SCALING ESTIMATES FOR SOLUTIONS AND DYNAMICAL LOWER BOUNDS ON WAVEPACKET SPREADING

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Abstract. We establish quantum dynamical lower bounds for discrete one-dimensional Schrödinger operators in situations where, in addition to power-law upper bounds on solutions corresponding to energies in the spectrum, one also has lower bounds following a scaling law. As a consequence, we obtain improved dynamical results for the Fibonacci Hamiltonian and related models.

1. Introduction

Consider a discrete one-dimensional Schrödinger operator,

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{1}$$

on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, \dots\}$. In the case of $\ell^2(\mathbb{Z}_+)$, we will work with a Dirichlet boundary condition, $\psi(0) = 0$, but our results easily extend to all other self-adjoint boundary conditions.

A number of recent papers (e.g., [DST, DT, JL1, JL2, JSS, KKL, T1, T2]) were devoted to proving lower bounds on the spreading of an initially localized wavepacket, say $\psi = \delta_1$, under the dynamics governed by H, typically in situations where the spectral measure of δ_1 with respect to H is purely singular and sometimes even pure point.

A standard quantity that is considered to measure the spreading of the wavefunction is the following: For $p > 0$, define

$$\langle |X|_{\delta_1}^p \rangle(T) = \sum_n |n|^p a(n,T), \tag{2}$$

where

$$a(n,T) = \frac{2}{T} \int_0^{+\infty} e^{-2t/T} |\langle e^{-itH}\delta_1, \delta_n \rangle|^2 \, dt. \tag{3}$$

Clearly, the faster $\langle |X|_{\delta_1}^p \rangle(T)$ grows, the faster $e^{-itH}\delta_1$ spreads out, at least averaged in time. One typically wants to prove power-law lower bounds on $\langle |X|_{\delta_1}^p \rangle(T)$ and hence it is natural to define the following quantity: For $p > 0$, define the lower growth exponent $\beta_{\delta_1}^-(p)$ by

$$\beta_{\delta_1}^-(p) = \liminf_{T \to +\infty} \frac{\log \langle |X|_{\delta_1}^p \rangle(T)}{\log T}.$$

When one wants to bound these exponents from below for specific models, it is useful to connect these quantities to the qualitative behavior of the solutions of the

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difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{4}$$

for energies E in the spectrum of the operator H. In fact, most of the known results are based on such a correspondence; compare [JL1, JL2] for an approach where the link is furnished by Hausdorff-dimensional properties of spectral measures, and [DST, DT] for a direct approach without intermediate step. The latter papers use power-law upper bounds on solutions corresponding to energies from a set S to derive lower bounds for $\beta^-_{\delta_1}(p)$. The set S can even be very small. One already gets non-trivial bounds when S is not empty. If S is not negligible with respect to the spectral measure of δ_1, the bounds are stronger, but there are situations of interest (e.g., random polymer models [JSS]), where the spectral measure assigns zero weight to S.

While both approaches yield bounds on $\beta^-_{\delta_1}(p)$ for all $p > 0$, in concrete applications there is a transition point, p_0, such that the method from [JL1, JL2] works better for $0 < p < p_0$, whereas the method from [DST, DT] gives better bounds for $p > p_0$.

Our goal here is to develop an approach that, whenever it applies, gives stronger lower bounds than both previous methods throughout the entire range of the powers p.

A model for which the exponents $\beta^-_{\delta_1}(p)$ have been heavily studied (e.g., [D, DKL, DST, DT, JL2, KKL]) is given by the Fibonacci Hamiltonian. This is the standard model of a one-dimensional quasicrystal and it is one of the few for which one can actually prove "anomalous" transport properties rigorously; compare the discussion in [KKL]. With this model in mind, we will refine the results from [DST, DT, KKL] in what follows. It turns out that the bounds on $\beta^-_{\delta_1}(p)$ can be considerably strengthened if, in addition to power-law upper bounds for solutions, one also assumes suitable lower bounds. The necessary input does in fact hold for the Fibonacci model and related ones, as we will show, and we thereby obtain improved dynamical results that are strictly better than all previously known ones. While we will discuss this issue in more detail later, we mention at this point that, for this particular model, the paper [KKL] had the best previous bounds for small values of p, while for large values of p, the best previous bounds were obtained in [DST, DT]. Here we will get stronger bounds for all values of p.

Let us now specify the assumptions we are going to work with. We consider real solutions u of the difference equation (4). If

$$|u(0)|^2 + |u(1)|^2 = 1, \tag{5}$$

we say that u is normalized. For $L \ge 1$, we define

$$\|u\|_L^2 = \sum_{n=1}^{[L]} |u(n)|^2 + (L - [L])|u([L]+1)|^2.$$

We assume that for some non-empty $A \subseteq \mathbb{R}$, the following conditions are satisfied:

(a) There exist constants $C, \alpha > 0$ such that for every $E \in A$, every normalized solution u of (4), and every $L \ge 1$,

$$||u||_L^2 \le CL^{2\alpha+1}.$$

(b) There exist constants $0 < k, \gamma < 1$ and $L_0 \geq 1$ such that for every $E \in A$, every solution u of (4), and every $L \geq L_0$,

$$||u||_L^2 \geq (1+\gamma)||u||_{kL}^2.$$

It is easy to see that (b) implies the following:

(c) There exist constants $D, \kappa > 0$ such that for every $E \in A$, every normalized solution u of (4), and every $L \geq 1$,

$$||u||_L \geq DL^\kappa.$$

We make this condition explicit since the constant κ is crucial in the dynamical lower bounds we will prove below, that is, it is desirable to find the largest possible value of κ such that (c) holds. In our general treatment we will only assume the conditions (a) and (b), but in concrete applications one can try to optimize κ.

Denote by F the Borel transform of the spectral measure μ associated with the operator H and vector δ_1. That is,

$$F(z) = \langle (H-z)^{-1}\delta_1, \delta_1\rangle = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}. \tag{6}$$

For $\Delta \subset \mathbb{R}$ and $\nu > 0$, we let

$$I(\Delta, \nu) = \nu \int_\Delta (\operatorname{Im} F(E+i\nu))^2 \, dE$$

and write Δ_ν for the ν-neighborhood of Δ.

Our first result establishes a lower bound for $\langle |X|_{\delta_1}^p\rangle(T)$ in terms of the integrals $I(A_{2\varepsilon}, \varepsilon)$, where $\varepsilon = 1/T$. We stress that (a)–(c) above only concern the behavior of the solutions on the right half-line, even in the case of a whole-line operator. This is natural from a physical point of view, for if there is "transport" on a half-line, there should also be "transport" for the whole-line model. On a mathematical level, this intuition does not always translate into an easy proof and some earlier papers needed certain symmetry assumptions to prove quantum dynamical lower bounds for whole-line operators (e.g., [D, JL2]). However, the Jitomirskaya-Last theory [JL1, JL2] does allow for a decent whole-line version that works with solution estimates on one half-line only; see [DKL].

Theorem 1. *Let H be a discrete Schrödinger operator, given by* (1)*, acting on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$. Assume that the conditions* (a) *and* (b) *are satisfied for some set $A \subseteq \mathbb{R}$. Then, for $0 < p \leq 2\alpha+1$,*

$$\langle |X|_{\delta_1}^p\rangle(T) \gtrsim T^{2(p+2\kappa)/(2\alpha+1+2\kappa)} I(A_{2\varepsilon}, \varepsilon), \tag{7}$$

whereas for $p > 2\alpha+1$, we have

$$\langle |X|_{\delta_1}^p\rangle(T) \gtrsim T^2 I(A_{2\varepsilon}, \varepsilon) \tag{8}$$

and

$$\langle |X|_{\delta_1}^p\rangle(T) \gtrsim T^{\frac{p+2\kappa}{2\alpha+1}} I(A_{2\varepsilon}, \varepsilon). \tag{9}$$

The bounds (7)–(9) *hold for $T \geq T_0$ with $\varepsilon = 1/T$.*

Remark. We write $f(T) \gtrsim g(T)$ if there is a positive, T-independent constant C such that $f(T) \geq Cg(T)$. In the bounds above, these constants depend on the values of $p, C, \alpha, D, \kappa, k, \gamma$ in (a)–(c). They are given by

$$\frac{\gamma k^p}{12\pi}\left(\frac{\gamma^2}{5184\,C}\right)^{\frac{p+2\kappa}{2\alpha+1+2\kappa}} D^{2-\frac{2(p+2\kappa)}{2\alpha+1+2\kappa}}$$

in (7)–(8) and

$$\frac{\gamma k^p}{12\pi}\left(\frac{\gamma^2}{5184\,C^2}\right)^{\frac{p+2\kappa}{2}} D^2$$

in (9), respectively.

In order to apply this theorem, we need to establish a lower bound for $I(A_{2\varepsilon}, \varepsilon)$. In general, this is a difficult problem. One can show that if $\mu(A) = 1$, then

$$\varepsilon \int_{A_{2\varepsilon}} (\operatorname{Im} F(E+i\varepsilon))^2 \, dE \sim \varepsilon \int_{\mathbb{R}} (\operatorname{Im} F(E+i\varepsilon))^2 \, dE,$$

where the last integral over $\mathbb{R}$ is closely related to the time-averaged return probability and the correlation dimension of the spectral measure [T1, T2].

If $\mu(A_\varepsilon) > 0$,

$$I(A_{2\varepsilon}, \varepsilon) \gtrsim \varepsilon \int_{A_{2\varepsilon}} \operatorname{Im} F(E+i\varepsilon) \, dE \gtrsim \varepsilon \mu(A_\varepsilon)$$

(for a proof of the last inequality, see, e.g., [KKL]). In particular, if $\mu(A) > 0$, then $I(A_{2\varepsilon}, \varepsilon) \gtrsim \varepsilon$. However, such a lower bound for I is not optimal in most cases of interest.

When considering the time-averaged moments, a method combining the Parseval formula and the classical Guarneri approach was proposed in [T2]. It yields better lower bounds without any information on I. As a particular consequence of this method, we can derive the following result:

Theorem 2. *Let H be a discrete Schrödinger operator, given by* (1)*, acting on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$. Suppose A is a set* (*possibly depending on T*) *such that $\mu(A) > 0$ and*

$$\langle |X|^p_{\delta_1} \rangle(T) \geq g_p(T) I(A_{2\varepsilon}, \varepsilon), \; \varepsilon = \frac{1}{T}, \; p > 0, \tag{10}$$

where $g_p(T)$ are positive functions of T. Then

$$\langle |X|^p_{\delta_1} \rangle(T) \gtrsim (g_p(T))^{p/(p+1)} (\mu(A))^{(1+2p)/(p+1)}.$$

In particular, if the set A does not depend on time, then

$$\langle |X|^p_{\delta_1} \rangle(T) \gtrsim (g_p(T))^{p/(p+1)}.$$

This theorem, combined with Theorem 1, allows us to derive the following lower bound for $\beta^-_{\delta_1}(p)$:

Theorem 3. *Let H be a discrete Schrödinger operator, given by* (1)*, acting on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$. Assume $A \subseteq \mathbb{R}$ is such that $\mu(A) > 0$ and conditions* (a) *and* (b) *hold. Then,*

$$\beta^-_{\delta_1}(p) \geq \begin{cases} \frac{p(p+2\kappa)}{(p+1)(\alpha+\kappa+1/2)} & p \leq 2\alpha+1, \\ \frac{p}{\alpha+1} & p > 2\alpha+1. \end{cases} \tag{11}$$

Proof. The inequality for $0 < p \le 2\alpha + 1$ follows directly from Theorems 1 and 2. The Jensen inequality implies that $(\langle |X|^p_{\delta_1} \rangle(T))^{1/p}$ is non-decreasing in p. Taking $p_0 = 2\alpha + 1$ and $p > p_0$, we get the inequality for $p > 2\alpha + 1$. □

Remarks. (i) Since $2\kappa \le 2\alpha + 1$, an application of Theorem 2 to (8) or (9) does not give better bounds for $p > 2\alpha + 1$.

(ii) If $\mu(A) > 0$, it follows from the existence of generalized eigenfunctions [B, Si1] that the constant κ in (c) cannot be larger than $1/2$.

(iii) The Jitomirskaya-Last approach [JL1, JL2] (see also [DKL, KKL]) gives the following effective way to obtain dynamical bounds from solution estimates: Assuming conditions (a) and (c) for a set A with $\mu(A) > 0$, one obtains that

$$\beta^-_{\delta_1}(p) \ge \frac{2p\kappa}{\alpha + \kappa + 1/2}. \tag{12}$$

If we assume (b) rather than (c), we can prove the bound (11). It is easy to check that (11) coincides with (12) when $\kappa = 1/2$ and (11) is strictly stronger than (12) for every $p > 0$ when $\kappa < 1/2$. Thus, by Remark (ii) above, our bounds are favorable to the ones obtained through the Jitomirskaya-Last approach in all situations where one has (a) and (b). While in general (b) is a stronger condition than (c), in concrete applications one often really proves (b) in order to show (c) (e.g., in our and previous studies of the Fibonacci Hamiltonian and related models).

(iv) The paper [DST] worked under the sole assumption (a) and derived the bound

$$\beta^-_{\delta_1}(p) \ge \frac{p - 3\alpha}{\alpha + 1}$$

for all $p > 0$. In particular, this statement is vacuous when $p \le 3\alpha$. Thus, under the additional assumption (b), Theorem 3 extends the range of relevant p to the entire interval $(0, \infty)$ and on top of that improves the lower bound on $\beta^-_{\delta_1}(p)$. Moreover, even if we only assume condition (a) for some set A with $\mu(A) > 0$, we can improve the bounds from [DST] by applying Theorem 2.

(v) It is possible to derive lower bounds for the outside probabilities,

$$P(|n| \ge K(T), T) = \sum_{|n| \ge K(T)} a(n, T),$$

from this result. Here, $K(T)$ is a suitable growing function of T. This is discussed in Section 5.

We will demonstrate how to apply our general dynamical results to operators with Sturmian potentials on $\mathbb{Z}$, that is,

$$V(n) = \lambda \chi_{[1-\omega,1)}(n\omega \mod 1), \; n \in \mathbb{Z}, \tag{13}$$

with coupling constant $\lambda > 0$ and irrational rotation number $\omega \in (0, 1)$. If $\omega = (\sqrt{5} - 1)/2$, then V is usually called the Fibonacci potential and the associated operator is called the Fibonacci Hamiltonian.

Consider the continued fraction expansion of ω,

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_n \in \mathbb{Z}_+$. The number ω is said to have bounded partial quotients if the sequence $\{a_n\}$ is bounded. In this case,

$$d(\omega) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k < \infty. \tag{14}$$

The set of such numbers ω is uncountable but has Lebesgue measure zero. Note that $\omega = (\sqrt{5}-1)/2$ is contained in this set since, in this case, $a_n = 1$ for every n.

Theorem 4. *Suppose $\lambda > 0$ and $\omega \in (0,1)$ is irrational with $a_n \le C$. Consider the operator* (1) *with potential given by* (13). *With*

$$\alpha = D\, d(\omega)\, \log(2 + \sqrt{8+\lambda^2})$$

(D *is some universal constant*) *and*

$$\kappa = \frac{\log(\sqrt{17}/4)}{(C+1)^5},$$

the following dynamical bounds hold true:

$$\beta^-_{\delta_1}(p) \ge \begin{cases} \frac{p(p+2\kappa)}{(p+1)(\alpha+\kappa+1/2)} & p \le 2\alpha+1, \\ \frac{p}{\alpha+1} & p > 2\alpha+1. \end{cases} \tag{15}$$

Let us compare these bounds with the ones that were previously known (the papers [D, DKL, DLa, DST, DT, JL2, KKL] prove dynamical results for Sturmian potentials).

We first consider the Fibonacci case, that is, $\omega = (\sqrt{5}-1)/2$.[1] For small values of $p > 0$, the best previously known bound was obtained in [KKL], using the Jitmorskaya-Last approach, and consequently reads

$$\beta^-_{\delta_1}(p) \ge \frac{2p\kappa}{\alpha+\kappa+1/2}.$$

This bound is valid for all values of $p > 0$, but for p large, the bound

$$\beta^-_{\delta_1}(p) \ge \frac{p-3\alpha}{\alpha+1}, \tag{16}$$

obtained in [DST], which also holds for all $p > 0$, is better. As discussed in Remarks (iii) and (iv) after Theorem 3, the bound (15) is strictly better than both of these bounds for all $p > 0$.

For other ω's with bounded partial quotients, the gap between our bound and previously known bounds is even bigger. For small $p > 0$, the best bound was [D] (see also [DKL])

$$\beta^-_{\delta_1}(p) \ge \frac{2p\kappa_\lambda}{\alpha+\kappa_\lambda+1/2} \tag{17}$$

with some small λ-dependent $\kappa_\lambda > 0$ which obeys $\kappa_\lambda \to 0$ as $\lambda \to \infty$, whereas for larger values of p, it is again better to use the bound (16). Again we see that (15) improves upon all previously known dynamical bounds for these potentials.

[1] As discussed in Section 6, the value of κ can often be slightly improved over what is stated above. In particular, Killip et al. can work with $\kappa = \log[\sqrt{17}/(20\log\omega^{-1})]$ in the Fibonacci case. For this particular case, we can use this value also in (15).

To summarize, we establish a new effective way to derive quantum dynamical lower bounds from solution estimates. As with previous methods, we require suitable upper and lower bounds on solutions to the difference equation associated with the operator. While the upper bounds we need are the same as in previous approaches, the lower bounds are slightly stronger. Whenever our result applies, it gives better dynamical results (except in extreme cases, where the derived bounds are the same). The particular case of the Fibonacci Hamiltonian, and the related Sturmian models, is discussed in detail.

The question of how to obtain any upper bounds on the fast part of the time evolution remains an important open problem. For example, it is in general not clear how to bound $\langle |X|^p_{\delta_1}\rangle(T)$ or $\beta^-_{\delta_1}(p)$ from above. (The only exception is the case of growing sparse potentials [CM, T2].)

The organization of the paper is as follows. We prove Theorem 1 for half-line and whole-line operators in Sections 2 and 3, respectively. Then we prove Theorem 2 in Section 4 and discuss some consequences of Theorem 3 for outside probabilities in Section 5. Finally, we prove Theorem 4 in Section 6 and present some simplifications and improvements of some central results within the Jitomirskaya-Last theory in the appendix.

2. Proof of Theorem 1 For Half-Line Operators

In this section we consider Schrödinger operators on the half-line. We first introduce notation and gather a few useful results. Then we prove Theorem 1 for half-line operators.

Let H be a discrete Schrödinger operator on $\ell^2(\mathbb{Z}_+)$ with potential V; compare (1). For $z \in \mathbb{C}$ and $\theta \in [0, 2\pi)$, denote by $u_\theta(n, z)$ the solution to the equation

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n) \tag{18}$$

that obeys $u_\theta(0, z) = \sin\theta$, $u_\theta(1, z) = \cos\theta$. With F from (6), one can verify [JL1] that for any $n \geq 1$,

$$u(n, z) := \langle (H - z)^{-1}\delta_1, \delta_n\rangle = F(z)u_0(n, z) - u_{\pi/2}(n, z). \tag{19}$$

Thus,

$$(u(n+1, z), u(n, z))^T = T(n, 0; z)(F(z), -1)^T, \ n \geq 1, \tag{20}$$

where T is the transfer matrix associated to the equation $Hu = zu$:

$$T(n, 0; z) = \begin{pmatrix} u_0(n+1, z) & u_{\pi/2}(n+1, z) \\ u_0(n, z) & u_{\pi/2}(n, z) \end{pmatrix}.$$

If one has good control of the functions $u(n, z)$ for complex z, then two kinds of results can be obtained.

The first is related with the study of the function $F(z)$. In particular, since $\mu([E - \varepsilon, E + \varepsilon]) \leq 2\varepsilon \operatorname{Im} F(E + i\varepsilon)$, an upper bound for the measure of intervals follows from an upper bound on $\operatorname{Im} F(z)$. Such a bound provides a lower bound for the lower Hausdorff or packing dimension of the spectral measure. On the other hand, lower bounds on $|F(z)|$ can be used to show singularity of the spectral measure; compare [JL1].

The second group of results is based on the Parseval formula. For any f, define

$$\langle f(t)\rangle(T) = \frac{2}{T}\int_0^\infty e^{-2t/T} f(t)dt.$$

Then (cf. [KKL, RS])

$$\langle |\langle e^{-itH}\psi, \delta_n\rangle|^2\rangle(T) = \frac{\varepsilon}{\pi}\int_{\mathbb{R}} |\langle (H - E - i\varepsilon)^{-1}\psi, \delta_n\rangle|^2 dE,$$

where $\psi \in \ell^2(\mathbb{Z}_+)$ and $\varepsilon = 1/T$. In particular, for $a(n,T)$ as defined in (3), we obtain

$$a(n,T) = \frac{\varepsilon}{\pi}\int_{\mathbb{R}} |u(n, E + i\varepsilon)|^2 dE. \tag{21}$$

This formula can be used to bound $a(n,T)$ from below or from above.

To apply the formula (21) directly, one should have good control of $|u(n, E+i\varepsilon)|$ depending on n, E, ε for small values of ε. However, in most applications, it is much easier to obtain information on solutions to the equation (18) with real $z = E$. (Sparse potentials represent a rare case where solutions with complex z can be studied directly; see [T2].) Therefore, one uses perturbative methods to link solutions to the equation (18) with real $z = E$ and complex $z = E + i\varepsilon$ (with small ε). This also allows one to study the Borel transform $F(z)$ of the spectral measure [JL1].

Probably the most obvious approach is based on upper bounds on the norm $||T(n, 0; z)||$. If one has such a bound for $||T(m, 0; E)||$, $m \le n$, one can apply the method of Simon [Si2] to get bounds for complex z (if n is not too large). Since $\det T = 1$, (20) implies that for $n \ge 1$,

$$|u(n+1, z)|^2 + |u(n, z)|^2 \ge ||T(n, 0; z)||^{-2}(|F(z)|^2 + 1). \tag{22}$$

Therefore, if one has a non-trivial upper bound for $||T(n, 0; E)||$ on some set of energies E, (21) and (22) yield a lower bound for the quantity $a(n+1,T)+a(n,T)$, and thus a lower bound for the time-averaged moments

$$\langle |X|^p\rangle(T) = \sum_n n^p a(n,T), \ p > 0,$$

and outside probabilities

$$\sum_{n \ge M} a(n,T)$$

(with M depending on T) [DST, DT]. Although this method gives good lower bounds for moments with large p, it is clearly not optimal since the left-hand side of (22) may be much larger than what the perturbative argument gives as a lower bound for the right-hand side (at least, for some values of E).

The upper bound on $||T(n, 0; z)||$ can be also used to bound $\operatorname{Im} F(z)$ from above. Since

$$\begin{aligned} \operatorname{Im} F(z) &= \varepsilon ||R(z)\delta_1||^2 \\ &= \varepsilon \sum_n |u(n,z)|^2 \\ &\ge \frac{\varepsilon}{2}(|F(z)|^2 + 1)\sum_n ||T(n, 0; z)||^{-2} \\ &\ge \frac{\varepsilon}{2}\left(\operatorname{Im} F(z)\right)^2 \sum_n ||T(n, 0; z)||^{-2}, \end{aligned}$$

it follows that

$$\operatorname{Im} F(z) \le \frac{2}{\varepsilon} \left(\sum_n ||T(n,0;z)||^{-2} \right)^{-1}.$$

However, in most applications, better bounds can be obtained using the Jitomirskaya-Last approach discussed next. This method was proposed in [JL1] and later developed in [KKL]. We will discuss certain improvements of this theory in the appendix. Let $E', E \in \mathbb{R}$, $\varepsilon > 0$, and $z = E' + i\varepsilon$. The starting point is the following formula [JL1, KKL]:

$$u(n,z) = F(z)u_0(n,E) - u_{\pi/2}(n,E) + (z-E)(K(E)u(z))(n),\ n \ge 1, \tag{23}$$

where

$$(K(E)\psi)(n) = \sum_{j=1}^{n} K(n,j,E)\psi(j)$$

and

$$K(n,j,E) = u_0(n,E)u_{\pi/2}(j,E) - u_{\pi/2}(n,E)u_0(j,E).$$

For $L \ge 1$, consider the ($[L]$ or $[L]+1$))-dimensional space with the inner product

$$\langle f,g\rangle = \sum_{n=1}^{[L]} f(n)\overline{g(n)} + (L-[L])f([L]+1)\overline{g([L]+1)}.$$

We will denote the corresponding norm by $||f||_L$. The Hilbert-Schmidt norm of $K(E)$ in this space is given by [JL1, KKL]:

$$\begin{aligned} |||K(E)|||_L^2 &= ||u_0(E)||_L^2\, ||u_{\pi/2}(E)||_L^2 - \langle u_0(E), u_{\pi/2}(E)\rangle_L^2 \\ &= \sup_\theta ||u_\theta(E)||_L^2 \inf_\theta ||u_\theta(E)||^2. \end{aligned} \tag{24}$$

Let

$$u(n,E) := F(z)u_0(n,E) - u_{\pi/2}(n,E).$$

Assume that $|z-E| \le \delta$. Since the operator norm does not exceed the Hilbert-Schmidt norm, it follows from (23) that for any L,

$$(1+\delta|||K(E)|||_L)^{-1}||u(E)||_L \le ||u(z)||_L \le (1-\delta|||K(E)|||_L)^{-1}||u(E)||_L \tag{25}$$

(the second inequality holds under the condition $\delta|||K(E)|||_L < 1$). Thus, if the norm $|||K(E)|||_L$ is not too large, in order to control $||u(z)||_L$, it is sufficient to control $||u(E)||_L$. Since both solutions $u_0(n,E)$, $u_{\pi/2}(n,E)$ are real,

$$|u(n,E)|^2 = (\operatorname{Re} F(z)u_0(n,E) - u_{\pi/2}(n,E))^2 + (\operatorname{Im} F(z)u_0(n,E))^2. \tag{26}$$

Having some information about solutions $u_0(n,E)$, $u_{\pi/2}(n,E)$, we have some control of $||u(z)||_L$. In particular, one can prove bounds on $|F(z)|$ or $\operatorname{Im} F(z)$. On the other hand, a lower bound on $||u(z)||_L$ for E from some set yields, via (21), a lower bound for the inside probabilities

$$\sum_{n \le L} a(n,T).$$

Results of this kind were obtained in [JL1, KKL]. As noted above, we present simplified and unified proofs of some of them in the appendix.

We are now ready to give the

Proof of Theorem 1 for half-line operators. We shall estimate from below the quantities

$$h_p(E'+i\varepsilon) = \sum_n n^p |u(n,E'+i\varepsilon)|^2$$

with $E' \in A_{2\varepsilon}$. It follows from (a) and (24) that, for every $E \in A$,

$$||K(E)||_L^2 \le |||K(E)|||_L^2 \le CL^{2\alpha+1} f(L,E), \tag{27}$$

where $||\cdot||_L$ is the operator norm and

$$f(L,E) = \inf_\theta ||u_\theta(n,E)||_L^2.$$

Let $E' \in A_{2\varepsilon}$. That is, there exists $E \in A$ such that $|E'-E| \le 2\varepsilon$. Define $z = E' + i\varepsilon$. The bound (25) holds with $\delta = 3\varepsilon$. From now on we shall assume that

$$\beta := 3\varepsilon |||K(E)|||_L \le 1/4. \tag{28}$$

By (27), this holds provided that $9\varepsilon^2 CL^{2\alpha+1} f(L,E) \le 1/16$. The latter condition is satisfied if L is not too large (depending on ε, E).

Let $0 < M < L$. We can estimate from below, using (25),

$$\begin{aligned}
P(M,L) &:= ||u(z)||_L^2 - ||u(z)||_M^2 \\
&\ge (1+\delta||K(E)||_L)^{-2}||u(E)||_L^2 - (1-\delta||K(E)||_M)^{-2}||u(E)||_M^2 \\
&\ge (1+\delta||K(E)||_L)^{-2}||u(E)||_L^2 - (1-\delta||K(E)||_L)^{-2}||u(E)||_M^2
\end{aligned}$$

since $||K(E)||_M \le ||K(E)||_L$. Thus, by (28),

$$\begin{aligned}
P(M,L) &\ge \frac{1}{(1+\beta)^2} \sum_{n=M+1}^{L} |u(n,E)|^2 + \left((1+\beta)^{-2} - (1-\beta)^{-2}\right) ||u(E)||_M^2 \\
&\ge \frac{1}{2} \sum_{n=M+1}^{L} |u(n,E)|^2 - 8\beta ||u(E)||_M^2.
\end{aligned} \tag{29}$$

The identity (26) implies

$$|u(n,E)|^2 = (w(n,E))^2 + (\operatorname{Im} F(z) u_0(n,E))^2,$$

where $w(n) = \operatorname{Re} F(z)\, u_0(n,E) - u_{\pi/2}(n,E)$ is a real solution to the equation (4). Thus, (29) implies

$$\begin{aligned}
P(M,L) \ge \frac{1}{2} &\sum_{n=M+1}^{L} (w(n,E))^2 - 8\beta ||w(E)||_M^2 \\
&+ (\operatorname{Im} F(z))^2 \left(\frac{1}{2} \sum_{n=M+1}^{L} (u_0(n,E))^2 - 8\beta ||u_0(E)||_M^2 \right).
\end{aligned} \tag{30}$$

Let k be the constant from condition (b). If $\beta \leq \gamma/24$, we get for every real solution v of (4),[2]

$$\begin{aligned}
(31)\qquad \frac{1}{2}\sum_{n=kL+1}^{L}(v(n,E))^2 - 8\beta\sum_{n=1}^{kL}(v(n,E))^2 &\geq \left(\frac{1}{2}-\frac{8\beta}{\gamma}\right)\sum_{n=kL+1}^{L}(v(n,E))^2 \\
&\geq \frac{1}{6}\sum_{n=kL+1}^{L}(v(n,E))^2 \\
&\geq \frac{\gamma}{12}\sum_{n=1}^{L}(v(n,E))^2.
\end{aligned}$$

Using this bound twice in (30), where we take $M = kL$, we obtain

$$P(kL,L) \geq \frac{\gamma}{12}\left((\operatorname{Im} F(z)||u_0(E)||_L)^2 + ||w(E)||_L^2\right).$$

Thus,

$$h_p(z) \geq (kL)^p P(kL,L) \geq \frac{\gamma k^p}{12} L^p \left((\operatorname{Im} F(z)||u_0(E)||_L)^2 + ||w(E)||_L^2\right)$$

for any L such that $\beta \leq \gamma/24$.

Next, we note that $||u_0(E)||_L^2 \geq f(L,E) = \inf_\theta ||u_\theta(E)||_L^2$ since u_0 is a particular solution, corresponding to $\theta = 0$. Therefore,

$$h_p(z) \geq \frac{\gamma k^p}{12} L^p (\operatorname{Im} F(z))^2 f(L,E). \tag{32}$$

Condition (c), which, as noted above, is a consequence of the assumptions (a) and (b), implies

$$f(L,E) \geq D^2 L^{2\kappa}. \tag{33}$$

Up to now, we have not fixed the value of L. The only condition is that $\beta \leq \gamma/24$, which holds if

$$9\varepsilon^2 C L^{2\alpha+1} f(L,E) \leq \left(\frac{\gamma}{24}\right)^2.$$

Since $L^{2\alpha+1} f(L,E)$ is a monotone continuous function of L, there exists $L_{\max}$ (depending on E,ε) such that

$$9\varepsilon^2 C L_{\max}^{2\alpha+1} f(L_{\max},E) = \left(\frac{\gamma}{24}\right)^2. \tag{34}$$

Assume first that $p \leq 2\alpha + 1$. It follows from (33) that

$$D(E,\varepsilon) := \frac{f(L_{\max},E)}{L_{\max}^{2\kappa}} \geq D^2 > 0.$$

The identity (34) yields

$$\varepsilon^2 L_{\max}^{2\alpha+1+2\kappa} D(E,\varepsilon) = \frac{\gamma^2}{5184\, C} =: \tau > 0.$$

Thus,

$$L_{\max} = \left(\tau\varepsilon^{-2} D^{-1}(E,\varepsilon)\right)^{\frac{1}{2\alpha+1+2\kappa}}.$$

---

[2]Here, we need L large enough, that is, ε small enough or, in other words, T large enough.

Inserting this expression in (32) with $L = L_{\max}$, we get

$$\begin{aligned} h_p(z) &\geq \frac{\gamma k^p}{12} (\operatorname{Im} F(z))^2 L_{\max}^{p+2\kappa} D(E,\varepsilon) \\ &\geq \left[\frac{\gamma k^p}{12} \tau^{\frac{p+2\kappa}{2\alpha+1+2\kappa}} \right] (\operatorname{Im} F(z))^2 \varepsilon^{\frac{-2(p+2\kappa)}{2\alpha+1+2\kappa}} D(E,\varepsilon)^{1-\frac{p+2\kappa}{2\alpha+1+2\kappa}}. \end{aligned}$$

Since $p \leq 2\alpha+1$, and $D(E,\varepsilon) \geq D^2 > 0$, we finally obtain

$$h_p(z) \geq \operatorname{const}(p, C, \alpha, D, \kappa, k, \gamma) (\operatorname{Im} F(z))^2 \varepsilon^{-\frac{2(p+2\kappa)}{2\alpha+1+2\kappa}} \tag{35}$$

for every $z = E' + i\varepsilon$ with $E' \in A_{2\varepsilon}$, where

$$\operatorname{const}(p, C, \alpha, D, \kappa, k, \gamma) = \frac{\gamma k^p}{12} \left(\frac{\gamma^2}{5184\, C} \right)^{\frac{p+2\kappa}{2\alpha+1+2\kappa}} D^{2-\frac{2(p+2\kappa)}{2\alpha+1+2\kappa}}. \tag{36}$$

The Parseval formula (21) implies

$$\langle |X|_{\delta_1}^p \rangle(T) = \frac{\varepsilon}{\pi} \int_{\mathbb{R}} dE' h_p(E' + i\varepsilon), \ \varepsilon = \frac{1}{T}.$$

Integrating only over the set $A_{2\varepsilon}$, we obtain (7). The bound (8) follows since $\langle |X|^p \rangle(T)$ are increasing functions of p.

In the case $p > 2\alpha+1$, one again defines $L_{\max}$ by (34). From the (very rough) estimate $f(L_{\max}, E) \leq C L_{\max}^{2\alpha+1}$, we get

$$L_{\max} \geq \left(\frac{\gamma^2}{5184\, C^2} \right)^{\frac{1}{2}} \varepsilon^{-\frac{1}{2\alpha+1}}.$$

The bounds (32) and (33) thus imply

$$h_p(z) \geq \left[\frac{\gamma k^p}{12} \left(\frac{\gamma^2}{5184\, C^2} \right)^{\frac{p+2\kappa}{2}} D^2 \right] (\operatorname{Im} F(z))^2 \varepsilon^{-\frac{p+2\kappa}{2\alpha+1}},$$

from which (9) follows. □

3. Proof of Theorem 1 For Whole-Line Operators

In this section we prove dynamical bounds for whole-line operators analogous to the ones in the half-line case from Section 2.

Let H be a discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential V; compare (1). For $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ and $n \geq 1$, we can write

$$u(n,z) = \langle (H-z)^{-1} \delta_1, \delta_n \rangle = F(z) u_0(n,z) + G(z) u_{\pi/2}(n,z).$$

(Recall that $u_\theta(z)$ denotes the solution to (18) with $u_\theta(0,z) = \sin\theta$, $u_\theta(1,z) = \cos\theta$.) Here, F is the Borel transform of the spectral measure μ associated with H and δ_1 and G is a suitable complex-valued function. Again we have, for $n \geq 1$,

$$a(n,T) = \langle |\langle e^{-itH} \delta_1, \delta_n \rangle|^2 \rangle(T) = \frac{1}{\pi T} \int_{\mathbb{R}} \left| u(n, E + \tfrac{i}{T}) \right|^2 dE. \tag{37}$$

Write

$$u(n,E) = F(z) u_0(n,E) + G(z) u_{\pi/2}(n,E).$$

The analog of (23) is then given by

$$u(n,z) = u(n,E) + (z-E)(K(E)u(z))(n), \ n \geq 1, \tag{38}$$

where

$$(K(E)\psi)(n) = \sum_{j=1}^{n} \left[u_0(n,E)u_{\pi/2}(j,E) - u_{\pi/2}(n,E)u_0(j,E)\right]\psi(j).$$

Therefore, (24) holds, that is,

$$|||K(E)|||_L^2 = \sup_\theta ||u_\theta(E)||_L^2 \inf_\theta ||u_\theta(E)||^2.$$

The analog of (26) now reads

$$|u(n,E)|^2 = (w(n,E))^2 + (v(n,E))^2, \tag{39}$$

with the two real solutions to the equation (4),

$$w(n) = \operatorname{Re} F(z)u_0(n,E) + \operatorname{Re} G(z)u_{\pi/2}(n,E),$$
$$v(n) = \operatorname{Im} F(z)u_0(n,E) + \operatorname{Im} G(z)u_{\pi/2}(n,E).$$

Let us now turn to the

Proof of Theorem 1 for whole-line operators. Write $\varepsilon = 1/T$ and

$$h_p(E'+i\varepsilon) = \sum_{n\geq 1} n^p |u(n,E'+i\varepsilon)|^2$$

with $E' \in A_{2\varepsilon}$. Choose $E \in A$ such that $|E'-E| \leq 2\varepsilon$. Now we can mimic the proof in the half-line situation all the way up to (29), which reads

$$\begin{aligned} P(M,L) &\geq \frac{1}{(1+\beta)^2}\sum_{n=M+1}^{L}|u(n,E)|^2 + \left((1+\beta)^{-2}-(1-\beta)^{-2}\right)||u(E)||_M^2 \\ &\geq \frac{1}{2}\sum_{n=M+1}^{L}|u(n,E)|^2 - 8\beta||u(E)||_M^2 \\ &\geq \frac{1}{2}\left[\sum_{n=M+1}^{L}(w(n,E))^2 - 8\beta||w(E)||_M^2\right] + \\ &\quad + \frac{1}{2}\left[\sum_{n=M+1}^{L}(v(n,E))^2 - 8\beta||v(E)||_M^2\right], \end{aligned} \tag{40}$$

where we used (39) in the last step.

Using (31) twice in (40) with $M = kL$, we obtain

$$\begin{aligned} P(kL,L) &\geq \frac{\gamma}{12}\left(||w(E)||_L^2 + ||v(E)||_L^2\right) \\ &\geq \frac{\gamma}{12}\left((v(0,E))^2 + (v(1,E))^2\right) f(L,E) \\ &\geq \frac{\gamma}{12}\left(\operatorname{Im} F(E'+i\varepsilon)\right)^2 f(L,E). \end{aligned}$$

Thus,

$$h_p(z) \geq (kL)^p P(kL,L) \geq \frac{\gamma k^p}{12} L^p \left(\operatorname{Im} F(E'+i\varepsilon)\right)^2 f(L,E)$$

for any L such that $\beta \leq \gamma/24$.

From this point on, we can follow the reasoning from the half-line proof and finally obtain

$$h_p(z) \geq \mathrm{const}(p,C,\alpha,D,\kappa,k,\gamma)\,(\operatorname{Im} F(z))^2 \varepsilon^{-\frac{2(p+2\kappa)}{2\alpha+1+2\kappa}},$$

where $E' \in A_{2\varepsilon}$, $z = E' + i\varepsilon$ and $\mathrm{const}(p, C, \alpha, D, \kappa, k, \gamma)$ is as in (36). Using (37), this allows us to conclude the proof as before. $\square$

4. Proof of Theorem 2

In this section we consider operators on the half-line and on the whole line simultaneously and prove Theorem 2. Thus, let H be the operator on $\ell^2(\mathbb{Z})$ or $\ell^2(Z_+)$ given by (1). Throughout this section, μ will denote the spectral measure of the vector δ_1 with respect to H, and F will denote its Borel transform,

$$F(z) = \langle (H-z)^{-1}\delta_1, \delta_1\rangle = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}.$$

We note that all implicit constants below are positive and, if not universal, depend only on p.

Proof of Theorem 2. Let T be given and write $\varepsilon = 1/T$. We use [T2, Lemma 2.1]. Its proof implies that for any set A with $\mu(A) > 0$,

$$\langle |X|^p_{\delta_1}\rangle(T) \gtrsim (\mu(A))^{1+2p} \left(J\left(A, 2\varepsilon\right)\right)^{-p}, \tag{41}$$

where

$$J(\Delta, \nu) = \int_\Delta \int_{\mathbb{R}} \frac{\nu^2}{\nu^2 + (x-y)^2}\, d\mu(y)\, d\mu(x).$$

The fact that 2ε occurs in (41), and not ε as in [T2], is due to the fact that we use a different definition of time-averaging.

Next, one can bound J from above by I following the proof of [T2, Lemma 2.2]. For any $\delta > 0$,

$$I(A_\delta, \delta) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,u,\delta)\, d\mu(u)\, d\mu(x) \geq \int_A \int_{\mathbb{R}} f(x,u,\delta)\, d\mu(u)\, d\mu(x), \tag{42}$$

where

$$f(x,u,\delta) := \delta^3 \int_{A_\delta} \frac{dE}{((u-E)^2+\delta^2)((x-E)^2+\delta^2)}.$$

If $x \in A$, then $[x-\delta, x+\delta] \subset A_\delta$, and thus

$$\begin{aligned} f(x,u,\delta) &\geq \delta^3 \int_{x-\delta}^{x+\delta} \frac{dE}{((u-E)^2+\delta^2)((x-E)^2+\delta^2)} \\ &= \int_{-1}^{1} \frac{dt}{(t^2+1)((t+s)^2+1)} \\ &\gtrsim \frac{1}{s^2+1}, \end{aligned}$$

where $s = (x-u)/\delta$. Inserting this bound into (42), we get

$$I(A_\delta, \delta) \gtrsim J(A, \delta). \tag{43}$$

One can easily see that $I(\Delta, \nu) \gtrsim I(\Delta, 2\nu)$. Therefore, (43) implies

$$I(A_{2\varepsilon}, \varepsilon) \gtrsim I\left(A_{2\varepsilon}, 2\varepsilon\right) \gtrsim J(A, 2\varepsilon).$$

This, in combination with (41), yields

$$\langle |X|^p_{\delta_1}\rangle(T) \gtrsim (\mu(A))^{1+2p} \left(I(A_{2\varepsilon}, \varepsilon)\right)^{-p}. \tag{44}$$

Together with (10), we obtain

$$\langle |X|^p_{\delta_1}\rangle(T) \gtrsim \left(g_p(T)X + (\mu(A))^{1+2p} X^{-p}\right),$$

where $X = I(A_{2\varepsilon}, \varepsilon),\ \varepsilon = 1/T$. Since the function

$$f(X) = aX + bX^{-p},\ X > 0$$

is bounded from below by

$$\left[p^{\frac{1}{p+1}} + p^{-\frac{p}{p+1}}\right] a^{\frac{p}{p+1}} b^{\frac{1}{p+1}},$$

the statement of the theorem follows. □

5. Bounds for Outside Probabilities

In this section we make a connection between lower bounds for moments and lower bounds for outside probabilities,

$$P(|n| \geq K(T), T) = \sum_{|n| \geq K(T)} a(n, T),$$

with an increasing function $K(T)$. We then use this to derive lower bounds for the latter from Theorem 3. Thus, we can control the polynomially decaying tails of the wave-packet (for a more detailed discussion; see [GKT]).

Lemma 1. *Suppose that for some $p > 0$, we have*

$$\langle |X|_{\delta_1}^p \rangle (T) \geq f_p(T) \tag{45}$$

with a function f_p satisfying $\lim_{T\to\infty} f_p(T) = \infty$. Then, for any $\delta > 0$, we have

$$P\left(|n| \geq \left(\frac{f_p(T)}{2}\right)^{\frac{1}{p}}, T\right) \gtrsim T^{-p(1+\delta)} f_p(T).$$

Proof. Write $K_p(T) = (f_p(T)/2)^{1/p}$. For given $\delta > 0$, consider the following sets,

$$\begin{aligned} S_1 &= \{n \in \mathbb{Z} : |n| \leq K_p(T)\}, \\ S_2 &= \{n \in \mathbb{Z} : K_p(T) < |n| \leq T^{1+\delta}\}, \\ S_3 &= \{n \in \mathbb{Z} : |n| > T^{1+\delta}\}, \end{aligned}$$

and denote by A_1, A_2, A_3 the corresponding partial sums in the definition of $\langle |X|_{\delta_1}^p \rangle (T)$, that is,

$$A_j = \sum_{n \in S_j} |n|^p a(n, T),\ \ 1 \leq j \leq 3.$$

Since for every T, $\sum_n a(n, T) = 1$, it follows that

$$A_1 \leq K_p^p(T) = \frac{f_p(T)}{2}. \tag{46}$$

For A_2, we have the obvious bound

$$A_2 \leq T^{p(1+\delta)} P(|n| \geq K_p(T), T). \tag{47}$$

For every $s > 0$, we can estimate A_3 as follows:

$$\begin{aligned} A_3 &= \sum_{|n| > T^{1+\delta}} |n|^p a(n, T) \\ &\leq T^{-s(1+\delta)} \sum_{|n| > T^{1+\delta}} |n|^{p+s} a(n, T) \\ &\leq T^{-s(1+\delta)} \langle |X|_{\delta_1}^{p+s} \rangle (T). \end{aligned}$$

As there is a ballistic upper bound for the moments, that is, $\langle |X|^r_{\delta_1}\rangle(T) \le C(r)T^r$, we obtain, taking $s = p/\delta$,

$$A_3 \le C, \tag{48}$$

with a T-independent constant C. The bounds (45)–(48) yield

$$\begin{aligned} T^{p(1+\delta)} P(|n| \ge K_p(T), T) \ge A_2 & \\ &\ge \langle |X|^p_{\delta_1}\rangle(T) - A_1 - A_3 \\ &\ge \frac{f_p(T)}{2} - C. \end{aligned}$$

Since $\lim_{T\to\infty} f_p(T) = \infty$, the lemma follows. □

Remark. The result holds, of course, for any well-localized initial state ψ, $||\psi|| = 1$, not necessarily δ_1.

Under the conditions of Theorem 3 this lemma gives lower bounds of the form $P(|n| \ge T^\gamma, T) \ge T^{-g(\gamma)}$, where

$$\gamma_1 = \frac{2\kappa}{\alpha + \kappa + 1/2} < \gamma < \frac{1}{\alpha + 1} = \gamma_2,$$

and $g(\gamma)$ is an explicit positive growing function. This can be achieved by taking appropriate values of the parameter p. In particular, if one takes p small, then γ is close to γ_1 and $g(\gamma)$ is close to 0. If p is close to $2\alpha + 1$, then γ is close to γ_2 and $g(\gamma)$ is close to $\alpha(2\alpha + 1)/(\alpha + 1)$.

6. Application to Quasicrystal Models

In this section we consider operators with Sturmian potentials (i.e., given by (13)) and prove Theorem 4.

Recall that the continued fraction expansion of ω is given by

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

with uniquely determined $a_n \in \mathbb{Z}_+$. The associated rational approximants p_n/q_n are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}. \end{aligned}$$

The number ω is said to have bounded partial quotients if the sequence $\{a_n\}$ is bounded. More generally, it is said to have bounded density if $d(\omega)$, as defined in (14), is finite. Both sets of numbers are uncountable and have Lebesgue measure zero.

Our goal is to establish solution estimates for all energies in the spectrum in order to apply our general dynamical result. The upper bound is known under certain assumptions on ω. Namely, the following proposition is a consequence of [IRT, Corollary 10]:

Proposition 1. *Suppose ω is a bounded density number. For every λ, there is a constant C such that for every $E \in \sigma(H)$, every normalized solution u of* (4)*, and every $L \geq 1$, $\|u(E)\|_L^2 \leq CL^{2\alpha+1}$, with*

$$\alpha = D\, d(\omega)\, \log C_\lambda, \tag{49}$$

where D is some universal constant, $C_\lambda = 2 + \sqrt{8+\lambda^2}$, and $d(\omega)$ is as in (14)*.*

It remains to prove suitable lower bounds for solutions. Define the words s_n over the alphabet $\mathcal{A} = \{0, \lambda\}$ by

$$s_{-1} = \lambda, \quad s_0 = 0, \quad s_1 = s_0^{a_1 - 1} s_{-1}, \quad s_n = s_{n-1}^{a_n} s_{n-2},\ n \geq 2. \tag{50}$$

In particular, the word s_n has length q_n for each $n \geq 0$. By definition, s_{n-1} is a prefix of s_n for each $n \geq 2$. Thus, the words have a one-sided infinite limit which, in fact, coincides with the restriction of V to the right half-line (see [DKL, DLe]).

For later use, we recall the following elementary formula [DLe, Proposition 2.3] which implies that the word $s_n s_{n+1}$ has s_{n+1} as a prefix:

$$s_n s_{n+1} = s_{n+1} s_{n-1}^{a_n - 1} s_{n-2} s_{n-1} \text{ for every } n \geq 2. \tag{51}$$

Another important ingredient is that for each energy E in the spectrum, the trace $x_n(E)$ of the transfer matrix $T(q_n, 0; E)$ from 0 to q_n obeys

$$\min\{|x_n(E)|, |x_{n+1}(E)|\} \leq 2. \tag{52}$$

This was shown by Bellissard et al. in [BIST] (see also Sütő [Sü] for this result in the Fibonacci case).

Such trace bounds are useful as shown by the following lemma, which has been used a number of times [D, DKL, JL2, KKL]. Given a solution u to (4), we write $U(n) = (u(n+1, E), u(n, E))^T$ for the associated 2-vector. Thus, $U(n) = T(n, 0; E)U(0)$ for every n. We define $\|U(n)\|^2 = |u(n)|^2 + |u(n+1)|^2$ and, as before,

$$\|U\|_L^2 = \sum_{n=1}^{[L]} \|U(n)\|^2 + (L - [L])\|U([L]+1)\|^2.$$

Lemma 2. *Suppose $p, q \in \mathbb{Z}_+$ are such that $p \geq q$ and $V(m+p) = V(m)$ for $1 \leq m \leq p+q$. Then, we have*

$$\|U\|_{2p+q}^2 \geq \left(1 + \left(\frac{1}{\max\{2, 2|\operatorname{tr} T(p,0;E)|\}}\right)^2\right) \|U\|_q^2$$

for every solution u to (4)*.*

In particular, if $|\operatorname{tr} T(p,0;E)| \leq 2$, then

$$\|U\|_{2p+q}^2 \geq \frac{17}{16}\, \|U\|_q^2$$

for every solution u to (4)*.*

Proof. As mentioned above, this lemma is known. However, for the convenience of the reader, we supply the short proof.

By the assumption, the cyclicity of the trace, and the Cayley-Hamilton theorem,

$$U(2p+m) - \operatorname{tr} T(p,0;E)\, U(p+m) + U(m) = 0$$

and hence

$$\max\{\|U(p+m)\|, \|U(2p+m)\|\} \geq \frac{1}{\max\{2, 2|\mathrm{tr}\, T(p,0;E)|\}} \|U(m)\|$$

for all $1 \leq m \leq q$. We can therefore proceed as follows,

$$\begin{aligned}
\|U\|_{2p+q}^2 &= \sum_{m=1}^{2p+q} \|U(m)\|^2 \\
&= \sum_{m=1}^{q} \|U(m)\|^2 + \sum_{m=q+1}^{2p+q} \|U(m)\|^2 \\
&\geq \sum_{m=1}^{q} \|U(m)\|^2 + \left(\tfrac{1}{\max\{2,2|\mathrm{tr}\, T(p,0;E)|\}}\right)^2 \sum_{m=1}^{q} \|U(m)\|^2 \\
&= \left(1 + \left(\tfrac{1}{\max\{2,2|\mathrm{tr}\, T(p,0;E)|\}}\right)^2\right) \|U\|_q^2.
\end{aligned}$$

This proves the assertion. □

With these tools at our disposal, we can prove the following scaling result for solutions along a sequence of the form $\{q_{5n+n_0}\}$.

Lemma 3. *For every $\lambda > 0$, $\omega \in (0,1)$ irrational, $E \in \sigma(H)$, and every normalized solution u of (4), we have*

$$\|U\|_{q_{n+5}}^2 \geq \frac{17}{16} \|U\|_{q_n}^2$$

for all $n \geq 0$.

Proof. By (51)–(52) and Lemma 2, we only need to produce two *consecutive* squares followed be a suitable prefix within five levels of the s_n-hierarchy.

Consider first the case $a_{n+5} \geq 2$:

$$\begin{aligned}
s_{n+5} &= s_{n+4}^2 s_{n+3} \ldots \\
&= s_{n+3}^2 s_n \ldots,
\end{aligned}$$

where one possibly has to use (51). This yields two consecutive squares and we can now apply Lemma 2 with trace bound 2.

Now, consider the case $a_{n+5} = 1$:

$$\begin{aligned}
s_{n+5} &= s_{n+4} s_{n+3} \\
&= s_{n+3}^{a_{n+4}} s_{n+2} s_{n+3} = s_{n+3}^{a_{n+4}+1} s_{n+1}^{a_{n+2}-1} s_n s_{n+1} \\
&= (s_{n+2}^{a_{n+3}} s_{n+1})^{a_{n+4}} s_{n+2} s_{n+3}.
\end{aligned}$$

Now, if $a_{n+2} \geq 2$, we are done. Otherwise, we apply (51) twice and find a suitable square. □

This shows that along the q_n scales, we find suitable exponential growth of solutions. If the q_n's have reasonable growth, this translates into nice bounds for all values of L.

Proposition 2. *Suppose $\lambda > 0$ and $\alpha \in (0,1)$ is irrational with $a_n \leq C$, $\omega = 0$. Consider the operator (1) with potential given by (13). Then, for every $E \in \sigma(H)$*

and every normalized real solution u of (4)*, the following hold true:*
(a) *With $k = (C+1)^{-6} \in (0,1)$ and $\gamma = 1/16$, we have for L large enough,*

$$\|U\|_L^2 \geq (1+\gamma)\|U\|_{kL}^2.$$

(b) *With*

$$\kappa = \frac{\log(\sqrt{17}/4)}{(C+1)^5} \tag{53}$$

and some (E *and u-independent*) *constant D, we have for $L \geq 1$,*

$$\|U\|_L \geq DL^\kappa.$$

Remark. "Large enough" L means, for example, $L \geq q_5$. If $\tilde{C} = \limsup a_n < C$, one can choose in (b) a larger κ (namely, with C in (53) replaced by $\tilde{C}$) if one requires the estimate only for $L \geq L_0$; L_0 depends on how close to the maximum possible value one wants to choose κ.

Proof. (a) Given L, define n by $q_n \leq L < q_{n+1}$. Then, by Lemma 3,

$$\begin{aligned}\|U\|_L^2 \geq \|U\|_{q_n}^2 &\geq \frac{17}{16}\|U\|_{q_{n-5}}^2 = (1+\gamma)\|U\|_{q_{n-5}}^2 \\ &\geq (1+\gamma)\|U\|_{kq_{n+1}}^2 \geq (1+\gamma)\|U\|_{kL}^2.\end{aligned}$$

(b) We know already that $\|U\|_{q_{5n}} \geq \mathrm{const} \cdot (\frac{\sqrt{17}}{4})^n$. Define n by $q_{5n} \leq L < q_{5(n+1)}$. Then

$$\begin{aligned}\|U\|_L \geq \|U\|_{q_{5n}} &\geq \mathrm{const} \cdot \left(\frac{\sqrt{17}}{4}\right)^n = \frac{4 \cdot \mathrm{const}}{\sqrt{17}} \left(\frac{\sqrt{17}}{4}\right)^{n+1} \\ &\geq \frac{4 \cdot \mathrm{const}}{\sqrt{17}} \left(q_{5(n+1)}\right)^\kappa \geq \frac{4 \cdot \mathrm{const}}{\sqrt{17}} L^\kappa,\end{aligned}$$

as claimed. □

Proof of Theorem 4. With $A = \sigma(H)$, we clearly have $\mu(A) > 0$, where, as before, μ denotes the spectral measure of δ_1 with respect to H. Thus, combining Propositions 1 and 2 with Theorem 3, we obtain the claimed lower bound (15) for $\beta_{\delta_1}^-(p)$. Note that $\|u\|_L$ and $\|U\|_L$ are, of course, comparable. □

Remark. We end this section with a brief discussion of the more general potentials

$$V(n) = \lambda \chi_{[1-\omega,1)}(n\omega + \theta \mod 1),\ n \in \mathbb{Z}, \tag{54}$$

where $\lambda > 0$, ω is an irrational number with bounded partial quotients, and $\theta \in [0,1)$. Essentially, these are the elements of the hulls generated by potentials of the form (13) and are the natural objects when studying these operators within the framework of ergodic families of operators; compare [CL, PF].

For the potentials in (54), we can perform an analysis almost completely parallel to the one above. The only difference is that it is not possible to obtain the exact analog of Lemma 3. Rather, the universal constant 17/16 has to be replaced by a λ-dependent constant that goes to 1 as $\lambda \to \infty$.

The net result is that we are able to prove

$$\beta_{\delta_1}^-(p) \geq \begin{cases} \frac{p(p+2\kappa)}{(p+1)(\alpha+\kappa+1/2)} & p \leq 2\alpha+1, \\ \frac{p}{\alpha+1} & p > 2\alpha+1 \end{cases}$$

with the same λ-dependent constants α and τ as in (17). In particular, our approach gives better dynamical results for potentials of the form (54) than previous ones. However, we are limited here to the class of ω's with bounded partial quotients, whereas [DKL] could work with the slightly larger class of bounded density numbers.

Appendix A. The Jitomirskaya-Last Method Revisited

The Jitomirskaya-Last inequality, (55) below, provides a link between two limiting procedures by a clever association of a length scale to a given small ε. This inequality immediately implies all results of Gilbert-Pearson theory [GP]. Moreover, it allows for a strengthening of this theory that is needed to study Hausdorff-dimensional properties of spectral measures (see, e.g., [D, DK, DKL, DLa, JL1, JL2, KKL, KLS, R, Z] for applications in this context). A variant of the Jitomirskaya-Last inequality was proven and further developed in [KKL]; see (56) and (57) below. These refinements were crucial in the approach to new dynamical bounds in [KKL]. In this appendix we recount these central results of Jitomirskaya-Last theory and provide simplified proofs for some of them.

Let us denote

$$\begin{aligned} a(L) &= ||u_{\pi/2}(E)||_L^2, \\ b(L) &= ||u_0(E)||_L^2, \\ d(L) &= \langle u_0(E), u_{\pi/2}(E)\rangle_L, \\ w^2(L) &= |||K(E)|||_L^2. \end{aligned}$$

From (24), we see that $(w(L))^2 = ab - d^2$. Of course, all these numbers also depend on E. We will leave this implicit throughout this section but remark that these quantities are associated with the energy E even in situations involving some additional energy E'.

For non-negative functions f and g, we write $f \sim g$ if we have both $f \gtrsim g$ and $g \gtrsim f$, that is,

$$f \sim g \; :\Leftrightarrow \; \exists C_1, C_2 > 0 \text{ such that } C_1 f(x) \le g(x) \le C_2 f(x) \text{ for all } x.$$

It was shown in [JL1] that

$$|F(E+i\varepsilon)|^2 \sim \frac{a(L_1(\varepsilon))}{b(L_1(\varepsilon)}, \tag{55}$$

where $L_1(\varepsilon)$ is defined by the equality

$$a(L_1(\varepsilon))b(L_1(\varepsilon)) = \frac{1}{4\varepsilon^2}.$$

Later on [KKL, Theorem 2.3], it was shown that

$$|F(E+i\varepsilon)|^2 \sim \frac{a(L_2(\varepsilon))}{b(L_2(\varepsilon)}, \tag{56}$$

where $L_2(\varepsilon)$ is defined by

$$w(L_2(\varepsilon)) = \frac{1}{\varepsilon}.$$

Although not stated explicitly, it follows also from [KKL, Theorem 2.3] that

$$\operatorname{Im} F(E+i\varepsilon) \sim \frac{w(L_2(\varepsilon))}{b(L_2(\varepsilon))}. \tag{57}$$

The constants $C_{1,2}$ in (55)–(57) are universal. The lower bound for $|F(z)|$ can be used to prove the singularity of the spectral measure and the upper bound to prove its continuity. In fact, when considering the continuity of the measure, only the upper bound on $\operatorname{Im} F(E+i\varepsilon)$ matters, since one can use the inequality

$$\mu([E-\varepsilon, E+\varepsilon]) \leq 2\varepsilon \operatorname{Im} F(E+i\varepsilon).$$

Another result proved in [KKL] that develops ideas of [JL1] concerns a lower bound for the inside probabilities $\sum_{n<L(T)} a(n,T)$ with some growing $L(T)$.

We shall present below some version of the Jitomirskaya-Last method which gives, in particular, a simplified proof of (57). We begin with the following technical result.

Lemma 4. *Suppose that $E, E' \in \mathbb{R}$ and $\varepsilon > 0$. Write $z = E' + i\varepsilon$. Then, for $L > 0$,*

$$\operatorname{Im} F(z) \geq \varepsilon ||u(z)||_L^2 \tag{58}$$

$$\geq \frac{\varepsilon}{(1+|z-E|w(L))^2}\left((\operatorname{Im} F(z))^2 b(L) + \frac{(w(L))^2}{b(L)}\right) \tag{59}$$

$$\geq \frac{2\varepsilon w(L)}{(1+|z'-E|w(L))^2}\operatorname{Im} F(z). \tag{60}$$

Proof. The inequality (58) is obvious (in fact, equality holds for $L=\infty$). From (25) and (26), we obtain

$$\begin{aligned}(1+|z-E|w(L))^2||u(z)||_L^2 &\geq a(L) + ((\operatorname{Im} F(z))^2 + (\operatorname{Re} F(z))^2)b(L) - 2\operatorname{Re} F(z)\, d(L)\\ &\geq (\operatorname{Im} F(z))^2 b(L) + (a(L)b(L) - d^2(L))/b(L)\\ &= (\operatorname{Im} F(z))^2 b(L) + w^2(L)/b(L),\end{aligned}$$

which implies (59). (The second step above uses $x^2 b - 2xd \geq -d^2/b$ for any $x \in \mathbb{R}$.) Finally, (60) follows from the elementary bound $bs^2 + w^2/b \geq 2ws,\ b > 0$. $\square$

As a first consequence of this lemma, one can obtain the equivalence (57) for $\operatorname{Im} F(E+i\varepsilon)$.

Proposition 3. *The following inequalities hold:*

$$\operatorname{Im} F(E+i\varepsilon) \leq \inf_L \frac{(1+\varepsilon w(L))^2}{\varepsilon b(L)} \leq \frac{4w(L_2(\varepsilon))}{b(L_2(\varepsilon))}, \tag{61}$$

$$\operatorname{Im} F(E+i\varepsilon) \geq \sup_L \frac{\varepsilon w^2(L)}{b(L)(1+\varepsilon w(L))^2} \geq \frac{w(L_2(\varepsilon))}{4b(L_2(\varepsilon))}. \tag{62}$$

Proof. The bounds (58)–(59) with $E' = E$ yield

$$\operatorname{Im} F(E+i\varepsilon) \geq \varepsilon \frac{(\operatorname{Im} F(E+i\varepsilon))^2 b(L)}{(1+\varepsilon w(L))^2}.$$

Thus,

$$\operatorname{Im} F(E+i\varepsilon) \leq \frac{(1+\varepsilon w(L))^2}{\varepsilon b(L)}$$

for any L and in particular for $L = L_2(\varepsilon)$. The bound (61) follows. On the other hand, again from (58)–(59),

$$\operatorname{Im} F(E+i\varepsilon) \geq \frac{\varepsilon w^2(L)}{b(L)(1+\varepsilon w(L))^2}$$

for any L, which yields (62). $\square$

Another consequence is the following result, which is Proposition 2.4 of [KKL]. Together with the Parseval identity, it yields a lower bound for the time-averaged inside probabilities; see Theorem 1.1 and its proof in [KKL]. The proof we give is simpler than the one in [KKL] and provides a better constant on the right-hand side (where we have $\sqrt{2}/4$, they have $(3-2\sqrt{2})/36$). Moreover, it is clear why should one take L large enough but not too large.

Proposition 4. *Suppose $E, E' \in \mathbb{R}$, $\varepsilon > 0$, and $|E - E'| < \varepsilon$. Write $z = E' + i\varepsilon$ and define $L_3(\varepsilon)$ by*

$$\sqrt{2}\varepsilon w(L_3(\varepsilon)) = 1.$$

Then

$$\varepsilon ||u(z)||^2_{L_3(\varepsilon)} \geq \frac{\sqrt{2}}{4} \operatorname{Im} F(z).$$

Remark. Note that both u and F are associated with $z = E' + i\varepsilon$, but the length scale $L_3(\varepsilon)$ is defined using quantities associated with energy E. This fact is important to the applications of this result; compare the remark after [KKL, Proposition 2.4] and the proof of [KKL, Theorem 1.1].

Proof. Clearly, $|z - E| \leq \sqrt{2}\varepsilon$. It follows from (59)–(60) that

$$\varepsilon ||u(z)||^2_L \geq \frac{2\varepsilon w(L)}{(1 + \sqrt{2}\varepsilon w(L))^2} \operatorname{Im} F(z)$$

for any L. Since the function $f(y) = \frac{2y}{(1+\sqrt{2}y)^2}$, $y > 0$, has its maximum at $y_0 = \frac{1}{\sqrt{2}}$, the result follows. $\square$

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