \varepsilon \sum_{\underline{R}\in\underline{\mathcal{R}}_j(g,0)} |\varphi_T(\underline{R})|\,\widetilde{\zeta}_{\underline{R}} \prod_{R\in\underline{R}}\ \prod_{(H,u)\in R} e^{-(m_{j-1}-2\delta_j)\widehat{\mathcal{T}}_{j-1}(H,u)} \\ &\le \varepsilon \sum_{h\in\mathcal{G}_j} \sum_{\substack{\underline{R}\in\underline{\mathcal{R}}_j(g,0)\\ \underline{R}\restriction_j\ni h}} |\varphi_T(\underline{R})|\,\widetilde{\zeta}_{\underline{R}} \prod_{R\in\underline{R}}\ \prod_{(H,u)\in R} e^{-(m_{j-1}-2\delta_j)\widehat{\mathcal{T}}_{j-1}(H,u)} \\ &\le \varepsilon \sum_{h\in\mathcal{G}_j} e^{-(m_{j-1}-2\delta_j)\mathrm{d}_1(g,h)/2} \sup_{h\in\mathcal{G}_j} \sum_{\substack{\underline{R}\in\underline{\mathcal{R}}_j\\ \underline{R}\restriction_j\ni h}} |\varphi_T(\underline{R})|\,\widetilde{\zeta}_{\underline{R}}\end{aligned} \tag{5.59}$$

where we used (5.22).

We next observe that for $h\in\mathcal{G}_j$, by the notion of j–incompatible j–polymers, we have that $\underline{R}\restriction_j\ni h$ implies $\underline{R}\,\mathrm{inc}_j\,(h,0)$. Therefore, by Lemma 5.15,

$$\sup_{h\in\mathcal{G}_j} \sum_{\substack{\underline{R}\in\underline{\mathcal{R}}_j\\ \underline{R}\restriction_j\ni h}} |\varphi_T(\underline{R})|\,\widetilde{\zeta}_{\underline{R}} \le 1 \tag{5.60}$$

Finally,

$$\begin{aligned}\sum_{h\in\mathcal{G}_j} e^{-\frac{m_{j-1}-2\delta_j}{2}\mathrm{d}_1(g,h)} &\le \sum_{y\in\mathbb{L}} e^{-\frac{m_{j-1}-2\delta_j}{2}\mathrm{d}_1(y,B_{\Gamma_i})} \\ &\le [2\,(2\Gamma_i+\gamma_j)+1]^d + \sum_{\substack{y\in\mathbb{L}:\\ y\notin B_{2\Gamma_i+\gamma_j}}} e^{-\frac{m_{j-1}-2\delta_j}{2}\mathrm{d}_1(y,B_{\Gamma_i})} \\ &\le 4^d[\Gamma_i+\gamma_j]^d + K\left((m_{j-1}-2\delta_j)/2\right)\, e^{-\frac{m_{j-1}-2\delta_j}{4}\gamma_j}\end{aligned} \tag{5.61}$$

where we used Lemma 5.7.

Noticing that item 2 in Definition 2.2 implies $\gamma_j \le \Gamma_{j+1} \le \Gamma_i$ and recalling item 8 in Lemma 5.9, the bound (5.58) follows. □

The recursive estimate on the effective interaction due the integration on scale j requires now only a little extra effort. Indeed, the proof of Proposition 5.12 is concluded by the following Lemma.

Lemma 5.17. *Let $G\subset\subset\mathcal{G}_{\ge i}$, $G\cap\mathcal{G}_i\neq\emptyset$, $s\ge 0$ and $\Phi^{(i,j)}_{G,s,\Lambda}$ as defined in (4.19). Then for each $i\ge j+1$*

$$\|\Phi^{(i,j)}_{G,s,\Lambda}\|_\infty \le \varepsilon^{|G|}\, e^{-(m_{j-1}-4\delta_j)\widehat{\mathcal{T}}_j(G,s)} \tag{5.62}$$

for any $\Lambda\subset\subset\mathbb{L}$.

Proof. Let $g\in G\cap\mathcal{G}_i$; recall definition (5.3), by applying (5.22), Lemmata 5.6, 5.13, and using the same bounds as in (5.59) we get

$$\begin{aligned}\|\Phi^{(i,j)}_{G,s,\Lambda}\|_\infty &\le \varepsilon^{|G|} e^{-(m_{j-1}-3\delta_j)\widehat{\mathcal{T}}_j(G,s)} \sum_{h\in\mathcal{G}_j} e^{-\delta_j \mathrm{d}_1(g,h)/2} \sup_{h\in\mathcal{G}_j} \sum_{\underline{R}\in\underline{\mathcal{R}}_j:\,\underline{R}\restriction_j\ni h} |\varphi_T(\underline{R})|\,\widetilde{\zeta}_{\underline{R}} \\ &\le \varepsilon^{|G|} e^{-(m_{j-1}-4\delta_j)\widehat{\mathcal{T}}_j(G,s)-\delta_j\gamma_i/4}\left[4^d[\Gamma_i+\gamma_j]^d + K(\delta_j/2)\, e^{-\delta_j\gamma_j/4}\right]\end{aligned} \tag{5.63}$$

where we used (5.51) and (5.60), and argued as in (5.61). Recalling the bounds 7 and 9 in Lemma 5.9 the estimate (5.62) is proven. □

With the proof of this Lemma it is also completed the proof of Proposition 5.12. We finally show how to get Theorem 3.2 from (5.42) and (5.43).

Proof of Theorem 3.2. Item 1: equation (3.8) has been formally obtained in Section 4; the absolute convergence, uniform with respect to Λ, of the series involved in (3.8) follows from Proposition 5.12. Item 2 follows immediately from the remarks below definitions (4.11) and (4.15). Item 3: to prove the bound (3.9) we recall (4.8), (4.11), Theorem 5.10 and $S := \sup_{x\in\mathbb{L}} |\mathcal{S}_x|$ to get

$$\|\log Z^{(j)}_{g,\Lambda}\|_\infty \leq |g|(\log S + \|U\|) + \sum_{h=0}^{j-1} A_h \tag{5.64}$$

which implies the thesis. Finally, to get the bound (3.10) we have to use equation (5.49) in definition (4.15), the obvious fact that $m_{j-1} - \delta_j > m_j = m_{j-1} - 4\delta_j$ (see (5.36)), (5.5), and the fact that $m_j \geq \alpha/8$, which follows from (5.36) and item 2 in Lemma 5.9. □

6. Proof of the main theorems

First of all we show that Theorem 2.5 is a consequence of cluster expansion stated in Theorem 3.2.

Proof of Theorem 2.5. Recalling (2.10) and the notation introduced in Section 4, for $\Lambda, X \subset\subset \mathbb{L}$ we set

$$\Psi_{X,\Lambda,0} := \begin{cases} V_{X,\Lambda} & \text{if } \mathrm{diam}_\infty(X) \leq \varrho,\ \xi(X) = \emptyset,\ \text{and } X \cap \Lambda \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

and

$$\Phi_{X,\Lambda,0} := \begin{cases} V_{X,\Lambda} & \text{if } \mathrm{diam}_\infty(X) > \varrho,\ \xi(X) = \emptyset,\ \text{and } X \cap \Lambda \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

Note that the families $\{\Psi_{X,\Lambda,0},\ \Lambda \subset\subset \mathbb{L}\}$ and $\{\Phi_{X,\Lambda,0},\ \Lambda \subset\subset \mathbb{L}\}$ are $(X,\emptyset)$–compatible. Moreover, for $j \geq 1$

$$\begin{aligned} \Psi_{X,\Lambda,j} &:= \sum_{g\in\mathcal{G}_j:\, Y_0(g)=X} \log Z^{(j)}_{g,\Lambda} \\ \Phi_{X,\Lambda,j} &:= \sum_{\substack{\underline{R}\in\underline{\mathcal{R}}_j: \\ \underline{R}_{\geq j+1}=\emptyset,\, \mathrm{supp}\,\underline{R}=X}} \varphi_T(\underline{R})\zeta_{\underline{R},\Lambda} \end{aligned} \tag{6.3}$$

We finally set $\Psi_{X,\Lambda} := \sum_{j=0}^{\varkappa} \Psi_{X,\Lambda,j}$ and $\Phi_{X,\Lambda} := \sum_{j=0}^{\varkappa} \Phi_{X,\Lambda,j}$, recall $\varkappa$ has been introduced in item 1 of Theorem 3.2.

From equation (3.8) and the previous definitions we have that the identity (2.11) holds. On the other hand, from Condition 2.1, the $(Y_0(g), \emptyset)$–compatibility of $Z_{g,\Lambda}$, and the $(\operatorname{supp} \underline{R}, \widehat{\underline{R}\restriction_{\geq j+1}})$–compatibility of $\zeta_{\underline{R},\Lambda}$ we easily get that item 2 holds true.

Now, from (6.1) and (6.3) it follows that if $\operatorname{diam}_\infty(X) > \varrho$ and $\nexists g \in \mathcal{G}_{\geq 1}$ such that $Y_0(g) = X$ then $\Psi_{X,\Lambda} = 0$. Moreover, recalling item 6 in Definition 2.4, for each $x \in \mathbb{L}$ we get

$$\sum_{X \ni x} \sup_{\Lambda \subset\subset \mathbb{L}} \|\Psi_{X,\Lambda}\|_\infty \leq \sum_{X \ni x} \sup_{\Lambda \subset\subset \mathbb{L}} \Big[\|\Psi_{X,\Lambda,0}\|_\infty + \sum_{j=1}^{k_x} \sum_{\substack{g \in \mathcal{G}_j: \\ Y_0(g) \ni x}} \|\log Z^{(j)}_{g,\Lambda}\|_\infty \Big] \tag{6.4}$$

By exploiting (2.5) in Condition 2.1, the first term on the right–hand side of (6.4) can be easily bounded as follows

$$\sum_{X \ni x} \sup_{\Lambda \subset\subset \mathbb{L}} \|\Psi_{X,\Lambda,0}\|_\infty \leq \sum_{X \ni x} \sup_{\Lambda \subset\subset \mathbb{L}} \|V_{X,\Lambda}\|_\infty \leq A \tag{6.5}$$

To bound the second term on the right–hand side of (6.4) we note that $\big|\{g \in \mathcal{G}_j : Y_0(g) \ni x\}\big| \leq [\Gamma_j + 1 + 2\vartheta_j]^d$. Hence the bound (2.12), which completes the proof of item 3, follows from the above inequality, (3.9), (6.4), and (6.5).

In order to prove item 4 let us first show that for $G \subset\subset \mathcal{G}_j$ and $s \geq 0$, if $(|G|, s) \neq (1, 0)$ we have

$$\widehat{\mathcal{T}}_{j-1}(G, s) \geq \frac{1}{12} \operatorname{diam}_\infty\big(Y_s(G)\big) \tag{6.6}$$

It is interesting to remark that the bound (6.6) might fail if it were $G \subset\subset \mathcal{G}_{\geq j}$ and $G \cap \mathcal{G}_{\geq j+1} \neq \emptyset$. If $|G| = 1$ then $G = \{g\}$ for some $g \in \mathcal{G}_j$; by recalling (2.9), (5.5) we get, since $s \geq 1$ and $\vartheta_j > 3\Gamma_j$,

$$\widehat{\mathcal{T}}_{j-1}(\{g\}, s) = \frac{1}{4} \mathrm{d}_1\left(\mathcal{Q}(g), y_s(g)\right) \geq \frac{1}{4}(\vartheta_j + s) \geq \frac{1}{12}(2\vartheta_j + 2s + \Gamma_j) \geq \frac{1}{12} \operatorname{diam}_\infty\left(Y_s(g)\right)$$

Let, now, $|G| \geq 2$ and $s = 0$. Recall $\vartheta_j < 2\gamma_j$ and $\mathcal{T}(G) \geq \gamma_j > 2\Gamma_j$. By applying (5.5) we get

$$\widehat{\mathcal{T}}_{j-1}(G, s) \geq \frac{1}{2}\mathcal{T}(G) \geq \frac{1}{3}\gamma_j + \frac{1}{6}\mathcal{T}(G) \geq \frac{1}{6}\vartheta_j + \frac{1}{12}\operatorname{diam}_\infty\big(\mathcal{Q}(\widehat{G})\big) \geq \frac{1}{12}\operatorname{diam}_\infty\big(Y_0(\widehat{G})\big)$$

Finally, in the case $|G| \geq 2$ and $s \geq 1$ by (5.5)

$$\widehat{\mathcal{T}}_{j-1}(G, s) \geq \frac{1}{6}\mathcal{T}(G) + \frac{1}{6}(\vartheta_j + s) \geq \frac{1}{12}\operatorname{diam}_\infty\big(\mathcal{Q}(\widehat{G})\big) + \frac{1}{12}(2\vartheta_j + 2s) \geq \frac{1}{12}\operatorname{diam}_\infty\big(Y_s(\widehat{G})\big)$$

From (6.6) we get that, given $X \subset\subset \mathbb{L}$, for any $\underline{R} \in \underline{\mathcal{R}}_j$ such that $\operatorname{supp} \underline{R} = X$ and $\underline{R}\restriction_{\geq j+1} = \emptyset$ we have

$$\sum_{R \in \underline{R}} \sum_{\substack{(G,s) \in R: \\ (|G|,s) \neq (1,0)}} \widehat{\mathcal{T}}_{j-1}(G, s) \geq \frac{1}{12}\operatorname{diam}_\infty(X) \tag{6.7}$$

Furthermore, given $g \in \mathcal{G}_j$ and $x \in \mathbb{L}$, for any $\underline{R} \in \underline{\mathcal{R}}_j$ such that $\operatorname{supp} \underline{R} \ni x$, $\underline{R}\restriction_j \ni g$, and $\underline{R}\restriction_{\geq j+1} = \emptyset$ we have that the left–hand side of (6.7) is bounded from below by $\mathrm{d}_\infty(x, g)/12$. Recalling (5.52), by applying Lemma 5.13, and noticing that $m_{j-1} - 2\delta_j \geq m_j$, we have that for each $x \in \mathbb{L}$

$$\begin{aligned}\sum_{X \ni x} e^{q\alpha \mathrm{diam}_\infty(X)} \sup_{\Lambda \subset\subset \mathbb{L}} \|\Phi_{X,\Lambda}\|_\infty \leq \sum_{X \ni x} e^{q\alpha \mathrm{diam}_\infty(X)} \Bigg[\sup_{\Lambda \subset\subset \mathbb{L}} \|\Phi_{X,\Lambda,0}\|_\infty \\ + \sum_{j \geq 1} \sum_{\substack{\underline{R} \in \underline{\mathcal{R}}_j: \\ \operatorname{supp} \underline{R} = X, \underline{R}\restriction_{\geq j+1} = \emptyset}} \big|\varphi_T(\underline{R})\big| \exp\Big\{ - m_j \sum_{R \in \underline{R}} \sum_{\substack{(G,s) \in R: \\ (|G|, s) \neq (1,0)}} \widehat{\mathcal{T}}_{j-1}(G, s) \Big\} \cdot \widetilde{\zeta}_{\underline{R}} \Bigg]\end{aligned} \tag{6.8}$$

Recalling (6.2), the first term on the right–hand side of (6.8) can be bounded as follows

$$\begin{aligned}\sum_{X \ni x} e^{q\alpha \mathrm{diam}_\infty(X)} \sup_{\Lambda \subset\subset \mathbb{L}} \|\Phi_{X,\Lambda,0}\|_\infty &\leq \sum_{\substack{X \ni x: \\ \mathrm{diam}_\infty(X) > \varrho}} e^{q\alpha \mathbb{T}(X)} \sup_{\Lambda \subset\subset \mathbb{L}} \|V_{X,\Lambda}\|_\infty \\ &\leq e^{-(1-q)\varrho\alpha} \sum_{X \ni x} e^{\alpha \mathbb{T}(X)} \sup_{\Lambda \subset\subset \mathbb{L}} \|V_{X,\Lambda}\|_\infty \leq A e^{-(1-q)\varrho\alpha} \leq e^{-\alpha}\end{aligned}$$

where we used $\mathbb{T}(X) \geq \mathrm{diam}_\infty(X)$, Condition 2.1, and definitions (2.10).

Recall $q = 2^{-5}3^{-2}$, by using (6.7), the remark below it, (5.51), and $m_j \geq \alpha/8$ we get, by simple computations, that the second term on the right–hand side of (6.8) can be bounded by

$$e^{-m_j\gamma_1/36} \sum_{X \ni x} \sum_{j \geq 1} e^{-m_j\gamma_j/18} \sum_{g \in \mathcal{G}_j} e^{-(m_j/36)\mathrm{d}_\infty(x,g)} \sum_{\substack{\underline{R} \in \underline{\mathcal{R}}_j: \, \underline{R}\restriction_j \ni g \\ \operatorname{supp} \underline{R} = X, \underline{R}\restriction_{\geq j+1} = \emptyset}} |\varphi_T(\underline{R})| \widetilde{\zeta}_{\underline{R}} \tag{6.9}$$

which, in turn, by Lemma 5.15 is bounded by

$$\begin{aligned}e^{-q\alpha\gamma_1} \sum_{j \geq 1} e^{-m_j\gamma_j/18} \sum_{g \in \mathcal{G}_j} e^{-(m_j/36)\mathrm{d}_\infty(x,g)} &\leq e^{-q\alpha\gamma_1} \sum_{j \geq 1} e^{-32j/18} \sum_{y \in \mathbb{L}} e^{-\alpha q \mathrm{d}_\infty(y,x)} \\ &\leq e^{-q\alpha\gamma_1} \frac{e^{-16/9}}{1 - e^{-16/9}} K\Big(\frac{q\alpha}{d}\Big) \leq e^{-q\alpha\gamma_1} K\Big(\frac{q\alpha}{d}\Big)\end{aligned} \tag{6.10}$$

where we used item 6 in Lemma 5.9, Lemma 5.7, and the bound $\mathrm{d}_\infty(y, x) \geq \mathrm{d}_1(y, x)/d$. Recalling the function K has been defined in (5.24), we have proven the bound (2.13) which completes the proof of the Theorem. □

Theorem 2.6 follows from Theorem 2.5 by the combinatorial techniques in [3]. We are, indeed, in a situation analogous to [3, Rem. 2.2] and it is not difficult to check that items 1 and 2 in the hypotheses of Theorem 2.6 on the geometry of the supports of the local functions $f_1, \dots, f_n$ imply that Lemma 3.2 in [3], which yields the bound (2.17), holds.

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