TRACE FORMULAS AND BORG-TYPE THEOREMS FOR MATRIX-VALUED JACOBI AND DIRAC FINITE DIFFERENCE OPERATORS

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ABSTRACT. Borg-type uniqueness theorems for matrix-valued Jacobi operators H and supersymmetric Dirac difference operators D are proved. More precisely, assuming reflectionless matrix coefficients A, B in the self-adjoint Jacobi operator $H = AS^+ + A^-S^- + B$ (with $S^\pm$ the right/left shift operators on the lattice $\mathbb{Z}$) and the spectrum of H to be a compact interval $[E_-, E_+]$, $E_- < E_+$, we prove that A and B are certain multiples of the identity matrix. An analogous result which, however, displays a certain novel nonuniqueness feature, is proved for supersymmetric self-adjoint Dirac difference operators D with spectrum given by $\big[-E_+^{1/2}, -E_-^{1/2}\big] \cup \big[E_-^{1/2}, E_+^{1/2}\big]$, $0 \leq E_- < E_+$.

Our approach is based on trace formulas and matrix-valued (exponential) Herglotz representation theorems. As a by-product of our techniques we obtain the extension of Flaschka's Borg-type result for periodic scalar Jacobi operators to the class of reflectionless matrix-valued Jacobi operators.

1. Introduction

As discussed in detail in [23], while various aspects of inverse spectral theory for scalar Schrödinger, Jacobi, and Dirac-type operators, and more generally, for 2×2 Hamiltonian systems, are well-understood by now, the corresponding theory for such operators and Hamiltonian systems with $m \times m$, $m \in \mathbb{N}$, matrix-valued coefficients is still largely a wide open field. A particular inverse spectral theory aspect we have in mind is that of determining isospectral sets (manifolds) of such systems. In this context it may, perhaps, come as a surprise that even determining the isospectral set of Hamiltonian systems with matrix-valued periodic coefficients is an open problem. The present paper is a modest attempt toward a closer investigation of inverse spectral problems in connection with Borg-type uniqueness theorems for matrix-valued Jacobi operators and Dirac-type finite difference systems. It should be mentioned that these problems are not just of interest in a spectral theoretic context, but due to their implications for other areas such as completely integrable systems (e.g., the nonabelian Toda and Kac–van Moerbeke hierarchies), are of interest to a larger audience.

Before we describe the content of this paper in more detail, we briefly comment on background literature for matrix-valued Jacobi and Dirac-type difference operators. Spectral and Weyl–Titchmarsh theory for Jacobi operators can be found in [4, Sect. VII.2], [20], [47], [49, Ch. 10] and the literature therein. The case of Dirac finite

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difference operators was discussed in detail in [12]. Deficiency indices of matrix-valued Jacobi operators are studied in [35]–[37]. Inverse spectral and scattering theory for matrix-valued finite difference systems and its intimate connection to matrix-valued orthogonal polynomials and the moment problem are treated in [1], [2], [4, Sect. VII.2], [16]–[18], [21], [39], [40], [45], [46], [47], [49, Ch. 8], [50]. A number of uniqueness theorems for matrix-valued Jacobi operators were proved in [23]. Finally, connections with nonabelian completely integrable systems are discussed in [5], [6], [44], [48], [49, Chs. 9, 10].

Let $\mathbb{C}(\mathbb{Z})^{r\times s}$ be the space of sequences of complex $r \times s$ matrices, $r, s \in \mathbb{N}$. We also denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in the following.

The matrix-valued Jacobi operators H in $\ell^2(\mathbb{Z})^m$ discussed in this paper are of the form

$$H = AS^+ + A^-S^- + B, \quad \mathcal{D}(H) = \ell^2(\mathbb{Z})^m. \tag{1.1}$$

Here $S^\pm$ denote the shift operators acting upon $\mathbb{C}(\mathbb{Z})^{r\times s}$, $r, s \in \mathbb{N}$, as

$$S^\pm f(\,\cdot\,) = f^\pm(\,\cdot\,) = f(\,\cdot\, \pm 1), \quad f \in \mathbb{C}(\mathbb{Z})^{r\times s}, \tag{1.2}$$

and $A = \{A(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}$, $A(k) > 0$ for all $k \in \mathbb{Z}$, $B = \{B(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}$, $B(k) = B(k)^*$ for all $k \in \mathbb{Z}$ and $m \in \mathbb{N}$.

The Dirac-type finite difference operators D in $\ell^2(\mathbb{Z})^m \oplus \ell^2(\mathbb{Z})^m$ studied in this paper are of the form

$$D = S_\rho + X = \begin{pmatrix} 0 & \rho S^+ + \chi^* \\ \rho^- S^- + \chi & 0 \end{pmatrix}, \quad \mathcal{D}(D) = \ell^2(\mathbb{Z})^m \oplus \ell^2(\mathbb{Z})^m, \tag{1.3}$$

where $\rho = \{\rho(k)\}_{k\in\mathbb{Z}} \subset \mathbb{C}(\mathbb{Z})^{m\times m}$ and S_ρ and X are of the block form

$$S_\rho = \begin{pmatrix} 0 & \rho S^+ \\ \rho^- S^- & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \chi^* \\ \chi & 0 \end{pmatrix} \tag{1.4}$$

with $\rho(k), \chi(k) \in \mathbb{C}^{m\times m}$ invertible for all $k \in \mathbb{Z}$. Moreover, following [12], we may assume without loss of generality that for all $k \in \mathbb{Z}$, $\rho(k)$ is a positive definite diagonal $m \times m$ matrix (cf. Remark 5.3 for details)).

Additional assumptions on the coefficients $A(k)$, $B(k)$, $\rho(k)$, and $\chi(k)$, $k \in \mathbb{Z}$, will be formulated in Sections 2 and 5, respectively.

The difference operators H and D represent the natural matrix-valued generalizations of Lax operators arising in connection with Kac-van Moerbeke and Toda lattices (cf. [8], [22] and the references therein) and hence lead to nonabelian Toda and Kac–van Moerbeke hierarchies of completely integrable nonlinear evolution equations.

Next we briefly describe the history of Borg-type theorems relevant to this paper. In 1946, Borg [7] proved, among a variety of other inverse spectral theorems, the following result. (We denote by $\sigma(\cdot)$ and $\sigma_{\rm ess}(\cdot)$ the spectrum and essential spectrum of a densely defined closed linear operator.)

Theorem 1.1 ([7]). *Let $q \in L^1_{\rm loc}(\mathbb{R})$ be real-valued and periodic. Let $h = -\frac{d^2}{dx^2} + q$ be the associated self-adjoint Schrödinger operator in $L^2(\mathbb{R})$ and suppose that $\sigma(h) = [e_0, \infty)$ for some $e_0 \in \mathbb{R}$. Then q is of the form,*

$$q(x) = e_0 \text{ for a.e. } x \in \mathbb{R}. \tag{1.5}$$

Remark 1.2. Traditionally, uniqueness results such as Theorem 1.1 are called Borg-type theorems. This terminology, although generally accepted, is a bit unfortunate as the same term is also used for other theorems Borg proved in his celebrated 1946 paper [7]. Indeed, inverse spectral results on finite intervals in which the potential coefficient(s) are recovered from several spectra were also pioneered by Borg in [7] and theorems of this kind are now also described as Borg-type theorems in the literature, see, e.g., [41]–[43].

A closer examination of the proof of Theorem 1.1 in [13] shows that periodicity of q is not the point for the uniqueness result (1.5). The key ingredient (besides $\sigma(h) = [e_0, \infty)$ and q real-valued) is the fact that

$$\text{for all } x \in \mathbb{R}, \ \xi(\lambda, x) = 1/2 \ \text{ for a.e. } \lambda \in \sigma_{\text{ess}}(h). \tag{1.6}$$

Here $\xi(\lambda, x)$, the argument of the boundary value $g(\lambda+i0, x)$ of the diagonal Green's function of h on the real axis (where $g(z, x) = (h - zI)^{-1}(x, x)$), $z \in \mathbb{C}\backslash\sigma(h)$), is defined by

$$\xi(\lambda, x) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im}(\ln(g(\lambda + i\varepsilon, x)) \ \text{ for a.e. } \lambda \in \mathbb{R}. \tag{1.7}$$

Real-valued periodic potentials are known to satisfy (1.6), but so do certain classes of real-valued quasi-periodic and almost-periodic potentials q. In particular, the class of real-valued algebro-geometric finite-gap KdV potentials q (a subclass of the set of real-valued quasi-periodic potentials) is a prime example satisfying (1.6) without necessarily being periodic. Traditionally, potentials q satisfying (1.6) are called *reflectionless* (see [11], [13] and the references therein).

The extension of Borg's Theorem 1.1 to periodic matrix-valued Schrödinger operators was proved by Dépres [15]. A new strategy of the proof based on exponential Herglotz representations and a trace formula for such potentials, as well as the extension to reflectionless matrix-valued potentials, was obtained in [13].

The direct analog of Borg's Theorem 1.1 for scalar (i.e., $m = 1$) periodic Jacobi operators was proved by Flaschka [19] in 1975.

Theorem 1.3 ([19])**.** *Suppose a and b are periodic real-valued sequences in $\ell^\infty(\mathbb{Z})$ with the same period and $a(k) > 0$, $k \in \mathbb{Z}$. Let $h = aS^+ + a^- S^- + b$ be the associated self-adjoint Jacobi operator in $\ell^2(\mathbb{Z})$ and suppose that $\sigma(h) = [E_-, E_+]$ for some $E_- < E_+$. Then a and b are of the form,*

$$a(k) = (E_+ - E_-)/4, \quad b(k) = (E_- + E_+)/2, \quad k \in \mathbb{Z}. \tag{1.8}$$

The extension of Theorem 1.3 to reflectionless scalar Jacobi operators is due to Teschl [53, Corollary 6.3] (see also [54, Corollary 8.6]). As one of the principal results in this paper we will extend Theorem 1.3 to matrix-valued reflectionless Jacobi operators H of the type (1.1) in Section 4. This extension (and the special case of periodic matrix-valued Jacobi operators) is new. In addition, we will prove a Borg-type theorem for the supersymmetric Dirac difference operator D in (1.3) which is new even in the simplest case $m = 1$. The latter displays an interesting nonuniqueness feature which has not previously been encountered with Borg-type theorems.

In Section 2 we review the basic Weyl–Titchmarsh theory and the corresponding Green's matrices for matrix-valued Jacobi operators on $\mathbb{Z}$ and on a half-lattice. Section 3 is devoted to asymptotic expansions of Weyl–Titchmarsh and Green's matrices as the (complex) spectral parameter tends to infinity. Section 4 contains the derivation of a trace formula for Jacobi operators and one of our principal new results, the proof of a Borg-type theorem for matrix-valued Jacobi operators. Finally, Section 5 presents a Borg-type theorem for supersymmetric Dirac-type difference operators.

2. Weyl–Titchmarsh and Green's Matrices for Matrix-Valued Jacobi Operators

In this section we consider Weyl–Titchmarsh and Green's matrices for matrix-valued Jacobi operators on $\mathbb{Z}$ and on a half-lattice.

We closely follow the treatment of matrix-valued Jacobi operators in [23]. As the basic hypothesis in this section we adopt the following set of asumptions.

Hypothesis 2.1. *Let $m \in \mathbb{N}$ and consider the sequences of self-adjoint $m \times m$ matrices*

$$\begin{aligned} &A = \{A(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}, \quad A(k) > 0,\ k \in \mathbb{Z}, \\ &B = \{B(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}, \quad B(k) = B(k)^*,\ k \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

Moreover, assume that $A(k)$ and $B(k)$ are uniformly bounded with respect to $k \in \mathbb{Z}$, that is, there exists a $C > 0$, such that

$$\|A(k)\|_{\mathbb{C}^{m\times m}} + \|B(k)\|_{\mathbb{C}^{m\times m}} \leq C, \quad k \in \mathbb{Z}. \tag{2.2}$$

Next, denote by $S^{\pm}$ the shift operators in $\mathbb{C}(\mathbb{Z})^{r\times s}$, $r, s \in \mathbb{N}$,

$$(S^{\pm}g)(k) = g^{\pm}(k) = g(k \pm 1), \quad g \in \mathbb{C}(\mathbb{Z})^{r\times s},\ k \in \mathbb{Z}. \tag{2.3}$$

Given Hypothesis 2.1, the matrix-valued self-adjoint Jacobi operator H with domain $\ell^2(\mathbb{Z})^m$ is then defined by

$$H = AS^+ + A^-S^- + B, \quad \mathcal{D}(H) = \ell^2(\mathbb{Z})^m. \tag{2.4}$$

Because of hypothesis (2.2), H is a bounded symmetric operator and hence self-adjoint. In particular, the difference expression $AS^+ + A^-S^- + B$ induced by (2.4) is in the limit point case at $\pm\infty$. We chose to adopt (2.2) for simplicity only. Our formalism extends to unbounded Jacobi operators and to cases where the difference expression associated with (2.4) is in the limit circle case at $+\infty$ and/or $-\infty$ (cf. [12]). We just note in passing that without assuming (2.2), the difference expression $AS^+ + A^-S^- + B$ is in the limit point case at $\pm\infty$ if $\sum_{k=k_0}^{\pm\infty} \|A(k)\|_{\mathbb{C}^{m\times m}}^{-1} = \infty$ (see, e.g., [4, Theorem VII.2.9]). Whenever we need to stress the dependence of H on A, B we will write $H(A,B)$ instead of H.

The Green's matrix associated with H will be denoted by $G(z,k,\ell)$ in the following,

$$G(z,k,\ell) = (H - zI_m)^{-1}(k,\ell), \quad z \in \mathbb{C}\backslash\mathbb{R},\ k,\ell \in \mathbb{Z}. \tag{2.5}$$

Next, fix a site $k_0 \in \mathbb{Z}$ and define $m \times m$ matrix-valued solutions $\phi(z,k,k_0)$ and $\theta(z,k,k_0)$ of the equation

$$A(k)\psi(z,k{+}1) + A(k{-}1)\psi(z,k{-}1) + (B(k){-}zI_m)\psi(z,k) = 0, \quad z \in \mathbb{C},\ k \in \mathbb{Z}, \tag{2.6}$$

satisfying the initial conditions

$$\theta(z,k_0,k_0) = \phi(z,k_0+1,k_0) = I_m, \quad \phi(z,k_0,k_0) = \theta(z,k_0+1,k_0) = 0. \tag{2.7}$$

One then introduces $m \times m$ matrix-valued Weyl–Titchmarsh solutions $\psi_{\pm}(z,k,k_0)$ associated with H defined by

$$\psi_{\pm}(z,k,k_0) = \theta(z,k,k_0) - \phi(z,k,k_0)A(k_0)^{-1}M_{\pm}(z,k_0), \quad z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z}, \tag{2.8}$$

with the properties

$$A(k)\psi_{\pm}(z,k+1,k_0) + A(k-1)\psi_{\pm}(z,k-1,k_0) + (B(k) - zI_m)\psi_{\pm}(z,k,k_0) = 0, \quad z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z}, \tag{2.9}$$

$$\psi_\pm(z,\cdot,k_0) \in \ell^2((k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}, \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.10}$$

where $M_\pm(z,k_0)$ denote the half-line Weyl–Titchmarsh matrices associated with H. Since by assumption $AS^+ + A^-S^- + B$ is in the limit point case at $\pm\infty$, $M_\pm(z,k_0)$ in (2.8) are uniquely determined by the requirement (2.10). We also note that by a standard argument,

$$\det(\phi(z,k,k_0)) \neq 0 \text{ for all } k\in\mathbb{Z}\backslash\{k_0\} \text{ and } z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.11}$$

since otherwise one could construct a Dirichlet-type eigenvalue $z\in\mathbb{C}\backslash\mathbb{R}$ for H restricted to the finite segment $k_0+1,\dots,k-1$ for $k\geq k_0+1$ of $\mathbb{Z}$ (and similarly for $k\leq k_0-1$). Thus, introducing

$$M_N(z,k_0) = -\phi(z,N,k_0)^{-1}\theta(z,N,k_0), \quad z\in\mathbb{C}\backslash\mathbb{R},\ N\in\mathbb{Z}\backslash\{k_0\}, \tag{2.12}$$

one can then compute $M_\pm(z,k_0)$ by the limiting relation

$$M_\pm(z,k_0) = \lim_{N\to\pm\infty} M_N(z,k_0), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.13}$$

the limit being unique since $AS^+ + A^-S^- + B$ is in the limit point case at $\pm\infty$. Alternatively, (2.8) yields

$$M_\pm(z,k_0) = -A(k_0)\psi_\pm(z,k_0+1,k_0), \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.14}$$

More generally, recalling

$$\det(\psi_\pm(z,k,k_0)) \neq 0 \text{ for all } k\in\mathbb{Z} \text{ and } z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.15}$$

by an argument analogous to that following (2.11), we now introduce

$$M_\pm(z,k) = -A(k)\psi_\pm(z,k+1,k_0)\psi_\pm(z,k,k_0)^{-1}, \quad z\in\mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{Z}. \tag{2.16}$$

One easily verifies that $M_\pm(z,k)$ represent the Weyl–Titchmarsh M-matrices associated with the reference point $k\in\mathbb{Z}$. Moreover, one obtains the Riccati-type equation for $M_\pm(z,k)$,

$$M_\pm(z,k) + A(k-1)M_\pm(z,k-1)^{-1}A(k-1) + zI_m - B(k) = 0, \quad z\in\mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{Z} \tag{2.17}$$

as a result of (2.9). For later reference we summarize the principal results on $M_\pm(z,k_0)$ in the following theorem. (We denote as usual $\mathrm{Re}(M) = (M+M^*)/2$, $\mathrm{Im}(M) = (M^*-M)/(2i)$, etc., for square matrices M.)

Theorem 2.2 ([3], [9], [30]–[33], [38])**.** *Assume Hypothesis* 2.1 *and suppose that* $z\in\mathbb{C}\backslash\mathbb{R}$*, and* $k_0\in\mathbb{Z}$*. Then,*
(i) $\pm M_\pm(z,k_0)$ *is a matrix-valued Herglotz function of maximal rank. In particular,*

$$\mathrm{Im}(\pm M_\pm(z,k_0)) > 0, \quad z\in\mathbb{C}_+, \tag{2.18}$$

$$M_\pm(\bar z,k_0) = M_\pm(z,k_0)^*, \tag{2.19}$$

$$\mathrm{rank}(M_\pm(z,k_0)) = m, \tag{2.20}$$

$$\lim_{\varepsilon\downarrow 0} M_\pm(\lambda+i\varepsilon,k_0) \text{ exists for a.e. } \lambda\in\mathbb{R}. \tag{2.21}$$

Isolated poles of $\pm M_\pm(z,k_0)$ *and* $\mp M_\pm(z,k_0)^{-1}$ *are at most of first order, are real, and have a nonpositive residue.*
(ii) $\pm M_\pm(z,k_0)$ *admit the representations*

$$\pm M_\pm(z,k_0) = K_\pm(k_0) + L_\pm(k_0)z + \int_{\mathbb{R}} d\Omega_\pm(\lambda,k_0)\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right) \tag{2.22}$$

$$= \exp\left[C_\pm(k_0) + \int_{\mathbb{R}} d\lambda\, \Xi_\pm(\lambda, k_0)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)\right], \tag{2.23}$$

where

$$K_\pm(k_0) = K_\pm(k_0)^*,\ L_+(k_0) = 0,\ L_-(k_0) = L_-(k_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega_\pm(\lambda, k_0)\|}{1+\lambda^2} < \infty, \tag{2.24}$$

$$C_\pm(k_0) = C_\pm(k_0)^*, \quad 0 \leqslant \Xi_\pm(\,\cdot\,, k_0) \leqslant I_m \text{ a.e.} \tag{2.25}$$

Moreover,

$$\Omega_\pm((\lambda, \mu], k_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(\pm M_\pm(\nu + i\varepsilon, k_0)), \tag{2.26}$$

$$\Xi_\pm(\lambda, k_0) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\mathrm{Im}(\ln(\pm M_\pm(\lambda + i\varepsilon, k_0)) \ \textit{for a.e. } \lambda \in \mathbb{R}. \tag{2.27}$$

Next, we define the self-adjoint half-line Jacobi operators $H_{\pm,k_0}$ on $\ell^2([k_0, \pm\infty)\cap \mathbb{Z})^m$ by

$$H_{\pm,k_0} = P_{\pm,k_0} H P_{\pm,k_0}\big|_{\ell^2((k_0,\pm\infty)\cap\mathbb{Z})^m}, \quad k_0 \in \mathbb{Z}, \tag{2.28}$$

where $P_{\pm,k_0}$ are the orthogonal projections onto the subspaces $\ell^2([k_0, \pm\infty) \cap \mathbb{Z})^m$. In addition, Dirichlet boundary conditions at $k_0 \mp 1$ are associated with $H_{\pm,k_0}$,

$$\begin{aligned}
&(H_{+,k_0} f)(k_0) = A(k_0) f^+(k_0) + B(k_0) f(k_0),\\
&(H_{-,k_0} f)(k_0) = A^-(k_0) f^-(k_0) + B(k_0) f(k_0),\\
&(H_{\pm,k_0} f)(k) = A(k) f^+(k) + A^-(k) f^-(k) + B(k) f(k), \quad k \gtreqless k_0 \pm 1,\\
&f \in \mathcal{D}(H_{\pm,k_0}) = \ell^2([k_0, \pm\infty) \cap \mathbb{Z})^m
\end{aligned} \tag{2.29}$$

(i.e., formally, $f(k_0 \mp 1) = 0$). We also introduce m-functions $m_\pm(z, k_0)$ associated with $H_{\pm,k_0}$ by

$$m_\pm(z, k_0) = Q_{k_0}(H_{\pm,k_0} - zI_m)^{-1} Q_{k_0}, \tag{2.30}$$

$$= G_{\pm,k_0}(z, k_0, k_0), \quad z \in \mathbb{C}\backslash\mathbb{R},\ k_0 \in \mathbb{Z}. \tag{2.31}$$

Here Q_{k_0} are orthogonal projections onto the m-dimensional subspaces $\ell^2(\{k_0\})^m$, $k_0 \in \mathbb{Z}$ and

$$G_{\pm,k_0}(z, k, \ell) = (H_{\pm,k_0} - zI_m)^{-1}(k, \ell), \quad z \in \mathbb{C}\backslash\mathbb{R},\ k, \ell \in \mathbb{Z} \cap [k_0, \pm\infty) \tag{2.32}$$

represent the Green's matrices of $H_{\pm,k_0}$.

In order to find the connection between $m_\pm(z, k_0)$ and $M_\pm(z, k_0)$ we briefly discuss the Green's matrices $G_{\pm,k_0}(z, k, \ell)$ and $G(z, k, \ell)$ associated with $H_{\pm,k_0}$ and H next.

First we recall the definition of the Wronskian $W(f, g)(k)$ of two sequences of matrices $f(\cdot), g(\cdot) \in \mathbb{C}(\mathbb{Z})^{m\times m}$ given by

$$W(f, g)(k) = f(k)A(k)g(k + 1) - f(k + 1)A(k)g(k), \quad k \in \mathbb{Z}. \tag{2.33}$$

We note that for any two matrix-valued solutions $\varphi(z, \cdot)$ and $\psi(z, \cdot)$ of (2.6) the Wronskian $W(\varphi(\overline{z}, \cdot)^*, \psi(z, \cdot))(k)$ is independent of $k \in \mathbb{Z}$.

In complete analogy to the scalar Jacobi case (i.e., $m = 1$) one verifies,

$$G_{+,k_0}(z, k, \ell) = \begin{cases} -\psi_+(z, k, k_0 - 1)A(k_0 - 1)^{-1}\phi(\overline{z}, \ell, k_0 - 1)^*, & \ell \le k,\\ -\phi(z, k, k_0 - 1)A(k_0 - 1)^{-1}\psi_+(\overline{z}, \ell, k_0 - 1)^*, & \ell \ge k, \end{cases} \tag{2.34}$$

$$z \in \mathbb{C}\backslash\mathbb{R},\ k,\ell \in \mathbb{Z} \cap [k_0,\infty).$$

$$G_{-,k_0}(z,k,\ell) = \begin{cases} \phi(z,k,k_0+1)A(k_0+1)^{-1}\psi_-(\overline{z},\ell,k_0+1)^*, & \ell \le k, \\ \psi_-(z,k,k_0+1)A(k_0+1)^{-1}\phi(\overline{z},\ell,k_0+1)^*, & \ell \ge k, \end{cases} \tag{2.35}$$
$$z \in \mathbb{C}\backslash\mathbb{R},\ k,\ell \in \mathbb{Z} \cap (-\infty,k_0].$$

Similarly, using the fact that

$$\begin{aligned} &\psi_+(z,k,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_-(\overline{z},k,k_0)^* \\ &= \psi_-(z,k,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_+(\overline{z},k,k_0)^* \\ &= [M_-(z,k) - M_+(z,k)]^{-1}, \quad k \in \mathbb{Z}, \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} &A(k)\psi_+(z,k+1,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_-(\overline{z},k,k_0)^* \\ &- A(k)\psi_-(z,k+1,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_+(\overline{z},k,k_0)^* = I_m, \quad k \in \mathbb{Z}, \end{aligned} \tag{2.37}$$

one verifies

$$G(z,k,\ell) = \begin{cases} \psi_+(z,k,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_-(\overline{z},\ell,k_0)^*, & \ell \le k, \\ \psi_-(z,k,k_0)[M_-(z,k_0) - M_+(z,k_0)]^{-1}\psi_+(\overline{z},\ell,k_0)^*, & \ell \ge k, \end{cases} \tag{2.38}$$
$$z \in \mathbb{C}\backslash\mathbb{R},\ k,\ell \in \mathbb{Z}.$$

Using (2.7), (2.9), (2.17), and (2.31), one infers that the Weyl–Titchmarsh matrices $M_\pm(z,k)$ introduced by (2.13) (resp. (2.14)) and the m-functions $m_\pm(z,k)$ defined in (2.30) are related by

$$M_+(z,k) = -m_+(z,k)^{-1} - zI_m + B(k), \quad z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z} \tag{2.39}$$

and

$$M_-(z,k) = m_-(z,k)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z}. \tag{2.40}$$

In analogy to (2.17), $m_\pm(z,k)$ also satisfy Riccati-type equations of the form

$$\begin{aligned} &A(k-1)m_+(z,k)A(k-1)m_+(z,k-1) + (zI_m - B(k-1))m_+(z,k-1) + I_m = 0, \\ &\hspace{20em} z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z} \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} &A(k-1)m_-(z,k-1)A(k-1)m_-(z,k) + (zI_m - B(k))m_-(z,k) + I_m = 0, \\ &\hspace{20em} z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z}. \end{aligned} \tag{2.42}$$

Next, we introduce the $2m \times 2m$ Weyl–Titchmarsh matrix $M(z,k)$ associated with the Jacobi operator H in $\ell^2(\mathbb{Z})^m$ by

$$M(z,k) = \big(M_{j,j'}(z,k)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ k \in \mathbb{Z}, \tag{2.43}$$

where

$$M_{1,1}(z,k) = [M_-(z,k) - M_+(z,k)]^{-1}, \tag{2.44}$$
$$M_{1,2}(z,k) = 2^{-1}[M_-(z,k) - M_+(z,k)]^{-1}[M_-(z,k) + M_+(z,k)], \tag{2.45}$$
$$M_{2,1}(z,k) = 2^{-1}[M_-(z,k) + M_+(z,k)][M_-(z,k) - M_+(z,k)]^{-1}, \tag{2.46}$$
$$M_{2,2}(z,k) = M_\pm(z,k)[M_-(z,k) - M_+(z,k)]^{-1}M_\mp(z,k). \tag{2.47}$$

One verifies that $M(z,k)$ is a $2m\times 2m$ Herglotz matrix with the following properties:

Theorem 2.3 ([3], [9], [30]–[33]). *Assume Hypothesis* 2.1, $z\in\mathbb{C}\backslash\mathbb{R}$, *and* $k_0\in\mathbb{Z}$. *Then,* $M(z,k_0)$ *is a matrix-valued Herglotz function of rank* $2m$ *with representations*

$$M(z,k_0) = K(k_0) + \int_{\mathbb{R}} d\Omega(\lambda,k_0)\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right) \tag{2.48}$$

$$= \exp\left[C(k_0) + \int_{\mathbb{R}} d\lambda\, \Upsilon(\lambda,k_0)\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right)\right], \tag{2.49}$$

where

$$K(k_0) = K(k_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega(\lambda,k_0)\|}{1+\lambda^2} < \infty, \tag{2.50}$$

$$C(k_0) = C(k_0)^*, \quad 0 \leqslant \Upsilon(\,\cdot\,,k_0) \leqslant I_{2m} \ \text{a.e.} \tag{2.51}$$

Moreover,

$$\Omega((\lambda,\mu],k_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(M(\nu+i\varepsilon,k_0)), \tag{2.52}$$

$$\Upsilon(\lambda,k_0) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}(\ln(M(\lambda+i\varepsilon,k_0))) \ \textit{for a.e. } \lambda\in\mathbb{R}. \tag{2.53}$$

Remark 2.4. We note that the Weyl–Titchmarsh matrix $M(z,k)$ is related to the Green's matrix associated with the Jacobi operator H by

$$M(z,k) = \begin{pmatrix} I_m & 0 \\ 0 & -A(k)\end{pmatrix}\mathcal{M}(z,k)\begin{pmatrix} I_m & 0 \\ 0 & -A(k)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & I_m \\ I_m & 0\end{pmatrix}, \tag{2.54}$$

where

$$\mathcal{M}(z,k) = \begin{pmatrix} G(z,k,k) & G(z,k,k+1) \\ G(z,k+1,k) & G(z,k+1,k+1)\end{pmatrix}. \tag{2.55}$$

With $M(z,k)$ defined in (2.43), the following uniqueness theorem holds.

Theorem 2.5. *Assume Hypothesis* 2.1 *and let* $k_0\in\mathbb{Z}$. *Then the* $2m\times 2m$ *Weyl–Titchmarsh matrix* $M(z,k_0)$ *for all* $z\in\mathbb{C}_+$ *uniquely determines the Jacobi operator* H *and hence* $A=\{A(k)\}_{k\in\mathbb{Z}}$ *and* $B=\{B(k)\}_{k\in\mathbb{Z}}$.

Perhaps the simplest way to prove Theorem 2.5 is to reduce it to knowledge of $M_\pm(z,k_0)$, and hence by (2.39) and (2.40) to that of $m_\pm(z,k_0)$ for all $z\in\mathbb{C}_+$. (We note that the knowledge of $B(k_0)$, which is required according to (2.39), can be determined from the asymptotics of $M_-(z,k_0)$ in (3.8).) Using the standard construction of orthogonal matrix-valued polynomials with respect to the normalized measure in the Herglotz representation of $m_\pm(z,k_0)$,

$$m_\pm(z,k_0) = \int_{\mathbb{R}} d\nu_\pm(\lambda,k_0)\,(\lambda-z)^{-1}, \quad z\in\mathbb{C}_+, \quad \int_{\mathbb{R}} d\nu_\pm(\lambda,k_0) = I_m, \tag{2.56}$$

allows one to reconstruct $A(k)$, $B(k)$, $k\in[k_0,\pm\infty)\cap\mathbb{Z}$ from the measures $d\nu_\pm(\lambda,k_0)$ (cf., e.g., [4, Section VII.2.8]). More precisely,

$$\begin{aligned} A(k_0\pm k) &= \int_{\mathbb{R}} \lambda P_{\pm,k}(\lambda,k_0) d\nu_\pm(\lambda,k_0)\, P_{\pm,k+1}(\lambda,k_0)^*, \\ B(k_0\pm k) &= \int_{\mathbb{R}} \lambda P_{\pm,k}(\lambda,k_0) d\nu_\pm(\lambda,k_0)\, P_{\pm,k}(\lambda,k_0)^*, \quad k\in\mathbb{N}_0, \end{aligned} \tag{2.57}$$

where $\{P_{\pm,k}(\lambda,k_0)\}_{k\in\mathbb{N}_0}$ is an orthonormal system of matrix-valued polynomials with respect to the spectral measure $d\nu_\pm(\lambda,k_0)$, with $P_{\pm,k}(z,k_0)$ of degree k in z, $P_{\pm,0}(z,k_0)=I_m$. One verifies,

$$P_{+,k}(z,k_0)=\phi(z,k_0+k,k_0-1), \tag{2.58}$$

$$P_{-,k}(z,k_0)=\theta(z,k_0-k,k_0), \quad k\in\{-1\}\cup\mathbb{N}_0,\ k_0\in\mathbb{Z},\ z\in\mathbb{C}, \tag{2.59}$$

with $\phi(z,k,k_0)$ and $\theta(z,k,k_0)$ defined in (2.6), (2.7).

Given these preliminaries and introducing the diagonal Green's matrix by

$$g(z,k)=G(z,k,k), \quad z\in\mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{Z}, \tag{2.60}$$

we can also formulate the following uniqueness result for Jacobi operators obtained in [23].

Theorem 2.6 ([23]). *Assume Hypothesis* 2.1 *and let* $k_0\in\mathbb{Z}$. *Then any of the following three sets of data*
(i) $g(z,k_0)$ *and* $G(z,k_0,k_0+1)$ *for all* $z\in\mathbb{C}_+$;
(ii) $g(z,k_0)$ *and* $[G(z,k_0,k_0+1)+G(z,k_0+1,k_0)]$ *for all* $z\in\mathbb{C}_+$;
(iii) $g(z,k_0)$, $g(z,k_0+1)$ *for all* $z\in\mathbb{C}_+$ *and* $A(k_0)$;
uniquely determines the matrix-valued Jacobi operator H *and hence* $A=\{A(k)\}_{k\in\mathbb{Z}}$ *and* $B=\{B(k)\}_{k\in\mathbb{Z}}$.

The special scalar case $m=1$ of Theorem 2.6 is known and has been derived in [53] (see also [54, Sect. 2.7]. Condition (iii) in Theorem 2.6 is specific to the Jacobi case. In the corresponding Schrödinger case, the corresponding set of data does not even uniquely determine the potential in the scalar case $m=1$ (see, e.g., [27], [53]).

3. Asymptotic Expansions of Weyl–Titchmarsh and Green's Matrices

In this section we use Riccati-type equations to systematically determine norm convergent expansions of Weyl–Titchmarsh and Green's matrices as the spectral parameter tends to infinity.

We start again with the case of Jacobi operators H, assuming Hypothesis 2.1.

Insertion of the norm convergent ansatz

$$m_\pm(z,k) \underset{|z|\to\infty}{=} \sum_{j=1}^{\infty} m_{\pm,j}(k)z^{-j}, \quad m_{\pm,1}(z,k)=-I_m \tag{3.1}$$

into (2.41) and (2.42) then yields the following recursion relation for the coefficients $m_{\pm,j}(k)$:

$$\begin{aligned} &m_{+,1}=-I_m, \quad m_{+,2}=-B,\\ &m_{+,j+1}=Bm_{+,j}-\sum_{\ell=1}^{j-1} Am^+_{+,j-\ell}Am_{+,\ell}, \quad j\geq 2. \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &m_{-,1}=-I_m, \quad m_{-,2}=-B,\\ &m_{-,j+1}=Bm_{-,j}-\sum_{\ell=1}^{j-1} A^-m^-_{-,j-\ell}A^-m_{-,\ell}, \quad j\geq 2. \end{aligned} \tag{3.3}$$

Next, rewriting (2.17) in the form

$$\begin{aligned}&A(k)^{-1}M_\pm(z,k+1)A(k)^{-1}M_\pm(z,k)\\&\quad+A(k)^{-1}(zI_m-B(k+1))A(k)^{-1}M_\pm(z,k)+I_m=0,\quad z\in\mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{Z},\end{aligned}\tag{3.4}$$

and inserting the norm convergent ansatz

$$M_+(z,k)\underset{|z|\to\infty}{=}\sum_{j=1}^\infty M_{+,j}(k)z^{-j}\tag{3.5}$$

and the asymptotic ansatz

$$M_-(z,k)\underset{|z|\to\infty}{=}-I_mz+\sum_{j=0}^\infty M_{-,j}(k)z^{-j}\tag{3.6}$$

into (3.4) then yields the following recursion relation for the coefficients $M_{\pm,j}(k)$:

$$\begin{aligned}&M_{+,1}=-A^2,\quad M_{+,2}=-AB^+A,\\&M_{+,j+1}=AB^+A^{-1}M_{+,j}-\sum_{\ell=1}^{j-1}AM^+_{+,j-\ell}A^{-1}M_{+,\ell},\quad j\geq 2\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&M_{-,0}=B,\quad M_{-,1}=(A^-)^2,\quad M_{-,2}=A^-B^-A^-,\\&M_{-,j+1}=-B(A^-)^{-1}M^-_{-,j}A^-+\sum_{\ell=0}^{j}M_{-,j-\ell}(A^-)^{-1}M^-_{-,\ell}A^-,\quad j\geq 2.\end{aligned}\tag{3.8}$$

Remark 3.1. In the continuous cases of Schrödinger and Dirac-type operators discussed in detail in [10] and [11], establishing the existence of appropriate asymptotic expansions was a highly nontrivial endeavor. Here in the discrete context, hypothesis (2.2) together with (2.30) immediately yields a norm convergent expansion of $m_\pm(z,k)$ as $|z|\to\infty$. By (2.39) and (2.40), this immediately yields the existence of an asymptotic expansion for $M_\pm(z,k)$ as $|z|\to\infty$. By inspection, these expansions are of the form (3.1), (3.5), and (3.6).

Given the asymptotic expansions (3.5) and (3.6) for $M_\pm(z,k)$ as $|z|\to\infty$, one can of course derive analogous asymptotic expansions for the $2m\times 2m$ Weyl–Titchmarsh matrix $M(z,k)$ in (2.43)–(2.47). For the $(1,1)$-block matrix element of $M(z,k)$ one obtains the norm convergent expansion for $|z|$ sufficiently large,

$$g(z,k)=G(z,k,k)=M_{1,1}(z,k)\underset{|z|\to\infty}{=}\sum_{j=1}^\infty r_j(k)z^{-j}\tag{3.9}$$

$$\begin{aligned}&\underset{|z|\to\infty}{=}-I_mz^{-1}-B(k)z^{-2}-[A(k-1)^2+A(k)^2+B(k)^2]z^{-3}\\&\qquad-[B(k)^3+A(k-1)B(k-1)A(k-1)+A(k)B(k+1)A(k)\\&\qquad+B(k)A(k)^2+B(k)A(k-1)^2+A(k)^2B(k)+A(k-1)^2B(k)]z^{-4}\\&\qquad+O(z^{-5}),\quad k\in\mathbb{Z}.\end{aligned}\tag{3.10}$$

Similarly,

$$G(z,k,k+1)\underset{|z|\to\infty}{=}-A(k)z^{-2}+O(|z|^{-3}),\quad k\in\mathbb{Z},\tag{3.11}$$

$$G(z,k+1,k) \underset{|z|\to\infty}{=} -A(k)z^{-2} + O(|z|^{-3}), \quad k \in \mathbb{Z}. \tag{3.12}$$

Next, we also recall that the $(1,1)$ and $(2,2)$-block matrices of $M(z,k_0)$ are $m\times m$ Herglotz matrices. In particular, in addition to (2.48), (2.49), (2.52), and (2.53) one obtains

$$\begin{aligned} g(z,k) &= G(z,k,k) = M_{1,1}(z,k) \\ &= K_{1,1}(k) + \int_{\mathbb{R}} d\Omega_{1,1}(\lambda,k)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) &(3.13)\\ &= \exp\left[\int_{\mathbb{R}} d\lambda\, \Xi(\lambda,k)\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right)\right], \quad z\in\mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{Z}, &(3.14)\end{aligned}$$

where

$$0 \leqslant \Xi(\,\cdot\,,k) \leqslant I_m \text{ a.e.} \tag{3.15}$$

and

$$\Xi(\lambda,k) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}(\ln(g(\lambda+i\varepsilon,k))) \text{ for a.e. } \lambda\in\mathbb{R}. \tag{3.16}$$

We note that the constant term in the exponent of (3.14) vanishes because of the asymptotics (3.10).

4. Borg-Type Theorems for Matrix-Valued Jacobi Operators

In this section we prove trace formulas and the discrete analog of Borg's uniqueness theorem for matrix-valued Jacobi operators.

In the following, $\sigma(T)$ and $\sigma_{\mathrm{ess}}(T)$ denote the spectrum and essential spectrum of a densely defined closed operator T in a complex separable Hilbert space.

We start with the case of matrix-valued self-adjoint Jacobi operators H assuming the following hypothesis:

Hypothesis 4.1. *In addition to Hypothesis* 2.1 *suppose that* $\sigma(H) \subseteq [E_-, E_+]$ *for some* $-\infty < E_- < E_+ < \infty$.

Assuming Hypothesis 4.1, trace formulas associated with Jacobi operators then can be derived as follows. First we note that (3.9) implies the expansion (convergent for $|z|$ sufficiently large)

$$-\frac{d}{dz}\ln(g(z,k)) \underset{|z|\to\infty}{=} \sum_{j=1}^{\infty} s_j(k) z^{-j}, \quad k\in\mathbb{Z}, \tag{4.1}$$

$$s_1(k) = I_m, \tag{4.2}$$

$$s_2(k) = B(k), \tag{4.3}$$

$$s_3(k) = 2A(k-1)^2 + 2A(k)^2 + B(k)^2, \text{ etc.} \tag{4.4}$$

Moreover, by (3.14),

$$\begin{aligned}\frac{d}{dz}\ln(g(z,k)) &= \int_{E_-}^{E^+} d\lambda\,(\lambda - z)^{-2}\,\Xi(\lambda,k) + \int_{E_+}^{\infty} d\lambda\,(\lambda-z)^{-2} I_m \\ &= (E_+ - z)^{-1} I_m + \int_{E_-}^{E^+} d\lambda\,(\lambda-z)^{-2}\,\Xi(\lambda,k), &(4.5)\end{aligned}$$

where we used

$$\Xi(\lambda,k)=\begin{cases}0, & \lambda<E_-,\\ I_m, & \lambda>E_+.\end{cases} \tag{4.6}$$

Theorem 4.2. *Assume Hypothesis* 4.1. *Then* (*cf.* (4.1)),

$$s_j(k)=\frac{1}{2}(E_-^{j-1}+E_+^{j-1})I_m+\frac{1}{2}(j-1)\int_{E_-}^{E_+}d\lambda\,\lambda^{j-2}[I_m-2\,\Xi(\lambda,k)], \quad j\in\mathbb{N},\ k\in\mathbb{Z}. \tag{4.7}$$

Explicitly, for all $k\in\mathbb{Z}$,

$$B(k)=\frac{1}{2}(E_-+E_+)I_m+\frac{1}{2}\int_{E_-}^{E_+}d\lambda\,[I_m-2\,\Xi(\lambda,k)], \tag{4.8}$$

$$2A(k-1)^2+2A(k)^2+B(k)^2=\frac{1}{2}(E_-^2+E_+^2)I_m+\int_{E_-}^{E_+}d\lambda\,\lambda[I_m-2\,\Xi(\lambda,k)], \ \textit{etc.} \tag{4.9}$$

Proof. By (4.1) and (4.5) one infers

$$-\frac{d}{dz}\ln(g(z,k))=\frac{1}{2}\bigg[\frac{1}{z-E_+}+\frac{1}{z-E_-}\bigg]I_m+\frac{1}{2}\int_{E_-}^{E_+}d\lambda\,(\lambda-z)^{-2}[I_m-2\,\Xi(\lambda,k)] \tag{4.10}$$

$$\underset{|z|\to\infty}{=}\sum_{j=1}^{\infty}s_j(k)z^{-j}. \tag{4.11}$$

Expanding (4.10) with respect to z^{-1} and comparing powers of z^{-j} then yields (4.7). Equations (4.8) and (4.9) are then clear from (4.3) and (4.4). □

Remark 4.3. In the scalar case $m=1$, Theorem 4.2 was first derived in [26] assuming $A(k)=1$, $k\in\mathbb{Z}$. The Jacobi case for half-lines was explicitly discussed in this vein in [28]. The trace formula (4.8) for full-line Jacobi operators in the case $m=1$ (and other trace formulas) can be found in [53], [54, Sect. 6.2]. The current matrix-valued trace formula for $m\geq 2$ is new.

Next we turn to a Borg-type theorem for matrix-valued Jacobi operators. To set the stage we first recall Flaschka's result [19] for scalar (i.e., $m=1$) periodic Jacobi operators, the direct analog of Borg's theorem for periodic one-dimensional Schrödinger operators originally proved in [7].

Theorem 4.4. ([19]) *Suppose a and b are periodic real-valued sequences in $\ell^\infty(\mathbb{Z})$ with the same period and $a(k)>0$, $k\in\mathbb{Z}$. Let $h(a,b)=aS^++a^-S^-+b$ be the associated self-adjoint Jacobi operator in $\ell^2(\mathbb{Z})$* (*cf.* (2.4) *for* $m=1$) *and suppose that $\sigma(h(a,b))=[E_-,E_+]$ for some $E_-<E_+$. Then,*

$$a(k)=(E_+-E_-)/4, \quad b(k)=(E_-+E_+)/2, \quad k\in\mathbb{Z}. \tag{4.12}$$

While uniqueness results such as Theorem 4.4 are described as Borg-type theorems, other types of inverse spectral results are also described as Borg-type theorems as mentioned for Schrödinger operators in Remark 1.2. We also note that Theorem 4.4 is quite different from a recent result by Killip and Simon [34, Thm. 10.1] which states that $a(k)=1$, $k\in\mathbb{Z}$ and $\sigma(h)\subseteq[-2,2]$ implies $b(k)=0$, $k\in\mathbb{Z}$.

As shown in [11] and [13], periodicity is not the key ingredient in Borg-type theorems such as Theorem 4.4. In fact, it was shown there that the more general notion of being reflectionless is sufficient for Borg-type theorems to hold and we will turn to this circle of ideas next. We note that the class of reflectionless interactions include periodic and certain cases of quasi-periodic and almost-periodic interactions.

Following [11] and [13] , we now define reflectionless matrix-valued Jacobi operators as follows:

Definition 4.5. Assume Hypothesis 4.1. Then the matrix-valued coefficients A, B are called *reflectionless* if for all $k \in \mathbb{Z}$,

$$\Xi(\lambda, k) = \frac{1}{2} I_m \text{ for a.e. } \lambda \in \sigma_{\mathrm{ess}}(H) \tag{4.13}$$

with $\Xi(\cdot, k)$ defined in (3.16).

Since hardly any confusion can arise, we will also call $H = H(A, B)$ reflectionless if (4.13) is satisfied.

Remark 4.6. In the next theorem we will prove an inverse spectral result for matrix-valued Jacobi operators $H(A, B)$. However, in the general matrix-valued context, where $m \geq 2$, one cannot expect that the spectrum of $H(A, B)$ will determine A and B uniquely. Indeed, assume that B is a multiple of the identity, $B(k) = b(k)I_m$ for some $b \in \ell^\infty(\mathbb{Z})$, $b(k) \in \mathbb{R}$, $k \in \mathbb{Z}$. In addition, let U be a unitary $m \times m$ matrix and consider $\widetilde{A}(k) = UA(k)U^{-1}$, $k \in \mathbb{Z}$. Then clearly $H(A, B)$ and $H(\widetilde{A}, B)$ are unitarily equivalent and hence the spectrum of H cannot uniquely determine its coefficients. The following result, however, will illustrate a special case where the spectrum of $H(A, B)$ does determine A and B uniquely.

Given Definition 4.5, we now turn to a Borg-type uniqueness theorem for H and formulate the analog of Theorem 4.4 for reflectionless matrix-valued Jacobi operators.

Theorem 4.7. *Assume Hypotheses* 4.1 *and suppose that A and B are reflectionless. Let $H(A, B) = AS^+ + A^- S^- + B$ be the associated self-adjoint Jacobi operator in $\ell^2(\mathbb{Z})^m$ (cf.* (2.4)*) and suppose that $\sigma(H(A, B)) = [E_-, E_+]$ for some $E_- < E_+$. Then A and B are of the form,*

$$A(k) = \frac{1}{4}(E_+ - E_-)I_m, \quad B(k) = \frac{1}{2}(E_- + E_+)I_m, \quad k \in \mathbb{Z}. \tag{4.14}$$

Proof. By hypothesis, $\Xi(\lambda, k) = (1/2)I_m$ for a.e. $\lambda \in [E_-, E_+]$ and all $k \in \mathbb{Z}$. Thus the trace formula (4.8) immediately yields (4.14) for B. Inserting the formula (4.14) for B into the second trace formula (4.9) one infers

$$A(k-1)^2 + A(k)^2 = \frac{1}{8}(E_+ - E_-)^2 I_m, \quad k \in \mathbb{Z}. \tag{4.15}$$

The first order difference equation (4.15) has the solution

$$A(2\ell)^2 = A(0)^2, \quad A(2\ell+1)^2 = \frac{1}{8}(E_+ - E_-)^2 I_m - A(0)^2, \quad \ell \in \mathbb{Z}. \tag{4.16}$$

Since by hypothesis $A(0) > 0$ is a self-adjoint $m \times m$ matrix, there exists a unitary $m \times m$ matrix U that diagonalizes $A(0)$ and by (4.16), U simultaneously diagonalizes $A(k)$ for all $k \in \mathbb{Z}$,

$$\widetilde{A}(k) = UA(k)U^{-1}, \quad k \in \mathbb{Z}, \tag{4.17}$$

where $\widetilde{A}(k)$ are diagonal matrices for all $k \in \mathbb{Z}$. By (3.5)–(3.8) followed by an analytic continuation to all of $\mathbb{C}_+$, U also diagonalizes $M_\pm(z,k)$,

$$\widetilde{M_\pm}(z,k) = UM_\pm(z,k)U^{-1}, \quad k \in \mathbb{Z}, \tag{4.18}$$

where $\widetilde{M_\pm}(z,k)$ are diagonal matrices for all $k \in \mathbb{Z}$. The same result of course follows from (2.16) taking into account that U also diagonalizes $\psi_\pm(z,k,k_0)$, $\theta(z,k,k_0)$, and $\phi(z,k,k_0)$. The resulting diagonal matrices will of course be denoted by $\widetilde{\psi}_\pm(z,k,k_0)$, $\widetilde{\theta}(z,k,k_0)$, and $\widetilde{\phi}(z,k,k_0)$ below.

Next, we will invoke some Herglotz function ideas. Since for all $k \in \mathbb{Z}$,

$$\Xi(\lambda,k) = \begin{cases} 0, & \lambda < E_-, \\ \frac{1}{2} I_m, & \lambda \in (E_-,E_+), \\ I_m, & \lambda > E_+, \end{cases} \tag{4.19}$$

we can compute $g(z,k)$ from (3.14) and obtain

$$\begin{aligned} g(z,k) &= \exp\bigg[\int_{E_-}^\infty d\lambda\, \Xi(\lambda,k)\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg)\bigg] \\ &= \exp\bigg[(1/2)\int_{E_-}^{E_+} d\lambda \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) I_m \\ &\qquad + \int_{E_+}^\infty d\lambda \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) I_m\bigg] \\ &= -C[(z-E_-)(z-E_+)]^{-1/2} I_m, \quad z \in \mathbb{C}_+,\ k \in \mathbb{Z} \end{aligned} \tag{4.20}$$

for some $C > 0$. However, the known asymptotic behavior (3.10) of $g(z,k)$ as $|z| \to \infty$ then proves $C = 1$ and hence

$$g(z,k) = -[(z-E_-)(z-E_+)]^{-1/2} I_m := g(z), \quad z \in \mathbb{C}_+,\ k \in \mathbb{Z}. \tag{4.21}$$

Next, we consider $-g(z)^{-1}$ and its Herglotz representation,

$$\begin{aligned} -g(z)^{-1} &= M_+(z,k) - M_-(z,k) = [(z-E_-)(z-E_+)]^{1/2} I_m \\ &= zI_m - B + \int_{\mathbb{R}} d\Gamma(\lambda)\,(\lambda - z)^{-1} \\ &= \int_{\mathbb{R}} d\Omega_+(\lambda,k)\,(\lambda - z)^{-1} + zI_m - B + \int_{\mathbb{R}} d\Omega_-(\lambda,k)\,(\lambda - z)^{-1}, \end{aligned} \tag{4.22}$$

where $d\Omega_\pm(\lambda,k)$ denote the measures in the Herglotz representations of $\pm M_\pm(z,k)$,

$$M_+(z,k) = \int_{\mathbb{R}} d\Omega_+(\lambda,k)\,(\lambda - z)^{-1}, \tag{4.23}$$

$$-M_-(z,k) = zI_m - B + \int_{\mathbb{R}} d\Omega_-(\lambda,k)\,(\lambda - z)^{-1}. \tag{4.24}$$

Actually, applying U from the left and U^{-1} from the right on either side in (4.22)–(4.24), we may replace $M_\pm$ and $d\Gamma$, $d\Omega_\pm$ by $\widetilde{M_\pm}$ and $d\widetilde{\Gamma}$, $d\widetilde{\Omega}_\pm$ (where in obvious notation $d\widetilde{\Gamma}$ and $d\widetilde{\Omega}_\pm$ denote the diagonal measures associated with the diagonal Herglotz matrices $\widetilde{M_+} - \widetilde{M_-}$ and $\widetilde{M_\pm}$, respectively). In the following we assume that all quantities in (4.22)–(4.24) have been replaced by their diagonal matrix counterparts.

Next, we follow a strategy employed in [24] in the scalar Jacobi case. First we note that $\Xi(\lambda,k)=(1/2)I_m$ for a.e. $\lambda\in(E_-,E_+)$ is equivalent to

$$g(\lambda+i0)=-g(\lambda+i0)^*,\quad \lambda\in(E_-,E_+) \tag{4.25}$$

and hence to

$$-g(\lambda+i0)^{-1}=[g(\lambda+i0)^{-1}]^*,\quad \lambda\in(E_-,E_+). \tag{4.26}$$

Here and in the following, $f(\lambda+i0)$ denotes the normal limit $\lim_{\varepsilon\downarrow 0} f(\lambda+i\varepsilon)$. The last result is easily seen to be equivalent to the following fact: For all $k\in\mathbb{Z}$,

$$\mathrm{Re}(M_+(\lambda+i0,k))=\mathrm{Re}(M_-(\lambda+i0,k)) \text{ for a.e. } \lambda\in(E_-,E_+). \tag{4.27}$$

By (2.47) and (2.55) one computes

$$\begin{aligned}
UA(k)g(z,k+1)A(k)U^{-1}&=\widetilde{A}(k)g(z,k+1)\widetilde{A}(k)\\
&=UM_\pm(z,k)[M_-(z,k)-M_+(z,k)]^{-1}M_\mp(z,k)U^{-1}\\
&=\widetilde{M}_\pm(z,k)[\widetilde{M}_-(z,k)-\widetilde{M}_+(z,k)]^{-1}\widetilde{M}_\mp(z,k)\\
&=\widetilde{M}_\pm(z,k)g(z,k)\widetilde{M}_\mp(z,k),\quad z\in\mathbb{C}_+,\ k\in\mathbb{Z}.
\end{aligned} \tag{4.28}$$

Since $g(z,k)=g(z)$ is independent of $k\in\mathbb{Z}$, (4.28) implies

$$\widetilde{A}(k)^2=\widetilde{M}_+(z,k)\widetilde{M}_-(z,k)=\widetilde{M}_-(z,k)\widetilde{M}_+(z,k),\quad z\in\mathbb{C}_+,\ k\in\mathbb{Z}. \tag{4.29}$$

Inserting $\widetilde{M}_\pm(z,k)=\mathrm{Re}(\widetilde{M}_\pm(z,k))+i\mathrm{Im}(\widetilde{M}_\pm(z,k))$ into (4.29) then explicitly yields

$$\mathrm{Im}(\widetilde{M}_+(z,k))\mathrm{Re}(\widetilde{M}_-(z,k))+\mathrm{Re}(\widetilde{M}_+(z,k))\mathrm{Im}(\widetilde{M}_-(z,k))=0 \tag{4.30}$$

and since $\widetilde{M}_\pm(z,k)$, $\mathrm{Re}(\widetilde{M}_\pm(z,k))$, and $\mathrm{Im}(\widetilde{M}_\pm(z,k))$ are all diagonal matrices, we note that all entries in (4.30) commute. Similarly, the fact

$$\begin{aligned}
&g(\lambda+i0,k_0+1)\\
&\quad=\widetilde{A}(k_0)^{-1}\widetilde{M}_\pm(\lambda+i0,k_0)\\
&\qquad\times[\widetilde{M}_-(\lambda+i0,k_0)-\widetilde{M}_+(\lambda+i0,k_0)]^{-1}\widetilde{M}_\mp(\lambda+i0,k_0)\widetilde{A}(k_0)^{-1}\\
&\quad=-g(\lambda+i0,k_0+1)^*=-\widetilde{A}(k_0)^{-1}\widetilde{M}_\mp(\lambda+i0,k_0)^*\\
&\qquad\times[\widetilde{M}_-(\lambda+i0,k_0)^*-\widetilde{M}_+(\lambda+i0,k_0)^*]^{-1}\widetilde{M}_\pm(\lambda+i0,k_0)^*\widetilde{A}(k_0)^{-1}\\
&\qquad\qquad\text{for a.e. } \lambda\in(E_-,E_+)
\end{aligned} \tag{4.31}$$

implies

$$\begin{aligned}
&\widetilde{M}_\pm(\lambda+i0,k_0)[\widetilde{M}_-(\lambda+i0,k_0)-\widetilde{M}_+(\lambda+i0,k_0)]^{-1}\widetilde{M}_\mp(\lambda+i0,k_0)\\
&\quad=\widetilde{M}_\mp(\lambda+i0,k_0)^*[\widetilde{M}_-(\lambda+i0,k_0)-\widetilde{M}_+(\lambda+i0,k_0)]^{-1}\widetilde{M}_\pm(\lambda+i0,k_0)^*\\
&\qquad\qquad\text{for a.e. } \lambda\in(E_-,E_+)
\end{aligned} \tag{4.32}$$

since by (4.21),

$$g(z,k)=[M_-(z,k_0)-M_+(z,k_0)]^{-1}=[\widetilde{M}_-(z,k_0)-\widetilde{M}_+(z,k_0)]^{-1} \tag{4.33}$$

and hence (4.25) also applies to $[\widetilde{M}_-(\lambda+i0,k_0)-\widetilde{M}_+(\lambda+i0,k_0)]^{-1}$, $\lambda\in(E_-,E_+)$.

Inserting expression (2.8) for $\psi_\pm$ in terms of θ, ϕ, and $M_\pm$ into (2.38) taking $\ell=k$, and inserting the result into (4.26) then yields,

$$-\phi(\lambda,k,k_0)A(k_0)^{-1}M_\pm(\lambda+i0,k_0)[M_-(\lambda+i0,k_0)-M_+(\lambda+i0,k_0)]^{-1}$$

$$
\begin{aligned}
&\quad \times \theta(\lambda,k,k_0)^* \\
&- \theta(\lambda,k,k_0)[M_-(\lambda+i0,k_0) - M_+(\lambda+i0,k_0)]^{-1} M_\mp(\lambda+i0,k_0)A(k_0)^{-1} \\
&\quad \times \phi(\lambda,k,k_0)^* \\
&= -\phi(\lambda,k,k_0)A(k_0)^{-1}M_\mp(\lambda+i0,k_0)^*[M_-(\lambda+i0,k_0) - M_+(\lambda+i0,k_0)]^{-1} \\
&\quad \times \theta(\lambda,k,k_0)^* \\
&\quad - \theta(\lambda,k,k_0)[M_-(\lambda+i0,k_0) - M_+(\lambda+i0,k_0)]^{-1} M_\pm(\lambda+i0,k_0)^* A(k_0)^{-1} \\
&\quad \times \phi(\lambda,k,k_0)^*, \qquad (4.34)
\end{aligned}
$$

where we also used (4.25) and (4.32). Applying U and U^{-1} from the left and right on either side in (4.34), using the fact that

$$\widetilde{\phi}(\overline{z},k,k_0)^* = \widetilde{\phi}(z,k,k_0), \quad \widetilde{\theta}(\overline{z},k,k_0)^* = \widetilde{\theta}(z,k,k_0), \tag{4.35}$$

and that all diagonal matrices commute, one can rewrite (4.34) in the form,

$$
\begin{aligned}
&2i\widetilde{\phi}(\lambda,k,k_0)\widetilde{\theta}(\lambda,k,k_0)A(k_0)^{-1}g(\lambda+i0) \qquad (4.36)\\
&\quad \times [\operatorname{Im}(\widetilde{M}_-(\lambda+i0,k_0)) + \operatorname{Im}(\widetilde{M}_+(\lambda+i0,k_0))] = 0 \text{ for a.e. } \lambda \in (E_-,E_+).
\end{aligned}
$$

Since $k \in \mathbb{Z}$ can be chosen arbitarily in (4.36), this implies

$$\operatorname{Im}(\widetilde{M}_-(\lambda+i0,k_0)) = -\operatorname{Im}(\widetilde{M}_+(\lambda+i0,k_0)) \text{ for a.e. } \lambda \in (E_-,E_+). \tag{4.37}$$

Finally, since $k_0 \in \mathbb{Z}$ in (4.37) is arbitrary, one obtains for all $k \in \mathbb{Z}$,

$$\operatorname{Im}(\widetilde{M}_-(\lambda+i0,k)) = -\operatorname{Im}(\widetilde{M}_+(\lambda+i0,k)) \text{ for a.e. } \lambda \in (E_-,E_+) \tag{4.38}$$

and hence also for all $k \in \mathbb{Z}$,

$$\operatorname{Im}(M_-(\lambda+i0,k)) = -\operatorname{Im}(M_+(\lambda+i0,k)) \text{ for a.e. } \lambda \in (E_-,E_+) \tag{4.39}$$

applying U^{-1} and U from the left and right on both sides in (4.38). Together with (4.27) this yields for all $k \in \mathbb{Z}$,

$$M_-(\lambda \mp i0,k) = M_-(\lambda \pm i0,k)^* = M_+(\lambda \pm i0,k) \text{ for a.e. } \lambda \in (E_-,E_+). \tag{4.40}$$

Thus, $M_-(\cdot,k)$ is the analytic continuation of $M_+(\cdot,k)$ (and vice versa) through the interval (E_-,E_+). Since $d\Gamma$ is purely absolutely continuous (cf. (4.22)),

$$d\Gamma = d\Gamma_{\mathrm{ac}}, \quad d\Gamma_{\mathrm{pp}} = d\Gamma_{\mathrm{sc}} = 0 \tag{4.41}$$

(where $d\mu_{\mathrm{ac}}$, $d\mu_{\mathrm{pp}}$, and $d\mu_{\mathrm{sc}}$ denote the absolutely continuous, pure point, and singularly continuous parts of a measure $d\mu$), one also infers

$$d\Omega_{\pm,\mathrm{pp}} = d\Omega_{\pm,\mathrm{sc}} = 0. \tag{4.42}$$

(This also follows from the fact that $M_\pm(\cdot,k)$ have analytic continuations through (E_-,E_+), see [30, Lemma 5.6].) Especially, (4.38) then implies the k-independence of $d\Omega_\pm(\cdot,k)$,

$$d\Omega_+(\lambda,k) = d\Omega_-(\lambda,k) = \frac{1}{2}d\Gamma(\lambda). \tag{4.43}$$

Equations (3.7), (3.8), (4.23), (4.24), and (4.43) then prove

$$\widetilde{A}(k)^2 = \int_{E_-}^{E_+} d\Omega_+(\lambda,k) = \frac{1}{2}\int_{E_-}^{E_+} d\Gamma(\lambda) = \int_{E_-}^{E_+} d\Omega_-(\lambda,k) = \widetilde{A}(k-1)^2, \quad k \in \mathbb{Z} \tag{4.44}$$

and hence also

$$A(k)^2 = A(k-1)^2, \quad k \in \mathbb{Z}, \tag{4.45}$$

which proves that A is independent of $k \in \mathbb{Z}$. The trace formula (4.15) for $\widetilde{A}$,

$$\widetilde{A}(k-1)^2 + \widetilde{A}(k)^2 = \frac{1}{8}(E_+ - E_-)^2 I_m, \quad k \in \mathbb{Z}. \tag{4.46}$$

then proves

$$\widetilde{A}(k)^2 = \frac{1}{16}(E_+ - E_-)^2 I_m, \quad k \in \mathbb{Z}. \tag{4.47}$$

Using (4.17) then completes the proof of (4.14). □

Because of (4.43) and

$$d\Gamma(\lambda) = \begin{cases} \frac{1}{\pi}[(\lambda - E_-)(E_+ - \lambda)]^{1/2}, & \lambda \in [E_-, E_+], \\ 0, & \lambda \in \mathbb{R}\backslash[E_-, E_+], \end{cases} \tag{4.48}$$

(cf. (4.22)), one obtains

$$\widetilde{M}_\pm(z,k) = \bigg\{ -\frac{1}{2}z + \frac{1}{4}(E_- + E_+) \pm \frac{1}{2}[(z-E_-)(z-E_+)]^{1/2} \bigg\} I_m, \quad z \in \mathbb{C}_+, \ k \in \mathbb{Z}. \tag{4.49}$$

Corollary 4.8. *Assume Hypothesis* 4.1 *in the special case* $m = 1$ *and suppose that* a *and* b *are reflectionless. Let* $h(a,b) = aS^+ + a^- S^- + b$ *be the associated self-adjoint Jacobi operator in* $\ell^2(\mathbb{Z})$ (*cf.* (2.4)) *and suppose that* $\sigma(h(a,b)) = [E_-, E_+]$ *for some* $E_- < E_+$. *Then,*

$$a(k) = \frac{1}{4}(E_+ - E_-), \quad b(k) = \frac{1}{2}(E_- + E_+), \quad k \in \mathbb{Z}. \tag{4.50}$$

While Theorem 4.7 is new, Corollary 4.8 in the scalar case $m = 1$ was noted in [53, Corollary 6.3] (see also [54, Corollary 8.6]).

The following result can be proved in analogy to Theorems 4.6 and 4.8 in [13], hence we state it here without proof.

Theorem 4.9. *In addition to Hypothesis* 4.1, *suppose that* A *and* B *are periodic with the same period. Let* $H(A,B) = AS^+ + A^- S^- + B$ *be the associated self-adjoint Jacobi operator in* $\ell^2(\mathbb{Z})^m$ *and suppose that* $H(A,B)$ *has uniform spectral multiplicity* $2m$. *Then* $H(A,B)$ *is reflectionless and hence for all* $k \in \mathbb{Z}$,

$$\Xi(\lambda, k) = \frac{1}{2} I_m \text{ for a.e. } \lambda \in \sigma_{\mathrm{ess}}(H(A,B)). \tag{4.51}$$

In particular, assume that A *and* B *are periodic with the same period, that* $H(A,B)$ *has uniform spectral multiplicity* $2m$, *and that* $\sigma(H(A,B)) = [E_-, E_+]$ *for some* $E_- < E_+$. *Then* A *and* B *are of the form,*

$$A(k) = \frac{1}{4}(E_+ - E_-) I_m, \quad B(k) = \frac{1}{2}(E_- + E_+) I_m, \quad k \in \mathbb{Z}. \tag{4.52}$$

In connection with the special case $m = 1$ in Theorem 4.4 we note that scalar Jacobi operators automatically have uniform spectral multiplicity 2 since the product of the two Floquet multipliers equals one.

5. Borg-Type Theorems for Supersymmetric Dirac Difference Operators

In our final section we turn to a Borg-type theorem for a class of supersymmetric Dirac difference operators. Rather than developing the results from first principles as in the case of Jacobi operators with matrix-valued coefficients, we will employ the underlying supersymmetric structure and reduce the case of Dirac difference operators to that of Jacobi operators.

We start with the following abstract result.

Theorem 5.1 ([14], [25]). *Let C be a densely defined closed operator in a separable complex Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(C)$ and introduce the operator*

$$Q = \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix}, \quad \mathcal{D}(Q) = \mathcal{D}(C) \oplus \mathcal{D}(C^*) \tag{5.1}$$

in $\mathcal{H} \oplus \mathcal{H}$. Then,

$$Q = Q^*, \tag{5.2}$$

$$Q^2 = \begin{pmatrix} C^*C & 0 \\ 0 & CC^* \end{pmatrix}, \tag{5.3}$$

$$\sigma_3 Q \sigma_3 = -Q, \quad \sigma_3 = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix}, \tag{5.4}$$

$$(Q - zI_{\mathcal{H}\oplus\mathcal{H}})^{-1} = \begin{pmatrix} z(C^*C - z^2 I_{\mathcal{H}})^{-1} & C^*(CC^* - z^2 I_{\mathcal{H}})^{-1} \\ C(C^*C - z^2 I_{\mathcal{H}})^{-1} & z(CC^* - z^2 I_{\mathcal{H}})^{-1} \end{pmatrix}, \tag{5.5}$$
$$z^2 \in \mathbb{C}\backslash\{\sigma(C^*C) \cup \sigma(CC^*)\}.$$

In addition, we mention the following facts:

$$I_{\mathcal{H}} + \zeta(CC^* - \zeta I_{\mathcal{H}})^{-1} \supseteq C(C^*C - \zeta I\mathcal{H})^{-1}C^*, \tag{5.6}$$
$$\zeta \in \mathbb{C}\backslash\{\sigma(C^*C) \cup \sigma(CC^*)\},$$

$$I_{\mathcal{H}} + \zeta(C^*C - \zeta I_{\mathcal{H}})^{-1} \supseteq C^*(CC^* - \zeta I\mathcal{H})^{-1}C, \tag{5.7}$$
$$\zeta \in \mathbb{C}\backslash\{\sigma(C^*C) \cup \sigma(CC^*)\}.$$

Moreover,

$$QU(z) = zU(z), \quad U(z) = \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} \tag{5.8}$$

implies

$$C^*u_2(z) = zu_1(z), \quad Cu_1(z) = zu_2(z) \tag{5.9}$$

and hence

$$C^*Cu_1(z) = z^2 u_1(z), \quad CC^*u_2(z) = z^2 u_2(z). \tag{5.10}$$

Conversely,

$$C^*Cu_1(z) = z^2 u_1(z), \quad z \neq 0, \tag{5.11}$$

implies

$$QU(z) = zU(z), \quad U(z) = \begin{pmatrix} u_1(z) \\ (1/z)Cu_1(z) \end{pmatrix} \tag{5.12}$$

and

$$CC^*u_2(z) = z^2 u_2(z), \quad z \neq 0, \tag{5.13}$$

implies

$$QU(z) = zU(z), \quad U(z) = \begin{pmatrix} (1/z)C^* u_2(z) \\ u_2(z) \end{pmatrix}. \tag{5.14}$$

In order to apply this setup to finite difference Dirac-type systems (cf. [12]), we introduce the following hypothesis.

Hypothesis 5.2. *Let $m \in \mathbb{N}$ and consider the sequence of invertible $m \times m$ matrices*

$$\rho = \{\rho(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}, \quad \rho(k) = \rho(k)^*, \ k \in \mathbb{Z}, \tag{5.15}$$

$$\chi = \{\chi(k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{m\times m}, \tag{5.16}$$

$$\det_{\mathbb{C}^m}(\rho(k)) \neq 0, \quad \det_{\mathbb{C}^m}(\chi(k)) \neq 0, \quad k \in \mathbb{Z}. \tag{5.17}$$

In addition, we assume that $\rho(k)$ is a positive definite diagonal $m \times m$ matrix

$$\rho(k) = \operatorname{diag}(\rho_1(k), \dots, \rho_m(k)), \quad \rho_j(k) > 0, \ 1 \le j \le m, \ k \in \mathbb{Z}, \tag{5.18}$$

and that $\rho(k)$ and $\chi(k)$ are uniformly bounded with respect to $k \in \mathbb{Z}$, that is, there exists a $C > 0$, such that

$$\|\rho(k)\|_{\mathbb{C}^{m\times m}} + \|\chi(k)\|_{\mathbb{C}^{m\times m}} \le C, \quad k \in \mathbb{Z}. \tag{5.19}$$

Finally, we suppose that $\rho\chi^+$ and $\chi\rho$ are positive definite,

$$\rho(k)\chi(k+1) > 0, \ \ \chi(k)\rho(k) > 0, \ k \in \mathbb{Z}. \tag{5.20}$$

Assuming Hypothesis 5.2, we thus introduce the bounded linear operator

$$E = \rho^- S^- + \chi, \quad \mathcal{D}(E) = \ell^2(\mathbb{Z})^m, \tag{5.21}$$

on $\ell^2(\mathbb{Z})^m$ and the bounded Dirac-type difference operator

$$D = \begin{pmatrix} 0 & E^* \\ E & 0 \end{pmatrix} = S_\rho + X, \quad \mathcal{D}(D) = \ell^2(\mathbb{Z})^m \oplus \ell^2(\mathbb{Z})^m \tag{5.22}$$

on $\ell^2(\mathbb{Z})^m \oplus \ell^2(\mathbb{Z})^m$, where

$$S_\rho = \begin{pmatrix} 0 & \rho S^+ \\ \rho^- S^- & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \chi^* \\ \chi & 0 \end{pmatrix}. \tag{5.23}$$

One then computes

$$H_1 = E^*E = A_1 S^+ + A_1^- S^- + B_1, \tag{5.24}$$

$$H_2 = EE^* = A_2 S^+ + A_2^- S^- + B_2, \tag{5.25}$$

where

$$A_1(k) = \rho(k)\chi^+(k) > 0, \quad B_1(k) = (\rho(k))^2 + \chi(k)^*\chi(k), \ k \in \mathbb{Z}, \tag{5.26}$$

$$A_2(k) = \chi(k)\rho(k) > 0, \quad B_2(k) = (\rho(k)^-)^2 + \chi(k)\chi(k)^* \tag{5.27}$$

and notes that $H_1 \ge 0$ and $H_2 \ge 0$ are matrix-valued Jacobi operators in $\ell^2(\mathbb{Z})^m$ of the form (2.4).

Remark 5.3. We note that D with a positive definite diagonal $m \times m$ matrix ρ in (5.22) represents a normal form of Dirac-type difference in the following sense: Assume Hypothesis 5.2 except for the condition that $\rho(k)$ is a positive definite diagonal matrix for all $k \in \mathbb{Z}$. Then, following [12, Lemma 2.3], there exists a sequence of unitary matrices $U(\rho) = \{U(\rho, k)\}_{k\in\mathbb{Z}} \in \mathbb{C}(\mathbb{Z})^{2m\times 2m}$ such that

$$U(\rho)(S_\rho + X)U(\rho)^{-1} = S_{\widehat{\rho}} + \widehat{X}, \tag{5.28}$$

where $\widehat{\rho}$ is diagonal and positive definite and $\widehat{X}$ is of the form

$$\widehat{X} = U(\rho)XU(\rho)^{-1} = \begin{pmatrix} 0 & \widehat{\chi}^* \\ \widehat{\chi} & 0 \end{pmatrix}, \tag{5.29}$$

with $\widehat{\chi} \in \mathbb{C}(\mathbb{Z})^{m\times m}$. Indeed, denote by $Q_\rho(k) \in \mathbb{C}^{m\times m}$ a unitary matrix such that $Q_\rho(k)\rho(k)Q_\rho(k)^{-1} = \widetilde{\rho}(k)$, where $\widetilde{\rho}(k) \in \mathbb{R}^{m\times m}$ is diagonal and self-adjoint for all $k \in \mathbb{Z}$. Then,

$$U_\rho(S_\rho + X)U_\rho^{-1} = S_{\widetilde{\rho}} + \widetilde{X}, \quad \widetilde{X} = U_\rho X U_\rho^{-1}, \quad U_\rho = \begin{pmatrix} Q_\rho & 0 \\ 0 & Q_\rho^- \end{pmatrix}. \tag{5.30}$$

Next, let $\widetilde{\varepsilon}(k) \in \mathbb{R}^{m\times m}$ be a diagonal matrix for which $(\widetilde{\varepsilon}(k))_{\ell,\ell} \in \{+1, -1\}$, $\ell = 1, \dots, m$. Define $\varepsilon(k) \in \mathbb{R}^{m\times m}$ by $\varepsilon(k) = \widetilde{\varepsilon}(k)\widetilde{\varepsilon}(k+1)$ and choose $\widetilde{\varepsilon}(k)$ so that $\widehat{\rho} = \varepsilon\widetilde{\rho} > 0$. Then,

$$U_\varepsilon(S_{\widetilde{\rho}} + \widetilde{X})U_\epsilon^{-1} = S_{\widehat{\rho}} + \widehat{X}, \quad \widehat{X} = U_\varepsilon \widetilde{X} U_\varepsilon^{-1}, \quad U_\varepsilon = \begin{pmatrix} \widetilde{\varepsilon} & 0 \\ 0 & \widetilde{\varepsilon} \end{pmatrix}. \tag{5.31}$$

Thus, one obtains

$$U(\rho) = U_\varepsilon U_\rho = \begin{pmatrix} \widetilde{\varepsilon} Q_\rho & 0 \\ 0 & \widetilde{\varepsilon} Q_\rho^- \end{pmatrix}, \tag{5.32}$$

$$\widehat{X} = \begin{pmatrix} 0 & \widehat{\chi}^* \\ \widehat{\chi} & 0 \end{pmatrix}, \quad \widehat{\chi} = \widetilde{\varepsilon} Q_\rho^- \chi Q_\rho^{-1} \widetilde{\varepsilon}. \tag{5.33}$$

Next, we introduce the $2m \times 2m$ Weyl–Titchmarsh matrices associated with D (cf. [12]) by

$$M^D(z,k) = \big(M^D_{j,j'}(z,k)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \ k \in \mathbb{Z}, \tag{5.34}$$

where

$$M^D_{1,1}(z,k) = \big[M^D_-(z,k) - M^D_+(z,k)\big]^{-1}, \tag{5.35}$$

$$M^D_{1,2}(z,k) = 2^{-1}\big[M^D_-(z,k) - M^D_+(z,k)\big]^{-1}\big[M^D_-(z,k) + M^D_+(z,k)\big], \tag{5.36}$$

$$M^D_{2,1}(z,k) = 2^{-1}\big[M^D_-(z,k) + M^D_+(z,k)\big]\big[M^D_-(z,k) - M^D_+(z,k)\big]^{-1}, \tag{5.37}$$

$$M^D_{2,2}(z,k) = M^D_\pm(z,k)\big[M^D_-(z,k) - M^D_+(z,k)\big]^{-1} M_\mp(z,k), \tag{5.38}$$

and similarly, the $2m \times 2m$ Weyl–Titchmarsh matrices associated with the Jacobi operators H_ℓ, $\ell = 1, 2$ (cf. (2.43)–(2.47)) by

$$M^{H_\ell}(z,k) = \big(M^{H_\ell}_{j,j'}(z,k)\big)_{j,j'=1,2}, \quad \ell = 1, 2, \ z \in \mathbb{C}\backslash\mathbb{R}, \ k \in \mathbb{Z}, \tag{5.39}$$

where

$$M^{H_\ell}_{1,1}(z,k) = \big[M^{H_\ell}_-(z,k) - M^{H_\ell}_+(z,k)\big]^{-1}, \tag{5.40}$$

$$M^{H_\ell}_{1,2}(z,k) = 2^{-1}\big[M^{H_\ell}_-(z,k) - M^{H_\ell}_+(z,k)\big]^{-1}\big[M^{H_\ell}_-(z,k) + M^{H_\ell}_+(z,k)\big], \tag{5.41}$$

$$M^{H_\ell}_{2,1}(z,k) = 2^{-1}\big[M^{H_\ell}_-(z,k) + M^{H_\ell}_+(z,k)\big]\big[M^{H_\ell}_-(z,k) - M^{H_\ell}_+(z,k)\big]^{-1}, \tag{5.42}$$

$$M^{H_\ell}_{2,2}(z,k) = M^{H_\ell}_\pm(z,k)\big[M^{H_\ell}_-(z,k) - M^{H_\ell}_+(z,k)\big]^{-1} M_\mp(z,k). \tag{5.43}$$

The supersymmetric formalism (5.1)–(5.5) then implies the following relations between $M^D(z)$ and $M^{H_\ell}(z)$, $\ell = 1, 2$.

Theorem 5.4. *Assume Hypothesis* 5.2 *and let* $z \in \mathbb{C}\backslash\sigma(D)$, $k \in \mathbb{Z}$. *Then,*

$$M_\pm^D(z,k) = -z^{-1}\rho(k) + z^{-1}\rho(k)^{-1/2}M_\pm^{H_1}(z^2,k)\rho(k)^{-1/2}, \tag{5.44}$$

$$M_\pm^D(z,k) = -z\rho(k)^{-1} - z\rho(k)^{-1/2}\big[\chi(k)^{-1}M_\pm^{H_2}(\chi(k)^{-1})^*\big]^{-1}\rho(k)^{-1/2}. \tag{5.45}$$

Proof. We freely employ the relations (5.8)–(5.14) and some results from [12]. Let

$$U_\pm(z,k,k_0) = \begin{pmatrix} u_{1,\pm}(z,k,k_0) \\ u_{2,\pm}(z,k,k_0)\end{pmatrix}, \quad u_{1,\pm}(z,k_0,k_0) = I_m \tag{5.46}$$

be the normalized Weyl–Titchmarsh solutions associated with D. Then by equations (2.35a)–(2.35c) and (2.96) in [12],

$$u_{2,\pm}(z,k+1,k_0)u_{1,\pm}(z,k,k_0)^{-1} = -\rho(k)^{-1/2}M_\pm^D(z,k)\rho(k)^{1/2}. \tag{5.47}$$

Moreover, using $(C^*u_{2,\pm})(z,k_0,k_0) = zu_{1,\pm}(z,k_0,k_0) = zI_m$ (cf. (5.9)), one derives

$$u_{2,\pm}(z,k_0,k_0) = (\chi(k_0)^*)^{-1}\big[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}\big]. \tag{5.48}$$

Next, let

$$\psi_{\ell,\pm}(z,k,k_0), \quad \psi_{\ell,\pm}(z,k_0,k_0) = I_m \tag{5.49}$$

be the normalized Weyl–Titchmarsh solutions associated with H_ℓ, $\ell = 1,2$. Then by (2.16),

$$M_\pm^{H_\ell}(z,k) = -A_\ell(k)\psi_{\ell,\pm}(z,k+1,k_0)\psi_{\ell,\pm}(z,k,k_0)^{-1}. \tag{5.50}$$

Given the uniqueness of Weyl–Titchmarsh solutions for D and H_ℓ, $\ell = 1,2$, (5.8)–(5.14) yield

$$u_{1,\pm}(z,k,k_0) = \psi_{1,\pm}(z^2,k,k_0), \tag{5.51}$$

$$\begin{aligned} u_{2,\pm}(z,k,k_0) &= (1/z)Eu_{1,\pm}(z,k,k_0) \\ &= (1/z)[\rho^-(k)u_{1,\pm}(z,k-1,k_0) + \chi(k)u_{1,\pm}(z,k,k_0)] \\ &= (1/z)[\rho^-(k)\psi_{1,\pm}(z^2,k-1,k_0) + \chi(k)\psi_{1,\pm}(z^2,k,k_0)]. \end{aligned} \tag{5.52}$$

Thus,

$$\begin{aligned} &u_{2,\pm}(z,k+1,k_0)u_{1,\pm}(z,k,k_0)^{-1} = -\rho(k)^{-1/2}M_\pm^D(z,k)\rho(k)^{1/2} \\ &\quad = (1/z)\big[\rho(k)\psi_{1,\pm}(z^2,k,k_0) + \chi(k+1)\psi_{1,\pm}(z^2,k+1,k_0)\big]\psi_{1,\pm}(z^2,k,k_0)^{-1} \\ &\quad = (1/z)\big[\rho(k) + \chi(k+1)(-A_1(k))^{-1}M_\pm^{H_1}(z^2,k)\big] \\ &\quad = (1/z)\big[\rho(k) - \rho(k)^{-1}M_\pm^{H_1}(z^2,k)\big] \end{aligned} \tag{5.53}$$

proving (5.44).

Similarly, the uniqueness of Weyl–Titchmarsh solutions also yields

$$u_{2,\pm}(z,k,k_0) = \psi_{2,\pm}(z^2,k,k_0)d_\pm(z,k_0) \tag{5.54}$$

for some constant $m \times m$ matrix $d_\pm(z,k_0)$. Thus,

$$d_\pm(z,k_0) = u_{2,\pm}(z,k_0,k_0) = (\chi(k_0)^*)^{-1}[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}]. \tag{5.55}$$

One then computes,

$$\begin{aligned} &u_{2,\pm}(z,k_0+1,k_0) = -\rho(k_0)^{-1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2} \\ &\quad = \psi_{2,\pm}(z^2,k_0+1,k_0)(\chi(k_0)^*)^{-1}[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}] \\ &\quad = -(A_2(k_0))^{-1}M_\pm^{H_2}(z^2,k_0)(\chi(k_0)^*)^{-1}[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}] \end{aligned}$$

$$= -\rho(k_0)^{-1}\chi(k_0)^{-1}M_\pm^{H_2}(z^2,k_0)(\chi(k_0)^*)^{-1}[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}]. \tag{5.56}$$

Hence,

$$\begin{aligned}&\rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}\\ &\quad = \chi(k_0)^{-1}M_\pm^{H_2}(z^2,k_0)(\chi(k_0)^*)^{-1}[zI_m + \rho(k_0)^{1/2}M_\pm^D(z,k_0)\rho(k_0)^{1/2}]\end{aligned} \tag{5.57}$$

and since $k_0 \in \mathbb{Z}$ is arbitrary,

$$\begin{aligned}&\rho(k)^{1/2}M_\pm^D(z,k)\rho(k)^{1/2}\\ &\quad = \chi(k)^{-1}M_\pm^{H_2}(z^2,k)(\chi(k)^*)^{-1}[zI_m + \rho(k)^{1/2}M_\pm^D(z,k)\rho(k)^{1/2}], \quad k \in \mathbb{Z}.\end{aligned} \tag{5.58}$$

Solving (5.58) for $M_\pm^D(z,k)$ then yields (5.45). □

According to (2.49), $M^D(z,k)$ is a matrix-valued Herglotz function of rank $2m$ with exponential Herglotz representation

$$M^D(z,k) = \exp\bigg[C^D(k) + \int_{\mathbb{R}} d\lambda\, \Upsilon^D(\lambda,k)\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg)\bigg], \tag{5.59}$$

where

$$C^D(k) = C^D(k)^*, \quad 0 \leqslant \Upsilon^D(\,\cdot\,,k) \leqslant I_{2m} \text{ a.e.}, \tag{5.60}$$

$$\Upsilon^D(\lambda,k) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}(\ln(M(\lambda+i\varepsilon,k))) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{5.61}$$

Following [11] we now define reflectionless matrix-valued Dirac-type operators as follows:

Definition 5.5. Assume Hypothesis 5.2. Then the matrix-valued coefficients ρ, χ are called *reflectionless* if for all $k \in \mathbb{Z}$,

$$\Upsilon^D(\lambda,k) = \frac{1}{2}I_{2m} \text{ for a.e. } \lambda \in \sigma_{\mathrm{ess}}(D). \tag{5.62}$$

We also call $D = D(\rho,\chi)$ reflectionless if (5.62) holds.

Remark 5.6. The definition (4.13) of reflectionless Jacobi operators H and the definition (5.62) of reflectionless Dirac operators D can be replaced by the more stringent requirement that for all $k \in \mathbb{Z}$,

$$M_+(\lambda + i0,k) = M_-(\lambda - i0,k) \text{ for a.e. } \lambda \in \sigma_{\mathrm{ess}}(H) \tag{5.63}$$

for Jacobi operators H and similarly for Dirac-type operators D replacing $M_\pm(z,k)$ by $M_\pm^D(z,k)$. This yields a unified definition of the notion of reflectionless matrix-valued Jacobi and Dirac-type operators. It is easy to see that (5.63) implies (4.13). The converse is more subtle and was proved by Sodin and Yuditskii in [51] and [52] for scalar Schrödinger and Jacobi operators H under the assumption that $\sigma(H)$ is a homogeneous set. Their proof extends to the present matrix-valued setting.

In the special case of a Borg-type situation with $\sigma(H) = [E_-,E_+]$, we explicitly derived (5.63) in (4.40). In this particular case, $M_-(\cdot,k)$ is the analytic continuation of $M_+(\cdot,k)$ (and vice versa) through the interval (E_-,E_+). In general, the homogeneous set $\sigma(H)$ may be a Cantor set (of positive Lebesgue measure) and then $M_\pm(\cdot,k)$ are called *pseudo-continuable* through $\sigma(H)$.

Lemma 5.7. *Assume Hypothesis* 5.2. *If D is reflectionless* (*in the sense of* (5.62)) *then H_ℓ, $\ell = 1,2$, are reflectionless* (*in the sense of* (4.13)).

Proof. D being reflectionless in the sense of (5.62) is equivalent to the assertion that for all $k \in \mathbb{Z}$, $M^D(\lambda + i0, k)$ is skew-adjoint for a.e. $\lambda \in \sigma_{\rm ess}(D)$. Equivalently, for all $k \in \mathbb{Z}$,

$$M^D(\lambda + i0, k) = iC(\lambda, k) \text{ with } C(\lambda, k) = C(\lambda, k)^* \text{ for a.e. } \lambda \in \sigma_{\rm ess}(D). \quad (5.64)$$

In fact, $C(\lambda, k) = \mathrm{Im}(M^D(\lambda + i0, k)) \geq 0$. As a consequence, also all block submatrices of $M^D(\lambda + i0, k)$, symmetric with respect to the diagonal of $M^D(\lambda + i0, k)$, are skew-adjoint. In particular, the two $m \times m$ diagonal blocks of $M^D(\lambda + i0, k)$ satisfy for all $k \in \mathbb{Z}$,

$$M^D_{\ell,\ell}(\lambda + i0, k) = iC_{\ell,\ell}(\lambda, k) \text{ with } C_{\ell,\ell}(\lambda, k) \geq 0 \text{ for a.e. } \lambda \in \sigma_{\rm ess}(D). \quad (5.65)$$

Equation (5.44) implies

$$[M^D_-(z, k) - M^D_+(z, k)]^{-1} = \rho(k)^{1/2} z\big[M^{H_1}_-(z^2, k) - M^{H_1}_+(z^2, k)\big]^{-1}\rho(k)^{1/2}, \quad k \in \mathbb{Z} \quad (5.66)$$

and hence, for all $k \in \mathbb{Z}$ and a.e. $\lambda \in \sigma_{\rm ess}(D)$,

$$\begin{aligned} M^D_{1,1}(\lambda + i0, k) &= \big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1} \\ &= \rho(k)^{1/2}\lambda\big[M^{H_1}_-(\lambda^2 + i0, k) - M^{H_1}_+(\lambda^2 + i0, k)\big]^{-1}\rho(k)^{1/2}. \quad (5.67) \end{aligned}$$

Thus, $M^{H_1}_{1,1}(\lambda^2+i0, k) = g^{H_1}(\lambda^2+i0, k)$ is skew-adjoint and hence H_1 is reflectionless by (4.13).

Similarly, applying (5.45) yields

$$\begin{aligned} &z\big[M^{H_2}_-(z^2, k) - M^{H_2}_+(z^2, k)\big]^{-1} \\ &\quad = (\chi(k)^{-1})^*\rho(k)^{1/2}\big[M^D_-(z, k) + z\rho(k)^{-1}\big]\big[M^D_-(z, k) - M^D_+(z, k)\big]^{-1} \\ &\qquad \times \big[M^D_+(z, k) + z\rho(k)^{-1}\big]\rho(k)^{1/2}\chi(k)^{-1}, \quad k \in \mathbb{Z} \quad (5.68) \end{aligned}$$

and hence, for all $k \in \mathbb{Z}$ and a.e. $\lambda \in \sigma_{\rm ess}(D)$,

$$\begin{aligned} &\lambda\big[M^{H_2}_-(\lambda^2 + i0, k) - M^{H_2}_+(\lambda^2 + i0, k)\big]^{-1} \\ &\quad = (\chi(k)^{-1})^*\rho(k)^{1/2}\big[M^D_-(\lambda + i0, k) + \lambda\rho(k)^{-1}\big] \\ &\qquad \times \big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1} \\ &\qquad \times \big[M^D_+(\lambda + i0, k) + \lambda\rho(k)^{-1}\big]\rho(k)^{1/2}\chi(k)^{-1}. \quad (5.69) \end{aligned}$$

Next, consider the Herglotz matrix

$$M_\alpha(z, k) = \big[M^D_-(z, k) + \alpha(k)\big]\big[M^D_-(z, k) - M^D_+(z, k)\big]^{-1}\big[M^D_+(z, k) + \alpha(k)\big] \quad (5.70)$$

for some self-adjoint $m \times m$ matrix $\alpha(k)$. We claim that for all $k \in \mathbb{Z}$ and a.e. $\lambda \in \sigma_{\rm ess}(D)$,

$$M_\alpha(\lambda + i0, k) = iC_\alpha(\lambda, k) \text{ with } C_\alpha(\lambda, k) = C_\alpha(\lambda, k)^*. \quad (5.71)$$

Indeed, one computes

$$\begin{aligned} M_\alpha(\lambda + i0, k) &= M^D_-(\lambda + i0, k)\big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1}M^D_+(\lambda + i0, k) \\ &\quad + \alpha(k)\big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1}\alpha(k) \\ &\quad + \alpha(k)\big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1}M^D_+(\lambda + i0, k) \\ &\quad + M^D_-(\lambda + i0, k)\big[M^D_-(\lambda + i0, k) - M^D_+(\lambda + i0, k)\big]^{-1}\alpha(k). \quad (5.72) \end{aligned}$$

The first two terms on the right-hand side of (5.72) are already of the required form (5.71). The last two terms on the right-hand side of (5.72) can be rewritten in the form

$$
\begin{aligned}
&\alpha(k)\big[M_-^D(\lambda+i0,k)-M_+^D(\lambda+i0,k)\big]^{-1}M_+^D(\lambda+i0,k)\\
&\quad + M_-^D(\lambda+i0,k)\big[M_-^D(\lambda+i0,k)-M_+^D(\lambda+i0,k)\big]^{-1}\alpha(k)\\
&= (1/2)\alpha(k)\big[M_-^D(\lambda+i0,k)-M_+^D(\lambda+i0,k)\big]^{-1}\\
&\quad \times \big[M_-^D(\lambda+i0,k)+M_+^D(\lambda+i0,k)\big]\\
&\quad + (1/2)\big[M_-^D(\lambda+i0,k)+M_+^D(\lambda+i0,k)\big]\\
&\quad \times \big[M_-^D(\lambda+i0,k)-M_+^D(\lambda+i0,k)\big]^{-1}\alpha(k)\\
&= (1/2)i[\alpha(k)C_{1,2}(\lambda,k)+C_{1,2}(\lambda,k)^*\alpha(k)]
\end{aligned} \tag{5.73}
$$

using (5.64). Thus, also the last two terms on the right-hand side of (5.72) are of the required form (5.71) which completes the proof of the claim (5.71). The result (5.71) applied to (5.69) then proves that $M_{1,1}^{H_2}(\lambda^2+i0,k)=g^{H_2}(\lambda^2+i0,k)$ is skew-adjoint and hence also H_2 is reflectionless. $\square$

The next result is a Borg-type theorem for supersymmetric Dirac difference operators D. However, unlike the Borg-type theorem for (matrix-valued) Schrödinger, Dirac, and Jacobi operators, this analog for Dirac difference operators displays a characteristic nonuniqueness feature.

Theorem 5.8. *Assume Hypothesis* 5.2 *and suppose that ρ and χ are reflectionless. Let $D(\rho,\chi)=\begin{pmatrix} 0 & \rho S^+ + \chi^* \\ \rho^- S^- + \chi & 0\end{pmatrix}$ be the associated self-adjoint Dirac difference operator in $\ell^2(\mathbb{Z})^m\oplus\ell^2(\mathbb{Z})^m$ (cf.* (5.22)*) and suppose that $\sigma(D(\rho,\chi))=\big[-E_+^{1/2},-E_-^{1/2}\big]\cup\big[E_-^{1/2},E_+^{1/2}\big]$ for some $0\le E_-<E_+$. Then ρ and χ are of the form,*

$$
\begin{aligned}
\rho(k)&=\operatorname{diag}(\rho_1(k),\dots,\rho_m(k)),\\
\rho_j(k)&=\frac{1}{2}\big(E_+^{1/2}-\varepsilon_j E_-^{1/2}\big), \quad 1\le j\le m,\ k\in\mathbb{Z},
\end{aligned} \tag{5.74}
$$

$$
\begin{aligned}
\chi(k)&=\operatorname{diag}(\chi_1(k),\dots,\chi_m(k)),\\
\chi_j(k)&=\frac{1}{2}\big(E_+^{1/2}+\varepsilon_j E_-^{1/2}\big), \quad 1\le j\le m,\ k\in\mathbb{Z},
\end{aligned} \tag{5.75}
$$

where

$$\varepsilon_j\in\{1,-1\}, \quad 1\le j\le m. \tag{5.76}$$

Proof. Since D is reflectionless by hypothesis, so are H_1 and H_2 by Lemma 5.7. By (5.3),

$$\sigma(D)=\big[-E_+^{1/2},-E_-^{1/2}\big]\cup\big[E_-^{1/2},E_+^{1/2}\big] \tag{5.77}$$

implies

$$\sigma(H_\ell)=\big[E_-,E_+\big], \quad \ell=1,2. \tag{5.78}$$

Applying Theorem 4.7 to H_1 and H_2 then yields

$$\frac{1}{4}(E_+-E_-)I_m=A_1=\rho\chi^+=A_2=\chi\rho, \tag{5.79}$$

$$\frac{1}{4}(E_++E_-)I_m=B_1=\rho^2+\chi^*\chi=B_2=(\rho^-)^2+\chi\chi^*, \tag{5.80}$$

By (5.79), χ satisfies

$$\chi = \chi^+ = \frac{1}{4}(E_+ - E_-)\rho^{-1} \tag{5.81}$$

and hence χ is a constant (i.e., k-independent) positive definite sequence of $m \times m$ matrices. By (5.80) (equivalently, by (5.81)), then also $\rho = \rho^-$ is a constant sequence of $m \times m$ matrices. Insertion of (5.81) into (5.80) then yields

$$\rho^4 - \frac{1}{4}(E_+ + E_-)\rho^2 + \frac{1}{16}(E_+ - E_-)^2 I_m = 0. \tag{5.82}$$

Since by hypothesis ρ is a positive definite diagonal matrix, solving the quadratic equation (5.82) for ρ_j, $1 \leq j \leq m$, yields (5.74) and hence also (5.75) using (5.81). $\square$

To the best of our knowledge this result is new even in the scalar case $m = 1$. In particular, the sign ambiguities displayed in (5.74) and (5.75), giving rise to 2^m isospectral supersymmetric Dirac difference operators, are new in this Borg-type context. The sign ambiguity disappears and hence uniqueness of the corresponding inverse spectral problem is restored only in the special case $E_- = 0$, that is, whenever the spectral gap $\big(-E_-^{1/2}, E_-^{1/2}\big)$ of $D(\rho,\chi)$ closes.

Using the supersymmetric formalism described in this section, the proof of following result can be reduced to that of Theorem 4.9.

Theorem 5.9. *In addition to assuming Hypothesis* 5.2*, suppose that ρ and χ are periodic with the same period. Let $D(\rho,\chi) = \begin{pmatrix} 0 & \rho S^+ + \chi^* \\ \rho^- S^- + \chi & 0 \end{pmatrix}$ be the associated self-adjoint Dirac difference operator in $\ell^2(\mathbb{Z})^m \oplus \ell^2(\mathbb{Z})^m$ and suppose that $D(\rho,\chi)$ has uniform spectral multiplicity $2m$. Then $D(\rho,\chi)$ is reflectionless and hence for all $k \in \mathbb{Z}$,*

$$\Upsilon^D(\lambda,k) = \frac{1}{2} I_{2m} \text{ for a.e. } \lambda \in \sigma_{\rm ess}(D(\rho,\chi)). \tag{5.83}$$

In particular, assume that ρ and χ are periodic with the same period, that $D(\rho,\chi)$ has uniform spectral multiplicity $2m$, and that $\sigma(D(\rho,\chi)) = [-E_+^{1/2}, -E_-^{1/2}] \cup [E_-^{1/2}, E_+^{1/2}]$ for some $0 \leq E_- < E_+$. Then ρ and χ are of the form,

$$\begin{aligned} \rho(k) &= \mathrm{diag}(\rho_1(k), \dots, \rho_m(k)), \\ \rho_j(k) &= \frac{1}{2}\big(E_+^{1/2} - \varepsilon_j E_-^{1/2}\big), \quad 1 \leq j \leq m, \ k \in \mathbb{Z}, \end{aligned} \tag{5.84}$$

$$\begin{aligned} \chi(k) &= \mathrm{diag}(\chi_1(k), \dots, \chi_m(k)), \\ \chi_j(k) &= \frac{1}{2}\big(E_+^{1/2} + \varepsilon_j E_-^{1/2}\big), \quad 1 \leq j \leq m, \ k \in \mathbb{Z}, \end{aligned} \tag{5.85}$$

where

$$\varepsilon_j \in \{1, -1\}, \quad 1 \leq j \leq m. \tag{5.86}$$

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