Square-Integrable Wave Packets from the Volkov Solutions

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July 29, 2004

Abstract

Rigorous mathematical proofs of some properties of the Volkov solutions are presented, which describe the motion of a relativistic charged Dirac particle in a classical, plane electromagnetic wave. The Volkov solutions are first rewritten in a convenient form, which clearly reveals some of the symmetries of the underlying Dirac equation. Assuming continuity and boundedness of the electromagnetic vector potential, it is shown how one may construct square-integrable wave packets from momentum distributions in the space $\mathcal{C}_0^\infty(\mathbb{R}^3)^4$. If, in addition, the vector potential is $\mathcal{C}^1$ and the derivative is bounded, these wave packets decay in space faster than any polynomial and fulfill the Dirac equation. The mapping which takes momentum distributions into wave packets is shown to be isometric with respect to the $L^2(\mathbb{R}^3)^4$ norm and may therefore be continuously extended to a mapping from $L^2(\mathbb{R}^3)^4$. For a momentum function in $L^1(\mathbb{R}^3)^4 \cap L^2(\mathbb{R}^3)^4$, an integral representation of this extension is presented.

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I. Introduction

There exist only few exact solutions of the Dirac equation, the relativistic equation which describes the motion of a charged particle with spin 1/2 in an external electromagnetic field. One important class was discovered by Volkov in 1935,[1] who treated the field of a classical, plane electromagnetic wave. Ever since then, the Volkov solutions have experienced a wide range of applications in the physics of interactions of electrons and positrons with a laser beam, of which we enumerate a few.

Brown and Kibble[2] as well as Gol'dman[3] studied the Compton scattering of a free electron in a high-intensity laser and used the Volkov solutions to calculate the cross section for this process. Nikishov and Ritus[4] and Narozhnyĭ *et al.*[5] analyzed the $\pi \to \mu + \nu$ decay in the presence of an electromagentic wave. The scattering of electrons by a scalar potential in the presence of an intense laser field was considered by Bergou and Varró.[6] Moreover, the production of electron-positron pairs in a laser was considered by Nikishov and Ritus[7] and by Narozhnyĭ *et al.*[5] In atomic physics, the Volkov solutions are used in the calculation of ionization rates in strong laser fields within the so-called strong-field approximation, see for example.[8,9] Szymanowski and coworkers examined laser-assisted scattering of electrons in a Coulomb potential[10] and in a Yukawa potential.[11] The cross section for pair-production may be enhanced in the presence of a bare atomic nucleus–a situation which was investigated by Yakovlev[12] and Müller *et al.*[13,14]

The Volkov solutions for given "asymptotic momenta", which will be written out explicitly in the subsequent section, are not square-integrable. Therefore a direct interpretation of a physical particle that is localized in space is not possible. In the case of the free Dirac equation, where the same problem occurs, square-integrable solutions–so-called *wave packets*–may be constructed from complex-valued functions in momentum space by multiplying these with the free plane-wave solutions and integrating over momentum space.[15,16] The construction of such wave packets from the Volkov solutions was suggested for the Klein-Gordon equation by Neville and Rohrlich[17] and for the Dirac equation by San Román and coworkers.[18–21]

Mathematically spoken, the integration over momentum space for the free Dirac equation represents nothing but the usual Fourier transformation with a minor modification, which accounts for the spinor structure of the Dirac equation.[16] This procedure automatically garuantees the "normalization" and "completeness" of the free plane-wave solutions, i. e. corresponding functions in momentum space and in spatial space have identical L^2 norms, and any L^2 function in spatial space may be represented as such a wave packet. In the literature, only peripheral attention has been paid to a study of the "normalization" and "completeness" properties for the Volkov solutions. There exist several treatments that offer physically motivated arguments for the normalization. Bergou and Varró[6] as well as Filipowicz,[22] for example, make use of delta functions and interchange the orders of divergent integrals in order to obtain the desired result. The standard textbook by Berestetskiĭ et al.[23] assumes electromagentic vector potentials that decrease far away from the origin, such that–according to the authors' reasoning–the normalization integral is practically not influenced by this potential.

However, there seem to be no rigorous mathematical proofs for the claims of "normalization" and "completeness". Nevertheless, many authors use these properties in their calculations more or less explicitly. For example, San Román and collaborators[18–21] determine the time evolution of a free electron wave packet in a laser beam by first calculating the momentum distribution of the initial wave packet at time $t = 0$ and then computing the resulting wave packets for positive times. The validity of this procedure relies on both "normalization" and "completeness". Furthermore, calculations of cross sections etc. with the help of the non-square-integrable Volkov solutions and "delta functions"–as performed, for example, in[2–14]–implicitly assume "normalization" and "completeness". To be more precise, an incorrect "normalization" would lead to wrong factors in the differential cross sections. On the other hand, the derivation of the total cross section requires an integration and summation of the differential cross section over *all* final states, i. e. these final states need to satisfy "completeness".

It is the purpose of this article to fill part of the gap indicated above and to provide a proof for the "normalization" of the Volkov solutions. In section II, we present the explicit form of the Volkov solutions. Some symmetries of the Dirac equation that will be useful in the further development are revealed in section III. Section IV is devoted to the construction of *regular wave packets* from $\mathcal{C}_0^\infty$ functions in momentum space in the case where the electromagnetic vector potential is continuous and bounded. Assuming also continuous differentiability of the vector potential and boundedness of its derivative, these regular wave packets are shown to decay rapidly in spatial space and to fulfill the Dirac equation. In section V, the central part of the paper, we state the main theorem asserting that the norms of a momentum distribution and of its corresponding wave packet are identical. The statement of the theorem is related to the physical concepts of "normalization to a delta function" and "orthogonality". The construction of wave packets from *any* L^2 functions in momentum space is exhibited in section VI, which is achieved by a continuous extension from the space $\mathcal{C}_0^\infty$. Finally, in Appendix A, we offer the proof for the main theorem from Section V.

II. The Volkov solutions

We treat a relativistic spin-1/2 particle with charge e and mass m that moves in the four-potential $A^\mu \equiv (A^0, \mathbf{A})$ of an electromagnetic field. The motion of the particle is described by the Dirac equation, which reads

$$\left(i\not{\partial} - e\not{A} - m\right)\psi = 0, \tag{1}$$

where we have used the Feynman slash notation

$$\not{a} := \gamma^\mu a_\mu \tag{2}$$

with the 4×4 Dirac matrices γ^μ.[15,16] These matrices fulfill

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}. \tag{3}$$

Here as in the entire article, we use natural units ($\hbar = c = 1$) and the signature $(+---)$ of the metric tensor $g_{\mu\nu}$. Scalar products shall be noted as

$$(ab) \equiv a^\mu b_\mu. \tag{4}$$

In Lorentz gauge,

$$(\partial A) \equiv \partial_\mu A^\mu(x) = 0, \tag{5}$$

a plane electromagnetic wave propagating in the direction of the unit vector $\mathbf{n} \in \mathbb{R}^3$ may be represented by the four-potential

$$A^\mu(x) = \tilde{A}^\mu(\xi)\ ,\ \xi = (nx)\ , \tag{6}$$

with the four-vector

$$n^\mu = (1, \mathbf{n}) \tag{7}$$

and the additional conditions

$$A^0 = \mathbf{A} \cdot \mathbf{n} \equiv 0. \tag{8}$$

The analytical Volkov solutions of the Dirac equation (1) with this potential are[6]

$$\psi_{\mathbf{p},r}(\mathbf{r}, t) = \mathbf{M}^{(\epsilon_r)}(\mathbf{p}, t - \mathbf{n}\cdot\mathbf{r})u_r(\mathbf{p})e^{iS^{(\epsilon_r)}(\mathbf{p},\mathbf{r},t)}, \tag{9}$$

where

$$\epsilon_r = \begin{cases} 1 & ,\quad r = 1, 2, \\ -1 & ,\quad r = 3, 4, \end{cases} \tag{10}$$

$$\mathbf{M}^{(\pm)}(\mathbf{p}, \xi) = \mathbf{1} \pm \frac{e\not{n}\tilde{\not{A}}(\xi)}{2(E(\mathbf{p}) - \mathbf{n}\cdot\mathbf{p})} \in \mathbb{C}^{4\times 4}\ , \tag{11}$$

$$S^{(\pm)}(\mathbf{p},\mathbf{r},t) = \pm\left(\mathbf{p}\cdot\mathbf{r} - E(\mathbf{p})t - \int_0^{t-\mathbf{n}\cdot\mathbf{r}} J_{\mathbf{p}}^{(\pm)}(\xi)\,d\xi\right), \tag{12}$$

$$\begin{aligned} J_{\mathbf{p}}^{(\pm)}(\xi) &= \frac{1}{E(\mathbf{p}) - \mathbf{n}\cdot\mathbf{p}}\left(\mp e\mathbf{p}\cdot\tilde{\mathbf{A}}(\xi) - \frac{e^2}{2}\tilde{A}(\xi)^2\right) \\ &= \frac{1}{E(\mathbf{p}) - \mathbf{n}\cdot\mathbf{p}}\left(\mp e\mathbf{p}\cdot\tilde{\mathbf{A}}(\xi) + \frac{e^2}{2}\left|\tilde{\mathbf{A}}(\xi)\right|^2\right), \end{aligned} \tag{13}$$

and

$$E(\mathbf{p}) = \sqrt{m^2 + |\mathbf{p}|^2} \tag{14}$$

is the energy of the particle. The spinors $u_r(\mathbf{p})$ are the free Dirac spinors, which can be written as columns of a certain matrix:[16]

$$\mathbf{u}(\mathbf{p}) \equiv (u_1(\mathbf{p})\ u_2(\mathbf{p})\ u_3(\mathbf{p})\ u_4(\mathbf{p})) = \sqrt{\frac{E(\mathbf{p})+m}{2E(\mathbf{p})}}\left(\mathbf{1} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E(\mathbf{p})+m}\right). \tag{15}$$

Here,

$$\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma} \tag{16}$$

is the three-vector of the Dirac α matrices. The functions $u_r(\mathbf{p})$ are $\mathcal{C}^\infty$ in the argument $\mathbf{p}$ and satisfy

$$\not{p}u_r(\mathbf{p}) = \epsilon_r m u_r(\mathbf{p}). \tag{17}$$

In addition, they obey the orthogonality relations

$$u_r(\epsilon_r\mathbf{p})^\dagger u_s(\epsilon_s\mathbf{p}) \equiv \langle u_r(\epsilon_r\mathbf{p}), u_s(\epsilon_s\mathbf{p})\rangle_{\mathbb{C}^4} = \delta_{r,s}. \tag{18}$$

We remark that equation (9) is actually not the original form that was published by Volkov,[1] who considered only the superposition of at most countably many unifrequent electromagentic waves. In his formulas, infinite series over the amplitudes and phases of the external potential occur. Brown and Kibble[2] and Nikishov and Ritus[7] seem to be the first to have written down the solutions in a form that involves a more or less arbitrary form of the electromagnetic potential which involves an integral in the phase factor. A recent derivation of the Volkov solutions can be found in,[18] where also the cases $r = 3, 4$ are treated. In addition, we mention that the authors cited above use $-\infty$ instead of 0 as the lower integration limit in the phase (12). Taking 0 (or any other finite number) does not require the vector potential to decay for large values $|\xi|$ of the argument ξ in order for the integral to converge.

For the sake of simplicity, we set $\mathbf{n} = \mathbf{e}_z$, so that

$$t - \mathbf{n} \cdot \mathbf{r} = t - z \tag{19}$$

and

$$E(\mathbf{p}) - \mathbf{n} \cdot \mathbf{p} = E(\mathbf{p}) - p_z. \tag{20}$$

The spatial part of the vector potential is then of the form

$$\mathbf{A}(\mathbf{r}, t) = \tilde{A}_x(t - z)\mathbf{e}_x + \tilde{A}_y(t - z)\mathbf{e}_y \equiv \tilde{\mathbf{A}}(t - z) \in \mathbb{R}^2. \tag{21}$$

Here as in the following, we make the identification

$$\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \tag{22}$$

in order to suppress an overduly complex notation. We also set

$$\mathbf{p}_\perp := (p_x, p_y) \in \mathbb{R}^2 \tag{23}$$

and

$$\mathbf{r}_\perp := (x, y) \in \mathbb{R}^2, \tag{24}$$

so that

$$\mathbf{p} \cdot \mathbf{A}(\mathbf{r}, t) = \mathbf{p}_\perp \cdot \tilde{\mathbf{A}}(t - z). \tag{25}$$

In addition, we will assume throughout that the functions $\tilde{A}_x, \tilde{A}_y : \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}_b$, i. e. continuous and bounded, and set

$$||\mathbf{A}|| := \sup_{\xi \in \mathbb{R}} \sqrt{\tilde{A}_x(\xi)^2 + \tilde{A}_y(\xi)^2} < \infty. \tag{26}$$

III. Symmetries of the Volkov solutions

The construction of the Volkov solutions (9) relies heavily on certain symmetries of the Dirac equation (1): The three differential operators ∂_x, ∂_y, and $\partial_t + \partial_z$ commute formally with $\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)$, which has already been noticed by Nikishov and Ritus.[7] These translational symmetries allow the separation of variables in the equation, which is inherited by the relations

$$\begin{aligned} -i\partial_x \psi_{\mathbf{p},r} &= \epsilon_r p_x \psi_{\mathbf{p},r}, && (27) \\ -i\partial_y \psi_{\mathbf{p},r} &= \epsilon_r p_y \psi_{\mathbf{p},r}, && (28) \\ -i\left(\partial_t + \partial_z\right)\psi_{\mathbf{p},r} &= \epsilon_r \left(p_z - E(\mathbf{p})\right)\psi_{\mathbf{p},r}. && (29) \end{aligned}$$

(For more symmetries of the Dirac equation (1) and a detailed group theoretical analysis, see the work of Janner and Janssen.[24]) Using the Hamiltonian form of (1), i. e.

$$i\partial_t\psi(\mathbf{r},t) = H(t)\psi(\mathbf{r},t) \tag{30}$$

with the formally self-adjoint Hamiltonian operator

$$H(t) = \boldsymbol{\alpha}\left(-i\boldsymbol{\nabla} - e\mathbf{A}\right) + \beta m, \tag{31}$$

we easily derive from (29) that for every t,

$$\left(H(t) + i\partial_z\right)\psi_{\mathbf{p},r} = \epsilon_r\left(E(\mathbf{p}) - p_z\right)\psi_{\mathbf{p},r} \equiv \epsilon_r\kappa(\mathbf{p})\psi_{\mathbf{p},r}, \tag{32}$$

where we have introduced

$$\kappa(\mathbf{p}) := E(\mathbf{p}) - p_z = \sqrt{m^2 + |\mathbf{p}|^2} - p_z > 0. \tag{33}$$

The Volkov solutions are thus simultaneous eigenfunctions of the three operators $-i\partial_x$, $-i\partial_y$, and $H(t) + i\partial_z$ with corresponding eigenvalues $\epsilon_r p_x$, $\epsilon_r p_y$, and $\epsilon_r\kappa(\mathbf{p})$, which do not depend on the chosen time t. It will be useful to define

$$\hat{\boldsymbol{\tau}} : \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}^+\ ,\ \mathbf{p} \mapsto \hat{\boldsymbol{\tau}}(\mathbf{p}) := \left(\mathbf{p}_\perp, \kappa(\mathbf{p})\right). \tag{34}$$

Note that the inverse is given by

$$\boldsymbol{\tau} := \hat{\boldsymbol{\tau}}^{-1} : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^3\ ,\ \boldsymbol{\tau}(\mathbf{p}_\perp, \lambda) := \left(\mathbf{p}_\perp, \frac{m^2 + |\mathbf{p}_\perp|^2 - \lambda^2}{2\lambda}\right). \tag{35}$$

The matrix part (11) and the phase (12) of the Volkov solutions may be rewritten in a simple form in terms of the eigenvalues:

$$\mathbf{M}^{(\epsilon_r)}(\mathbf{p},\xi) = \mathbf{1} + \frac{e\not{n}\tilde{\not{A}}(\xi)}{2\epsilon_r\kappa(\mathbf{p})} \in \mathbb{C}^{4\times 4} \tag{36}$$

and

$$\begin{aligned} S^{(\epsilon_r)}(\mathbf{p},\mathbf{r},t) &= \epsilon_r\mathbf{p}_\perp\cdot\mathbf{r}_\perp - \epsilon_r\kappa(\mathbf{p})t - \epsilon_r p_z(t-z) - \epsilon_r\int_0^{t-z} J_{\mathbf{p}}^{(\epsilon_r)}(\xi)\,d\xi \\ &= \epsilon_r\mathbf{p}_\perp\cdot\mathbf{r}_\perp - \epsilon_r\kappa(\mathbf{p})t - \epsilon_r\left[\left(p_z - \frac{|\mathbf{p}_\perp|^2}{2(E(\mathbf{p}) - p_z)}\right)(t-z)\right. \\ &\quad \left. + \frac{1}{2(E(\mathbf{p}) - p_z)}\int_0^{t-z}\left|\epsilon_r\mathbf{p}_\perp - e\tilde{\mathbf{A}}(\xi)\right|^2 d\xi\right] \\ &= \epsilon_r\mathbf{p}_\perp\cdot\mathbf{r}_\perp - \epsilon_r\kappa(\mathbf{p})t - \frac{1}{2}\left(\frac{m^2}{\epsilon_r\kappa(\mathbf{p})} - \epsilon_r\kappa(\mathbf{p})\right)(t-z) \\ &\quad - \frac{1}{2\epsilon_r\kappa(\mathbf{p})}\int_0^{t-z}\left|\epsilon_r\mathbf{p}_\perp - e\tilde{\mathbf{A}}(\xi)\right|^2 d\xi. \end{aligned} \tag{37}$$

A similar form of the phase $S^{(\epsilon_r)}$ has already been found by Gol'dman.[3] Defining

$$\pi_\perp(\xi, \mathbf{p}_\perp) := \begin{cases} \left(\dfrac{1}{\xi}\displaystyle\int_0^\xi \left|\mathbf{p}_\perp - e\tilde{\mathbf{A}}(\xi')\right|^2 d\xi'\right)^{1/2} & , \ \xi \neq 0, \\ \left|\mathbf{p}_\perp - e\tilde{\mathbf{A}}(0)\right| & , \ \xi = 0, \end{cases} \tag{38}$$

$$\boldsymbol{\nu}_\xi : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^3 \ , \ \boldsymbol{\nu}_\xi(\mathbf{p}_\perp, \lambda) := \left(\mathbf{p}_\perp, \frac{1}{2}\left[\frac{m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2}{\lambda} - \lambda\right]\right), \tag{39}$$

$$\begin{aligned} \mathbf{q}_\xi^{(\pm)}(\mathbf{p}) &:= \boldsymbol{\nu}_\xi\left(\pm\hat{\boldsymbol{\tau}}(\mathbf{p})\right) \\ &= \pm\left(\mathbf{p}_\perp, \frac{1}{2}\left[\frac{m^2 + \pi_\perp(\xi, \pm\mathbf{p}_\perp)^2}{\kappa(\mathbf{p})} - \kappa(\mathbf{p})\right]\right), \end{aligned} \tag{40}$$

and

$$\mathbf{s}_t(\mathbf{r}) := (x, y, z - t), \tag{41}$$

we may write this as

$$S^{(\epsilon_r)}(\mathbf{p}, \mathbf{r}, t) = \mathbf{q}_{t-z}^{(\epsilon_r)}(\mathbf{p}) \cdot \mathbf{s}_t(\mathbf{r}) - \epsilon_r \kappa(\mathbf{p}) t. \tag{42}$$

Note that

$$0 \leq \pi_\perp(\xi, \mathbf{p}_\perp) \leq |\mathbf{p}_\perp| + |e| \, ||\mathbf{A}|| \, . \tag{43}$$

Moreover, for $\tilde{\mathbf{A}} \equiv 0$, $\pi_\perp(\xi, \mathbf{p}_\perp) = |\mathbf{p}_\perp|$, so that $\pi_\perp(\xi, \mathbf{p}_\perp)$ may be thought of as a modified "perpendicular momentum".

IV. Regular wave packets

In this section, we shall construct square-integrable wave packets from the Volkov solutions. These wave packets will be of the form

$$\psi(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{4} \int_{\mathbb{R}^3} \phi_r(\mathbf{p}) \psi_{\mathbf{p},r}(\mathbf{r}, t) \, d^3p. \tag{44}$$

This integral representation was used, for example, by San Román *et al.*[18–21] in their calculations of the time evolution of wave packets. It will be our task to make sense of (44), i. e. we will ensure the existence of this integral for as many functions ϕ_r as possible and to show that the resulting wave packet ψ fulfills the Dirac equation (1).

In this section, we will start with momentum distributions ϕ_r that are in $\mathcal{C}_0^\infty$ and show that the wave packets defined by these decay faster than any power of $|\mathbf{r}|$ for $|\mathbf{r}| \to \infty$. In order to control the $\mathbf{p}$ dependence of the phase (42), we need to determine the inverses of the mappings $\mathbf{q}_\xi^{(\pm)}$ from (40):

$$\begin{aligned} \tilde{\mathbf{q}}_\xi^{(\pm)}(\tilde{\mathbf{p}}) &:= \left(\mathbf{q}_\xi^{(\pm)}\right)^{-1}(\tilde{\mathbf{p}}) \\ &= \boldsymbol{\tau}\left(\pm\boldsymbol{\nu}_\xi^{-1}(\tilde{\mathbf{p}})\right) \\ &= \boldsymbol{\tau}\left(\pm\left(\mathbf{p}_\perp, \sqrt{m^2+\pi_\perp\left(\xi,\pm\tilde{\mathbf{p}}_\perp\right)^2+\tilde{p}_z^2}\mp\tilde{p}_z\right)\right), \end{aligned} \tag{45}$$

which are $\mathcal{C}^\infty$. We claim:

Lemma 1. *For every compact set $K \subset \mathbb{R}^3$ and every $n_x, n_y, n_z \in \mathbb{N}_0$, there is a constant $c_{n_x,n_y,n_z}(K)$ satisfying*

$$\left|\left|\left(\frac{\partial}{\partial\tilde{p}_x}\right)^{n_x}\left(\frac{\partial}{\partial\tilde{p}_y}\right)^{n_y}\left(\frac{\partial}{\partial\tilde{p}_z}\right)^{n_z}\tilde{\mathbf{q}}_\xi^{(\pm)}(\tilde{\mathbf{p}})\right|\right| \le c_{n_x,n_y,n_z}(K)\;,\;\tilde{\mathbf{p}}\in K\;,\;\xi\in\mathbb{R}. \tag{46}$$

Proof. By the chain rule, one sees that the derivatives of $\tilde{\mathbf{q}}_\xi^{(\pm)}$ depend continuously on $\tilde{\mathbf{p}}$ as well as on $\pi_\perp(\xi,\tilde{\mathbf{p}}_\perp)^2$, $\partial_{\tilde{p}_x}\left[\pi_\perp(\xi,\tilde{\mathbf{p}}_\perp)\right]^2$, and $\partial_{\tilde{p}_y}\left[\pi_\perp(\xi,\tilde{\mathbf{p}}_\perp)\right]^2$. (Note that higher-order derivatives of $\left[\pi_\perp(\xi,\tilde{\mathbf{p}}_\perp)\right]^2$ vanish, see the definition (38).) The last three expressions are bounded for $\xi \in \mathbb{R}$ and $\tilde{\mathbf{p}}_\perp$ in a fixed compact set, which is due to the boundedness of $\tilde{\mathbf{A}}$ and (43). The derivatives of $\tilde{\mathbf{q}}_\xi^{(\pm)}$ are therefore bounded on the compact set K. □

We may now state the first main theorem, which allows the construction of L^2 wave packets:

Theorem 2. *Let $\phi \equiv (\phi_1,\ldots,\phi_4) \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4$.*

a) The wave packet ψ defined by

$$\psi(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}}\sum_{r=1}^{4}\int_{\mathbb{R}^3}\phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\,d^3p \tag{47}$$

is continuous in $\mathbf{r}$ and t.

b) For any $n \in \mathbb{N}_0$, there is a constant c_n such that

$$||\psi(\mathbf{r},t)||_{\mathbb{C}^4} \le c_n\frac{1+|t|^n}{1+|\mathbf{s}_t(\mathbf{r})|^n} = c_n\frac{1+|t|^n}{1+\left[x^2+y^2+(z-t)^2\right]^{n/2}}. \tag{48}$$

In particular $\psi(\cdot,t) \in L^p(\mathbb{R}^3)^4$ for all $t \in \mathbb{R}$ and $p \in [1,\infty]$.

Proof. *a)* It suffices to show the result for the summands r independently. Let $\mathbf{r}_n \to \mathbf{r}$ and $t_n \to t$. Then $\psi_{\mathbf{p},r}(\mathbf{r}_n, t_n) \to \psi_{\mathbf{p},r}(\mathbf{r}, t)$ by continuity of the Volkov solutions (9), so the integrand in (47) converges pointwise. Furthermore, since $\psi_{\mathbf{p},r}(\mathbf{r}, t)$ is bounded in $\mathbf{p} \in \operatorname{supp} \phi_r$ as well as in $\mathbf{r}$ and t, continuity of ψ is established by dominated convergence, see, e. g., Theorem I.11 in.[25]

b) We will show that for any $n_x, n_y, n_z \in \mathbb{N}_0$, there is a constant c_{n_x,n_y,n_z} such that

$$|x|^{n_x}|y|^{n_y}|z-t|^{n_z}||\psi(\mathbf{r},t)||_{\mathbb{C}^4} \leq c_{n_x,n_y,n_z}\left(1+|t|^{n_x+n_y+n_z}\right). \tag{49}$$

The estimate in (48) follows from this by using

$$\begin{aligned}\left[x^2+y^2+(z-t)^2\right]^{n/2} = \left(\sqrt{x^2+y^2+(z-t)^2}\right)^n \leq \\ \left(\sqrt{(|x|+|y|+|z-t|)^2}\right)^n = (|x|+|y|+|z-t|)^n,\end{aligned} \tag{50}$$

where the right-hand side is a polynomial of degree n in $|x|$, $|y|$, and $|z-t|$.

In order to prove the estimate (49), we shall apply the stationary-phase method (see, for example, Theorem XI.14 in[26]). We only need to look at the case $\mathbf{r} \equiv (x, y, z) \in \mathbb{R}^3$ with $x, y, z-t \neq 0$; for x, y, or $z-t$ equal to zero, (49) follows by continuity. Setting $\epsilon := \epsilon_r$ for abbreviation, we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\, d^3p \\
&= \int_{\mathbb{R}^3} \phi_r(\mathbf{p})\mathbf{M}^{(\epsilon)}(\mathbf{p},t-z)u_r(\mathbf{p}) \exp\left[i\left(\mathbf{q}^{(\epsilon)}_{t-z}(\mathbf{p})\cdot\mathbf{s}_t(\mathbf{r}) - \epsilon\kappa(\mathbf{p})t\right)\right] d^3p \\
&= \int_{\mathbb{R}^3} \phi_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))\mathbf{M}^{(\epsilon)}(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}),t-z)u_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})) \\
&\quad \times \exp\left[i\left(\tilde{\mathbf{p}}\cdot\mathbf{s}_t(\mathbf{r}) - \epsilon\kappa(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))t\right)\right] \left|\det\left(D\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})\right)\right| d^3\tilde{p} \\
&= \frac{(-i)^{n_x+n_y+n_z}}{x^{n_x}y^{n_y}(z-t)^{n_z}} \int_{\mathbb{R}^3} \phi_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))\mathbf{M}^{(\epsilon)}(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}),t-z)u_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})) \\
&\quad \times \exp\left(-i\epsilon\kappa(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))t\right) \left|\det\left(D\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})\right)\right| \\
&\quad \times \left[\left(\frac{\partial}{\partial\tilde{p}_x}\right)^{n_x}\left(\frac{\partial}{\partial\tilde{p}_y}\right)^{n_y}\left(\frac{\partial}{\partial\tilde{p}_z}\right)^{n_z} \exp\left(i\tilde{\mathbf{p}}\cdot\mathbf{s}_t(\mathbf{r})\right)\right] d^3\tilde{p},
\end{aligned} \tag{51}$$

where (42) has been used and the transformation formula for integrals has been applied. Since $\phi_r \in \mathcal{C}_0^\infty$ and all other $\tilde{\mathbf{p}}$ dependences are $\mathcal{C}^\infty$, we may perform integrations by parts and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\, d^3p \\
&= \frac{i^{n_x+n_y+n_z}}{x^{n_x}y^{n_y}(z-t)^{n_z}} \int_{\mathbb{R}^3} \exp\left(i\tilde{\mathbf{p}}\cdot\mathbf{s}_t(\mathbf{r})\right) \left(\frac{\partial}{\partial\tilde{p}_x}\right)^{n_x} \left(\frac{\partial}{\partial\tilde{p}_y}\right)^{n_y} \left(\frac{\partial}{\partial\tilde{p}_z}\right)^{n_z} \\
&\quad \Big[\phi_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))\mathbf{M}^{(\epsilon)}(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}),t-z)u_r(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})) \\
&\quad \times \exp\left(-i\epsilon\kappa(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))t\right)\left|\det\left(D\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}})\right)\right|\Big]\, d^3\tilde{p}. \qquad (52)
\end{aligned}$$

Observe that by the chain rule,

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial\tilde{p}_x}\right)^{m_x}\left(\frac{\partial}{\partial\tilde{p}_y}\right)^{m_y}\left(\frac{\partial}{\partial\tilde{p}_z}\right)^{m_z}\exp\left(-i\epsilon\kappa(\tilde{\mathbf{q}}^{(\epsilon)}_{t-z}(\tilde{\mathbf{p}}))t\right)\right| \leq \\
&\qquad d_{m_x,m_y,m_z}\left(1+|t|^{m_x+m_y+m_z}\right)\ ,\ \tilde{\mathbf{p}}\in\operatorname{supp}\phi_r\ , \qquad (53)
\end{aligned}$$

for some constants d_{m_x,m_y,m_z}. Again by the chain rule and by Lemma 1, the derivatives in (52) are uniformly bounded in $\mathbf{r}$, and we may conclude that the integral is bounded in $\mathbf{r}$ and that the t dependence is governed by (53). Thus,

$$|x|^{n_x}|y|^{n_y}|z-t|^{n_z}\left|\left|\int_{\mathbb{R}^3}\phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\,d^3p\right|\right|_{\mathbb{C}^4} \leq \tilde{c}_{r,n_x,n_y,n_z}\left(1+|t|^{n_x+n_y+n_z}\right) \qquad (54)$$

for some constants $\tilde{c}_{r,n_x,n_y,n_z}$, from which (49) follows after summation over r.
□

Theorem 2 allows us to define regular wave packets:

Definition 1. *For $t\in\mathbb{R}$, define*

$$\mathcal{F}_t : \mathcal{C}_0^\infty(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4\cap\mathcal{C}_\infty(\mathbb{R}^3)^4\ ,\ \phi\mapsto\mathcal{F}_t(\phi), \qquad (55)$$

where

$$\left(\mathcal{F}_t(\phi)\right)(\mathbf{r}) := \frac{1}{(2\pi)^{3/2}}\sum_{r=1}^{4}\int_{\mathbb{R}^3}\phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\,d^3p \qquad (56)$$

and $\mathcal{C}_\infty(\mathbb{R}^3)^4\subset\mathcal{C}_b(\mathbb{R}^3)^4$ is the space of continuous functions $\mathbb{R}^3\to\mathbb{C}^4$ that vanish at infinity. The elements of the range $\mathcal{F}_t\left(\mathcal{C}_0^\infty(\mathbb{R}^3)^4\right)$ will be called regular (Volkov) wave packets.

The following theorem assures that under weak assumptions, the wave packets defined above actually fulfill the Dirac equation in the Hamiltonian form (30). We only need the additional requirement that $\tilde{\mathbf{A}} \in \mathcal{C}_b^1$, i. e. $\tilde{\mathbf{A}}$ is continuously differentiable and both $\tilde{\mathbf{A}}$ and the derivative $\tilde{\mathbf{A}}'$ are bounded.

Theorem 3. *Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4$, and set*

$$\psi(\mathbf{r}, t) := \mathcal{F}_t(\phi)(\mathbf{r}). \tag{57}$$

Assume that $\tilde{\mathbf{A}} \in \mathcal{C}_b^1$. Then the following assertions hold:

a) $\psi \in \mathcal{C}^1(\mathbb{R}^4)^4$ and

$$i\partial_t \psi(\mathbf{r}, t) = H(t)\psi(\mathbf{r}, t) \equiv \left[\boldsymbol{\alpha}\left(-i\boldsymbol{\nabla} - e\tilde{\mathbf{A}}(t - z)\right) + \beta m\right]\psi(\mathbf{r}, t). \tag{58}$$

b) There are constants c_n, $n \in \mathbb{N}_0$, satisfying

$$||\left(H(t)\psi(\cdot, t)\right)(\mathbf{r})|| \leq c_n \frac{1 + |t|^n}{1 + [x^2 + y^2 + (z - t)^2]^{n/2}}. \tag{59}$$

Therefore, $H(t)\psi(\cdot, t) \in L^2(\mathbb{R}^3)^4$ for all t.

c) The mapping $\mathbb{R} \to L^2(\mathbb{R}^3)^4$, $t \mapsto \psi(\cdot, t)$, is $\mathcal{C}^1$, and

$$i\frac{d}{dt}\psi(\cdot, t) = H(t)\psi(\cdot, t), \tag{60}$$

where the derivative is taken in the space $L^2(\mathbb{R}^3)^4$.

Proof. a) Since the Volkov solutions $\psi_{\mathbf{p},r}$ are $\mathcal{C}^1$ and satisfy the Dirac equation (30), we only have to show that the differentiations in (58) may be interchanged with the integration in (56). This is possible since the derivatives of $\psi_{\mathbf{p},r}(\mathbf{r}, t)$ with respect to t, x, y, and z are continuous functions of $\mathbf{p}$, $\mathbf{r}$, and t and since the support of ϕ is compact.

b) Upon interchange of differentiation and integration and using the boundedness of the derivative $\tilde{\mathbf{A}}'$, this statement follows as in the proof of Theorem 2.

c) We need to show that for every t_0,

$$i \lim_{\tau \to 0} \frac{\psi(\cdot, t_0 + \tau) - \psi(\cdot, t_0)}{\tau} = H(t_0)\psi(\cdot, t_0), \tag{61}$$

where the limit is to be understood in the L^2 norm. In view of part a), pointwise convergence for every $\mathbf{r} \in \mathbb{R}^3$ is given. It remains to show that the left-hand side of (61) is bounded by a commen L^2 function for $|\tau| < 1$; the statement follows then by dominated convergence. For $\tau > 0$,

$$\begin{aligned}\left|\left|\frac{\psi(\mathbf{r},t_0+\tau)-\psi(\mathbf{r},t_0)}{\tau}\right|\right| &= \sup_{\sigma\in(0,\tau)} \left|\left|\,\partial_t\psi(\mathbf{r},t)|_{t=t_0+\sigma}\right|\right| \\ &= \sup_{\sigma\in(0,\tau)} \left|\left|\left(H(t_0+\sigma)\psi(\cdot,t_0+\sigma)\right)(\mathbf{r})\right|\right| \\ &\leq C \sup_{\sigma\in(0,\tau)} \frac{1+|t_0+\sigma|^2}{1+x^2+y^2+(z-(t_0+\sigma))^2}\end{aligned} \tag{62}$$

for a certain constant C, which is clear by part b). For $\tau < 0$, we obtain a similar result, in which $\sup_{\sigma\in(0,\tau)}$ is replaced by $\sup_{\sigma\in(\tau,0)}$. With little effort, we see that for $|\tau| \leq 1$,

$$\left|\left|\frac{\psi(\mathbf{r},t_0+\tau)-\psi(\mathbf{r},t_0)}{\tau}\right|\right| \leq C\,\frac{1+(|t_0|+1)^2}{1+x^2+y^2+\max\{|z-t_0|-1,0\}^2}, \tag{63}$$

where the right-hand side is an L^2 function of x, y, and z. This confirms (61) and concludes the proof. $\square$

For the usual Fourier transformation, the Riemann-Lebesgue lemma holds, which states that the Fourier transformation defined on $\mathcal{C}_0^\infty$ extends uniquely to a bounded map from L^1 into $\mathcal{C}_\infty$, the continuous functions vanishing at infinity, see Theorem IX.7 in.[27] In the case of the Volkov solutions, a similar result is true:

Theorem 4.

a) There is a constant C with

$$||\mathcal{F}_t(\phi)||_\infty \leq C\,||\phi||_{L^1(\mathbb{R}^3)^4} \tag{64}$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4 \subset L^1(\mathbb{R}^3)^4$ and all $t \in \mathbb{R}$.

b) The unique bounded extension $\mathcal{F}_t^{(1)}$ to $L^1(\mathbb{R}^3)^4$ fulfills

$$\mathcal{F}_t^{(1)} : L^1(\mathbb{R}^3)^4 \to \mathcal{C}_\infty(\mathbb{R}^3)^4, \tag{65}$$

i. e. $\mathcal{F}_t^{(1)}(\phi)$ vanishes at infinity for all $\phi \in L^1(\mathbb{R}^3)^4$.

c) For $\phi \in L^1(\mathbb{R}^3)^4$,

$$\left(\mathcal{F}_t^{(1)}(\phi)\right)(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\sum_{r=1}^{4}\int_{\mathbb{R}^3}\phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\,d^3p, \tag{66}$$

i. e. the same formula as in (56) holds for all $\phi \in L^1(\mathbb{R}^3)^4$.

Proof. a) In order to adapt the proof for the usual Fourier transformation in,[27] we have to check first that the Volkov solutions are bounded in $\mathbf{p}$. Since this is not directly clear from the defining equation (9), we shall first rewrite the Volkov solutions. To this end, observe that

$$\not{n}\tilde{\not{A}}(\xi) = -\tilde{\not{A}}(\xi)\not{n}. \tag{67}$$

Employing (17), we can write

$$\psi_{\mathbf{p},r}(\mathbf{r},t) = \tilde{\mathbf{M}}^{(\epsilon_r)}(\mathbf{p}, t-z) u_r(\mathbf{p}) e^{iS^{(\epsilon_r)}(\mathbf{p},\mathbf{r},t)} \tag{68}$$

with

$$\begin{aligned} \tilde{\mathbf{M}}^{(\pm)}(\mathbf{p},\xi) &:= \mathbf{1} \mp e\tilde{\not{A}}(\xi)\,\frac{E(\mathbf{p})\not{n} \pm m - \not{p}}{2E(\mathbf{p})(E(\mathbf{p})-p_z)} \\ &= \mathbf{1} \mp e\tilde{\not{A}}(\xi)\left(\frac{\pm m + p_x\gamma^1 + p_y\gamma^2}{2E(\mathbf{p})(E(\mathbf{p})-p_z)} - \frac{1}{2E(\mathbf{p})}\gamma^3\right). \end{aligned} \tag{69}$$

Using the equality

$$\frac{1}{E(\mathbf{p})(E(\mathbf{p})-p_z)} + \frac{1}{E(\mathbf{p})(E(\mathbf{p})+p_z)} = \frac{2E(\mathbf{p})}{E(\mathbf{p})(m^2+|\mathbf{p}_\perp|^2)} = \frac{2}{m^2+|\mathbf{p}_\perp|^2}, \tag{70}$$

we find the estimate

$$\frac{m^2+|\mathbf{p}_\perp|^2}{E(\mathbf{p})(E(\mathbf{p})-p_z)} \leq 2 \tag{71}$$

and conclude that $\tilde{\mathbf{M}}^{(\pm)}(\mathbf{p},\xi)\;\psi_{\mathbf{p},r}(\mathbf{r},t)$ are bounded with respect to $\mathbf{p}$ and ξ. Consequently, $\psi_{\mathbf{p},r}(\mathbf{r},t)$ is bounded in $\mathbf{p}$, $\mathbf{r}$, and t:

$$||\psi_{\mathbf{p},r}(\mathbf{r},t)|| \leq c_r \tag{72}$$

for some constants $c_r \geq 0$. For $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4$, this results in the estimate

$$\begin{aligned} ||(\mathcal{F}_t(\phi))(\mathbf{r})|| &\leq \frac{1}{(2\pi)^{3/2}}\sum_{r=1}^{4}\int_{\mathbb{R}^3}|\phi_r(\mathbf{p})|\,||\psi_{\mathbf{p},r}(\mathbf{r},t)||\;d^3p \\ &\leq \frac{1}{(2\pi)^{3/2}}\sum_{r=1}^{4} c_r\,||\phi_r||_{L^1}\,. \end{aligned} \tag{73}$$

b) This follows as for the Riemann-Lebesgue lemma: For a given $\phi \in L^1(\mathbb{R}^3)^4$, choose a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty(\mathbb{R}^3)^4$ with $\phi_n \to \phi$, $n\to\infty$, in the $L^1(\mathbb{R}^3)^4$ norm. By Theorem 2, $\mathcal{F}_t^{(1)}(\phi_n) = \mathcal{F}_t(\phi_n) \in \mathcal{C}_\infty(\mathbb{R}^3)^4$ for all

$n \in \mathbb{N}$. On the other hand, part a) of the present theorem ensures that $\mathcal{F}_t^{(1)}(\phi_n) \to \mathcal{F}_t^{(1)}(\phi)$, $n \to \infty$, uniformly on $\mathbb{R}^3$. Therefore, we also have $\mathcal{F}_t(\phi) \in \mathcal{C}_\infty(\mathbb{R}^3)^4$.

c) In view of (72), the integral on the right-hand side of (66) exists for $\phi \in L^1(\mathbb{R}^3)^4$. Taking again a sequence as in the proof of part b), the statement follows by dominated convergence. □

V. "Normalization"

The following theorem may be considered as the central point of this article:

Theorem 5. *Let $\tilde{\mathbf{A}} \in \mathcal{C}_b^1$. Then for every $t \in \mathbb{R}$, $\mathcal{F}_t$ is an isometry with respect to the L^2 norm, i. e. for every $\phi \in L^2(\mathbb{R}^3)^4$,*

$$||\mathcal{F}_t(\phi)||_{L^2(\mathbb{R}^3)^4} = ||\phi||_{L^2(\mathbb{R}^3)^4} . \tag{74}$$

As a simple conclusion, we note:

Corollary 6. *Under the assumptions of Theorem 5,*

$$\langle \mathcal{F}_t(\phi), \mathcal{F}_t(\chi) \rangle_{L^2(\mathbb{R}^3)^4} = \langle \phi, \chi \rangle_{L^2(\mathbb{R}^3)^4} \tag{75}$$

for every $\phi, \chi \in L^2(\mathbb{R}^3)^4$.

The lengthy proof of Theorem 5 will be deferred to Appendix A. Here we shall only verify Corollary 6 and then point out the connection with the "normalization to a delta function" and "orthogonality" of the Volkow solutions.

Proof of Corollary 6. This corollary is just the application of a general theorem valid for any isometry between two complex Hilbert spaces: We use the polarization identity

$$\langle x, y \rangle = \frac{1}{4} \left(||x+y||^2 + ||x-y||^2 + i\,||x-iy||^2 - i\,||x+iy||^2 \right) , \tag{76}$$

which follows from taking appropriate linear combinations from

$$||x+y||^2 = \langle x+y, x+y \rangle = ||x||^2 + ||y||^2 + 2\,\mathrm{Re}\,\langle x, y \rangle \tag{77}$$

and similar expressions for $||x-y||^2$, $||x+iy||^2$, and $||x-iy||^2$. For an isometry T (satisfying $||Tz|| = ||z||$ for all z), this leads to

$$\begin{aligned} &\langle Tx, Ty \rangle \\ &= \frac{1}{4} \left(||T(x+y)||^2 + ||T(x-y)||^2 + i\,||T(x-iy)||^2 - i\,||T(x+iy)||^2 \right) \\ &= \frac{1}{4} \left(||x+y||^2 + ||x-y||^2 + i\,||x-iy||^2 - i\,||x+iy||^2 \right) = \langle x, y \rangle . \end{aligned} \tag{78}$$

□

In the physical literature, e. g.,[7,13,23,28] the "normalization" and "orthogonality" property is laxly written as follows:

$$\int_{\mathbb{R}^3} \psi^\dagger_{\mathbf{p},r}(\mathbf{r},t)\psi_{\mathbf{p}',r'}(\mathbf{r},t)\, d^3r = (2\pi)^3 \delta_{r,r'}\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \tag{79}$$

where $\delta^{(3)}$ is the three-dimensional delta function. The property (79) is used for the verification of (75):

$$\begin{aligned}
\langle \mathcal{F}_t(\phi), \mathcal{F}_t(\chi)\rangle &= \frac{1}{(2\pi)^3}\sum_{r,r'=1}^{4}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\overline{\phi_r(\mathbf{p})}\chi_{r'}(\mathbf{p}')\right.\\
&\qquad \left.\times \langle \psi_{\mathbf{p},r}(\mathbf{r},t), \psi_{\mathbf{p}'r'}(\mathbf{r},t)\rangle_{\mathbb{C}^4}\; d^3p\, d^3p'\right) d^3r\\
&= \frac{1}{(2\pi)^3}\sum_{r,r'=1}^{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\overline{\phi_r(\mathbf{p})}\chi_{r'}(\mathbf{p}')\\
&\qquad \times\left(\int_{\mathbb{R}^3}\langle \psi_{\mathbf{p},r}(\mathbf{r},t), \psi_{\mathbf{p}'r'}(\mathbf{r},t)\rangle_{\mathbb{C}^4}\; d^3r\right) d^3p\, d^3p'\\
&= \sum_{r=1}^{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\overline{\phi_r(\mathbf{p})}\chi_r(\mathbf{p}')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\, d^3p\, d^3p'\\
&= \sum_{r=1}^{4}\int_{\mathbb{R}^3}\overline{\phi_r(\mathbf{p})}\chi_r(\mathbf{p})\, d^3p\\
&= \langle \phi, \chi\rangle_{L^2(\mathbb{R}^3)^4}\,.
\end{aligned} \tag{80}$$

This calculation certainly involves an unjustified interchange of integration orders and is therefore rather questionable from a mathematical point of view.

We mention that a different "normalization" has been employed by Neville and Rohrlich[17] and Bergou and Varró.[6] These authors use a scalar product that is calculated on those hyperplanes in Minkowski space-time where the vector potential $\mathbf{A}$ is constant, i. e. their considerations involve integrals of the form

$$\int_{\{(\mathbf{r},t)\in\mathbb{R}^4 : t-z=u\}} \cdots\, d^3r\, dt \tag{81}$$

for fixed $u \in \mathbb{R}$. However, the L^2 norms determined from this integration cannot be regarded as a normalization for the probability to find a particle. For a probability interpretation, one would have to integrate over space for a fixed time, at least for a certain specified inertial frame. That is, one would need a space-like hyperplane, whereas the integration in (81) is over a light-like hyperplane. Even though the light-like hyperplane and the related coordinates seem more adopted to the symmetry of the problem, an integration over a space-like hyperplane as in (75) or (79) appears much more "physical".

VI. Extension to L^2

Theorem 2 allows us to pronounce the following definition:

Definition 2. *In the case* $\tilde{\mathbf{A}} \in \mathcal{C}_b^1$, *let*

$$\mathcal{F}_t^{(2)} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4 \tag{82}$$

be the unique isometric extension of $\mathcal{F}_t$ *from Def. 1 to* $L^2(\mathbb{R}^3)^4$.

It will be profitable to have an integral representation of $\mathcal{F}_t^{(2)}$ for a larger subspace of L^2 than just $\mathcal{C}_0^\infty$. This is indeed possible for amplitude functions in $L^1 \cap L^2$:

Theorem 7. *For* $\phi \in L^1(\mathbb{R}^3)^4 \cap L^2(\mathbb{R}^3)^4$, *we have*

$$\mathcal{F}_t^{(2)}(\phi) = \mathcal{F}_t^{(1)}(\phi) \in \mathcal{C}_\infty(\mathbb{R}^3)^4, \tag{83}$$

or more explicitly,

$$\left(\mathcal{F}_t^{(2)}(\phi)\right)(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{4} \int_{\mathbb{R}^3} \phi_r(\mathbf{p}) \psi_{\mathbf{p},r}(\mathbf{r},t)\, d^3p. \tag{84}$$

Proof. We adapt here a proof from the usual Fourier transformation, which can be found in the book of Weidmann.[29] Let $(\phi_n)_n$ be a sequence in $\mathcal{C}_0^\infty(\mathbb{R}^3)^4$ with $\phi_n \to \phi$, $n \to \infty$, in $L^1(\mathbb{R}^3)^4$ and in $L^2(\mathbb{R}^3)^4$ simultaneously. Fix $t \in \mathbb{R}$ and set $\psi_n := \mathcal{F}_t(\phi_n)$. Theorem 2 yields $\psi_n \in L^1(\mathbb{R}^3)^4 \cap \mathcal{C}_\infty(\mathbb{R}^3)^4$. We have

$$\begin{aligned} \mathcal{F}_t^{(1)}(\phi) &= \lim_{n\to\infty} \psi_n \in \mathcal{C}_b(\mathbb{R}^3)^4 \quad \text{and} \\ \mathcal{F}_t^{(2)}(\phi) &= \lim_{n\to\infty} \psi_n \in L^2(\mathbb{R}^3)^4, \end{aligned} \tag{85}$$

where the limits are understood in $\mathcal{C}_b(\mathbb{R}^3)^4$ and $L^2(\mathbb{R}^3)^4$, respectively. Choose a subsequence $(\psi_{n_k})_k$ such that

$$\psi_{n_k}(\mathbf{r}) \to \mathcal{F}_t^{(2)}(\phi)(\mathbf{r})\ ,\ k \to \infty\ , \quad \text{a.e.} \tag{86}$$

But since

$$\psi_{n_k}(\mathbf{r}) \to \mathcal{F}_t^{(1)}(\phi)(\mathbf{r})\ ,\ k \to \infty, \tag{87}$$

for all $\mathbf{r}$, we conclude that

$$\mathcal{F}_t^{(1)}(\phi)(\mathbf{r}) = \mathcal{F}_t^{(2)}(\phi)(\mathbf{r}) \quad \text{a. e.} \tag{88}$$

□

Corollary 8. *Let $\phi \in L^2(\mathbb{R}^3)^4$ and $K_n := \{x \in \mathbb{R}^3 : ||x||_2 \leq n\}$. Then*

$$(\mathcal{F}_t^{(2)}\phi)(\mathbf{r}) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{4} \int_{K_n} \phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\, d^3p. \tag{89}$$

Proof. Again adapting a proof from Weidmann,[29] let χ_n be the characteristic function of the set K_n and fix $t \in \mathbb{R}$. Then $\chi_n\phi \in L^1(\mathbb{R}^3)^4 \cap L^2(\mathbb{R}^3)^4$, and we have $\chi_n\phi \to \phi$ in $L^2(\mathbb{R}^3)^4$. According to Theorem 7,

$$(\mathcal{F}_t^{(2)}(\chi_n\phi))(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{4} \int_{K_n} \phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)\, d^3p. \tag{90}$$

The claim follows from the fact that $\mathcal{F}_t^{(2)}(\chi_n\phi) \to \mathcal{F}_t^{(2)}(\phi)$ in $L^2(\mathbb{R}^3)^4$. □

VII. Summary and outlook

In this article, we have put some assertions concerning the relativistic Volkov solutions on solid mathematical ground. As a general assumption, we took the electromagnetic vector potential to be continuous and bounded. In Theorem 2, we have justified the construction of regular wave packets from Volkov solutions as used, for example, in[18] and have proved their rapid decay in spatial space. Theorem 3 shows that these wave packets satisfy the Dirac equation, as long as the vector potential is also $\mathcal{C}^1$ and has a bounded derivative. The "normalization" property is verified in the central Theorem 5 and allows the definition of wave packets for any L^2 function in momentum space. Finally, Theorem 7 provides a useful integral representation for the momentum-space functions contained in $L^1 \cap L^2$.

As mentioned in the introduction, it is still desirable to find a rigorous proof of the "completeness" of the Volkov solutions, i. e. for the hypothesis that for any initial wave packet $\psi_0 \in L^2(\mathbb{R}^3)^4$ at time $t = 0$, there is a momentum distribution $\phi \in L^2(\mathbb{R}^3)^4$ from which ψ_0 and also the time-dependent solution $\psi(\mathbf{r}, t)$ with $\psi(\cdot, 0) = \psi_0$ follows. In other words, the mapping $\mathcal{F}_t^{(2)}$

from (82) should be surjective and therefore unitary for a “reasonable” vector potential $\tilde{\mathbf{A}}$. A comparison with the field-free case $\tilde{\mathbf{A}} \equiv 0$ suggests that the inverse

$$\mathcal{G}_t := \left(\mathcal{F}_t^{(2)}\right)^{-1} \tag{91}$$

might be obtainable by the formula

$$\phi(\mathbf{p}) := \left(\mathcal{G}_t(\psi)\right)(\mathbf{p}) = \left(\frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{4} \int_{\mathbb{R}^3} \langle \psi_{\mathbf{p},r}(\mathbf{r},t), \psi(\mathbf{r}) \rangle \; d^3p \right)_{r=1,\dots,4}, \tag{92}$$

at least when $\psi \in \mathcal{C}_0^\infty$. The “physical” version of the above inversion formula (92) is

$$\sum_{r=1}^{4} \int_{\mathbb{R}^3} \psi^\dagger_{\mathbf{p},r}(\mathbf{r},t) \psi_{\mathbf{p},r}(\mathbf{r}',t) \; d^3p = (2\pi)^3 \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{93}$$

The validity of the inversion formula (92) would allow one to write the unitary propagator for the Dirac equation (30) as

$$U(t,t') = \mathcal{F}_t^{(2)} \circ \mathcal{G}_{t'}. \tag{94}$$

This conjecture is used by San Román and coworkers in[18–21] for the calculation of the time evolution of a Volkov wave packet. Having established (92), one might also try to understand on a more analytical level the squeezing and distortion of wave packets observed in.[18,19]

In order to prove (92), one should (i) verify the existence of the integral for $\psi \in \mathcal{C}_0^\infty$, (ii) show that $\phi \in L^1 \cap L^2$, and (iii) demonstrate with the help of Theorem 7 that

$$\psi = \mathcal{F}_t^{(2)} \left(\mathcal{G}_t(\psi)\right). \tag{95}$$

There are strong indications to the author that the assertions (i) and (ii) are correct for a bounded vector potential $\tilde{\mathbf{A}} \in \mathcal{C}^2$ with bounded first and second derivates. More details and results shall be supplied in a future publication.

Acknowledgment

The author is grateful to Carsten Müller for valuable discussions and for his pointing out several mistakes in a previous version of this manuscript.

Appendix A: Proof of Theorem 5

In this appendix, we provide the proof for the main Theorem 5.

Proof. Fix $t \in \mathbb{R}$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)^4$. We have to show that

$$||\mathcal{F}_t(\phi)||^2_{L^2(\mathbb{R}^3)^4} \equiv \frac{1}{(2\pi)^3} \sum_{r,r'=1}^{4} \int_{\mathbb{R}^3} I_{r,r'}(\mathbf{r})\; d^3r = ||\phi||^2_{L^2(\mathbb{R}^3)^4} \tag{A1}$$

with

$$I_{r,r'}(\mathbf{r}) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \overline{\phi_r(\mathbf{p})} \phi_{r'}(\mathbf{p}') \left\langle \psi_{\mathbf{p},r}(\mathbf{r},t), \psi_{\mathbf{p}'r'}(\mathbf{r},t) \right\rangle_{\mathbb{C}^4}\; d^3p\, d^3p', \tag{A2}$$

where the bar denotes complex conjugation. Note that all integrals appearing above exist since $\mathcal{F}_t(\phi) \in L^2(\mathbb{R}^3)^4$ by virtue of Theorem 2.

Inserting the Volkov solutions (9), we find

$$\begin{aligned} I_{r,r'}(\mathbf{r}) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \overline{\phi_r(\mathbf{p})} \phi_{r'}(\mathbf{p}') \exp\left[i \left(S^{(\epsilon_{r'})}(\mathbf{p}', \mathbf{r}, t) - S^{(\epsilon_r)}(\mathbf{p}, \mathbf{r}, t) \right) \right] \\ &\quad \times \left\langle \mathbf{M}^{(\epsilon_r)}(\mathbf{p}, t-z) u_r(\mathbf{p}), \mathbf{M}^{(\epsilon_{r'})}(\mathbf{p}', t-z) u_{r'}(\mathbf{p}') \right\rangle\; d^3p\, d^3p' \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \overline{\phi_r(\mathbf{p})} \phi_{r'}(\mathbf{p}') \exp\left[i \left(\epsilon_r \kappa(\mathbf{p}) - \epsilon_{r'} \kappa(\mathbf{p}') \right) t \right] \\ &\quad \times \left\langle \mathbf{M}^{(\epsilon_r)}(\mathbf{p}, t-z) u_r(\mathbf{p}), \mathbf{M}^{(\epsilon_{r'})}(\mathbf{p}', t-z) u_{r'}(\mathbf{p}') \right\rangle \\ &\quad \times \exp\left[i \left(\mathbf{q}^{(\epsilon_{r'})}_{t-z}(\mathbf{p}') - \mathbf{q}^{(\epsilon_r)}_{t-z}(\mathbf{p}) \right) \cdot \mathbf{s}_t(\mathbf{r}) \right]\; d^3p\, d^3p'. \end{aligned} \tag{A3}$$

Note that the only x and y dependence is contained in the term $\mathbf{s}_t(\mathbf{r})$. This fact is intimately connected with the validity of the eigenvalue equations (27) and (28). We thus perform the integration over these two variables $\mathbf{r}_\perp \equiv (x, y)$ first and obtain

$$\begin{aligned} \int_{\mathbb{R}^2} I_{r,r'}(\mathbf{r})\; d^2r_\perp &= \int_{\mathbb{R}^2} \Bigg(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left\langle \tilde{\phi}_r(\mathbf{p}_\perp, z), \tilde{\phi}_{r'}(\mathbf{p}'_\perp, z) \right\rangle_{\mathbb{C}^4} \\ &\qquad \times e^{i(\epsilon_{r'}\mathbf{p}'_\perp - \epsilon_r \mathbf{p}_\perp)\cdot \mathbf{r}_\perp}\; d^2p_\perp\, d^2p'_\perp \Bigg)\; d^2r_\perp, \end{aligned} \tag{A4}$$

where

$$\begin{aligned}\tilde{\phi}_r(\mathbf{p}_\perp, z) &:= \int_{\mathbb{R}} \phi_r(\mathbf{p})\psi_{\mathbf{p},r}(\mathbf{r},t)e^{-i\epsilon_r \mathbf{p}_\perp\cdot\mathbf{r}_\perp}\, dp_z \\ &= \int_{\mathbb{R}} \phi_r(\mathbf{p})\mathbf{M}^{(\epsilon_r)}(\mathbf{p}, t-z)u_r(\mathbf{p})e^{-i\epsilon_r\kappa(\mathbf{p})t} \\ &\quad \times \exp\left[i\frac{\epsilon_r}{2}\left(\frac{m^2+\pi_\perp(t-z,\epsilon_r\mathbf{p}_\perp)^2}{\kappa(\mathbf{p})} - \kappa(\mathbf{p})\right)(z-t)\right] dp_z.\end{aligned} \tag{A5}$$

Since $\phi_r \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, we see that for every z,

$$\tilde{\phi}_r(\cdot, z) \in \mathcal{C}_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2). \tag{A6}$$

Using Parseval's identity, we find

$$\int_{\mathbb{R}^2} I_{r,r'}(\mathbf{r})\, d^2r_\perp = (2\pi)^2 \int_{\mathbb{R}^2} \left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, z)\right\rangle_{\mathbb{C}^4}\, d^2p_\perp. \tag{A7}$$

Moreover, with the method applied in the proof of Theorem 2, we can even show that

$$\tilde{\phi}_r \in L^2(\mathbb{R}^3). \tag{A8}$$

We may therefore interchange the order of integrations by Fubini's theorem:

$$\begin{aligned}&||\mathcal{F}_t(\phi)||_{L^2(\mathbb{R}^3)^4} \\ &= \frac{1}{2\pi}\sum_{r,r'=1}^{4}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}\left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, z)\right\rangle_{\mathbb{C}^4}\, d^2p_\perp\right) dz \\ &= \frac{1}{2\pi}\sum_{r,r'=1}^{4}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}\left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, z)\right\rangle_{\mathbb{C}^4}\, dz\right) d^2p_\perp.\end{aligned} \tag{A9}$$

The identity (A1)–i. e. the statement of the theorem–is true if for every $\mathbf{p}_\perp$,

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}}\left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, z)\right\rangle_{\mathbb{C}^4}\, dz \equiv \\ &\qquad \frac{1}{2\pi}\left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, \cdot), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, \cdot)\right\rangle_{L^2} = \delta_{r,r'}\int_{\mathbb{R}}\left|\phi_r(\epsilon_r\mathbf{p}_\perp, p_z)\right|^2\, dp_z.\end{aligned} \tag{A10}$$

For the rest of the proof, we shall fix $\mathbf{p}_\perp$ and verify (A10). In preparation for exploiting the symmetry (32) of the Volkov solutions, we set

$$\tilde{\boldsymbol{\tau}}^{(\pm)}(\lambda) := \boldsymbol{\tau}\left(\pm\mathbf{p}_\perp, \lambda\right) = \left(\pm\mathbf{p}_\perp, \frac{m^2+|\mathbf{p}_\perp|^2-\lambda^2}{2\lambda}\right) \tag{A11}$$

and find

$$\tilde{E}(\lambda) := E\big(\tilde{\boldsymbol{\tau}}^{(\pm)}(\lambda)\big) = \left(m^2 + |\mathbf{p}_\perp|^2 + \frac{(m^2 + |\mathbf{p}_\perp|^2 - \lambda^2)^2}{4\lambda^2}\right)^{1/2} = \frac{m^2 + |\mathbf{p}_\perp|^2 + \lambda^2}{2\lambda} \quad \text{(A12)}$$

for $\lambda > 0$. In view of the equalities

$$\tilde{\boldsymbol{\tau}}^{(\pm)}(\kappa(\mathbf{p})) = (\pm\mathbf{p}_\perp, p_z) \quad \text{(A13)}$$

and

$$\kappa\left(\tilde{\boldsymbol{\tau}}^{(\pm)}(\lambda)\right) = \lambda\ ,\ \lambda > 0, \quad \text{(A14)}$$

we may write

$$\tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z) = \int_{\mathbb{R}} \phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\kappa(\mathbf{p}))\right) \tilde{\psi}_{\kappa(\mathbf{p}),r}(z)\, dp_z. \quad \text{(A15)}$$

Here,

$$\begin{aligned} \tilde{\psi}_{\lambda,r}(z) &:= \psi_{\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda),r}(\mathbf{r},t)e^{-i\mathbf{p}_\perp\cdot\mathbf{r}_\perp} \\ &= \left(\mathbf{1} + \frac{e\not{n}\tilde{\not{A}}(t-z)}{2\epsilon_r\lambda}\right)\tilde{u}_r(\lambda)e^{-i\epsilon_r\lambda t}\exp\left[i\epsilon_r(z-t)\tilde{\nu}_{t-z}(\lambda)\right], \end{aligned} \quad \text{(A16)}$$

where

$$\tilde{\nu}_\xi(\lambda) := \frac{1}{2}\left(\frac{m^2 + \pi_\perp(\xi,\mathbf{p}_\perp)^2}{\lambda} - \lambda\right) \quad \text{(A17)}$$

is the third component of $\boldsymbol{\nu}_\xi$ defined in (39). Furthermore,

$$\tilde{u}_r(\lambda) := u_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right), \quad \text{(A18)}$$

or, more explicitly,

$$\tilde{u}_r(\lambda) = \sqrt{\frac{\tilde{E}(\lambda)+m}{2\tilde{E}(\lambda)}}\left(\mathbf{1} + \frac{\boldsymbol{\alpha}\cdot\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)}{\tilde{E}(\lambda)+m}\right)_r, \quad \text{(A19)}$$

where the index r on the right-hand side denotes the rth column of the enclosed matrix.

The functions $\tilde{\psi}_{\lambda,r}$ fulfill an eigenvalue equation closely related to (32):

$$\Xi\tilde{\psi}_{\lambda,r}(z) = \epsilon_r\lambda\tilde{\psi}_{\lambda,r}(z), \quad \text{(A20)}$$

where

$$\begin{aligned}\Xi &:= \alpha_x p_x + \alpha_y p_y - i\alpha_z \partial_z - e\boldsymbol{\alpha}\tilde{\mathbf{A}}(t-z) + \beta m + i\partial_z \\ &= i\,(\mathbf{1} - \alpha_z)\,\partial_z + \alpha_x p_x + \alpha_y p_y - e\boldsymbol{\alpha}\tilde{\mathbf{A}}(t-z) + \beta m \qquad \text{(A21)}\end{aligned}$$

is a formally self-adjoint ordinary differential operator. The validity of (A20) is shown easily: From (A16) we infer that

$$\psi_{\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda),r}(\mathbf{r},t) = e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\tilde{\psi}_{\lambda,r}(z), \qquad \text{(A22)}$$

so that by (32),

$$\begin{aligned}(H(t) + i\partial_z)\left(e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\tilde{\psi}_{\lambda,r}(z)\right) &= \epsilon_r\kappa(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda))\psi_{\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda),r}(\mathbf{r},t) \\ &= \epsilon_r\lambda\, e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\tilde{\psi}_{\lambda,r}(z). \qquad \text{(A23)}\end{aligned}$$

Since

$$(-i\alpha_x\partial_x - i\alpha_y\partial_y)\left(e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\tilde{\psi}_{\lambda,r}(z)\right) = (\alpha_x p_x + \alpha_y p_y)\left(e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\tilde{\psi}_{\lambda,r}(z)\right), \qquad \text{(A24)}$$

equation (A20) follows immediately.

Performing the substitution

$$\sigma : p_z \mapsto \kappa(\mathbf{p}) \equiv \lambda, \qquad \text{(A25)}$$

we are led to

$$\tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, z) = \frac{1}{2}\int_0^\infty \phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right)\tilde{\psi}_{\lambda,r}(z)\left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\lambda^2} + 1\right)d\lambda, \qquad \text{(A26)}$$

where we have used

$$\frac{d}{d\lambda}\sigma^{-1}(\lambda) = \frac{d}{d\lambda}\frac{m^2 + |\mathbf{p}_\perp|^2 - \lambda^2}{2\lambda} = -\frac{1}{2}\left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\lambda^2} + 1\right). \qquad \text{(A27)}$$

An attempt to prove (A10) with $\tilde{\phi}_r$ from (A26) directly by interchanging the orders of integration fails because the assumptions of Fubini's theorem are not fulfilled. In order to apply the theorem, the term $\tilde{\psi}_{\lambda,r}(z)$ would have to decay for large values of $|z|$. As we shall see, we can avoid this problem by constructing an analytic continuation of (A16) to values $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, and then in a limiting procedure, the desired result will follow. For this purpose we need to examine the Dirac spinor $\tilde{u}_r$ because its definition (A19) might produce a singularity and a branch cut at certain values of λ. However, for $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\operatorname{Re} \tilde{E}(\lambda) = \left(\frac{m^2 + |\mathbf{p}_\perp|^2}{2|\lambda|^2} + \frac{1}{2} \right) \operatorname{Re} \lambda. \tag{A28}$$

So whenever $\operatorname{Re} \lambda > 0$, we also have $\operatorname{Re} \tilde{E}(\lambda) > 0$ and

$$\operatorname{Re} \frac{\tilde{E}(\lambda) + m}{2\tilde{E}(\lambda)} > 0. \tag{A29}$$

Consequently, we may extend the definition (A19) also to complex arguments λ with $\operatorname{Re} \lambda > 0$ if we choose the branch cut of the square root to lie on the negative real axis. Observing that both sides of (A20) are analytic with respect to λ and identical for real λ, we deduce the validity of this equation for *all* complex λ with $\operatorname{Re} \lambda > 0$.

Next inspect the function $\tilde{\nu}_\xi$ for complex arguments $\mu = \lambda + i\delta$ with $\lambda > 0$ and $\delta \in \mathbb{R}$:

$$\tilde{\nu}_\xi(\lambda + i\delta) = \frac{\lambda}{2} \left(\frac{m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2}{\lambda^2 + \delta^2} - 1 \right) - \frac{i\delta}{2} \left(\frac{m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2}{\lambda^2 + \delta^2} + 1 \right), \tag{A30}$$

so that we obtain for the exponential in (A16):

$$\begin{aligned} &\left| \exp\left[i\epsilon_r (z - t) \tilde{\nu}_{t-z}(\lambda + i\delta) \right] \right| \\ &\quad = \exp\left[-\epsilon_r (z - t) \operatorname{Im} \tilde{\nu}_{t-z}(\lambda + i\delta) \right] \\ &\quad = \exp\left[\epsilon_r (z - t) \, \frac{\delta}{2} \left(\frac{m^2 + \pi_\perp(t - z, \mathbf{p}_\perp)^2}{\lambda^2 + \delta^2} + 1 \right) \right]. \end{aligned} \tag{A31}$$

In addition,

$$\frac{\partial}{\partial \lambda} \tilde{\nu}_\xi(\lambda + i\delta) = \tilde{\nu}'_\xi(\lambda + i\delta) = -\frac{1}{2} \left(\frac{m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2}{(\lambda + i\delta)^2} + 1 \right), \tag{A32}$$

for which we find

$$\begin{aligned} \left| \tilde{\nu}'_\xi(\lambda + i\delta) \right| &= \frac{\left| m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2 + (\lambda + i\delta)^2 \right|}{2\left| \lambda + i\delta \right|^2} = \\ &\frac{\sqrt{\left[m^2 + \pi_\perp(\xi, \mathbf{p}_\perp)^2 + \lambda^2 - \delta^2 \right]^2 + 4\lambda^2\delta^2}}{2\left(\lambda^2 + \delta^2 \right)} \neq 0 \,, \ \lambda > 0. \end{aligned} \tag{A33}$$

So for $\lambda > 0$,

$$(z-t)\tilde{\psi}_{\lambda+i\delta,r}(z) = \left(\mathbf{1} + \frac{e\not{n}\tilde{\not{A}}(t-z)}{2\epsilon_r(\lambda+i\delta)}\right)\tilde{u}_r(\lambda+i\delta)e^{-i\epsilon_r(\lambda+i\delta)t} \times \frac{-i\epsilon_r}{\tilde{\nu}'_{t-z}(\lambda+i\delta)}\frac{\partial}{\partial\lambda}\exp\left[i\epsilon_r(z-t)\tilde{\nu}_{t-z}(\lambda+i\delta)\right]. \tag{A34}$$

Now define

$$\hat{\phi}_{r,\delta}(z) := \frac{1}{2}\int_0^\infty \phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right)\tilde{\psi}_{\lambda-i\delta\cdot\mathrm{sgn}[\epsilon_r(z-t)],r}(z)\left(\frac{m^2+|\mathbf{p}_\perp|^2}{\lambda^2}+1\right)d\lambda \tag{A35}$$

for $\delta \geq 0$. This integral exists because the integrand is continuous and $\phi_r \circ \tilde{\boldsymbol{\tau}}^{(\epsilon_r)}$ is compactly supported in $(0,\infty)$. Comparing with (A26), we recognize that for $\delta = 0$,

$$\hat{\phi}_{r,0} = \tilde{\phi}_r(\epsilon_r\mathbf{p},\cdot). \tag{A36}$$

The sign of the imaginary part in (A35) was chosen such that for $\delta > 0$, we observe exponential decay with z:

$$\begin{aligned}\left|\left|\tilde{\psi}_{\lambda-i\delta\cdot\mathrm{sgn}[\epsilon_r(z-t)],r}(z)\right|\right| &\leq \\ &\left(\mathbf{1} + \frac{|e|}{2\left|\lambda-i\delta\cdot\mathrm{sgn}\left[\epsilon_r(z-t)\right]\right|}\left|\left|\not{n}\tilde{\not{A}}(t-z)\right|\right|\right) \\ &\times \left|\left|\tilde{u}_r(\lambda-i\delta\cdot\mathrm{sgn}\left[\epsilon_r(z-t)\right]\right|\right| e^{-\epsilon_r\delta\cdot\mathrm{sgn}[\epsilon_r(z-t)]t} \\ &\times \exp\left[-\epsilon_r\left|z-t\right|\frac{\delta}{2}\left(\frac{m^2+\pi_\perp(t-z,\mathbf{p}_\perp)^2}{\lambda^2+\delta^2}+1\right)\right],\end{aligned} \tag{A37}$$

where (A31) was used. From this we obtain that

$$\left(z \mapsto \tilde{\psi}_{\lambda-i\delta\cdot\mathrm{sgn}[\epsilon_r(z-t)],r}(z)\right) \in L^2(\mathbb{R})^4\ ,\ \delta > 0. \tag{A38}$$

This enables us to inspect the decay properties of $\hat{\phi}_{r,\delta}$ for $|z-t| \to \infty$. We first see that

$$\left|\left|\hat{\phi}_{r,\delta}(z)\right|\right| \leq C_0(\phi)\ ,\ \delta \in [0,1], \tag{A39}$$

for a constant $C_0(\phi)$. This is clear since the integrand in (A35) is continuous with respect to δ as well as λ and again since $\phi_r \circ \tilde{\boldsymbol{\tau}}^{(\epsilon_r)}$ has compact support.

Furthermore, application of (A34) gives

$$
\begin{aligned}
(z-t)\hat{\phi}_{r,\delta}(z) \;=\; & \frac{1}{2}\int_0^\infty \phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right)\left(\frac{m^2+|\mathbf{p}_\perp|^2}{\lambda^2}+1\right) \\
& \times\left(\mathbf{1}+\frac{e\not{p}\tilde{\not{A}}(t-z)}{2\epsilon_r\left(\lambda-i\delta\cdot\operatorname{sgn}\left[\epsilon_r(z-t)\right]\right)}\right) \\
& \times\tilde{u}_r(\lambda-i\delta\cdot\operatorname{sgn}\left[\epsilon_r(z-t)\right])\exp\left[-i\epsilon_r\lambda t\right] \\
& \times\exp\left\{-\epsilon_r\delta\cdot\operatorname{sgn}\left[\epsilon_r(z-t)\right]t\right\}\frac{-i\epsilon_r}{\tilde{\nu}'_{t-z}(\lambda+i\delta)} \\
& \times\frac{\partial}{\partial\lambda}\exp\left[i\epsilon_r(z-t)\tilde{\nu}_{t-z}(\lambda-i\delta\cdot\operatorname{sgn}\left[\epsilon_r(z-t)\right])\right]d\lambda.
\end{aligned}
\tag{A40}
$$

Performing an integration by parts, we notice that also

$$|z-t|\left|\left|\hat{\phi}_{r,\delta}(z)\right|\right| \leq C_1(\phi)\ ,\ \delta\in[0,1]. \tag{A41}$$

Concludingly,

$$\left|\left|\hat{\phi}_{r,\delta}(z)\right|\right| \leq \frac{C_0(\phi)+C_1(\phi)}{1+|z-t|} \equiv \frac{C(\phi)}{1+|z-t|}\ ,\ \delta\in[0,1], \tag{A42}$$

so that the functions $\hat{\phi}_{r,\delta}$, $\delta\in[0,1]$, are uniformly bounded in L^2. In addition, we easily see by dominated convergence that

$$\forall z\in\mathbb{R}\ :\ \hat{\phi}_{r,\delta}(z)\to\hat{\phi}_{r,0}(z)=\tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp,z)\ ,\ \delta\to 0^+. \tag{A43}$$

Therefore,

$$\hat{\phi}_{r,\delta}\to\tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp,\cdot)\ ,\ \delta\to 0^+\ ,\ \text{in } L^2. \tag{A44}$$

The above considerations lead us to

$$\left\langle\tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp,\cdot),\tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp,\cdot)\right\rangle_{L^2} = \lim_{\delta\to 0^+}\left\langle\hat{\phi}_{r,\delta},\hat{\phi}_{r',\delta}\right\rangle_{L^2}, \tag{A45}$$

which necessitates the evaluation of the right-hand side for $\delta>0$. Due to the decay property (A37) that is fulfilled for positive δ, we may now invoke Fubini's theorem:

$$
\begin{aligned}
\left\langle\hat{\phi}_{r,\delta},\hat{\phi}_{r',\delta}\right\rangle_{L^2} \;=\; & \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right)}\,\phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_{r'})}(\lambda')\right) \\
& \times\left(\frac{m^2+|\mathbf{p}_\perp|^2}{\lambda^2}+1\right)\left(\frac{m^2+|\mathbf{p}_\perp|^2}{\lambda'^2}+1\right) \\
& \times J_{r,r',\delta}(\lambda,\lambda')\,d\lambda\,d\lambda',
\end{aligned}
\tag{A46}
$$

where

$$\begin{aligned}
J_{r,r',\delta}(\lambda,\lambda') &:= \int_{\mathbb{R}} \left\langle \tilde{\psi}_{\lambda - i\delta\cdot\mathrm{sgn}[\epsilon_r(z-t)],r}(z), \tilde{\psi}_{\lambda' - i\delta\cdot\mathrm{sgn}[\epsilon_{r'}(z-t)],r'}(z) \right\rangle dz \\
&= \int_{-\infty}^{t} \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&\quad + \int_{t}^{\infty} \left\langle \tilde{\psi}_{\lambda-i\epsilon_r\delta,r}(z), \tilde{\psi}_{\lambda'-i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&\equiv J^{(-)}_{r,r',\delta}(\lambda,\lambda') + J^{(+)}_{r,r',\delta}(\lambda,\lambda'). \qquad \text{(A47)}
\end{aligned}$$

These integrals can be evaluated by employing the eigenvalue equation (A20) as follows:

$$\begin{aligned}
\int_{-\infty}^{t} \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \Xi\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz &= \epsilon_{r'}\left(\lambda' + i\epsilon_{r'}\delta\right) J^{(-)}_{r,r',\delta}(\lambda,\lambda') \\
&= \left(\epsilon_{r'}\lambda' + i\delta\right) J^{(-)}_{r,r',\delta}(\lambda,\lambda') \qquad \text{(A48)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{t} \left\langle \Xi\tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle &= \overline{\epsilon_r\left(\lambda + i\epsilon_r\delta\right)} J^{(-)}_{r,r',\delta}(\lambda,\lambda') \\
&= \left(\epsilon_r\lambda - i\delta\right) J^{(-)}_{r,r',\delta}(\lambda,\lambda'). \qquad \text{(A49)}
\end{aligned}$$

Remembering the formal self-adjointness of Ξ and performing an integration by parts, we see that

$$\begin{aligned}
&\int_{-\infty}^{t} \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \Xi\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&- \int_{-\infty}^{t} \left\langle \Xi\tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&= \int_{-\infty}^{t} \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), i\left(\mathbf{1} - \alpha_z\right)\partial_z \tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&\quad - \int_{-\infty}^{t} \left\langle i\left(\mathbf{1} - \alpha_z\right)\partial_z \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(z), \tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(z) \right\rangle dz \\
&= i\left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), \left(\mathbf{1} - \alpha_z\right)\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t) \right\rangle. \qquad \text{(A50)}
\end{aligned}$$

From the last three equations we derive that

$$J^{(-)}_{r,r',\delta}(\lambda,\lambda') = \frac{i}{\epsilon_{r'}\lambda' - \epsilon_r\lambda + 2i\delta} \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), \left(\mathbf{1} - \alpha_z\right)\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t) \right\rangle. \quad \text{(A51)}$$

In a similar way, one may show that

$$J^{(+)}_{r,r',\delta}(\lambda,\lambda') = -\frac{i}{\epsilon_{r'}\lambda' - \epsilon_r\lambda_r - 2i\delta}\left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t)\right\rangle, \tag{A52}$$

so that

$$\begin{aligned} & J_{r,r',\delta}(\lambda,\lambda') \\ &= \left(\frac{i}{\epsilon_{r'}\lambda' - \epsilon_r\lambda_r + 2i\delta} - \frac{i}{\epsilon_{r'}\lambda' - \epsilon_r\lambda_r - 2i\delta}\right) \\ &\quad \times \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t)\right\rangle \\ &= \frac{4\delta}{(\epsilon_r\lambda - \epsilon_{r'}\lambda')^2 + 4\delta^2}\left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t)\right\rangle. \end{aligned} \tag{A53}$$

This leads to

$$\begin{aligned} \left\langle \hat{\phi}_{r,\delta}, \hat{\phi}_{r',\delta}\right\rangle_{L^2} &= \frac{\pi}{2}\int_0^\infty\int_0^\infty \eta_\delta(\epsilon_r\lambda - \epsilon_{r'}\lambda')\,\overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_r)}(\lambda)\right)}\,\phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon_{r'})}(\lambda')\right) \\ &\quad \times \left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\lambda^2} + 1\right)\left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\lambda'^2} + 1\right) \\ &\quad \times \left\langle \tilde{\psi}_{\lambda+i\epsilon_r\delta,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda'+i\epsilon_{r'}\delta,r'}(t)\right\rangle\, d\lambda\, d\lambda' \end{aligned} \tag{A54}$$

with

$$\eta_\delta(x) := \frac{2}{\pi}\frac{\delta}{x^2 + 4\delta^2}. \tag{A55}$$

The integral in (A54) is of the form

$$\left\langle \hat{\phi}_{r,\delta}, \hat{\phi}_{r',\delta}\right\rangle_{L^2} = \frac{\pi}{2}\int_0^\infty\int_0^\infty \eta_\delta(\epsilon_r\lambda - \epsilon_{r'}\lambda') f(\delta,\lambda,\lambda')\, d\lambda\, d\lambda' \tag{A56}$$

with a function f depending continuously on δ, λ, and λ'. The restriction $f|_{[0,1]\times[0,\infty)\times[0,\infty)}$ has compact support in $[0,1]\times(0,\infty)\times(0,\infty)$. The trivial extension to $[0,1]\times\mathbb{R}^2$, i. e.

$$\tilde{f}(\delta,\lambda,\lambda') := \begin{cases} f(\delta,\lambda,\lambda') & ,\lambda,\lambda' > 0, \\ 0 & ,\text{otherwise}, \end{cases} \tag{A57}$$

is therefore continuous, and we may write

$$\begin{aligned} \left\langle \hat{\phi}_{r,\delta}, \hat{\phi}_{r',\delta}\right\rangle_{L^2} &= \frac{\pi}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\delta(\epsilon_r\lambda - \epsilon_{r'}\lambda')\tilde{f}(\delta,\lambda,\lambda')\, d\lambda\, d\lambda' \\ &= \frac{\pi}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\delta(\lambda - \lambda')\tilde{f}(\delta,\epsilon_r\lambda,\epsilon_{r'}\lambda')\, d\lambda\, d\lambda'. \end{aligned} \tag{A58}$$

We are required to determine the limit of (A54) for $\delta \to 0^+$, see (A45). This can be accomplished with the help of Lemma 9 below. The result is

$$\lim_{\delta \to 0^+} \left\langle \hat{\phi}_{r,\delta}, \hat{\phi}_{r',\delta} \right\rangle_{L^2} = \frac{\pi}{2} \int_{\mathbb{R}} \tilde{f}(0, \epsilon_r \lambda, \epsilon_{r'} \lambda)\, d\lambda. \tag{A59}$$

Owing to (A45) and (A57), we have

$$\left\langle \tilde{\phi}_r(\epsilon_r \mathbf{p}_\perp, \cdot), \tilde{\phi}_{r'}(\epsilon_{r'} \mathbf{p}_\perp, \cdot) \right\rangle_{L^2} = \lim_{\delta \to 0^+} \left\langle \hat{\phi}_{r,\delta}, \hat{\phi}_{r',\delta} \right\rangle_{L^2} = 0 \tag{A60}$$

for $\epsilon_r \neq \epsilon_{r'}$, which validates (A10) in this case.

Now assume $\epsilon_r = \epsilon_{r'} =: \epsilon$. Then again by (A45) and Lemma 9,

$$\begin{aligned}
&\left\langle \tilde{\phi}_r(\epsilon_r \mathbf{p}_\perp, \cdot), \tilde{\phi}_{r'}(\epsilon_{r'} \mathbf{p}_\perp, \cdot) \right\rangle_{L^2} \\
&= \frac{\pi}{2} \int_0^\infty \overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\lambda)\right)}\, \phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\lambda)\right) \left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\lambda^2} + 1\right)^2 \\
&\quad \times \left\langle \tilde{\psi}_{\lambda,r}(t), (\mathbf{1} - \alpha_z)\, \tilde{\psi}_{\lambda,r'}(t) \right\rangle d\lambda \\
&= \pi \int_{\mathbb{R}} \overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right)}\, \phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right) \left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\kappa(\mathbf{p})^2} + 1\right) \\
&\quad \times \left\langle \tilde{\psi}_{\kappa(\mathbf{p}),r}(t), (\mathbf{1} - \alpha_z)\, \tilde{\psi}_{\kappa(\mathbf{p}),r'}(t) \right\rangle dp_z,
\end{aligned} \tag{A61}$$

where we have substituted back from λ to p_z. By virtue of the identities

$$\mathbf{1} - \alpha_z = \gamma^0 \left(\gamma^0 - \gamma^3\right) = \gamma^0 \not{n}, \tag{A62}$$

which is a hermitean matrix, and

$$\not{n}^2 = n^2 = 0, \tag{A63}$$

we find

$$
\begin{aligned}
&\left\langle \tilde{\psi}_{\lambda,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda,r'}(t)\right\rangle \\
&= \left\langle \left(\mathbf{1}+\frac{e\not{n}\tilde{\not{A}}(0)}{2\epsilon\lambda}\right)\tilde{u}_r(\lambda), \gamma^0\not{n}\left(\mathbf{1}+\frac{e\not{n}\tilde{\not{A}}(0)}{2\epsilon\lambda}\right)\tilde{u}_{r'}(\lambda)\right\rangle \\
&= \left\langle \left(\mathbf{1}+\frac{e\not{n}\tilde{\not{A}}(0)}{2\epsilon\lambda}\right)\tilde{u}_r(\lambda), \gamma^0\not{n}\tilde{u}_{r'}(\lambda)\right\rangle \\
&= \left\langle \gamma^0\not{n}\left(\mathbf{1}+\frac{e\not{n}\tilde{\not{A}}(0)}{2\epsilon\lambda}\right)\tilde{u}_r(\lambda), \tilde{u}_{r'}(\lambda)\right\rangle \\
&= \left\langle \gamma^0\not{n}\tilde{u}_r(\lambda), \tilde{u}_{r'}(\lambda)\right\rangle \\
&= \left\langle \tilde{u}_r(\lambda), \gamma^0\not{n}\tilde{u}_{r'}(\lambda)\right\rangle \\
&= \left\langle u_r(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\lambda)), \gamma^0\not{n}u_{r'}(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\lambda))\right\rangle. \qquad \text{(A64)}
\end{aligned}
$$

In order to evaluate this scalar product, fix λ and set $\mathbf{p} := \tilde{\boldsymbol{\tau}}^{(\epsilon)}(\lambda)$. We will make use of the 4×4 matrices[15]

$$\Lambda_{\pm}(\mathbf{p}) := \frac{\pm\not{p}+m}{2m} \equiv \frac{\pm(\gamma^0 E(\mathbf{p}) - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z) + m}{2m}, \qquad \text{(A65)}$$

which obey

$$\Lambda_{\epsilon_r}(\mathbf{p})u_r(\mathbf{p}) = u_r(\mathbf{p}), \qquad \text{(A66)}$$

and the fact that for any four-vector a,

$$\not{a}^\dagger = \gamma^0\not{a}\gamma^0. \qquad \text{(A67)}$$

This gives

$$
\begin{aligned}
\left\langle \tilde{\psi}_{\lambda,r}(t), (\mathbf{1}-\alpha_z)\,\tilde{\psi}_{\lambda,r'}(t)\right\rangle &= \left\langle \Lambda_\epsilon(\mathbf{p})u_r(\mathbf{p}), \gamma^0\not{n}\Lambda_\epsilon(\mathbf{p})u_{r'}(\mathbf{p})\right\rangle \\
&= \left\langle u_r(\mathbf{p}), \gamma^0\Lambda_\epsilon(\mathbf{p})\not{n}\Lambda_\epsilon(\mathbf{p})u_{r'}(\mathbf{p})\right\rangle \\
&= \left\langle u_r(\mathbf{p}), \gamma^0 M u_{r'}(\mathbf{p})\right\rangle \qquad \text{(A68)}
\end{aligned}
$$

with the 4×4 matrix

$$
\begin{aligned}
M &:= \Lambda_\epsilon(\mathbf{p})\not{n}\Lambda_\epsilon(\mathbf{p}) \\
&= \frac{\epsilon\not{p}+m}{2m}\not{n}\frac{\epsilon\not{p}+m}{2m} \\
&= \frac{1}{4m^2}\left[\not{p}\not{n}\not{p} + \epsilon m\left(\not{p}\not{n}+\not{n}\not{p}\right) + m^2\not{n}\right]. \qquad \text{(A69)}
\end{aligned}
$$

Using

$$\not{a}\not{b} + \not{b}\not{a} = 2(ab)\mathbf{1}, \tag{A70}$$

which is valid for any four-vectors a and b of real numbers, and

$$\not{p}^2 = p^2 = m^2, \tag{A71}$$

we continue:

$$M = \frac{1}{4m^2}\left[2(np)\not{p} - m^2\not{n} + 2\epsilon m(np) + m^2\not{n}\right] = \frac{\epsilon(np)}{m}\Lambda_\epsilon(\mathbf{p}) = \frac{\epsilon\kappa(\mathbf{p})}{m}\Lambda_\epsilon(\mathbf{p}). \tag{A72}$$

Hence,

$$\begin{aligned}\left\langle \tilde{\psi}_{\lambda,r}(t), (\mathbf{1} - \alpha_z)\, \tilde{\psi}_{\lambda,r'}(t)\right\rangle = \frac{\epsilon\kappa(\mathbf{p})}{m}\left\langle u_r(\mathbf{p}), \gamma^0\Lambda_\epsilon(\mathbf{p})u_{r'}(\mathbf{p})\right\rangle = \\ \frac{\epsilon\kappa(\mathbf{p})}{m}\left\langle u_r(\mathbf{p}), \gamma^0 u_{r'}(\mathbf{p})\right\rangle = \delta_{r,r'}\frac{\kappa(\mathbf{p})}{E(\mathbf{p})},\end{aligned} \tag{A73}$$

where

$$\left\langle u_r(\mathbf{p}), \gamma^0 u_{r'}(\mathbf{p})\right\rangle \equiv u_r(\mathbf{p})^\dagger\gamma^0 u_{r'}(\mathbf{p}) = \epsilon_r\delta_{r,r'}\frac{m}{E(\mathbf{p})} \tag{A74}$$

was exploited.[15] Therefore, we obtain from (A61):

$$\begin{aligned}&\left\langle \tilde{\phi}_r(\epsilon_r\mathbf{p}_\perp, \cdot), \tilde{\phi}_{r'}(\epsilon_{r'}\mathbf{p}_\perp, \cdot)\right\rangle_{L^2} \\ &= \delta_{r,r'}\,\pi \int_{\mathbb{R}} \overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right)}\,\phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right) \\ &\quad\times\left(\frac{m^2 + |\mathbf{p}_\perp|^2}{\kappa(\mathbf{p})^2} + 1\right)\frac{\kappa(\mathbf{p})}{E(\mathbf{p})}\,dp_z \\ &= 2\pi\delta_{r,r'}\int_{\mathbb{R}} \overline{\phi_r\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right)}\,\phi_{r'}\left(\tilde{\boldsymbol{\tau}}^{(\epsilon)}(\kappa(\mathbf{p}))\right)\,dp_z \\ &= 2\pi\delta_{r,r'}\int_{\mathbb{R}} \left|\phi_r(\epsilon_r\mathbf{p}_\perp, p_z)\right|^2\,dp_z,\end{aligned} \tag{A75}$$

which confirms (A10) also in the case $\epsilon_r = \epsilon_{r'}$. This completes the proof. □

We are left with showing:

Lemma 9. *Let* $f \in \mathcal{C}_0\left([0,1]\times\mathbb{R}^2, \mathbb{C}\right)$. *Then*

$$I_\delta := \int_{\mathbb{R}^2} \eta_\delta(x - x')f(\delta, x, x')\,dx\,dx' \to \int_{\mathbb{R}} f(0,x,x)\,dx\ ,\ \delta \to 0^+, \tag{A76}$$

with η_δ *from* (A55).

Proof. We first collect some properties of the functions η_δ:

$$\forall \delta > 0\,,\; x \in \mathbb{R} \;:\; \eta_\delta(x) > 0, \tag{A77}$$

$$\forall \delta > 0 \;:\; \int_{\mathbb{R}} \eta_\delta(x)\,dx = 1, \tag{A78}$$

and

$$\forall \gamma > 0 \;:\; \int_{\mathbb{R}\setminus[-\gamma,\gamma]} \eta_\delta(x)\,dx \to 0\,,\;\; \delta \to 0^+. \tag{A79}$$

Set

$$g(\delta, x) := \int_{\mathbb{R}} \eta_\delta(x - x') f(\delta, x, x')\,dx' \tag{A80}$$

for $\delta > 0$, so that

$$I_\delta = \int_{\mathbb{R}} g(\delta, x)\;dx. \tag{A81}$$

The compactness of $\operatorname{supp} f$ implies the existence of a compact interval $[a, b]$ such that

$$g(\delta, x) = 0 \quad \text{for} \quad \delta > 0\,,\; x \notin [a, b]. \tag{A82}$$

Therefore,

$$I_\delta = \int_a^b g(\delta, x)\;dx. \tag{A83}$$

Note that

$$|g(\delta, x)| \le \int_{\mathbb{R}} \eta_\delta(x - x')\,|f(\delta, x, x')|\;dx' \le ||f||_\infty < \infty. \tag{A84}$$

We claim that

$$\forall x \in [a, b] \;:\; \lim_{\delta \to 0^+} g(\delta, x) = f(0, x, x). \tag{A85}$$

The statement in the lemma follows then by dominated convergence.

In order to prove (A85), fix $x \in [a, b]$. For any $\gamma, \delta > 0$, we estimate

$$\begin{aligned}
|g(\delta, x) - f(0, x, x)| &= \left| \int_{\mathbb{R}} \eta_\delta(x - x')\,[f(\delta, x, x') - f(0, x, x)]\;dx' \right| \\
&= \left| \int_{\mathbb{R}} \eta_\delta(y)\,[f(\delta, x, x - y) - f(0, x, x)]\;dy \right| \\
&\le 2\,||f||_\infty \int_{\mathbb{R}\setminus[-\gamma,\gamma]} \eta_\delta(y)\,dy \\
&\quad + \sup_{r \in [-\gamma,\gamma]} |f(\delta, x, x + r) - f(0, x, x)|\,,
\end{aligned} \tag{A86}$$

where (A78) was employed twice. Choose any $\epsilon > 0$. Since f is continuous, we find $\gamma, \alpha > 0$ such that

$$\sup_{r\in[-\gamma,\gamma]} |f(\delta, x, x + r) - f(0, x, x)| < \frac{\epsilon}{2} \quad \text{for} \quad \delta \in (0, \alpha). \tag{A87}$$

In view of (A79), we find $\beta \in (0, \alpha)$ with

$$2\,||f||_\infty \int_{\mathbb{R}\setminus[-\gamma,\gamma]} \eta_\delta(y)\, dy < \frac{\epsilon}{2} \quad \text{for} \quad \delta \in (0, \beta). \tag{A88}$$

Therefore,

$$|g(\delta, x) - f(0, x, x)| < \epsilon \quad \text{for} \quad \delta \in (0, \beta), \tag{A89}$$

and the lemma is proved. □

References

[1] D. M. Wolkow, Z. Phys. **94**, 250 (1935).

[2] T. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

[3] I. I. Gol'dman, Zh. Eksp. Teor. Fiz. **46**, 1412 (1964) [Sov. Phys. JETP **19**, 954 (1964)].

[4] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 1768 (1964) [Sov. Phys. JETP **19**,1191 (1964)].

[5] N. B. Narozhnyĭ, A. I. Nikishov, and V. I. Ritus, Zh. Eksp. Teor. Fiz. **47**, 930 (1964) [Sov. Phys. JETP **20**, 622 (1965)].

[6] J. Bergou, S. Varró, J. Phys. A **13**, 2823 (1980).

[7] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964) [Sov. Phys. JETP **19**, 529 (1964)].

[8] H. R. Reiss, J. Opt. Soc. Am. B **7**, 574 (1990).

[9] D. P. Crawford and H. R. Reiss, Opt. Express **2**, 289 (1998).

[10] C. Szymanowski, V. Véniard, R. Taïeb, A. Maquet, and C. H. Keitel, Phys. Rev. A **56**, 3846 (1997).

[11] C. Szymanowski and A. Maquet, Opt. Express **2**, 262 (1998).

[12] V. P. Yakovlev, Zh. Eksp. Teor. Fiz. **49**, 318 (1965) [Sov. Phys. JETP **22**, 223 (1966)].

[13] C. Müller, A. B. Voitkiv, and N. Grün, Phys. Rev. A **67**, 063407 (2003).

[14] C. Müller, A. B. Voitkiv, and N. Grün, Phys. Rev. Lett. **91**, 223601 (2003).

[15] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (MacGraw-Hill, New York, 1964).

[16] B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).

[17] R. A. Neville and F. Rohrlich, Phys. Rev. D **3**, 1692 (1971).

[18] J. San Román, L. Roso, and H. R. Reiss, J. Phys. B **33**, 1869 (2000).

[19] J. San Román, L. Plaja, and L. Roso, Phys. Rev A **64**, 063402 (2001).

[20] J. San Román, L. Roso, and L. Plaja, J. Phys. B **36**, 2253 (2003).

[21] J. San Román, L. Roso, and L. Plaja, J. Phys. B **37**, 435 (2004).

[22] P. Filipowicz, J. Phys. A. **18**, 1675 (1985).

[23] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Relativistic Quantum Theory* (Pergamon Press, Oxford, 1971).

[24] A. Janner and T. Janssen, Physica **60**, 292 (1972).

[25] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis* (Academic Press, New York-London, 1972).

[26] M. Reed and B. Simon, *Methods of modern mathematical physics. III. Scattering theory* (Academic Press, New York-London, 1979).

[27] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness* (Academic Press, New York-London, 1975).

[28] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (MacGraw-Hill, New York, 1980).

[29] J. Weidmann, *Linear operators in Hilbert spaces* (Springer, New York-Berlin, 1980).