Dynamical Casimir effect in a periodically changing domain: A dynamical systems approach

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Abstract. We study the problem of the behavior of a quantum massless scalar field in the space between two parallel infinite perfectly conducting plates, one of them stationary, the other moving periodically. We reformulate the physical problem into a problem about the asymptotic behavior of the iterates of a map of the circle, and then apply results from theory of dynamical systems to study the properties of the map. Many of the general mathematical properties of maps of the circle translate into properties of the field in the cavity. For example, we give a complete classification of the possible resonances in the system, and show that small enough perturbations do not destroy the resonances. We use some mathematical identities to give transparent physical interpretation of the processes of creation and amplification of the quantum field due to the motion of the boundary and to elucidate the similarities and the differences between the classical and quantum fields in domains with moving boundaries.

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1. Introduction

Recently the problem of the behavior of the fields in a cavity with a (periodically) moving boundary has received significant attention. From a mathematical point of view, it constitutes an instructive example of parametrically driven system that exhibits interesting resonant effects. From point of view of physics, besides its fundamental importance as a modification of Casimir effect, it can be used as a model for the mechanism of formation of wave packet in lasers, processes in atomic physics, and even interstellar flight [47]! We would like to draw reader's attention to the reviews of Casimir effect by Bordag *et al* [3], the books by Milton [50], and Mostepanenko and Trunov [52], and to the recent review by Dodonov [20] devoted specifically to the dynamical (or nonstationary) Casimir effect (i.e., the quantum effect in a pulsating cavity) which contains more than 300 references.

In this paper we will apply a method we have developed in [45] (and generalized in [54]) to study the behavior of the classical electromagnetic field in a one-dimensional cavity with a moving wall by using methods of dynamical systems. Here we will employ this methodology to analyze the quantum problem. Our approach is based on studying the collective behavior of the characteristics of the wave equation by applying theory of circle maps. The general theorems allow us to predict the behavior of the system without solving partial differential equations. We show that the mechanism of the resonant amplification of the quantum field is the same as in the classical case – it is due to Doppler effect at reflection from the moving mirror.

In the quantum case, however, the motion of the mirror "creates" new field which is then amplified by the Doppler effect. Using some mathematical identities, we give simple physical interpretation of the different contributions to the energy density.

In the rest of the Introduction, we review very briefly some of the literature related to the mathematical and physical aspects of our approach, referring the reader to the review [20].

The mathematical theory of the solutions of the wave equation in presence of (periodically) moving boundaries in one or more spatial dimensions has been developed by Cooper [6, 7, 8], Cooper and Koch [9], Yamaguchi and collaborators [59, 60, 62, 61], Dittrich *et al* [17]. In the physics literature, the classical version of the problem was studied by Dittrich *et al* [18], Cole and Schieve [5], Mèplan and Gignoux [49], who used the geometric method of solving the wave equation (i.e., the method of characteristics), and recognized that under in some cases the field develops wave packets that become narrower with time.

The foundations of the quantum theory of the problem of a 1-dimensional resonator with a moving wall were laid by Moore [51], and developed by Fulling and Davies [28, 12]. Dodonov *et al* [23, 24, 22] considered the case of resonant motion of the mirror within Moore's formalism; they and Jaekel and Reynaud [34], Méplan and Gignoux [49], and others predicted that the force between the mirrors can be enhanced significantly in the resonant case.

In [42], Law proposed an exact analytic solution for a particular choice of the motion of the mirror, for which the field in the cavity develops two wave packets which become narrower in time and whose energy grows. Law's solution was generalized by Ying Wu *et al* [58], who constructed motions of the mirror for which the field develops several wave packets.

Dodonov [19] noticed that small enough "detuning" from the exact resonance conditions does not change the qualitative features of the behavior of the field in the cavity. Within our approach, one can find explicitly for what detuning the behavior of the field will change dramatically. The packet formation in a resonantly pulsating resonator was studied in detail by Dodonov and Andreata [21, 1].

Ideas from dynamical systems (i.e., study of the iterates of a certain map) have been used by Cooper [7, 8] (maps of the interval), Méplan and Gignoux [49] (area-preserving maps of the cylinder), Dittrich *et al* [18, 17], Yamaguchi [59, 60, 62] (circle maps).

2. Method of characteristics and dynamical systems

In this section, we explain the physical setup, pose the mathematical problem, and explain how to analyze it. We recommend that the reader consult our paper [45] for details.

We will refer often to the electromagnetic (EM) field in the cavity (meaning the classical, not quantum, field); as it turns out, some aspects of the problem are similar in the classical and in the quantum problem case.

2.1. Description of the physical system

Consider the EM field in the empty space (no medium, electric charges, or currents) between two parallel perfectly reflecting mirrors, one stationary at $x = 0$, the other moving according to $x = a(t)$. The function a must satisfy the physically natural conditions $a(t) > 0$ for each t (the resonator never collapses to zero length), $|a'(t)| < 1$ (the speed of the moving mirror never exceeds the speed of light). To avoid technicalities, we assume that $a(t)$ is a smooth (C^∞) function. We will focus on the case in which the motion of the mirror is 1-periodic (i.e., periodic of period 1):

$$a(t+1) = a(t) \quad \text{for each } t \ . \tag{1}$$

Examples of such functions are

$$a(t) = \frac{\alpha}{2} + \frac{\beta}{2\pi}\sin 2\pi t \ , \quad \frac{\alpha}{2} > \frac{|\beta|}{2\pi} \ , \quad |\beta| < 1 \ ;$$

$$a(t) = \frac{\alpha}{2} + \frac{\beta}{2\pi}\sin\left(2\pi t + \gamma(\sin 4\pi t)^2\right) \ , \quad \frac{\alpha}{2} > \frac{|\beta|}{2\pi} \ , \quad |\beta|(1+2|\gamma|) < 1 \ . \tag{2}$$

In our numerical simulations we will use the function $a(t)$ from (2).

2.2. Boundary-value problem

Since there are no charges and currents in the cavity, we impose Coulomb gauge $A_0 = 0$, $\nabla \cdot \mathbf{A} = 0$ on the EM 4-potential $A_\mu = (A_0, \mathbf{A})$, and obtain that $\mathbf{A}$ satisfies the wave equation. We consider linearly polarized plane waves propagating in x direction, so we can assume without loss of generality that the vector potential has the form

$$\mathbf{A}(t,x) = A(t,x)\, \mathbf{e}_y \ .$$

The function $A(t,x)$ satisfies the $(1+1)$-dimensional wave equation

$$A_{tt}(t,x) - A_{xx}(t,x) = 0 \tag{3}$$

in the spatio-temporal domain $\Lambda := \{(t,x) \in \mathbb{R}^2 | 0 < x < a(t),\ t > 0\}$. The boundary conditions (BCs) come from the fact that in the coordinate frame instantaneously co-moving with the mirror, the tangential to the mirror component of the electric field must vanish at the mirror, which yields the "perfect reflection" BCs

$$A_t(t,0) = 0 \ , \quad A_t(t,a(t)) + a'(t)\, A_x(t,a(t)) = 0 \tag{4}$$

for each $t \geq 0$. Geometrically, the BCs (4) mean that the derivative of $A(t,x)$ along the world line of the mirror (i.e., the line $\{(t,a(t)) \mid t \in \mathbb{R}\}$ in the space-time diagram) must be 0. Note that the Dirichlet BCs

$$A(t,0) = c_1 = \text{const} \ , \qquad A(t,a(t)) = c_2 = \text{const} \tag{5}$$

are equivalent to (4). Parenthetically, we would like to note that Neumann BCs are not Lorenz covariant, so they are not physically natural for the case of EM fields. If, nevertheless, one imposes Neumann BCs, the predictions of the theory are dramatically different from those of the "perfect reflection" BCs (4) (see [54, Section 5.6] or Dittrich *et al* [17, Section 4]).

In the quantum case (Moore [51]), one obtains the boundary value problem consisting of the equation (3), the Dirichlet BCs (5) with $c_1 = c_2 = 0$, and initial conditions.

2.3. Method of characteristics

In absence of spatial boundaries (i.e., if $\Lambda = \{(t,x) \in \mathbb{R}^2 | t > 0\}$), the solution of the wave equation (3) with "perfect reflection" (4) or Dirichlet (5) BCs, and initial conditions $A(0,x) = \psi_1(x)$, $A_t(0,x) = \psi_2(x)$, $x \in \mathbb{R}$, is a superposition of waves propagating to the left and to the right: $A(t,x) = \Psi^+(x_0^+) + \Psi^-(x_0^-)$, where $x_0^\pm = x \pm t$, and $\Psi^\pm(s) = \frac{1}{2}\left[\psi_1(s) \pm \int_c^s \psi_2(s')\,\mathrm{d}x\right]$ (c is an arbitrary constant, the same for Ψ^+ and Ψ^-). Geometrically, in the space-time diagram, the waves propagate along the *characteristics*, $x \pm t = \text{const}$.

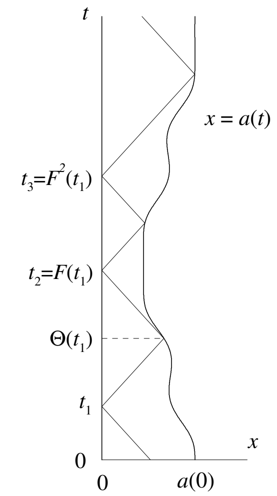

Figure 1. Characteristics of the wave equation.

In the presence of spatial boundaries (stationary or moving), the characteristics are no more straight lines, but are piecewise linear, each part of them being a straight line at 45° with respect to the t axis (see Figure 1). In order for the BCs (4) to be satisfied, the field changes sign at each reflection, so that

$$A(t,x) = (-1)^{N_+}\Psi^+(x_0^+) + (-1)^{N_-}\Psi^-(x_0^-) \ ,$$

where $N_\pm$ is the number of reflections of the corresponding broken characteristic by a mirror between the initial moment $t = 0$ and the present time t. For a proof that this algorithm works, see [45, Section II.B].

2.4. The importance of the reflections; Doppler effect

Loosely speaking, the "density" of the characteristics in the space-time diagram is proportional to the energy density of the EM field. Consider two characteristics corresponding to the ends of a narrow wave packet. During free propagation (no reflection), the width of the wave packet and the vector potential $\mathbf{A}$ do not change. At reflection from the moving mirror, however, not only does A change sign, but also the width of the wave packet changes. If the width of the wave packet before the reflection was Δ, and the reflection occurs at time t (assume that the wave packet is so narrow that the reflection happens almost instantaneously), simple trigonometry shows that after the reflection the width Δ' of the wave packet is $\Delta' = \Delta/D(t)$, where

$$D(t) = \frac{1 - a'(t)}{1 + a'(t)} \tag{6}$$

is the *Doppler factor* at reflection at time t. The term "Doppler factor" comes from the fact that the initial (classical) EM energy of the wave packet, $\frac{1}{8\pi}\int_\Delta \left[A_t(t,x)^2 + A_x(t,x)^2\right] \mathrm{d}x$, increases by a factor of $D(t)$ at reflection from the moving mirror (see [54, Section 2.4] for a simple proof). Clearly, if the mirror is moving inwards (outwards) at the time of reflection, the energy of the wave packet will increase (decrease). If it happens that every time a certain group of nearby characteristics (representing a wave packet) is reflected from the moving mirror while the mirror is moving inwards, then they are going to get closer together, which will lead to squeezing of the wave packet (see Dodonov *et al* [23] and Jaekel and Reynaud [34]) and to exponential growth of the energy of the field, as we will see below.

Since the motion of the boundary is 1-periodic (1), the position $a(t)$ and the velocity $a'(t)$ of the mirror, as well as the Doppler factor $D(t)$ (6), do not depend on the integer part of t, but only on its fractional part,

$$\{t\} := t - \lfloor t \rfloor \ , \tag{7}$$

which we will refer to as the *phase* of the motion of the mirror. To make the phase of t change continuously as t increases, we will think of $\{t\}$ as belonging to a circle of length 1, i.e., to the interval $[0, 1]$ with its ends identified. The long-time behavior of the field depends on the asymptotic behavior of the characteristics, which in turn can be analyzed by invoking the mathematical theory of circle maps, as explained below.

2.5. From characteristics to circle maps

Since characteristics belong to a very simple class of plane curves – namely, piecewise linear at a 45° angle with the t axis, – to reconstruct a particular characteristic, it is enough to know only one moment at which this characteristic is reflected from, say, the stationary mirror.

To study the collective behavior of the characteristics, we introduce the *time advance map* $F : \mathbb{R} \to \mathbb{R}$ such that if certain characteristic is reflected from the stationary mirror at time t_1, the next reflection from the same mirror occurs at time $t_2 = F(t_1)$ (see Figure 1). To derive an expression for F in terms of the function a giving the motion of the mirror, we notice that the time $\Theta(t_1)$ between t_1 and t_2 at which the characteristic is reflected from the moving mirror satisfies $\Theta(t_1) - a(\Theta(t_1)) = t_1$, which can be written as $(\mathrm{Id} - a)(\Theta(t_1)) = t_1$, therefore

$$\Theta = (\mathrm{Id} - a)^{-1} \ . \tag{8}$$

On the other hand, $F(t_1) = \Theta(t_1) + a(\Theta(t_1)) = (\mathrm{Id} + a)(\Theta(t_1))$, thus

$$F = (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1} \ . \tag{9}$$

The conditions $a(t) > 0$ and $|a'(t)| < 1$ guarantee the invertibility of $(\mathrm{Id} \pm a)$ (hence the existence of Θ and F) as well as the fact that F is strictly increasing and, therefore, invertible. We leave to the reader to check that a can be expressed in terms of F as

$$a = \tfrac{1}{2}(F - \mathrm{Id}) \circ \left[\tfrac{1}{2}(F + \mathrm{Id})\right]^{-1} \ .$$

The 1-periodicity (1) of a guarantees that F satisfies the property

$$F(t + 1) = F(t) + 1 \ . \tag{10}$$

Since only the phase $\{t\}$ (7) is physically important, instead of considering the function $F : \mathbb{R} \to \mathbb{R}$, we define the function

$$f : S^1 \to S^1 : \{t\} \mapsto \{F(\{t\})\} \tag{11}$$

that maps the phase $\{t\}$ at some reflection from the stationary mirror to the phase $\{F(\{t\})\}$ at the next reflection from the same mirror. Here S^1 stands for the "circle", i.e., the interval $[0, 1]$ with its ends identified (in mathematical notations, this can be written as $S^1 = \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ stands for the integers). The function f is well-defined due to (10).

If the first reflection of a particular characteristic from the stationary mirror occurs at time t_1, the times of the subsequent reflections are $F(t_1)$, $F^2(t_1)$, $F^3(t_1)$, . . ., where

$$F^n := \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$$

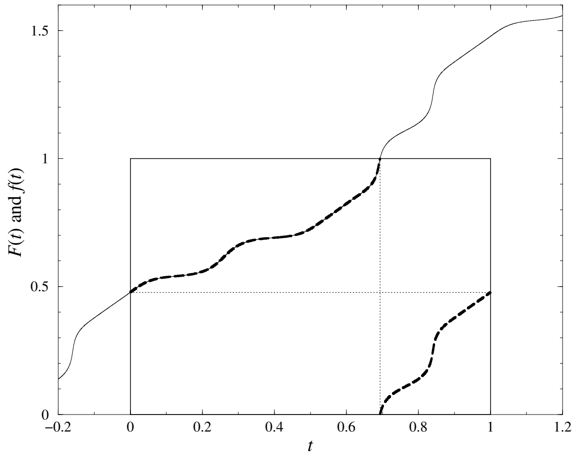

Figure 2. Graph of the functions F (thin solid line) and f (thick dashed line) corresponding to the function a (2) for $\alpha = 0.35$, $\beta = 0.4$, $\gamma = 0.7$.

is the nth *iterate* of the function F. Since the asymptotic behavior of the characteristics is completely determined by the asymptotic behavior of the phases at reflection, the long-time behavior of the system can be studied by analyzing the high iterates of f. The branch of mathematics that studies the behavior of highly iterated functions is called *theory of dynamical systems*. Traditionally, the functions that are going to be iterated – like F and f – are called *maps*. In particular, the map f (11) is an example of a *circle map* (CM), i.e., a map from the circle S^1 to itself. Theory of CMs is a prominent part of theory of dynamical systems; it was initiated by Poincaré in 1880s, and nowadays is a highly developed field of mathematics with many physical applications.

The relationship between the time advance map $F : \mathbb{R} \to \mathbb{R}$ and the CM $f : S^1 \to S^1$ is shown pictorially in Figure 2. Note that although f looks discontinuous in the figure, it is continuous as a function on the circle S^1 because of the identifications of 0 and 1 shown in the figure with dotted lines. The map F is called a *lift* of f, while f is sometimes called the *projection* of F. Clearly, F determines f uniquely; on the other hand, each CM f has infinitely many lifts that differ by an additive integer constant (in our case, however, the lift F is defined uniquely by (9)).

3. Circle maps and wave packet formation

In this section we collect some facts about the *dynamics*, i.e., the behavior of the high iterates f^n, of CMs. For more information the reader can consult the introductory expositions in Hasselblatt and Katok [31, Chapter 4] or Devaney [14, Section 1.14], or the more sophisticated treatments in Katok and Hasselblatt [39, Chapters 11 and 12], de Melo and van Strien [13, Chapter I]. Section III of our paper [45] contains a selection of mathematical facts adapted to the problem of the resonator. In Section 3.3 we will give interpretation of the mathematical results in terms of the asymptotic behavior of the field in the resonator.

3.1. Circle maps – basic definitions

By a circle map (CM), we will always mean a smooth (C^∞) invertible map of the circle whose inverse is also smooth – this is exactly the class of CMs that correspond to motions of the boundary a satisfying the conditions from Section 2.1. For the map f to be invertible, we have to assume that the cavity is not too long, or, more concretely, that $F(t) - t < 1$ (for which it is enough to assume that $a(t) < \frac{1}{2}$ for all t). This condition only helps to avoid clumsy sentences, but is not a restriction of the generality – our ideas can be easily applied *mutatis mutandis* to the case of a longer cavity.

The most important characteristic of a CM is its *rotation number* defined as the "average amount of rotation":

$$\tau(f) \equiv \tau(F) := \lim_{n\to\infty} \frac{F^n(t) - t}{n} \tag{12}$$

(with the above restriction on the length of the cavity, $\tau(f) \in [0, 1)$). It can be proved that $\tau(f)$ always exists and does not depend on the value of t in (12).

An *orbit* of a point $t \in S^1$ is the set $\{f^n(t)\}_{n=0}^{\infty}$ of all future (i.e., for $n \geq 0$) iterates of t. If for some point $t^* \in S^1$ there exists an integer q such that $f^q(t^*) = t^*$, then we say that t^* is a *periodic point* of period q (or a q*-periodic point*) and call the orbit $\{f^n(t^*)\}_{n=0}^{q-1}$ of this point a q*-periodic orbit*.

The simplest example of a CM is the *rigid rotation* $r_\sigma : S^1 \to S^1$ (where $\sigma \in [0, 1)$) defined through its lift R_σ,

$$\begin{aligned} R_\sigma(t) &:= t + \sigma \ , \quad t \in \mathbb{R} \ , \\ r_\sigma(t) &:= \{t + \sigma\} \ , \quad t \in S^1 \ . \end{aligned} \tag{13}$$

Clearly, $\tau(r_\sigma) = \sigma$. The dynamics of r_σ is very simple:

- if σ is a rational number, i.e., $\sigma = p/q$ for some integers p and q (we will always assume that p and q do not have common factors), then after q iterations any point $t \in S^1$ returns to its initial position, having traversed the circle p times, i.e., each point $t \in S^1$ is a q-periodic point:

$$\begin{aligned} R^q_{p/q}(t) &= t + p \quad \forall t \in \mathbb{R} \ , \\ r^q_{p/q}(t) &= t \quad \forall t \in S^1 \ ; \end{aligned}$$

- if σ is not a rational number, then the orbit $\{r^n_\sigma(t)\}_{n=0}^{\infty}$ of any point $t \in S^1$ fills the circle densely, and will never return to the initial point t, thus, in this case there are no periodic orbits.

3.2. Phase locking, Arnol'd tongues, devil's staircase

Here we will describe in detail the case of a general CM f with a rational rotation number, $\tau(f) = p/q$, in which case the map f is said to be *phase locked* (*frequency locked*, *mode locked*).

If $\tau(f) = p/q$, then generically f has an attracting q-periodic orbit $\{t_j^{(\mathrm{a})}\}_{j=1}^q$, and a repelling q-periodic orbit $\{t_j^{(\mathrm{r})}\}_{j=1}^q$. "Attracting" means that the orbit of each point $t \in S^1$ that is not one of the of the repelling periodic points $\{t_j^{(\mathrm{r})}\}_{j=1}^q$ tends asymptotically to the attracting periodic orbit $\{t_j^{(\mathrm{a})}\}_{j=1}^q$. The repelling periodic orbit "repels" the iterates of f; it is an attracting periodic orbit for the inverse map f^{-1} (which is also a CM). The attracting and repelling periodic orbits of the CM f give rise to attracting and repelling characteristics of the wave equation, and to formation of wave packets (see Section 3.3).

A very important for the physics of the problem question is how "generic" the case of phase locking is. In Figure 3(a) we show in the (α, β)-plane the regions of values of the parameters α and β of the motion of the mirror that correspond to phase-locking of several rotation numbers $\tau(f)$. These "phase-locked" regions in the (α, β)-plane are called *Arnol'd tongues* in honor of Arnol'd who studied them in his famous paper on CMs [2]. The Arnol'd tongue corresponding to a p/q phase locking emanates (i.e., starts as $\beta \to 0^+$) from $\alpha = p/q$, and becomes thicker as β increases. Tongues with large p and q are very thin (see the 9/20 tongue in Figure 3(a)).

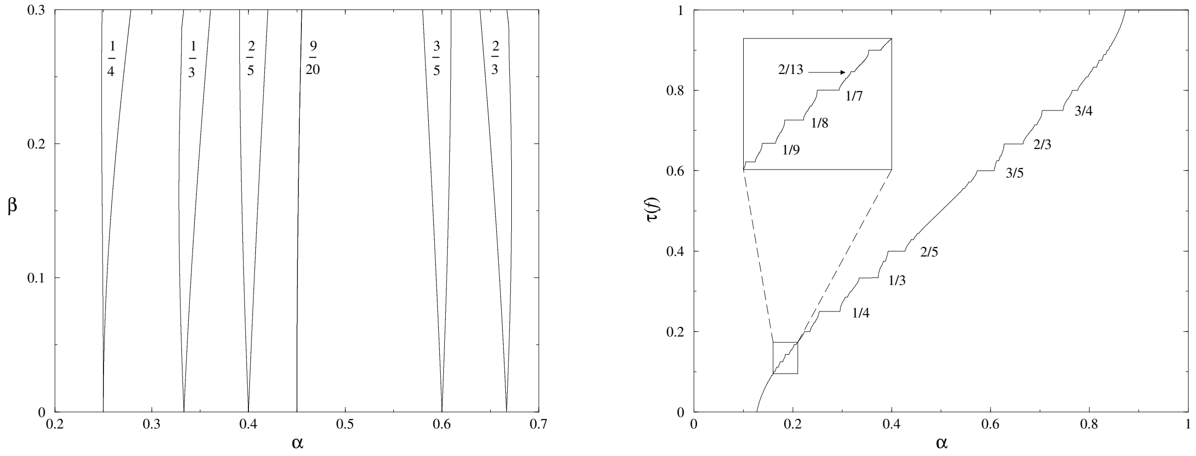

Figure 3. (a) Arnol'd tongues with rotation numbers 1/4, 1/3, 2/5, 9/20, 3/5, and 2/3 for the CM f corresponding to motion of the boundary (2) with $\gamma = 0.7$. (b) Graph of the rotation number $\tau(f)$ for (2) with $\beta = 0.4$, $\gamma = 0.7$.

Another illustration of the abundance of phase-locking is the graph of the rotation number $\tau(f)$ versus α (all other parameters fixed), shown in Figure 3(b). It can be proved that this function is continuous, and it is locally constant if $\tau(f)$ is rational, and strictly increasing if $\tau(f)$ is irrational, i.e., the graph of such a function contains infinitely many densely interspersed horizontal pieces, each corresponding to a particular type of phase locking. Such a graph is called a *devil's staircase*. Several of the horizontal pieces in the figure are labeled with p/q showing the type of phase locking. The behavior of the widths of the Arnol'd tongues as $\beta \to 0^+$ is studied by Jonker [38] and Davie [11].

It is worth noting that, in some sense, phase locking is more "generic" than the unlocked case. Namely, if $\tau(f)$ is irrational, then there exists an arbitrarily small smooth perturbation of f such that the perturbed map is phase locked. On the other hand, if $\tau(f) = p/q$, then the parameter values are either strictly inside or on the boundary of the q/p-Arnol'd tongue. If the parameters are strictly inside the tongue, then a small enough (but otherwise arbitrary) smooth perturbation will not change its rotation number, i.e., the perturbed map will have rotation number p/q. Practically, however, one should not forget that the width of the Arnol'd tongues decreases fast when p and q increase or when β decreases.

If the CM f is in p/q phase locking and the parameters are strictly inside the p/q tongue, f has an attracting q-periodic orbit and a repelling one. In this case, if t^* belongs to the attracting periodic orbit, then $f^q(t^*) = t^*$ and $f'(t^*) < 1$. In the following, we will say that a phase locking is "generic" if the parameters of the CM are strictly inside the Arnol'd tongue, in which case there exist an attracting and a repelling orbit. If the parameters of the map are on the boundary of the Arnol'd tongue (i.e., at some end on the corresponding horizontal piece of the graph in Figure 3(b)), then there exists a q-periodic orbit which is neither attracting, nor repelling; in this case $f'(t^*) = 1$ where t^* is a q-periodic point.

When β reaches the critical value at which the boundary is moving at the speed of light at some moment in each period (i.e., if $|a'(t)| = 1$ for some t), then the total length of the phase locking intervals (i.e., of the horizontal pieces in Figure 3(b)) becomes equal to 1, which means physically that the probability of phase locking is 1. This mathematical problem is studied numerically by Jensen *et al* [36] and Lanford [41], and proved rigorously for general CMs by Graczyk and Świątek [30].

3.3. Derivative of the circle map, Doppler factor, formation of wave packets

Now we translate the mathematical facts about the dynamics of the CM f (11) and the time advance map F (9) into asymptotic properties of the field in the cavity.

If a particular characteristic is reflected by the stationary mirror at time t, then the times of the subsequent reflections from the same mirror are given by $F^n(t)$ $(n = 1, 2, \ldots)$; the phases (7) of the motion of the mirror at these times are given by the iterates of the corresponding CM, $f^n(\{t\})$. If f is phase locked with rotation number $\tau(f) = p/q$, then, generically, there exists an attracting periodic orbit $\{t_j^{(\mathrm{a})}\}_{j=1}^q$ which attracts the iterates of any point in S^1 (except the repelling periodic points) under the map f. Physically, each attracting periodic point $t_j^{(\mathrm{a})} \in S^1$ of the CM f corresponds to an infinite sequence of times of the form $n + t_j^{(\mathrm{a})}$, where n is any integer, such that the characteristics that are reflected from the stationary mirror at these times attract the nearby characteristics. We will call times of the form $n + t_j^{(\mathrm{a})}$ "attracting q-periodic times", and the corresponding characteristics "attracting q-periodic characteristics". The presence of q attracting periodic characteristics means physically that the field in the cavity develops (at most) q wave packets, whose widths decrease exponentially, and whose energies increase exponentially with time.

In the case of generic p/q phase locking, the rate at which the characteristics get closer together is related to the first derivative of F (or, equivalently, f), which in turn is related (according to (6), (8), (9)) to the Doppler factor $D(\Theta(t))$ at the time $\Theta(t)$ of the first reflection from the moving mirror after t:

$$F'(t) = \frac{1 + a'(\Theta(t))}{1 - a'(\Theta(t))} = \frac{1}{D(\Theta(t))} \; .$$

Asymptotically, the wave packets are very narrow, so that they are reflected from the moving mirror practically instantaneously, at times of the form $n + t_j^{(\mathrm{a})}$. The asymptotic "cumulative" Doppler factor $\mathcal{D}_q$ over a sequence of q consecutive reflections of the packet from the moving mirror (which takes total time p according to the fact that $F^q(t_j^{(\mathrm{a})}) = t_j^{(\mathrm{a})} + p$) is equal to the product of Doppler factors at each of these q reflections:

$$\mathcal{D}_q := \prod_{j=1}^{q} D(\Theta(t_j^{(\mathrm{a})})) = \left[\prod_{j=1}^{q} F'(t_j^{(\mathrm{a})})\right]^{-1} = \left[(F^q)'(t_1^{(\mathrm{a})})\right]^{-1} \; . \tag{14}$$

This formula holds exactly only in the case of classical EM field, when the motion of the mirror only amplifies the field through Doppler effect at reflection. In Section 4.2, we will see that in the quantum case the energy is not only amplified, but also created by the motion of the mirror, which introduces corrections to the rate of change of the energy.

In the case when the rotation number $\tau(f)$ is irrational, the field does not develop wave packets, and its energy changes with time, but does not have a tendency towards steady grow or decay.

4. Quantum effects in a periodically pulsating resonator

4.1. Moore's functional equation

Moore [51] was the first to consider the problem of quantizing the electromagnetic field in a one-dimensional resonator with a moving wall. We leave out all the complications that he had to overcome in the development of a quantization scheme, and focus on one particular aspect of his treatment (adapting his equations to our approach). Let the motion of the mirror correspond to time advance map F (9) of rotation number σ. Moore showed that in this case one has to look for an expansion of the field operator $A(t, x)$ in mode functions

$$A_k(t, x) = \mathrm{e}^{-2\pi \mathrm{i} k \frac{1}{\sigma} \Sigma(t-x)} - \mathrm{e}^{-2\pi \mathrm{i} k \frac{1}{\sigma} \Sigma(t+x)} \; , \quad k = 1, 2, 3, \ldots \; ,$$

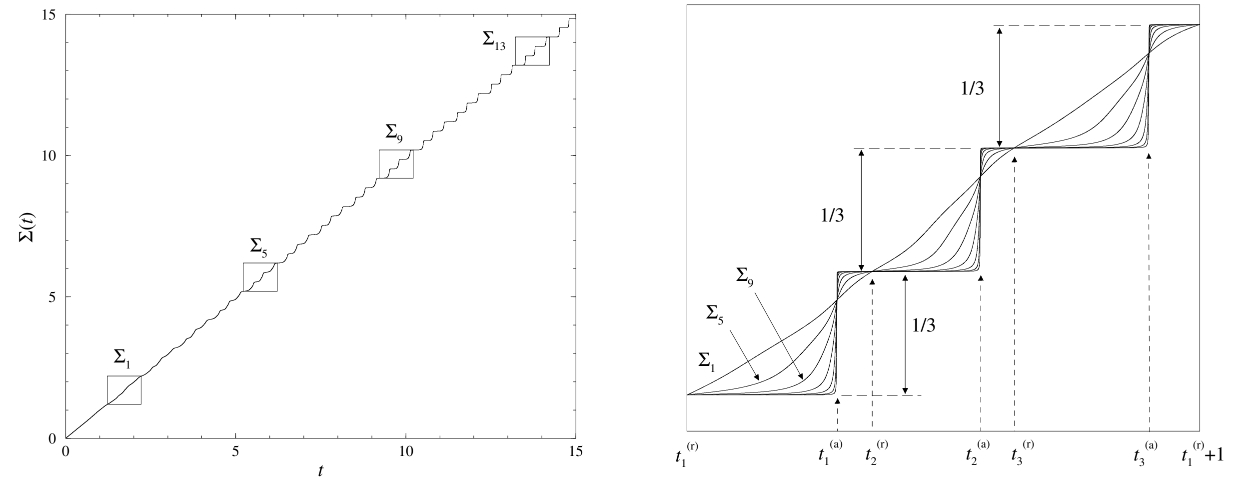

Figure 4. (a) Development of the staircase-like structure of the function Σ for motion of the mirror given by (16) with $\alpha = 0.34$, $\beta = 0.2$, $\gamma = 0.3$, corresponding to 1/3-phase locking. (b) Graphs of the functions Σ_n (17), $n = 1, 5, 9, 13, 17, 21, 25$, for the same parameter values.

where the function $\Sigma : \mathbb{R} \to \mathbb{R}$ satisfies Moore's functional equation

$$\Sigma(t + a(t)) = \Sigma(t - a(t)) + \sigma \ ,$$

which ensures that $A_k(t, x)$ satisfy the Dirichlet BCs (5) with $c_1 = c_2 = 0$. This equation can be rewritten in terms of the map F as $\Sigma \circ F(t) = \Sigma(t) + \sigma$, or, equivalently, as

$$\Sigma \circ F = R_\sigma \circ \Sigma \tag{15}$$

(where R_σ (13) is the rigid rotation by σ), and interpreted as the fact that the value of Σ changes between two consecutive reflections from the stationary mirror by σ. This implies, in particular, that if for some particular value $\bar{t}$ we know the values of $\Sigma(t)$ in the interval $[\bar{t}, F(\bar{t}))$, then we can reconstruct the function Σ for all $t \in \mathbb{R}$ by using (15).

Moore's functional equation is easy to solve numerically. Let us assume that before $t = 0$, the two mirrors were at rest, and at $t = 0$ the right mirror started moving:

$$a(t) = \begin{cases} \frac{\alpha}{2} \ , & t \leq 0 \\ \frac{\alpha}{2} + \frac{\beta}{2\pi}(1 - \mathrm{e}^{-t^4}) \sin\left(2\pi t + \gamma (\sin 4\pi t)^2\right) \ , & t > 0 \ . \end{cases} \tag{16}$$

The motion of the boundary for $t > 0$ is very similar to the one in (2); we have used the factor $1 - \mathrm{e}^{-t^4}$ to smooth out the transition (i.e., to ensure that Σ', Σ'', and Σ''' are continuous, the reason for which will become clear in Section 4.2) the factor $(1 - \mathrm{e}^{-t^4})$ tends to 1 very quickly as t grows, so it does not affect the asymptotic behavior of the system. If the mirrors are stationary, it is natural to take the function $\Sigma(t)$ to be linear, so for $t \in [-\alpha, 0)$, we take $\Sigma(t) = \frac{\sigma}{\alpha} t + \mathrm{const}$ (the constant is immaterial since only the derivatives of Σ have physical meaning), and then use (15) to find $\Sigma(t)$ for $t \geq 0$.

Now we will apply our knowledge about the dynamics of the CM f – and, hence, about the time advance map F – to draw conclusions about the asymptotic behavior of the function Σ, which will allow us to make predictions about the long-time behavior of the energy density of the field. Since the case of a rational $\sigma = p/q$ is especially interesting because of the occurrence of resonant phenomena (phase locking), we focus on this case in the rest of this subsection. In the case of a generic p/q phase locking, the CM f

has an attracting and a repelling q-periodic orbits. Let $t_1^{(\mathrm{a})}, \ldots, t_q^{(\mathrm{a})}$ be a sequence of q consecutive attractive q-periodic times, and let the ordering be such that

$$t_1^{(\mathrm{a})} \overset{F}{\mapsto} t_2^{(\mathrm{a})} \overset{F}{\mapsto} \cdots \overset{F}{\mapsto} t_q^{(\mathrm{a})} \overset{F}{\mapsto} t_1^{(\mathrm{a})} + p$$

(this implies that the fractional parts of these times satisfy $\{t_1^{(\mathrm{a})}\} \overset{f}{\mapsto} \cdots \overset{f}{\mapsto} \{t_q^{(\mathrm{a})}\} \overset{f}{\mapsto} \{t_1^{(\mathrm{a})}\}$). If n is a positive integer, then (15) iterated nq times reads

$$\Sigma \circ F^{nq}(t) = \Sigma(t) + np \; .$$

Using that $F^{nq}(t_1^{(\mathrm{r})}) = t_1^{(\mathrm{r})} + np$, we obtain

$$\Sigma(t_1^{(\mathrm{r})} + np) = \Sigma(t_1^{(\mathrm{r})}) + np \; .$$

This allows us to define a sequence of functions (for $n = 0,\, 1,\, \ldots$)

$$\Sigma_n : [t_1^{(\mathrm{r})}, t_1^{(\mathrm{r})} + np) \to [\Sigma(t_1^{(\mathrm{r})}), \Sigma(t_1^{(\mathrm{r})}) + np) \tag{17}$$

$$\Sigma_n(t) := \Sigma(t + np) - np \; ,$$

and study the behavior of $\Sigma(t)$ for very large t by analyzing the behavior of $\Sigma_n(t)$ (where $t \in [t_1^{(\mathrm{r})}, t_1^{(\mathrm{r})} + np)$) for $n \to \infty$. The graph of Σ can be assembled from translates of the graphs of Σ_n as shown in Figure 4(a), for parameters corresponding to 1/3 phase locking. In Figure 4(b), we show the graphs of several Σ_n for the same parameter values.

In the case of generic p/q phase locking, the times of reflection, t, $F(t)$, $F^2(t)$, …, of a particular characteristic from the stationary mirror accumulate at the attracting periodic times (of the form $n + t_j^{(\mathrm{a})}$), while the values of Σ at two consecutive reflections differ by the constant value σ. This difference between the behavior of the arguments and the values of the function Σ (cf. (15)) explains the occurrence of exactly q steep parts of the graphs of Σ_n at $t = t_j^{(\mathrm{a})}$ $(j = 1, \ldots, q)$, and q almost horizontal parts for large n. Physically, the steep parts correspond to the times of reflection of the packets of the field; the "widths" of these packets are given approximately by (cf. (14))

$$\Delta_j^{(n)} \sim \mathrm{const} \cdot [(F^q)'(t_1^{(\mathrm{a})})]^n = \frac{\mathrm{const}}{\mathcal{D}_q^n} \; .$$

4.2. Energy of the quantum field

Fulling and Davies [28] computed the energy density of the quantum field in a one-dimensional cavity with one stationary and one moving mirror using the "point-splitting" method of DeWitt [15]. They found that the regularized energy density in the space between the mirrors (i.e., the energy minus an infinite constant) is a superposition of the energies of left- and right-propagating disturbances:

$$\langle T_{00}(t,x)\rangle_{\mathrm{reg}} = -\frac{1}{24\pi}\left[\Phi(t+x) + \Phi(t-x)\right] \; , \tag{18}$$

where

$$\Phi(\xi) = \mathscr{S}_\Sigma(\xi) + \frac{2\pi^2}{\sigma^2}\left[\Sigma'(\xi)\right]^2 \; ,$$

and $\mathscr{S}_\Sigma$ is the *Schwarzian derivative* of the function Σ, defined as

$$\mathscr{S}_\Sigma(z) := \frac{\Sigma'''(z)}{\Sigma'(z)} - \frac{3}{2}\left[\frac{\Sigma''(z)}{\Sigma'(z)}\right]^2 \; . \tag{19}$$

The Schwarzian derivative is a remarkable (highly nonlinear!) differential operator, first introduced in complex analysis. If ϕ is a complex analytic function, then vanishing of $\mathscr{S}_\phi$ is a necessary and sufficient condition for ϕ to be a Möbius (i.e., fractional linear) transformation, $M(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$ (see, e.g., Nehari [53, Chapter V]). The Schwarzian derivative is invariant with respect to a composition with a Möbius transformation, $\mathscr{S}_{M\circ F} = \mathscr{S}_F$, which follows from the identity

$$\mathscr{S}_{G\circ H}(z) = \mathscr{S}_G(H(z))\,\left[H'(z)\right]^2 + \mathscr{S}_H(z)\ . \tag{20}$$

The Schwarzian derivative appears in many branches of mathematics – dynamical systems (Singer [55], de Melo and van Strien [13, Chapter 1], Graczyk *et al* [29]), Lorentzian geometry (Kostant and Sternberg [40], Duval and Guieu [25], Duval and Ovsienko [26], Singer [56]), theory of differential equations (Hille [33, Chapter 10]), integrable systems (Burstall *et al* [4]), among many others. Even more interestingly, Schwarzian derivative is widely used as a tool in theory of CMs (Herman [32], Graczyk and Świątek [30]).

One can use the the property (15) and the composition rule (20) to predict the long-time behavior of the energy density (18). To this end, differentiate both sides of $\Sigma \circ F^j = R_{j\sigma} \circ \Sigma$ (which is (15) iterated j times) to obtain

$$\Sigma'(F^j(t)) = \frac{\Sigma'(t)}{(F^j)'(t)}\ .$$

On the other hand, taking the Schwarzian derivative of the same relationship and using (20) with $G = \Sigma$, $H = F^j$, we have

$$\mathscr{S}_\Sigma(F^j(t)) = \frac{1}{\left[(F^j)'(t)\right]^2}\left[\mathscr{S}_\Sigma(t) - \mathscr{S}_{F^j}(t)\right]\ .$$

These expressions yield

$$\Phi(F^j(t)) = \frac{1}{\left[(F^j)'(t)\right]^2}\left[\Phi(t) - \mathscr{S}_{F^j}(t)\right]\ . \tag{21}$$

Using this equation, we can compute the energy of the field at an arbitrary space-time point if the function $\Sigma(t)$ is known for $t \in [\bar{t}, F(\bar{t}))$ where $\bar{t}$ is an arbitrary value; in particular, if for $t < 0$ the mirrors are at rest, we can take $\Sigma(t)$ as in the discussion after (16) (there we used the smoothing factor $1 - \mathrm{e}^{-t^4}$ because $\mathscr{S}_\Sigma$ contains third derivatives of Σ). In the case of classical EM field, the evolution of the energy density is similar to (21) except for the term $\mathscr{S}_{F^j}(t)$, which corresponds to the purely quantum effect of creation of field by the motion of the mirror. Of course, in absence of electromagnetic field in the cavity at $t = 0$, the classical energy is zero for all $t \geq 0$, while the energy of the quantum field is non-zero for $t > 0$ even if at $t = 0$ it was zero.

The composition rule (20) implies

$$\mathscr{S}_{F^j}(t) = \sum_{k=0}^{j-1} \mathscr{S}_F(F^k(t))\ [(F^k)'(t)]^2 = \sum_{k=0}^{j-1} \mathscr{S}_F(F^k(t)) \prod_{i=0}^{k-1} [F'(F^i(t))]^2\ ,$$

which allows us to rewrite equation (21) as

$$\Phi(F^j(t)) = \frac{\Phi(t)}{\left[(F^j)'(t)\right]^2} - \sum_{k=0}^{j-1} \frac{\mathscr{S}_F(F^k(t))}{F'(F^{j-1}(t))^2\ \cdots\ F'(F^k(t))^2}\ . \tag{22}$$

4.3. Physical mechanism of the energy changes

In this subsection, we give a transparent physical interpretation of the terms in the right-hand side of (22). Namely, the first term in the right-hand side of (22) is the initial energy density amplified in the j reflections from the moving mirror between t and $F^j(t)$, while the term with summation index k in the sum corresponds to the field created due to the motion of the mirror at time $F^k(t)$ and subsequently amplified at each of the following reflections. To prove this, we will use the following result concerning the energy density emitted by a single moving mirror. Fulling and Davies [28] proved that if a perfect mirror is moving in vacuum according to $x = a(t)$, then the regularized energy density to the right of the mirror (i.e., for $x > a(t)$) is given by

$$\langle T_{00}(t,x)\rangle_{\rm reg} = -\frac{1}{24\pi}\mathscr{S}_F(t-x) = -\frac{1}{24\pi}\mathscr{S}_F\big((\mathrm{Id}-a)(t_e)\big) \ ,$$

where F is given by (9), and t_e is the time of emission. To the left of the mirror (for $x < a(t)$), the energy density is given by

$$\langle T_{00}(t,x)\rangle_{\rm reg} = -\frac{1}{24\pi}\mathscr{S}_{\widetilde{F}}(t+x) = -\frac{1}{24\pi}\mathscr{S}_{\widetilde{F}}\big((\mathrm{Id}+a)(t_e)\big) \ , \tag{23}$$

where $\widetilde{F} = (\mathrm{Id}-a)\circ(\mathrm{Id}+a)^{-1}$ is defined similarly to F (9), but with a replaced by $-a$. Similarly to (8), we define the function $\widetilde{\Theta} = (\mathrm{Id}+a)^{-1}$.

To rewrite (23) in another form, we will need the following formulae:

$$\mathscr{S}_{\mathrm{Id}\pm a}(\xi) = \frac{a'''(\xi)\,[\pm 1 + a'(\xi)] - \frac{3}{2}a''(\xi)^2}{[1\pm a'(\xi)]^2} \ ,$$

(which follows directly from (19)), and

$$\mathscr{S}_G(G^{-1}(\xi)) = -[G'(G^{-1}(\xi))]^2\,\mathscr{S}_{G^{-1}}(\xi)$$

(a consequence of (20)), which in turn implies

$$\mathscr{S}_{\widetilde{\Theta}}\big((\mathrm{Id}+a)(t_e)\big) = -\widetilde{\Theta}'\big((\mathrm{Id}+a)(t_e)\big)^2\,\mathscr{S}_{\mathrm{Id}+a}(t_e) = -\frac{\mathscr{S}_{\mathrm{Id}+a}(t_e)}{[1+a'(t_e)]^2} \ .$$

Now we have

$$\begin{aligned}
\mathscr{S}_{\widetilde{F}}\big((\mathrm{Id}+a)(t_e)\big) &= \mathscr{S}_{(\mathrm{Id}-a)\circ\widetilde{\Theta}}\big((\mathrm{Id}+a)(t_e)\big) \\
&= \big[\widetilde{\Theta}'\big((\mathrm{Id}+a)(t_e)\big)\big]^2\,\mathscr{S}_{\mathrm{Id}-a}(t_e) + \mathscr{S}_{\widetilde{\Theta}}\big((\mathrm{Id}+a)(t_e)\big) \\
&= \big[\widetilde{\Theta}'\big((\mathrm{Id}+a)(t_e)\big)\big]^2\,[\mathscr{S}_{\mathrm{Id}-a}(t_e) - \mathscr{S}_{\mathrm{Id}+a}(t_e)] \\
&= -\,2\,\frac{a'''(t_e)\,\big[1-a'(t_e)^2\big] + 3a'(t_e)a''(t_e)^2}{[1-a'(t_e)]^4\,\,[1+a'(t_e)]^2} \ .
\end{aligned} \tag{24}$$

Now we will use (24) to understand the physical meaning of (22). For simplicity, we consider the evolution of the energy density along a characteristic which at time t_0 passes through the point x_0 while moving to the left. At time $t_+ := t_0 + x_0$ this characteristic is reflected from the stationary mirror, then at time $\Theta(t_+)$ it is reflected from the moving mirror; the next three reflections occur at times $F(t_+)$, $\Theta(F(t_+))$, and $F^2(t_+)$, respectively. Let t_1 be time after $F^2(t_+)$ but before the next reflection, and $x_1 = t_1 - F^2(t_+)$ be the spatial coordinate of the characteristic at time t_1. For the energy density we obtain from (22)

$$\begin{aligned}
\langle T_{00}(t_1,x_1)\rangle_{\rm reg} &= D(\Theta(F(t_+)))^2\,D(\Theta(t_+))^2\,\langle T_{00}(t_0,x_0)\rangle_{\rm reg} \\
&\quad + D(\Theta(F(t_+)))^2\,\frac{1}{24\pi}\frac{\mathscr{S}_F(t_+)}{F'(t_+)^2} + \frac{1}{24\pi}\frac{\mathscr{S}_F(F(t_+))}{F'(F(t_+))^2} \ .
\end{aligned} \tag{25}$$

The first term in the right-hand side of (25) is the initial energy density $\langle T_{00}(t_0,x_0)\rangle_{\rm reg}$ amplified by the factor of $D(\Theta(F(t_+)))^2 = [F'(t_+)]^{-2}$ at the reflection from the moving mirror at time $\Theta(t_+)$, and by the factor of $D(\Theta(F(t_+)))^2 = [F'(F(t_+))]^{-2}$ at the next reflection from the moving mirror at time $\Theta(F(t_+))$.

The second term in the right-hand side of (25) is the energy density created at the reflection of the characteristic from the moving mirror at time $\Theta(t_+)$, and consequently amplified by the factor of $D(\Theta(F(t_+)))^2$ at the next reflection from the moving mirror. Indeed, we have

$$\begin{aligned}\mathscr{S}_F(t_+) &= \mathscr{S}_{(\mathrm{Id}+a)\circ\Theta}(t_+) = [\Theta'(t_+)]^2\,\mathscr{S}_{\mathrm{Id}+a}(\Theta(t_+)) + \mathscr{S}_\Theta(t_+)\\ &= [\Theta'(t_+)]^2\,\left[\mathscr{S}_{\mathrm{Id}+a}(\Theta(t_+)) - \mathscr{S}_{\mathrm{Id}-a}(\Theta(t_+))\right]\\ &= 2\left.\frac{a'''(\xi)\left[1-a'(\xi)^2\right]+3a'(\xi)a''(\xi)^2}{\left[1-a'(\xi)\right]^4\,\left[1+a'(\xi)\right]^2}\right|_{\xi=\Theta(t_+)} .\end{aligned}$$

Together with (24), this equality implies

$$\begin{aligned}\frac{1}{24\pi}\frac{\mathscr{S}_F(t_+)}{F'(t_+)^2} &= \frac{1}{12\pi}\left.\frac{a'''(\xi)\left[1-a'(\xi)^2\right]+3a'(\xi)a''(\xi)^2}{\left[1-a'(\xi)\right]^2\,\left[1+a'(\xi)\right]^4}\right|_{\xi=\Theta(t_+)}\\ &= -\frac{1}{24\pi}\mathscr{S}_{\tilde{F}}\big((\mathrm{Id}+a)(\Theta(t_+))\big)\ ,\end{aligned} \tag{26}$$

and a comparison with (26) proves the correctness of our interpretation of the term $\frac{1}{24\pi}\frac{\mathscr{S}_F(t_+)}{F'(t_+)^2}$.

Similarly, the third term in the right-hand side of (25) is the energy density created by the moving mirror at time $\Theta(F(t_+))$.

This discussion elucidates the difference between the classical and the quantum cases – in the classical case the moving mirror amplifies the wave packets by squeezing them, while in the quantum case the moving mirror not only amplifies the already existing wave packets, but also creates new field which subsequently is amplified at each reflection from the moving mirror.

4.4. Resonant amplification in a periodically pulsating cavity

Now we will consider the particular case of periodic motion of the mirror when the corresponding CM f has rational rotation number, $\tau(f) = \sigma = p/q$, i.e., is in p/q phase locking. In this case the classical EM field in the cavity develops wave packets whose number can be anywhere between 1 and q depending on the initial conditions. The wave packets become narrower at each reflection, and their energy increases at each reflection.

In the quantum case, the motion of the mirror itself creates energy which is subsequently concentrated in narrow wave packets. If the acceleration of the the mirror is not zero except at isolated times, then the mirror is emitting energy all the time, and the number of wave packets developed is *exactly* q.

In the case of p/q phase locking, the CM f generically has an attracting q-periodic orbit, which corresponds to the times of reflection of the attracting characteristics. If t^* is such a time (i.e., if the fractional part of t^* belongs to the attracting q-periodic orbit of f), then

$$F^q(t^*) = t^* + p\ , \quad 0 < (F^q)'(t^*) < 1\ ,$$

which implies that asymptotically the cumulative Doppler factor $\mathcal{D}_q = [(F^q)'(t^*)]^{-1}$ (see (14)) is greater than 1. In the case of p/q phase locking, (22) implies (for any integer n)

$$\Phi(F^{nq}(t)) = \frac{\Phi(t)}{\left[(F^{nq})'(t)\right]^2} - \sum_{j=0}^{n-1}\frac{\mathscr{S}_{F^q}(F^{jq}(t))}{\left[(F^{(n-j)q})'(F^{jq}(t))\right]^2}\ . \tag{27}$$

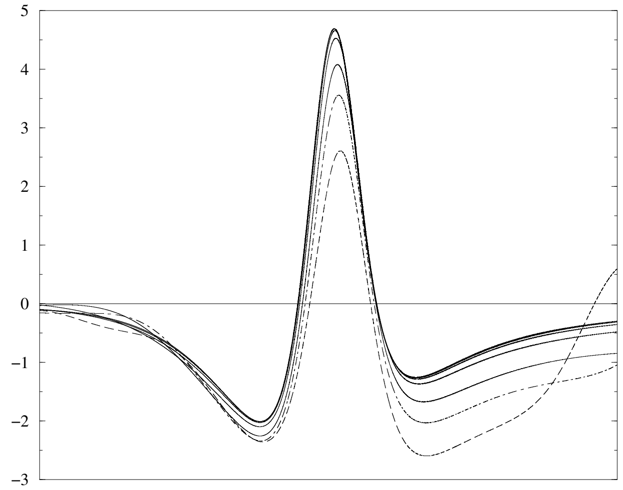

Figure 5. Rescaled "shape" of a wave packet at times 2 (dashed line), 3 (dot-dashed line), 4, 6, 8, 10, 12 (solid lines) (see the text).

If t^* is a time of reflection of an attracting characteristic, (27) yields

$$\begin{aligned}\Phi(t^*+np) &= \mathcal{D}_q^{2n}\left(\Phi(t^*) - \mathscr{S}_{F^q}(t^*)\sum_{j=0}^{n-1}\mathcal{D}_q^{-2j}\right)\\ &= \mathcal{D}_q^{2n}\left(\Phi(t^*) - \frac{1-\mathcal{D}_q^{-2n}}{1-\mathcal{D}_q^{-2}}\mathscr{S}_{F^q}(t^*)\right) \ . \end{aligned} \tag{28}$$

The applicability of this expression is not restricted to times like t^* – since asymptotically all characteristics are very close to the attracting ones, (28) gives approximately the asymptotic behavior of the energy density of the wave packets. Parenthetically, we remark that using this expression, one can prove that for motion of the mirror with parameters corresponding to the ends of the phase-locking intervals (i.e., the ends of the horizontal parts of the graph in Figure 3(b)), the energy density grows not exponentially, but polynomially:

$$\lim_{\mathcal{D}_q\to 1^+}\Phi(t^*+np) = \Phi(t^*) - n\mathscr{S}_{F^q}(t^*) \ .$$

The energy of the jth wave packet at time t is given by

$$\mathcal{E}_j(t) = \int_{\Delta_j(t)}\langle T_{00}(t,x)\rangle_{\mathrm{reg}}\,\mathrm{d}x = -\frac{1}{24\pi}\int_{\Delta_j(t)}\left[\Phi(t-x)+\Phi(t+x)\right]\,\mathrm{d}x \ ,$$

where $\Delta_j(t)$ is the support of the jth wave packet at time t. Recalling that at reflection at time t the width of the wave packet decreases $D(t)$ times, we obtain that asymptotically the energy of the field in the cavity increases exponentially: for large t,

$$\mathcal{E}_{\mathrm{total}}(t) \sim \mathrm{const}\cdot\mathcal{D}_q^{t/p} \ .$$

The energy density of the wave packets changes after each reflection from the moving mirror, but after appropriate rescaling, the its "shape" at times np as $n\to\infty$ tends to a some constant profile which depends on the motion of the mirror. In Figure 5, we show the evolution of the "shape" of a wave packet for mirror's motion same as in Figure 4(a). In the figure we show the rescaled energy density, $\mathcal{D}_q^{-2n}\cdot\langle T_{00}(n,x)\rangle_{\mathrm{reg}}$, on the vertical axis versus the shifted and rescaled spatial coordinate $\mathcal{D}_q^n\cdot(x-x_n^*)$ on the horizontal axis, at times n for several values of n; here x_n^* is the spatial coordinate of the attracting characteristic corresponding to this packet, at time n.

5. Concluding remarks

The power of the methods of theory of dynamical systems is due to their generality. The predictions we have made about the behavior of the field in the cavity are applicable to any motion of the mirror, not only to particular examples. Within our approach, we gave a complete classification of the possible resonances (phase locking) in the system, predicted that, generically, small detuning does not destroy the resonance, gave a simple explanation of the squeezing of the wave packets, interpreted the origin of the different contributions to the energy density in the cavity. We would also like to emphasize that our technique is non-perturbative.

In the case of resonance, the standard numerical methods for solving partial differential equations would be very difficult to apply because of the concentration of the field in narrow packets. The proposed method, however, relies on iterating one-dimensional maps, so that resonances do not present any additional difficulty. The computer programs used to produce the pictures in this paper took minutes to run on a PC.

Our methodology easily generalizes (see our paper [54]) to the case of quasiperiodic motion of the mirror, and the case of two moving mirrors (studied previously by Ji *et al* [37], Dodonov [19], Dalvit and Mazzitelli [10], Li and Li [43]). Recently, similar ideas from dynamical systems have been applied to the study of waves in a fluid in two-dimensional basin by Manders *et al* [48].

Interestingly, the behavior of the field of the cavity (described by a partial differential equation) is easier to analyze than the behavior of a particle bouncing back and forth between two perfectly reflecting walls (assuming that the reflections are perfectly elastic). The latter system, suggested by Fermi [27] as a possible mechanism for acceleration of the particles in the cosmic rays, reveals a much richer dynamical behavior (see, e.g., the book of Lichtenberg and Lieberman [44]).

There are many questions that deserve a further study. An interesting question is whether theory of dynamical systems can be applied to the case of a constant-length cavity filled with dielectric with changing properties. Another problem is the absence of resonances of certain type (noticed in our paper [45] and discussed by Węgrzyn [57]) – is it generic, and which resonances are forbidden? Can methods of dynamical systems be used to study the problem in higher dimensions? Can our methodology be applied to the recent suggestion by Jaffe and Scardicchio [35] to apply methods of geometric optics to the study of Casimir effect? Can similar methods be applied to other fields (see, e.g., the study of a classical massive field in a pulsating resonator by Dittrich and Duclos [16])?

The appearance of the Schwarzian derivative hints at possible deeper connections between the quantum problem of a moving mirror, partial differential equations, and dynamical systems.

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References

[1] M. A. Andreata and V. V. Dodonov. Energy density and packet formation in a vibrating cavity. *J. Phys. A*, 33(16):3209–3223, 2000.

[2] V. I. Arnol'd. Small denominators. I. Mapping the circle onto itself. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:21–86, 1961. Correction: 28:479–480, 1964; English translation: *Amer. Math. Soc. Transl. (2)*, 46:213–284, 1965.

[3] M. Bordag, U. Mohideen, and V. M. Mostepanenko. New developments in the Casimir effect. *Phys. Rep.*, 353(1-3):1–205, 2001.

[4] F. Burstall, F. Pedit, and U. Pinkall. Schwarzian derivatives and flows of surfaces. In *Differential Geometry and Integrable Systems (Tokyo, 2000)*, volume 308 of *Contemp. Math.*, pages 39–61. Amer. Math. Soc., Providence, RI, 2002. math.DG/0111169.

[5] C. K. Cole and W. C. Schieve. Radiation modes of a cavity with a moving boundary. *Phys. Rev. A*, 52(6):4405–4415, 1995.

[6] J. Cooper. Scattering of electromagnetic fields by a moving boundary: the one-dimensional case. *IEEE Trans. Antennas and Propagation*, 28(6):791–795, 1980.

[7] J. Cooper. Asymptotic behavior for the vibrating string with a moving boundary. *J. Math. Anal. Appl.*, 174(1):67–87, 1993.

[8] J. Cooper. Long-time behavior and energy growth for electromagnetic waves reflected by a moving boundary. *IEEE Trans. Antennas and Propagation*, 41(10):1365–1370, 1993.

[9] J. Cooper and H. Koch. The spectrum of a hyperbolic evolution operator. *J. Funct. Anal.*, 133(2):301–328, 1995.

[10] D. A. R. Dalvit and F. D. Mazzitelli. Creation of photons in an oscillating cavity with two moving mirrors. *Phys. Rev. A*, 59(4):3049–3059, 1999.

[11] A. M. Davie. The width of Arnold tongues for the sine circle map. *Nonlinearity*, 9(2):421–432, 1996.

[12] P. C. W. Davies and S. A. Fulling. Quantum vacuum energy in two dimensional space-times. *Proc. Roy. Soc. London Ser. A*, 354(1676):59–77, 1977.

[13] W. de Melo and S. van Strien. *One-Dimensional Dynamics.* Springer-Verlag, Berlin, 1993.

[14] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley, Redwood City, CA, second edition, 1989.

[15] B. S. DeWitt. Quantum field theory in curved space-time. *Phys. Rep.*, 19(6):295–357, 1975.

[16] J. Dittrich and P. Duclos. Massive scalar field in a one-dimensional oscillating region. *J. Phys. A*, 35(39):8213–8230, 2002.

[17] J. Dittrich, P. Duclos, and N. Gonzalez. Stability and instability of the wave equation solutions in a pulsating domain. *Rev. Math. Phys.*, 10(7):925–962, 1998.

[18] J. Dittrich, P. Duclos, and P. Šeba. Instability in a classical periodically driven string. *Phys. Rev. E*, 49(4):3535–3538, 1994.

[19] V. V. Dodonov. Resonance photon generation in a vibrating cavity. *J. Phys. A*, 31(49):9835–9854, 1998.

[20] V. V. Dodonov. Nonstationary Casimir effect and analytical solutions for quantum fields in cavities with moving boundaries. In *Contemporary Optics and Electrodynamics, part 1. Advances in Chemical Physics, vol. 119, part 1 (M. Evans, ed.)*, pages 309–394. John Wiley & Sons, New York, 2001. quant-ph/0106081.

[21] V. V. Dodonov and M. A. Andreata. Squeezing and photon distribution in a vibrating cavity. *J. Phys. A*, 32(39):6711–6726, 1999.

[22] V. V. Dodonov and A. B. Klimov. Generation and detection of photons in a cavity with a resonantly oscillating boundary. *Phys. Rev. A*, 53(4):2664–2682, 1996.

[23] V. V. Dodonov, A. B. Klimov, and V. I. Man'ko. Generation of squeesed states in a resonator with a moving wall. *Phys. Lett. A*, 149(4):225–228, 1990.

[24] V. V. Dodonov, A. B. Klimov, and D. E. Nikonov. Quantum phenomena in resonators with moving walls. *J. Math. Phys.*, 34(7):2742–2756, 1993.

[25] C. Duval and L. Guieu. The Virasoro group and Lorentzian surfaces: the hyperboloid of one sheet. *J. Geom. Phys.*, 33(1-2):103–127, 2000.

[26] C. Duval and V. Yu. Ovsienko. Lorentz world lines and the Schwarzian derivative. *Funktsional. Anal. i Prilozhen.*, 34(2):69–72, 2000. English translation: *Funct. Anal. Appl.*, 34(2):135–137 (2000).

[27] E. Fermi. On the origin of the cosmic radiation. *Phys. Rev. (2)*, 75(8):1169–1174, 1949.

[28] S. A. Fulling and P. C. W. Davies. Radiation from a moving mirror in two dimensional space-time: conformal anomaly. *Proc. Roy. Soc. London Ser. A*, 348(1654):393–414, 1976.

[29] J. Graczyk, D. Sands, and G. Świątek. La dérivée schwarzienne en dynamique unimodale. *C. R. Acad. Sci. Paris Sér. I*

Math., 332(4):329–332, 2001.

[30] J. Graczyk and G. Świątek. Critical circle maps near bifurcation. *Comm. Math. Phys.*, 176(2):227–260, 1996.

[31] B. Hasselblatt and A. Katok. *A First Course in Dynamics: With a Panorama of Recent Developments*. Cambridge University Press, New York, 2003.

[32] M.-R. Herman. Simple proofs of local conjugacy theorems for diffeomorphisms of the circle with almost every rotation number. *Bol. Soc. Brasil. Mat.*, 16(1):45–83, 1985.

[33] E. Hille. *Ordinary Differential Equations in the Complex Domain*. John Wiley & Sons, New York, 1976.

[34] M.-T. Jaekel and S. Reynaud. Motional Casimir force. *J. Physique I*, 2(2):142–165, 1992.

[35] R. L. Jaffe and A. Scardicchio. Casimir effect and geometric optics. *Phys. Rev. Lett.*, 92(7):070402, 4, 2004.

[36] M. H. Jensen, P. Bak, and T. Bohr. Transition to chaos by interaction of resonances in dissipative systems. I. Circle maps. *Phys. Rev. A (3)*, 30(4):1960–1969, 1984.

[37] J.-Y. Ji, H.-H. Jung, and K.-S. Soh. Interference phenomena in the photon production between two moving mirrors. *Phys. Rev. A*, 57(6):4952–4955, 1998.

[38] L. B. Jonker. The scaling of Arnol′d tongues for differentiable homeomorphisms of the circle. *Comm. Math. Phys.*, 129(1):1–25, 1990.

[39] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.

[40] B. Kostant and S. Sternberg. The Schwartzian [Schwarzian] derivative and the conformal geometry of the Lorentz hyperboloid. In *Quantum Theories and Geometry (Les Treilles, 1987)*, volume 10 of *Math. Phys. Stud.*, pages 113–125. Kluwer Acad. Publ., Dordrecht, 1988.

[41] O. E. Lanford, III. A numerical study of the likelihood of phase locking. *Phys. D*, 14(3):403–408, 1985.

[42] C. K. Law. Resonance responce of the quantum vacuum to an oscillating boundary. *Phys. Rev. Lett.*, 73(4):1931–1934, 1994.

[43] Ling Li and Bo-Zang Li. Numerical solutions of the generalized moore's equations for a one-dimensional cavity with two moving mirrors. *Phys. Lett. A*, 300(1):27–32, 2002.

[44] A. J. Lichtenberg and M. A. Lieberman. *Regular and Chaotic Dynamics*. Springer-Verlag, New York, second edition, 1992.

[45] R. de la Llave and N. P. Petrov. Theory of circle maps and the problem of one-dimensional optical resonator with a periodically moving wall. *Phys. Rev. E (3)*, 59(6):6637–6651, 1999.

[46] R. de la Llave and N. P. Petrov. Regularity of conjugacies between critical circle maps: An experimental study. *Experiment. Math.*, 11(2):219–241, 2002.

[47] G. J. Maclay and R. L. Forward. A *gedanken* spacecraft that operates using the quantum vacuum (dynamic Casimir effect). *Found. Phys.*, 34(3):477–500, 2004.

[48] A. M. M. Manders, J. J. Duistermaat, and L. R. M. Maas. Wave attractors in a smooth convex enclosed geometry. *Phys. D*, 186(3-4):109–132, 2003.

[49] O. Méplan and C. Gignoux. Exponential growth of a wave in a 1D vibrating cavity: Application to the quantum vacuum. *Phys. Rev. Lett.*, 76(3):408–410, 1996.

[50] K. A. Milton. *The Casimir effect: Physical Manifestations of Zero-Point Energy*. World Scientific, River Edge, NJ, 2001.

[51] G. T. Moore. Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity. *J. Math. Phys.*, 11(9):2679–2691, 1970.

[52] V. M. Mostepanenko and N. N. Trunov. *The Casimir Effect and Its Applications*. Clarendon Press, Oxford, 1997.

[53] Z. Nehari. *Conformal Mapping*. McGraw-Hill, New York, Toronto, London, 1952.

[54] N. P. Petrov, R. de la Llave, and J. A. Vano. Torus maps and the problem of one-dimensional optical resonator with a quasiperiodically moving wall. *Phys. D*, 180(3–4):140–184, 2003.

[55] D. Singer. Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.*, 35(2):260–267, 1978.

[56] D. A. Singer. Diffeomorphisms of the circle and hyperbolic curvature. *Conform. Geom. Dyn.*, 5:1–5 (electronic), 2001.

[57] P. Węgrzyn. Parametric resonance in a vibrating cavity. *Phys. Lett. A*, 322(5–6):263–269, 2004.

[58] Ying Wu, K. W. Chan, M.-C. Chu, and P. T. Leung. Radiation modes of a cavity with a resonantly oscillating boundary. *Phys. Rev. A*, 59(2):1662–1666, 1999.

[59] M. Yamaguchi. Quasiperiodic motions of vibrating string with periodically moving boundaries. *J. Differential Equations*, 135(1):1–15, 1997.

[60] M. Yamaguchi. Periodic motions of vibrating string with a periodically moving boundary. *Discrete Contin. Dynam. Systems*, (Added Volume II):303–314, 1998. Dynamical Systems and Differential Equations, Vol. II (Springfield, MO, 1996).

[61] M. Yamaguchi. Periodic solutions of nonlinear equations of string with periodically oscillating boundaries. *Funkcial. Ekvac.*, 45(3):397–416, 2002.

[62] M. Yamaguchi and H. Yoshida. Nonhomogeneous string problem with periodically moving boundaries. In *Operator Theory and Its Applications (Winnipeg, MB, 1998)*, pages 565–574. Amer. Math. Soc., Providence, RI, 2000.