Existence of Critical Invariant Tori

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Abstract. We consider analytic Hamiltonian systems with two degrees of freedom, and prove that every Hamiltonian on the strong local stable manifold of the renormalization group fixed point obtained in [26] has a non-differentiable golden invariant torus (conjugacy to a linear flow).

1. Introduction and main results

In families of Hamiltonian systems with two degrees of freedom, the breakup of invariant tori with quadratic irrational rotation numbers is believed to occur at points where the family intersects a certain "critical surface". This is part of the renormalization group (RG) picture that was proposed first in [1,2]. Further developments, and other applications of RG ideas in Hamiltonian systems can be found in [3–27]. Renormalization suggests that the above-mentioned critical surface is the stable manifold $\mathcal{W}^s$ of a RG fixed point, and that this fixed point exhibits nontrivial scaling, causing Hamiltonians on $\mathcal{W}^s$ to have non-smooth invariant tori. Our goal is to confirm part of this prediction, by using the fixed point obtained recently in [26], and proving that every Hamiltonian on its local stable manifold has a golden invariant torus that is not of class C^1.

If H is a Hamiltonian defined in an open neighborhood of $\mathcal{D} = \mathbb{T}^2 \times \{0\}$, where 0 denotes the zero vector in $\mathbb{R}^2$, and if Φ denotes the flow for H, then a golden invariant torus for H is a continuous map Γ, from $\mathcal{D}$ to the domain of H, such that

$$\Phi^t \circ \Gamma = \Gamma \circ \Psi^t\,, \qquad \Psi^t(q,0) = (q + t\omega, 0)\,. \tag{1.1}$$

Here, ω denotes a vector in $\mathbb{R}^2$ whose component ratio ω_2/ω_1 is equal to the golden mean $\vartheta = (1+\sqrt{5})/2$. Notice that, by virtue of (1.1), the torus Γ is in fact differentiable in the direction of the flow. A different semi-conjugacy $\Lambda \circ \Gamma = \Gamma \circ \mathcal{T}_1$ will be used later to investigate the regularity of invariant tori in a direction transverse to the flow.

Consider the matrix $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which has ω as one of its eigenvectors. The renormalization group transformation considered in [26] is of the form

$$\mathcal{R}(H) = \frac{\theta}{\mu} H' \circ U_{H'} - \varepsilon\,, \quad H' = H \circ \mathcal{T}_\mu\,, \qquad \mathcal{T}_\mu(q,p) = \bigl(Tq, \mu T^{-1} p\bigr)\,, \tag{1.2}$$

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where θ, μ, and ε are normalization constants depending on H, and where $U_{H'}$ is a suitable canonical transformation (homotopic to the identity) that will be described later. In what follows, we will ignore constant terms in Hamiltonians and set $\varepsilon = 0$. Among the fixed points of $\mathcal{R}$ are the Hamiltonians

$$K_m(q,p) = \omega \cdot p + \frac{m}{2}(\Omega \cdot p)^2 , \qquad \omega = (\vartheta^{-1}, 1) , \quad \Omega = (1, -\vartheta^{-1}) . \tag{1.3}$$

For these Hamiltonians, $U_{K'_m}$ is simply the identity map. The normalization constants are $\theta = \vartheta$, and $\mu = \vartheta^{-3}$ in the case $m \neq 0$. Similar fixed points exist in higher dimensions. A RG analysis near such Hamiltonians was carried out in [23], and used e.g. to construct smooth invariant tori for near-integrable Hamiltonians.

For an analysis of critical tori, we restrict the space of Hamiltonians considered to functions $H = K_0 + h$, where $h(q,p)$ depends on q and $z = \Omega \cdot p$ only, and is invariant under $q \mapsto -q$. A Banach space $\mathcal{B}_\varrho$ of such functions h will be introduced in Section 2. Notice that for the flow of such a Hamiltonian H, the coordinate $\omega' \cdot q$ represents time, in the sense that its time derivative is 1. Here, and in what follows, ω' denotes the constant multiple of ω with normalization $\omega' \cdot \omega = 1$. Naturally, the canonical transformation $U_{H'}$ will be chosen in such a way that this property is preserved. As a result, the renormalization of time becomes trivial, in the sense that we can fix $\theta = \vartheta$ in the definition (1.2) of $\mathcal{R}$. The scaling constant $\mu = \mu(H)$ is chosen in such a way that the second partial derivative of $\mathcal{R}(H)$ with respect to z is equal to one.

Consider the affine space $\mathcal{H}_\varrho = K_0 + \mathcal{B}_\varrho$. A function in $\mathcal{H}_\varrho$ will be referred to as *real* if it takes real values for real arguments. Our main input from [26] is the following.

Theorem 1.1. [26] *The transformation $\mathcal{R}$ is well defined, analytic, and compact on some open domain in $\mathcal{H}_\varrho$. The image of a real Hamiltonian under $\mathcal{R}$ is real. The transformation $\mathcal{R}$ has a real analytic fixed point H_∞ in its domain, and this fixed point exhibits nontrivial scaling, in the sense that $0 < \mu(H_\infty) < \vartheta^{-3}$.*

Due to compactness, $\mathcal{R}$ contracts in all but finitely many directions. Numerically, the derivative $D\mathcal{R}(H_\infty)$ of $\mathcal{R}$ at the fixed point H_∞ has exactly two non-contracting eigenvalues. One of them, $\delta_1 = 1/(\vartheta\mu(H_\infty))$, is trivial and could be eliminated by incorporating a suitable translation $p \mapsto p + w(H)$ into the definition of $\mathcal{R}$. The other expanding eigenvalue, $\delta_2 \approx 1.627950$, describes e.g. the universal accumulation of bifurcation points along a family of Hamiltonians approaching the critical surface.

Let $\eta < 1$ be a real number larger than the absolute value of the largest contracting eigenvalue of $L = D\mathcal{R}(H_\infty)$. The strong stable manifold $\mathcal{W}^s$ of $\mathcal{R}$ at H_∞ can be defined as the set of all Hamiltonians H in the domain of $\mathcal{R}$, whose iterates $H_n = \mathcal{R}^n(H)$ all belong to the domain of $\mathcal{R}$ and approach H_∞ at a rate $\mathcal{O}(\eta^n)$. This manifold is the graph of an analytic function, and it is tangent to the corresponding affine manifold of the linearized map $H \mapsto H_\infty + L(H - H_\infty)$; see e.g. [31]. The main result of this paper is the following.

Theorem 1.2. *In some open neighborhood of H_∞, every Hamiltonian that lies on $\mathcal{W}^s$ has a golden invariant torus that is not continuously differentiable.*

The proof of this theorem relies on some estimates that have been carried out with the aid of a computer. These estimates will be described in Sections 4 and 5. The other parts of the proof are given in Sections 2 and 3.

2. A golden invariant torus for H_∞

Our construction of an invariant torus for the fixed point Hamiltonian H_∞ follows a procedure proposed in [25]. In order to simplify the description of symmetries and norms, we note that most functions considered in this paper, if defined on or near $\mathcal{D} = \mathbb{T}^2 \times \{0\}$, admit a representation

$$f(q,p) = \sum_{(\nu,k)\in I} \bigl[f_{\nu,k}\cos(\nu\cdot q) + f'_{\nu,k}\sin(\nu\cdot q)\bigr] z^k\,, \qquad z = (\Omega\cdot p)\,, \tag{2.1}$$

possibly after subtracting some trivial part, such as K_0 in the case of Hamiltonians. Here, $I = \mathcal{V}\times\mathbb{N}$ and $\mathcal{V} = \{\nu\in\mathbb{Z}\times\mathbb{N} : \nu_2 > 0 \text{ or } \nu_1 \geq 0\}$. We assume of course that $f'_{\nu,k} = 0$ for $\nu = 0$. We will call f *real* if the coefficients $f_{\nu,k}$ and $f'_{\nu,k}$ are all real. A function f whose coefficients $f'_{\nu,k}$ or $f_{\nu,k}$ are all zero will be called *even* or *odd*, respectively.

Given a vector-valued function $u = (u^1, u^2, u^3, u^4)$ with components u^i of the form (2.1), we can define a map $U = \mathrm{I} + u$ from (a neighborhood of) the domain $\mathcal{D}$ to some neighborhood of $\mathcal{D}$. If the components of $u_q = (u^1, u^2)$ are odd and those of $u_p = (u^3, u^4)$ are even, then u and U will be called *parity preserving*. We will call u and U *time preserving* if $\omega'\cdot u_q = 0$. The composition of parity preserving maps U_n, if defined, is again parity preserving. Similarly with time preserving maps.

As mentioned in the introduction, we only consider Hamiltonians $H = K_0 + h$ with h an even function of the form (2.1). We preserve this property under renormalization by choosing the canonical transformation $U_{H'}$ to be time and parity preserving.

The flow $\widetilde{\Phi}$ for the renormalized Hamiltonian $\widetilde{H} = \mathcal{R}(H)$ is related to the flow Φ for H by the equation

$$\Lambda_H \circ \widetilde{\Phi}^t = \Phi^{\vartheta t}\circ\Lambda_H\,, \qquad \Lambda_H = \mathcal{T}_\mu\circ U_{H'}\,, \tag{2.2}$$

on any domain where the composed maps in this equation are well defined. This follows from the fact that $U_{H'}$ is symplectic, and from an explicit computation, showing that (2.2) holds in the case where $\widetilde{H}$ is defined as $\vartheta\mu^{-1}H\circ\mathcal{T}_\mu$. Modulo questions of domains, the relation (2.2) implies that if Γ is a golden invariant torus for H, then

$$\mathcal{M}_H(\Gamma) = \Lambda_H\circ\Gamma\circ\mathcal{T}_1^{-1} \tag{2.3}$$

is a golden invariant torus for $\mathcal{R}(H)$. In particular, if H is a fixed point of $\mathcal{R}$ that has a unique golden invariant torus, then this torus is a fixed point for $\mathcal{M}_H$. A partial converse of this has been proved in [25]. When applied to the fixed point H_∞ described in Theorem 1.1, the result is the following.

Lemma 2.1. [25] *Let Γ_∞ be a time and parity preserving real map from $\mathbb{T}^2\times\{0\}$ into the domain of Λ_{H_∞} that satisfies the fixed point equation for $\mathcal{M}_{H_\infty}$. Assume that the derivative of Λ_{H_∞} at $\Gamma_\infty(0)$ has exactly one non-contracting direction, and that Γ_∞ is continuously differentiable with respect to the variable $\omega'\cdot q$. Then Γ_∞ is a golden invariant torus for H_∞.*

Our first goal now is to solve the fixed point equation for $\mathcal{M}_{H_\infty}$. We start by defining some specific domains and function spaces. Consider the set of variables

$$\tau = \omega'\cdot q\,, \quad x = \Omega'\cdot q\,, \quad y = \omega\cdot p\,, \quad z = \Omega\cdot p\,, \tag{2.4}$$

where Ω' the constant multiple of Ω that satisfies $\Omega' \cdot \Omega = 1$. In what follows, δ denotes a (small) positive real number that will be specified later. Let $\rho = (\rho_x, \rho_z)$ be a vector in $\mathbb{R}^2$ with positive components. We define $\mathcal{D}_\rho$ to be the complex neighborhood of $\mathcal{D}$, which is obtained by extending all variables into the complex plane, subject to the constraints $|\operatorname{Im}\tau| < \delta$, $|\operatorname{Im} x| < \rho_x$ and $|z| < \rho_z$. Denote by $\mathcal{F}_\rho$ the Banach space of all functions (2.1) that are analytic on the domain $\mathcal{D}_\rho$ and extend continuously to its boundary, and for which the norm

$$\|f\|_\rho = \sum_{(\nu,k)\in I} \bigl(|f_{\nu,k}| + |f'_{\nu,k}|\bigr) e^{\delta|\omega\cdot\nu|} \cosh(\rho_x \Omega \cdot \nu) \rho_z^k \tag{2.5}$$

is finite. The even and odd subspaces of $\mathcal{F}_\rho$ will be denoted by $\mathcal{B}_\rho$ and $\mathcal{C}_\rho$, respectively.

We note that most (if not all) of our analysis could be carried out with a zero value for δ. In this case, the domain condition $|\operatorname{Im}\tau| < \delta$ has to be replaced by $|\operatorname{Im}\tau| = 0$. A general Hamiltonian in $\mathcal{H}_\rho$ is then no longer differentiable in directions along which τ changes. This does not present any difficulty in defining the time evolution of the dual quantity y, since $y = E - h$ on a surface of fixed energy $H = E$. But other parts of our analysis would become more tedious, despite the fact that $\mathcal{R}$ actually generates analyticity in the variable τ (as explained next).

Define I^+ to be the collection of all pairs (ν, k) in I for which $|\omega \cdot \nu|$ does not exceed $\sigma|\Omega \cdot \nu|$ or κk, where σ and κ are fixed positive constants. (We will use here the same values as in [26], which are approximately 0.85001 and 2.125025, respectively.) To every function $f \in \mathcal{F}_\rho$, we now associate a new function $\mathbb{I}^+ f$, referred to as the "resonant" part of f, by restricting the sum (2.1) for f to the index set I^+. The "nonresonant" part of f is defined as $\mathbb{I}^- f = f - \mathbb{I}^+ f$. In addition, we declare K_0 to be resonant, and extend $\mathbb{I}^\pm$ to Hamiltonians by linearity.

One of the important property of the projection $\mathbb{I}^+$ is that it is analyticity improving in the variable τ. In fact, the resonant part of a function $f \in \mathcal{F}_\rho$ is analytic in τ even if we choose δ to be zero. This follows from the fact that the weight $\cosh(\rho_x \Omega \cdot \nu)\rho_z^k$, used in the norm (2.5), increases exponentially in every direction within I^+. The second important fact about the projections $\mathbb{I}^\pm$ concerns Hamiltonians that (possibly after some change of coordinates) belong to $\mathcal{H}_\rho$ and are close to being resonant. As was shown in [26], the nonresonant part of such Hamiltonians can be eliminated by a canonical change of coordinates. Thus, we regard resonant Hamiltonians to be in "normal form" and define the normalization step $H' \mapsto U_{H'}$ in such a way that

$$\mathbb{I}^-\bigl(H' \circ U_{H'}\bigr) = 0\,. \tag{2.6}$$

This equation can be solved by a Nash-Moser type iteration; for details, the reader is referred to [26]. Notice that, as a result of equation (2.6), the range of $\mathcal{R}$ consists of purely resonant Hamiltonians.

If $u = (u^1, u^2, u^3, u^4)$ is a vector-valued function on D_ρ such that $u_x = \Omega' \cdot u_q$ and $u_z = \Omega \cdot u_p$ belong to $\mathcal{F}_\rho$, we define

$$\|u\|_\rho = \max\bigl\{\|u_x\|_\rho\,, b\|u_z\|_\rho\bigr\}\,, \tag{2.7}$$

where b is some fixed positive constant that will be specified later. As mentioned earlier, we only consider canonical transformations (besides $\mathcal{T}_1$) that are time preserving. If $U = \mathrm{I} + u$ is such a transformation, then $u_\tau = \omega' \cdot u_q$ is identically zero. Furthermore, u does not depend on the variable y. As a result, the component $u_y = \omega \cdot u_p$ enters trivially in any composition of such transformation. In order to simplify notation, we will therefore identify u with (u_x, u_z), unless stated otherwise. The affine space of time preserving maps $U = \mathrm{I} + u$, with the metric defined by (2.7), will be denoted by $\mathcal{A}_\rho$.

For the same reason, u and U will be identified with functions of the variables (q, z) only. The corresponding restriction of the domains $\mathcal{D}$ and $\mathcal{D}_\rho$ will be denoted by D and D_ρ, respectively. We note that if U is parity preserving, then the line $q = 0$ is invariant under U. Thus, for the scaling maps $\Lambda_H = \mathcal{T}_\mu \circ U_{H'}$ we have

$$\Lambda_H(0, z) = (0, \ell_H(z)) , \tag{2.8}$$

where ℓ_H is some function of one variable. The following lemma describes some of the properties of this function, in the case $H = H_\infty$. In addition, it summarizes some relevant results from [26].

The domain parameter ϱ used here is approximately $(0.85, 0.15)$. For the precise values, the reader is referred to [26]. Let $\varrho' = (\vartheta\varrho_x, \vartheta^2\varrho_z)$.

Lemma 2.2. *If $\delta > 0$ is chosen sufficiently small, then there exists an analytic function $H \mapsto \mu(H)$ on an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, and an analytic map $H' \mapsto U_{H'}$ from an open neighborhood V' of H'_∞ in $\mathcal{H}_{\varrho'}$ to $\mathcal{A}_\varrho$, such that the following holds. For every Hamiltonian H in V, Λ_H maps the domain D_ϱ into a compact subset of itself, H' belongs to V', and equation (2.6) holds. Furthermore, the transformation $\mathcal{R}$ defined by (1.2) has the properties described in Theorem 1.1, and $\mu(H_\infty) = \mu_\infty = 0.230460196\ldots$. In addition, ℓ_{H_∞} maps the interval $[-\varrho_z/2, \varrho_z/2]$ into its interior, has a globally attracting fixed point z_∞, and the derivative of ℓ_{H_∞} at this fixed point is $\lambda_z = -0.326063\ldots$.*

This lemma was proved in [26], except for the results concerning ℓ_{H_∞}. To be more precise, the analysis in [26] was carried out for $\delta = 0$. But the results remain valid for sufficiently small $\delta > 0$, as will be explained in Section 5.

A more detailed discussion of the scaling map Λ_{H_∞} will be given in Section 4. At this point, we can already conclude that this map has a fixed point $(q, p) = (0, p_\infty)$ with $\Omega \cdot p_\infty = z_\infty$, and that its derivative at this fixed point has λ_z as one of its eigenvalues. We note that the corresponding eigenvector is not $(0, \Omega)$, as equation (2.8) might suggest, but it has a nonzero component in the y-direction that has been suppressed in (2.8). The remaining three eigenvalues of $D\Lambda_{H_\infty}(0, p_\infty)$ are obtained by using that U_{H_∞} is a time and parity preserving symplectic map: they are $\lambda_\tau = \vartheta$, $\lambda_x = \mu_\infty/\lambda_z = -0.706795\ldots$, and $\lambda_y = \mu_\infty/\lambda_\tau = 0.1424322345\ldots$. The corresponding eigenvectors are the direction of the flow at $(0, p_\infty)$, and the vectors $(\Omega, 0)$ and $(0, \omega)$, respectively.

Next, we consider the fixed point equation for the transformation $\mathcal{M}_{H_\infty}$ defined by equation (2.3). We note that if Γ is a time and parity preserving fixed point of this transformation, then $\Gamma(0)$ is a fixed point of Λ_{H_∞} that lies in the plane $q = 0$. As a result, $\Gamma(0) = (0, p_\infty)$, and our bounds on the eigenvalues λ_j show that the derivative of Λ_{H_∞} at $\Gamma(0)$ has exactly one non-contracting direction, as required in Lemma 2.1.

A continuous function on $\mathcal{D}$ can be represented by a Fourier series analogous to (2.1), but restricted to $k = 0$. Denote by $\mathcal{F}_0$ the Banach space of all such functions f, that can be continued analytically in τ to the strip $|\operatorname{Im}\tau| < \delta$, and for which the norm

$$\|f\|_0 = \sum_{\nu\in\mathcal{V}} \bigl(|f_\nu| + |f'_\nu|\bigr) e^{\delta|\omega\cdot\nu|}(1 + |\Omega\cdot\nu|)^r \tag{2.9}$$

is finite. Here, r is a fixed positive real number to be specified later. The domain characterized by $p = 0$, $|\operatorname{Im}\tau| < \delta$ and $|\operatorname{Im} x| = 0$ will be denoted by $\mathcal{D}_0$. We can consider $\mathcal{F}_0$ to be the $\rho = 0$ analogue of the spaces $\mathcal{F}_\rho$ defined earlier. Thus we extend all of our previous definitions to this case, including the norm (2.7) for vector-valued functions and the affine space $\mathcal{A}_0$. Since all of our tori $\Gamma = \mathrm{I} + \gamma$ are obtained as limits of time and parity preserving symplectic maps, they are themselves time and parity preserving. Thus, we will ignore the trivial components γ_y and $\gamma_\tau = 0$. When constructing an invariant torus for a Hamiltonian $H = K_0 + h$ on a surface of constant energy $H = E$ (the default value of E being 0), the component γ_y is simply defined by $\gamma_y = E - h\circ\Gamma$. It should be noted that the right hand side of this equation does not depend on γ_y.

Lemma 2.3. *There exist $b, r > 0$ and $0 < a < 1$ such that the following holds. There is a bounded open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, an open ball B in $\mathcal{A}_0$, and a concentric closed ball $B_0 \subset B$, such that for every $H \in V$, the transformation $\mathcal{M}_H$ is well defined on B, maps B into B_0, and contracts distances by a factor $\le a$. The image of a parity preserving map under $\mathcal{M}_H$ is again parity preserving. Furthermore, $(H,\Gamma) \mapsto \mathcal{M}_H(\Gamma)$ is analytic on $V \times B$.*

The proof of this lemma is based on a bound on the scaling maps Λ_H, for Hamiltonians H near the fixed point H_∞ of $\mathcal{R}$. For details we refer to Section 4. As in Lemma 2.2, the assumption here is that $\delta > 0$ has been chosen sufficiently small.

One of the implications of Lemma 2.3 is that $\mathcal{M}_{H_\infty}$ has a fixed point in B_0. By Lemma 2.1, this fixed point Γ_∞ is an invariant torus for H_∞. We note that the equation $\Gamma_\infty \circ \mathcal{T}_1 = \Lambda_{H_\infty} \circ \Gamma_\infty$ establishes a semi-conjugacy between the action of $\mathcal{T}_1$ on the torus $\mathbb{T}^2$, and the action of Λ_{H_∞} on the range of Γ_∞. Since the contracting eigenvalues of these two actions differ, Γ_∞ cannot be of class C^1 at 0, unless its derivative with respect to x vanishes at this point. By using e.g. that contractions can be linearized [30], it is not hard to see that Γ_∞ cannot have a zero x-derivative at 0 without being trivial. This argument does not generalize to other Hamiltonians, however, and thus we will use a different approach in Section 3. The following facts will be useful.

Let G be the range of Γ_∞. In Section 4 we will show that G is contained in $D_{\varrho'}$, where $\varrho' = (\varrho_x, \varrho_z/2)$.

Lemma 2.4. *There exist open neighborhoods S, S' of G, whose closures are contained in $D_{\varrho'}$, such that if $H \in \mathcal{H}_\varrho$ is sufficiently close (but not necessarily equal) to H_∞, then Λ_H maps S in a one-to-one fashion onto an open subset of $D_{\varrho'}$ containing S'.*

Proof. Consider the map $\Lambda = \Lambda_{H_\infty}$. Our first goal is to show that the restriction of Λ to G is one-to-one. We start by considering possible self-intersections of G.

Define $u' \sim u$ to mean $\Gamma_\infty(u') = \Gamma_\infty(u)$, and suppose that $u' \sim u$. By equation (1.1), we have $\Psi^t(u') \sim \Psi^t(u)$ for all t. Since orbits under the flow Ψ are dense in $\mathbb{T}^2 \times \{0\}$, and Γ_∞ is continuous, it follows that $u' + v \sim u + v$, for every v in the additive group $\mathbb{T}^2 \times \{0\}$. This in turn implies that $v + u_0 \sim v$ for every v, where $u_0 = u' - u$. In particular, $u_0 \sim 0$, and since $\Gamma_\infty \circ \mathcal{T}_1 = \Lambda \circ \Gamma_\infty$, we find that $\mathcal{T}_1^k u_0 \sim 0$ for all positive integers k.

Assume for contradiction that u_0 is not a periodic point of $\mathcal{T}_1$. Then some sequence $\{u_n\}$ of distinct points from the orbit $\{\mathcal{T}_1^k u_0\}$ converges. By the same arguments that were used to derive $v + u_0 \sim v$, we have

$$\Gamma_\infty\bigl(v + (u_{n+1} - u_n)\bigr) = \Gamma_\infty(v)\,, \qquad v \in \mathbb{T}^2 \times \{0\}\,. \tag{2.10}$$

Since $u_{n+1} - u_n \to 0$, this implies that Γ_∞ is constant in at least one direction. But this is incompatible with the fact that Γ_∞ is continuous and homotopic to the inclusion map $\mathrm{I} : \mathbb{T}^2 \times \{0\} \to \mathcal{D}_0$. As a result, $\mathcal{T}_1^k u_0 = u_0$ for some $k > 0$. This shows e.g. that $\mathcal{T}_1^{-1} u' \sim \mathcal{T}_1^{-1} u$.

Assume now that $\Gamma_\infty(v')$ and $\Gamma_\infty(v)$ have the same image under Λ. Since $\Lambda \circ \Gamma_\infty = \Gamma_\infty \circ \mathcal{T}_1$, we have $\Gamma_\infty(\mathcal{T}_1 v') = \Gamma_\infty(\mathcal{T}_1 v)$. But as shown above, this implies $\Gamma_\infty(v') = \Gamma_\infty(v)$. In other words, the restriction of Λ to G is one-to-one.

In order to see that Λ is also one-to-one on an open set containing G, consider three sequences, $\{u_n\}$, $\{u_n'\}$, and $\{u_n''\}$, accumulating at G, such that $\Lambda(u_n'') = \Lambda(u_n') = u_n$. Since G is compact and Λ locally invertible, we may assume that the given sequences converge to points u, u', and u'', respectively, belonging to G. By continuity, we have $\Lambda(u'') = \Lambda(u') = u$, and thus $u'' = u'$, as was proved above. This fact, together with the local invertibility of Λ, implies that $u_n'' = u_n'$ for large n. This allows us to conclude that Λ is one-to-one on some open neighborhood S of G. The image of S under Λ is open, since Λ is locally invertible, and we can choose S in such a way that both S and $\Lambda(S)$ have compact closures in $D_{\varrho'}$.

This proves that for $H = H_\infty$, the map Λ_H has the property stated in Lemma 2.4, with S' any open neighborhood of G whose closure is contained in $\Lambda(S)$. The same holds now for all H in some open neighborhood of H_∞, since by Lemma 2.2, $U_{H'}$ depends continuously on H in a space of analytic maps. **QED**

3. Non-differentiable golden invariant tori

In this section, we prove Theorem 1.2, using the results stated in Section 2. Let $V \subset \mathcal{H}_\varrho$ and $B \subset \mathcal{A}_0$ be as in Lemma 2.3, and let $F_0, F_1, \ldots$ be arbitrary maps in B. Given a sequence of Hamiltonians $H_0, H_1, \ldots$ in V, define

$$\begin{aligned}\Gamma_{n,m} &= \bigl(\mathcal{M}_n \circ \mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_{m-1}\bigr)(F_m) \\ &= \Lambda_n \circ \Lambda_{n+1} \circ \ldots \circ \Lambda_{m-1} \circ F_m \circ \mathcal{T}_1^{-m+n}, \qquad 0 \le n < m.\end{aligned} \tag{3.1}$$

Here, and in what follows, Λ_n and $\mathcal{M}_n$ are abbreviations for Λ_{H_n} and $\mathcal{M}_{H_n}$, respectively.

Theorem 3.1. *There is a constant $C > 0$ such that the following holds. Let $H_0, H_1, \ldots$ be Hamiltonians in V. Then the limits $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$ exist in $\mathcal{A}_0$, are time and parity preserving, do not depend on the choice of the maps F_m, and satisfy the bounds*

$$\|\Gamma_n - \Gamma_\infty\|_0 \le C \sup_{m \ge n} \|H_m - H_\infty\|_\varrho, \qquad n = 0, 1, \ldots. \tag{3.2}$$

If $H_n = \mathcal{R}^n(H_0)$ for all $n > 0$, then

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_1^{-1}, \qquad n = 0, 1, \ldots. \tag{3.3}$$

If in addition, V has been chosen sufficiently small, and H_0 belongs to $\mathcal{W}^s \cap V$, then Γ_n is a golden invariant torus for H_n, for each $n \ge 0$.

Proof. Lemma 2.3 shows that if $n < m < k$, then the difference $\Gamma_{n,k} - \Gamma_{n,m}$ is bounded in norm by da^{m-n}, where d is the diameter of B. Thus, the sequence $m \mapsto \Gamma_{n,m}$ converges in $\mathcal{A}_0$ to a limit Γ_n, and this limit is independent of the choice of the maps F_m. By taking $F_m = \mathrm{I}$ for all m, and using that the canonical transformations U_{H_n} are all time and parity preserving, we find that Γ_n is time and parity preserving as well. If $H_n = \mathcal{R}^n(H_0)$ for all $n > 0$, then (3.3) is obtained by taking $F_m = \Gamma_m$ for all m, and using that the maps $\mathcal{M}_n$ are continuous.

In order to prove (3.2), it suffices to consider the case $n = 0$. Let $R > 0$ be the distance between H_∞ and $\mathcal{H}_\varrho \setminus V$, and let $r = \sup_m \|H_m - H_\infty\|_\varrho$. If $r \ge R$, then (3.2) follows from the fact that both B and V are bounded. Assume now that $r < R$. For every complex number s in the disk $|s| < R/r$, consider the sequence of Hamiltonians $H_m(s) = H_\infty + s(H_m - H_\infty)$, and denote by $\Gamma_m(s)$ the corresponding maps, constructed as described in the last paragraph. By Lemma 2.2 and uniform convergence, $s \mapsto (\Gamma_0(s) - \Gamma_\infty)$ is an analytic function from the domain $|s| < R/r$ to B. Since this function vanishes at $s = 0$, Schwarz's lemma implies that $\Gamma_0(1) - \Gamma_\infty$ is bounded in norm by dr/R. Thus, (3.2) is proved.

Consider now the case where H_0 belongs to $\mathcal{W}^s$. We choose two open neighborhoods $V'' \subset V' \subset V$ of H_∞ in such a way that for every choice of H_0 in $\mathcal{W}^s \cap V''$, the Hamiltonians $H_n = \mathcal{R}^n(H_0)$ all belong to V', and the closure K of $\bigcup_n \mathrm{Range}(\Gamma_n)$ is contained in D_ϱ. This is possible by (3.2), and by the fact that $\mathcal{W}^s$ is the graph of an analytic function. Consider now the equation (2.2), viewed as an identity between two functions on K. If t is restricted to some small open interval J containing 0, then the range of these two functions is contained in D_ϱ, for any Hamiltonian $H \in V'$. Thus, for $t \in J$ and $0 \leq n < m$, we have

$$\begin{aligned} \Phi_n^t \circ \Gamma_{n,m} \circ \Psi^{-t} &= \Lambda_n \circ \Lambda_{n+1} \circ \ldots \circ \Lambda_{m-1} \circ \Phi_m^{t_{m-n}} \circ \Phi_\infty^{-t_{m-n}} \circ \Gamma_\infty \circ \mathcal{T}_1^{-m+n} \\ &= \bigl(\mathcal{M}_n \circ \mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_{m-1}\bigr)\bigl[\Phi_m^{t_{m-n}} \circ \Phi_\infty^{-t_{m-n}} \circ \Gamma_\infty\bigr], \end{aligned} \tag{3.4}$$

where $t_k = \vartheta^{-k} t$, and where Φ_k denotes the flow for H_k. Here, we have also used that $\mathcal{T}_1 \circ \Psi^s = \Psi^{\vartheta s} \circ \mathcal{T}_1$, and that Γ_∞ is an invariant torus for H_∞. Notice that the map $[\ldots]$ in equation (3.4) is time preserving, since the two flows acting on Γ_∞ change the τ-component of Γ_∞ by opposite amounts. Thus, equation (3.4) is an identity between maps in $\mathcal{A}_0$. As $m \to \infty$, both sides converge in $\mathcal{A}_0$ to Γ_n. But convergence in $\mathcal{A}_0$ implies pointwise convergence, and since Φ_n^t and Ψ^t are both continuous and invertible, we conclude that $\Phi_n^t \circ \Gamma_n \circ \Psi^{-t} = \Gamma_n$. This shows that Γ_n is a golden invariant torus for H_n, as claimed. **QED**

We note that the method used here to construct invariant tori can also be applied to near-integrable Hamiltonians, as an alternative to the method introduced in [23]. Such an analysis is being carried out in [29], for a general class of diophantine rotation numbers.

The following result implies Theorem 1.2.

Theorem 3.2. *There exists an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$ such that if H_0 belongs to $\mathcal{W}^s \cap V$, then the torus Γ_0 defined in Theorem 3.1 is not of class C^1.*

Proof. The idea is to use the fact that the transformations $\mathcal{M}_n$ worsen regularity in the x-direction. This is best exploited by considering the inverse transformations.

We choose V sufficiently small, such that (our previous results apply and) the closure of $\bigcup_n \mathrm{Range}(\Gamma_n)$ is contained in the two sets S and S' described in Lemma 2.4. This is possible by (3.2), and by the fact that $\mathcal{W}^s$ is the graph of an analytic function. Thus, we have

$$\Gamma_n = \Lambda_{n-1}^{-1} \circ \ldots \circ \Lambda_1^{-1} \circ \Lambda_0^{-1} \circ \Gamma_0 \circ \mathcal{T}_1^n, \tag{3.5}$$

for all $n \geq 0$, where the inverse scalings are defined in an unambiguous way.

In what follows, we replace $\mathbb{T}^2$ by its universal covering $\mathbb{R}^2$, and lift the functions Λ_n and Γ_n accordingly. In addition, we restrict our analysis to the surface $\tau = 0$, which is left invariant by the time preserving maps Λ_n and Γ_n. In order to simplify notation, we will write Λ_n and Γ_n as functions of x and y. Let $\varrho' = (\varrho_x, \varrho_z/2)$.

Since the map Λ_n is parity preserving, it can be written in the form

$$\Lambda_n(x, z) = \bigl(f_n(x, z)x,\ \ell_n(z) + g_n(x, z)x^2\bigr). \tag{3.6}$$

We will only (need to) consider points in S or S'. Thus, given that S and S' have compact closure in $D_{\varrho'}$, we have $|z| < b$, for some positive real number $b < \varrho_z/2$. By Lemma 2.2, there exists a real analytic function φ on $[-b.b]$, with non-vanishing derivative, such that $\varphi(\ell_\infty(z)) = \lambda_z \varphi(z)$. Consider now the coordinates $\tilde{z} = \varphi(z)$ and $\tilde{x} = x/\varphi'(z)$. Since $U_{H'_\infty}$ is symplectic, the functions ℓ_∞ and f_∞ in these new coordinates, when restricted to $\tilde{x} = 0$, are simply multiplication by λ_z and λ_x, respectively.

The inverse of Λ_n is also parity preserving, and thus admits a representation analogous to (3.6). In the coordinates $\tilde{x}$ and $\tilde{z}$, we have

$$\widetilde{\Lambda}_n^{-1}(\tilde{x},\tilde{z}) = \bigl(\phi_n(\tilde{x},\tilde{z})\tilde{x}, \psi_n(\tilde{x},\tilde{z})\bigr)\,, \tag{3.7}$$

with ϕ_n, ψ_n analytic in S'. Since all derivatives of Λ_n are bounded on S, uniformly in n, we have $\phi_n(\tilde{x},\tilde{z}) \to \phi_\infty(\tilde{x},\tilde{z})$, uniformly on S'. Furthermore, $\phi_\infty(\tilde{x},\tilde{z}) \to \lambda_x^{-1}$, as $\tilde{x} \to 0$, uniformly in $\tilde{z}$. Thus, if V has been chosen sufficiently small, we can find positive real numbers α and $\beta < 1$, such that $|\phi_n(\tilde{x},\tilde{y})| \le \beta\vartheta$ for all $n \ge 0$, whenever $|\tilde{x}| < \alpha$. Here, we have used the crucial fact that the eigenvalue λ_x of $D\Lambda_\infty$ at $\Gamma_\infty(0)$ is larger in modulus than the corresponding eigenvalue $-\vartheta^{-1}$ of $\mathcal{T}_1$.

By equation (3.5), the points $\Gamma_n(s,0)$, expressed in the coordinates $\tilde{x}$ and $\tilde{y}$, are given by

$$\widetilde{\Gamma}_n(s,0) = \bigl(\widetilde{\Lambda}_{n-1}^{-1} \circ \ldots \circ \widetilde{\Lambda}_1^{-1} \circ \widetilde{\Lambda}_0^{-1} \circ \widetilde{\Gamma}_0\bigr)(s_n,0)\,, \qquad s_n = (-\vartheta^{-1})^n s\,. \tag{3.8}$$

Assume now for contradiction that Γ_0 is continuously differentiable near 0. Then there exists a constant $c > 0$ such that for any given $s \in \mathbb{R}$, the angular ($\tilde{x}$) component of $\widetilde{\Gamma}_0(s_n,0)$ is bounded in modulus by $c|s_n|$, for large n. Thus, by using the abovementioned bound on the functions ϕ_m, we find that the angular component of $\widetilde{\Gamma}_n(s,0)$ is bounded in modulus by $c\beta^n|s| < \alpha$, if n is sufficiently large.

By using now that $\Gamma_n \to \Gamma_\infty$ in $\mathcal{A}_0$, and the fact that evaluation is continuous on $\mathcal{A}_0$, we conclude that the angular component of $\Gamma_\infty(s,0)$ is zero for all $s \in \mathbb{R}$. Thus, since the curve $s \mapsto s\Omega$ is dense in $\mathbb{T}^2$, the angular component of Γ_∞ is identically zero. But this is impossible, since Γ_∞ is continuous and homotopic to I. This shows that Γ_0 cannot be of class C^1. **QED**

4. The maps $\mathcal{M}_H$

In this section, we follow the strategy proposed in [25] and reduce the proof of Lemma 2.3 to the proof of a bound on the function $u = U_{H'} - \mathrm{I}$. In what follows, H is some fixed but arbitrary Hamiltonian in the domain of $\mathcal{R}$. Consider the transformation $\mathcal{N}_H$ that maps $\Gamma - \mathrm{I}$ to $\mathcal{M}_H(\Gamma) - \mathrm{I}$. Explicit expressions for this transformation and its derivative are

$$\begin{aligned} \mathcal{N}_H(\gamma) &= \mathcal{T}_\mu \circ \bigl[\gamma + u\circ(\mathrm{I}+\gamma)\bigr] \circ \mathcal{T}_1^{-1}\,,\\ D\mathcal{N}_H(\gamma)v &= \mathcal{T}_\mu\bigl[v + ((Du)\circ(\mathrm{I}+\gamma))v\bigr] \circ \mathcal{T}_1^{-1}\,. \end{aligned} \tag{4.1}$$

We note that u and μ depend on the Hamiltonian H.

The following lemma provides bounds on the individual steps that appear in (4.1). As was done earlier, we identify u with the pair (u_x,u_z) in the space $\mathcal{C}_\varrho \times \mathcal{B}_\varrho$. Similarly, γ is identified with an element of $\mathcal{F}_0^2$. The norm in these two spaces is given by equation (2.7), with ρ replaced by ϱ and 0, respectively. Define

$$\beta(c,s) = \sup_{t\ge 0} \frac{(1+t)^r}{\cosh(ct)}\, e^{st}\,, \qquad s < c\,. \tag{4.2}$$

Proposition 4.1. *Assume $f,g\in\mathcal{F}_0$, $h\in\mathcal{F}_\rho$, and $\gamma\in\mathcal{F}_0^2$. Then*
(*a*) $|f(q,0)|\le\|f\|_0$, *for all* $q\in\mathbb{T}^2$.
(*b*) $\|fg\|_0\le\|f\|_0\|g\|_0$.
(*c*) $\|f\circ\mathcal{T}_1^{-1}\|_0\le\vartheta^r\|f\|_0$.
(*d*) $\|h\circ(\mathrm{I}+\gamma)\|_0\le\beta(\rho_x,\rho'_x)\|h\|_\rho$, *if* $\|\gamma_x\|_0\le\rho'_x<\rho_x$ *and* $\|\gamma_z\|_0\le\rho_z$.

The proof of these inequalities is straightforward; see also [25].

As a domain for $\mathcal{N}_H$, we choose an open ball of radius $R>0$ in $\mathcal{F}_0^2$, centered at an approximate fixed point γ' of $\mathcal{M}_{H_\infty}$. The approximate fixed point γ' was determined numerically. In order to ensure that part (c) of Proposition 4.1 applies to each γ in this ball, with h a component of u or Du, we require that

$$\|\gamma'_x\|_0+R\le\rho'_x<\rho_x<\varrho_x,\qquad \|\gamma'_z\|_0+R/b\le\rho'_z=\rho_z\,, \tag{4.3}$$

and $\rho_z<\varrho_z$, for some appropriate domain parameters ρ and ρ'. In particular, the condition $\rho<\varrho$ is needed to ensure that the components of Du belong to $\mathcal{F}_\rho$. A useful measure for the size of Du is given by the seminorm

$$\|u\|'_\rho=\max\bigl\{\|\partial_x u_x\|_\rho+b^{-1}\|\partial_z u_x\|_\rho\,,\,b\|\partial_x u_z\|_\rho+\|\partial_z u_z\|_\rho\bigr\}\,. \tag{4.4}$$

Here, and in what follows, $b=2$. Our main technical estimate is

Lemma 4.2. *There exists a Fourier polynomial γ', real numbers $r,R>0$, and domain parameters ρ,ρ' satisfying (4.3) and $\rho_z<\varrho_z/2$, such that if $H=H_\infty$ and*

$$\varepsilon=\|\mathcal{N}_H(\gamma')-\gamma'\|_0\,,\qquad K_H=\vartheta^{r-1}+\vartheta^r\beta(\rho_x,\rho'_x)\|\mathcal{T}_\mu u\|'_\rho\,, \tag{4.5}$$

then $\varepsilon<(1-K_H)R$.

The proof of this lemma will be described in the next section. The constants r and R used in this proof are $r\approx10^{-4}$ and $R\approx2\times10^{-3}$. For the precise values we refer to [26].

Proof of Lemma 2.3. Assume that the hypotheses and conclusions of Lemma 4.2 hold. Since μ and u depend analytically on H, as described in Lemma 2.2, there exists a positive real number $a<1-\varepsilon/R$, and an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, such that $\vartheta|\mu|<1$ and $K_H<a$, for all Hamiltonians $H\in V$. Here, we have used that differentiation with respect to x and z is bounded from $\mathcal{F}_\varrho$ to $\mathcal{F}_\rho$. By (4.3) and Proposition 4.1, $\mathcal{N}_H$ is well defined, and analytic, on an open ball $B\subset\mathcal{F}_0^2$ with radius R and center γ'. The same proposition also shows that a is an upper bound on the operator norm of $D\mathcal{N}_H(\gamma)$, for every γ in B. Thus, by Lemma 4.2, the transformation $\mathcal{N}_H$ contracts distances by a factor $\le a$, and it maps B into a ball of radius $\varepsilon+aR<R$, centered at γ'. Since $U_{H'}$ is parity preserving for any $H\in V$, the subspace $\mathcal{C}_0\times\mathcal{B}_0$ is invariant under $\mathcal{N}_H$. The analyticity of $(H,\gamma)\mapsto\mathcal{N}_H(\gamma)$ follows from Lemma 2.2 and the chain rule. **QED**

We note that the bound $\rho_z<\varrho_z/2$ in Lemma 4.2 ensures that the range of Γ_∞ is contained in $D_{(\varrho_x,\varrho_z/2)}$, which was used in the proof of Lemma 2.4.

5. Remaining proofs

What remains to be proved are the estimates on $U_{H'_\infty}$ that are described in Lemma 4.2, and in the second part of Lemma 2.2. This is done with the assistance of a computer, by decomposing the desired bounds in a systematic way into trivial inequalities, and by verifying these inequalities. The decomposition process is based largely on bounds developed in [26]. In fact, our programs are essentially a superset of those used in [26]. Thus, in order to avoid undue repetition, we will limit our discussion to the parts that are new, and assume that the reader is familiar with the techniques described in [26]. As usual with computer-assisted proofs, the ultimate reference for details is the source code of our programs [28].

Our first step extends the results in [26], from $\delta = 0$, to values of δ in a nontrivial interval (defined implicitly) of non-negative real numbers. This extension is rather straightforward: As was explained in [25], general estimates that do not involve any specific function carry over to $\delta > 0$. This is due to the fact that $\mathcal{F}_\rho$ is a Banach algebra, for any $\delta \geq 0$, and that the transformations $U_{H'}$ are time preserving. The only relevant effect of increasing δ is to increase the norm of any explicitly given function (in our case Fourier-Taylor polynomials of degrees $|\nu| \leq 40$ and $k \leq 16$). Thus, it is sufficient to repeat the computer part of [26], with all norm weights $\cosh(\rho_x \Omega \cdot \nu)\rho_z^k$ increased by a fixed factor slightly larger than 1, which we did. As expected, the the results did not change in any significant way; but of course, any explicit bound used in the present paper stems from this generalization, not from [26] directly, and the new programs and data are included in [28].

Next, we consider the task of obtaining an upper bound on the quantity K_H, defined in equation (4.5). Here, and in what follows, $H = H_\infty$. The canonical transformation $U_{H'}$ solving equation (2.6) is composed of elementary canonical transformations of the form

$$U(q,p) = \Bigl(q + \bigl[\partial_\tau^{-1}\partial_z\psi\bigr](q,p+P)\Omega,\ p + P - \psi(q,p+P)\omega'\Bigr)\,, \tag{5.1}$$

that are associated with nonresonant generating functions ψ. Here, $P = g(q,p)\Omega'$, and g is defined implicitly by the equation $g(q,p) = -[\partial_x\psi](q,p+P)$. To be more precise, we have $U_{H'} = U'_0 \circ U'_1 \circ U'_2 \circ \ldots$, where the U'_n are canonical transformations of the form (5.1), obtained by iteratively solving (2.6). The transformation U'_0 is fixed, and its generating function is part of the data in [26,28]. The remaining transformations $U'_1, U_2, \ldots$ are corrections and very close to the identity. In [26], we obtained bounds on each individual U'_n, on the composed map $U'' = U_6 \circ U_7 \circ \ldots$, and on the Hamiltonian $H' \circ U_{H'}$, but not on the canonical transformation $U_{H'}$ by itself.

Our first step was to combine the existing bounds on the transformation U'_n and U'' into a bound on $\mathcal{U}_{H'} = U'_1 \circ U'_2 \circ \ldots$. More specifically, we duplicated some procedures in the package `RG_Ops`, and modified these copies (as indicated in comments) to perform this additional task. Since $\mathcal{U}_{H'}$ is very close to the identity, these compositions are carried out within a simple data type `UBall`, that only keeps bounds on the norms of u_x and u_z, for the canonical transformation $U = \mathrm{I} + u$ being represented. Bounds on the components of U_0 and $\mathcal{U}_{H'}$ are obtained and saved by running the program `UComp`.

For additional operations involving canonical transformations, we use a new data type, `Canonical`, which consists of a pair of type `Fourier`, representing a "standard set" in one of the spaces $\mathcal{F}_\rho^2$. We refer to [26] for the definition of such standard sets. A similar data type, `Torus`, consists of a pair of type `Sobolev`, and is used to represents standard sets in $\mathcal{F}_0^2$. Both are defined in a package named `Torus_Ops`, which also implements bounds on some operations involving these types. These bounds are obtained easily from the corresponding bounds on the components. In particular, a bound on the composition of functions in $\mathcal{F}_\rho$ is already available in `Fouriers.Ops`, as described in [26]. The resulting bound for the composition of two canonical transformations is used at the beginning of the program `CheckTorus`, to estimate $U_{H'} = U_0 \circ \mathcal{U}_{H'}$. Obtaining a bound on K_H is then straightforward. We find that $K_H < 0.9$, and consequently $(1 - K_H)R > 2 \times 10^{-4}$. The norm parameter ρ_x used in this estimate is $\rho_x \approx 0.7$; see [28] for its precise value.

The same estimate on $U_{H'}$ is used also to verify the claims in Lemma 2.2 regarding the function ℓ_H. The first step is to check that our upper bound ρ_z on $\|\gamma_z'\|_0 + R/b$ is smaller than $\varrho_z/2$, as stated in Lemma 4.2. Then, we show that the function ℓ_H has a non-vanishing derivative on $J_0 = [-\rho_z/2, \rho_z/2]$, that its 25-th iterate maps J_0 into a small subinterval J, and that both of these intervals are invariant under ℓ_H. Finally, we verify that $\ell_H'(z) = -0.326063\ldots$ for every $z \in J$. This is done by running the program `Scaling`.

The main new aspect of our computer-assisted proof, compared to [26], is the need to work with non-differentiable functions. Fortunately, the map $\mathcal{N}_H$ that needs to be estimated is much simpler than $\mathcal{R}$. As it turns out, representing a function in $\mathcal{F}_0$ by a Fourier polynomial and upper bounds on "higher order terms", yields estimates that are sufficiently good for our purpose. Our choice of standard sets for $\mathcal{F}_0$ is completely analogous to those for $\mathcal{F}_\rho$. These sets are represented by the data type `Sobolev`. For reasons of efficiency, all `Sobolev`-related definitions and bounds were added to the existing package `Fouriers` and its children. This includes bounds on the product `Sobolev` $*$ `Sobolev` and the composition `Fourier` $\circ$ `Sobolev`, which use Proposition 4.1 and are described in `Fouriers.Ops`.

After estimating K_H as mentioned above, the program `CheckTorus` uses the procedure `Torus_Ops.NN` to obtain a bound on $\mathcal{N}_H(\gamma')$. Here, γ' is an approximate fixed point of $\mathcal{N}_H$, which was obtained by iterating a purely numerical version (defining `Scalar` to be `Numeric` instead of `Interval`) of the procedure `Torus_Ops.NN`, starting with $\gamma = 0$. The Fourier coefficients for γ' can be found in [28]. Our bound on $\mathcal{N}_H(\gamma')$ yields $\varepsilon < 6 \times 10^{-5}$, which is less than $(1 - K_H)R$, as claimed in Lemma 4.2.

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