DIFFERENTIATING THE ABSOLUTELY CONTINUOUS INVARIANT MEASURE OF AN INTERVAL MAP f WITH RESPECT TO f.

by David Ruelle*.

Abstract. *Let the map* $f : [-1,1] \to [-1,1]$ *have a.c.i.m.* ρ *(absolutely continuous* f*-invariant measure with respect to Lebesgue). Let* $\delta\rho$ *be the change of* ρ *corresponding to a perturbation* $X = \delta f \circ f^{-1}$ *of* f*. Formally we have, for differentiable* A*,*

$$\delta\rho(A) = \sum_{n=0}^{\infty} \int \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

but this expression does not converge in general. For f *real-analytic and Markovian in the sense of covering* $(-1,1)$ m *times, and assuming an* analytic expanding *condition, we show that*

$$\lambda \mapsto \Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

is meromorphic in $\mathbf{C}$*, and has no pole at* $\lambda = 1$*. We can thus formally write* $\delta\rho(A) = \Psi(1)$*.*

* Mathematics Dept., Rutgers University, and IHES. 91440 Bures sur Yvette, France. <ruelle@ihes.fr>

We postpone a discussion of the significance of our result, and start to describe the conditions under which we prove it. Note that these conditions are certainly too strong: suitable differentiability should replace analyticity, and a weaker Markov property should be sufficient. But the point of the present note is to show how it is that $\Psi(\lambda)$ has no pole at $\lambda = 1$, rather than deriving a very general theorem.

Setup.

We assume that $f : [-1, 1] \to [-1, 1]$ is real analytic and piecewise monotone on $[-1, 1]$ in the following sense: there are points c_j ($j = 0, \ldots, m$, with $m \geq 2$) such that $-1 = c_0 < c_1 < \ldots < c_{m-1} < c_m = 1$ and, for $j = 0, \ldots, m$,

$$f(c_j) = (-1)^{j+1}$$

We assume that on $[-1, 1]$ the derivative f' vanishes only on $Z = \{c_1, \ldots, c_{m-1}\}$, and that f'' does not vanish on Z. For $j = 1, \ldots, m$, we have $f[c_{j-1}, c_j] = [-1, 1]$. In particular, f is Markovian. We shall also assume that f is *analytically expanding* in the sense of Assumption A below. The purpose of this note is to prove the following:

Theorem. *Under the above conditions, and Assumption A stated later, there is a unique f-invariant probability measure ρ absolutely continuous with respect to Lebesgue on $[-1, 1]$. If X is real-analytic on $[-1, 1]$, and $A \in \mathcal{C}^1[-1, 1]$, then*

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

extends to a meromorphic function in $\mathbf{C}$, without pole at $\lambda = 1$.

Our proof depends on a change of variable which we now explain. We choose a holomorphic function ω from a small open neighborhood U_0 of $[-1, 1]$ in $\mathbf{C}$ to a small open neighborhood W of $[-1, 1]$ in a Riemann surface which is 2-sheeted over $\mathbf{C}$ near -1 and 1. We call $\varpi = \omega^{-1} : W \to U_0$ the inverse of ω. We assume that $\omega(-x) = -\omega(x)$, $\omega(\pm 1) = \pm 1$, $\omega[-1, 1] = [-1, 1]$, $\omega'(\pm 1) = \omega'''(\pm 1) = 0$. We have thus

$$\omega(\pm(1 - \xi)) = \pm(1 - C\xi^2 + D\xi^4 \ldots)$$

with $C > 0$ and, if $a > 0$,

$$\varpi(\pm(1 - a\xi^2 + b\xi^3 \ldots)) = \pm(1 - \sqrt{\frac{a}{C}}\xi + \frac{b}{2\sqrt{aC}}\xi^2 \ldots)$$

[We may for instance take

$$\omega(x) = \sin\frac{\pi x}{2} \qquad , \qquad \varpi(x) = \frac{2}{\pi}\arcsin x$$

or

$$\omega(x) = \frac{1}{16}(25x - 10x^3 + x^5) \qquad , \qquad \varpi(x) = \frac{16}{25}x \ldots \qquad]$$

The function $g : \varpi \circ f \circ \omega$ from $[-1,1]$ to $[-1,1]$ has monotone restrictions to the intervals $\varpi[c_{j-1}, c_j] = [d_{j-1}, d_j]$. It is readily seen that g_j extends to a holomorphic function in a neighborhood of $[d_{j-1}, d_j]$, and that

$$g_1(-1+\xi) = -1 + \sqrt{f'(-1)}\xi + \alpha_-\xi^3 \ldots$$

$$g_m(1-\xi) = (-1)^{m+1}(1 - \sqrt{|f'(1)|}\xi - \alpha_+\xi^3 \ldots)$$

with no ξ^2 terms in the right-hand sides [this follows from our choice of ω, which has no ξ^3 term]. One also finds that, for $j = 1, \ldots, m-1$

$$g_j(d_j - \xi) = (-1)^{j+1}(1 - \sqrt{\frac{|f''(c_j)|}{2C}}\omega'(d_j)\xi + \gamma_j\xi^2 \ldots)$$

$$g_{j+1}(d_j + \xi) = (-1)^{j+1}(1 - \sqrt{\frac{|f''(c_j)|}{2C}}\omega'(d_j)\xi - \gamma_j\xi^2 \ldots)$$

where γ_j is the same in the two relations. We note the following easy consequences of the above developments:

Lemma 1. *Let $\psi_j : [-1,1] \to [d_{j-1}, d_j]$ be the inverse of g_j for $j = 1, \ldots, m$ (increasing for j odd, decreasing for j even). Then*

$$\psi_1(-1+\xi) = -1 + \frac{1}{\sqrt{f'(-1)}}\xi + \beta_-\xi^3$$

$$\psi_m((-1)^{m+1}(1-\xi)) = 1 - \frac{1}{\sqrt{|f'(1)|}}\xi + \beta_+\xi^3$$

(there are no ξ^2 terms in the right-hand sides). If $j < m$,

$$\psi_j((-1)^{j+1}(1-\xi)) = d_j - \sqrt{\frac{2C}{|f''(c_j)|}}\frac{1}{\omega'(d_j)}\xi + \delta_j\xi^2$$

$$\psi_{j+1}((-1)^{j+1}(1-\xi)) = d_j + \sqrt{\frac{2C}{|f''(c_j)|}}\frac{1}{\omega'(d_j)}\xi + \delta_j\xi^2$$

(with the same coefficient δ_j). $\square$

As inverses of the g_j, the functions ψ_j extend to holomorphic functions on a neighborhood of $[-1,1]$. We impose now the condition that f is *analytically expanding* in the following sense:

Assumption A *We have $[-1,1] \subset U \subset \mathbf{C}$, with U bounded open connected, such that the ψ_j extend to continuous functions $\bar{U} \mapsto \mathbf{C}$, holomorphic in U, and with $\psi_j\bar{U} \subset U$. [$\bar{U}$ denotes the closure of U].*

Let ϕ be holomorphic on a neighborhood of $\bar{U}$. Given a sequence $\mathbf{j} = (j_1, \ldots, j_\ell, \ldots)$ we define $\phi_{\mathbf{j}\ell} = \phi \circ \psi_{j_1} \cdots \circ \psi_{j_\ell}$ and note that the $\phi_{\mathbf{j}\ell}$ are uniformly bounded in a neighborhood of $\bar{U}$. We may thus choose $\ell(r)$ for $r = 1, 2 \ldots$ such that the subsequence $(\phi_{\mathbf{j}\ell(r)})_{r=1}^{\infty}$ converges uniformly on $\bar{U}$ to a limit $\tilde{\phi}_{\mathbf{j}}$. Writing $\tilde{U} = \cup_{j=1}^{m} \psi_j \bar{U}$ we have

$$\max_{z \in \bar{U}} |\phi_{\mathbf{j}\ell(r)}| \geq \max_{z \in \tilde{U}} |\phi_{\mathbf{j}\ell(r)}| \geq \max_{z \in \bar{U}} |\phi_{\mathbf{j}\ell(r+1)}|$$

so that $\max_{z \in \bar{U}} |\tilde{\phi}_{\mathbf{j}}| = \max_{z \in \tilde{U}} |\tilde{\phi}_{\mathbf{j}}|$ and, since $\tilde{U}$ is compact $\subset U$ connected, $\tilde{\phi}_{\mathbf{j}}$ is constant. Therefore ϕ is constant on $\cap_{\ell=0}^{\infty} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U}$. Since this is true for all ϕ, the intersection $\cap_{\ell=0}^{\infty} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U}$ consists of a single point $\tilde{z}(\mathbf{j})$. Given $\epsilon > 0$ we can thus, for each $\mathbf{j}$, find ℓ such that $\mathrm{diam} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U} < \epsilon$. Hence (using the compactness of the Cantor set of sequences $\mathbf{j}$) one can choose L so that the m^L sets

$$\psi_{j_1} \circ \cdots \psi_{j_L} \bar{U}$$

have diameter $< \epsilon$. The open connected set

$$V = \cup_{j_1, \ldots, j_L} \psi_{j_1} \circ \cdots \psi_{j_L} U$$

satisfies $[-1, 1] \subset V \subset U$, and $\psi_j \bar{V} = \cup_{j_1, \ldots, j_L} \psi_j \circ \psi_{j_1} \circ \cdots \circ \psi_{j_L} \bar{U} \subset \cup_{j_0, j_1, \ldots, j_{L-1}} \psi_{j_0} \circ \psi_{j_1} \circ \psi_{i_{L-1}} U = V$. This shows that U can be replaced in Assumption A by a set V contained in an ϵ-neighborhood of $[-1, 1]$.

Since we have shown above that $\mathrm{diam} \psi_{j_1} \circ \cdots \psi_{j_L} \bar{U} < \epsilon$, we see that ψ_1^L maps a small circle around -1 strictly inside itself. We have thus $\psi_1'(-1) < 1$ (*i.e.*, $f'(-1) > 1$) and similarly, if m is odd, $\psi_m'(1) < 1$ (*i.e.*, $f'(1) > 1$).

The following two lemmas state some easy facts to be used later.

Lemma 2. *Let H be the Hilbert space of functions $\bar{U} \to \mathbf{C}$ which are square integrable (with respect to Lebesgue) and holomorphic in U. The operator $\mathcal{L}$ on H defined by*

$$(\mathcal{L}\Phi)(z) = \sum_{j=1}^{m} (-1)^{j+1} \psi_j'(z) \Phi(\psi_j(z))$$

is holomorphy improving. In particular $\mathcal{L}$ is compact and trace-class. ∎

Lemma 3. *On $[-1, 1]$ we have*

$$(\mathcal{L}\Phi)(x) = \sum_j |\psi_j'(x)| \Phi(\psi_j(x))$$

hence $\Phi \geq 0$ implies $\mathcal{L}\Phi \geq 0$ ($\mathcal{L}$ preserves positivity) and

$$\int_{-1}^{1} dx\, (\mathcal{L}\Phi)(x) = \int_{-1}^{1} dx\, \Phi(x)$$

($\mathcal{L}$ preserves total mass). $\square$

Lemma 4. *$\mathcal{L}$ has a simple eigenvalue $\mu_0 = 1$ corresponding to an eigenfunction $\sigma_0 > 0$. The other eigenvalues μ_k ($k \geq 1$) satisfy $|\mu_k| < 1$, and their (generalized) eigenfunctions σ_k satisfy $\int_{-1}^{1} dx\, \sigma_k(x) = 0$.*

Let (μ_k, σ_k) be a listing of the eigenvalues and generalized eigenfunctions of the trace-class operator $\mathcal{L}$. For each μ_k there is some σ_k such that $\mathcal{L}\sigma_k = \mu_k \sigma_k$, hence

$$|\mu_k| \int_{-1}^{1} dx\, |\sigma_k(x)| = \int_{-1}^{1} dx\, |\mu_k \sigma_k(x)| = \int_{-1}^{1} dx\, |(\mathcal{L}\sigma_k)(x)|$$

$$\leq \int_{-1}^{1} dx\, (\mathcal{L}|\sigma_k|)(x) = \int_{-1}^{1} dx\, |\sigma_k(x)|$$

hence $|\mu_k| \leq 1$. Denote by $S_<$ and S_1 the spectral spaces of $\mathcal{L}$ corresponding to eigenvalues μ_k with $|\mu_k| < 1$, and $|\mu_k| = 1$ respectively. If $\sigma_k \in S_<$ then, for some $n \geq 1$,

$$0 = \int_{-1}^{1} dx\, ((\mathcal{L} - \mu_k)^n \sigma_k)(x) = \int_{-1}^{1} dx\, (1 - \mu_k)^n \sigma_k(x)$$

hence $\int_{-1}^{1} dx\, \sigma_k(x) = 0$.

On the finite dimensional space S_1, there is a basis of eigenvectors σ_k diagonalizing $\mathcal{L}$ (if $\mathcal{L}|S_1$ had non-diagonal normal form, $||\mathcal{L}^n|S_1||$ would tend to infinity with n, in contradiction with $\int_{-1}^{1} dx\, |(\mathcal{L}^n\Phi)(x)| \leq \int_{-1}^{1} dx\, |\Phi(x)|$). We shall now show that, up to multiplication by a constant $\neq 0$, we may assume $\sigma_k \geq 0$. If not, because σ_k is continuous and the intervals $\psi_{j_1} \circ \cdots \circ \psi_{j_n}[-1, 1]$ are small for large n (mixing), we would have $|(\mathcal{L}^n\sigma_k)(x)| < (\mathcal{L}^n|\sigma_k|)(x)$ for some n and x. This would imply $\int_{-1}^{1} dx\, |(\mathcal{L}^n\sigma_k)(x)| < \int_{-1}^{1} dx\, |\sigma_k(x)|$ in contradiction with $\mathcal{L}\sigma_k = \mu_k\sigma_k$ and $|\mu_k| = 1$. From $\sigma_k \geq 0$ we get $\mu_k = 1$, and the corresponding eigenspace is at most one dimensional (otherwise it would contain functions not ≥ 0). But we have $1 \notin S_<$ because $\int_{-1}^{1} dx, 1 \neq 0$, so that $S_1 \neq \{0\}$. Thus S_1 is spanned by an eigenfunction, which we call σ_0, to the eigenvalue $\mu_0 = 1$. Finally, $\sigma_0 > 0$ because if $\sigma_0(x) = 0$ we would have also $\sigma_0(y) = 0$ whenever $g^n(y) = x$, which is not compatible with σ_0 continuous $\neq 0$. $\square$

Lemma 5. *If we normalize σ_0 by $\int_{-1}^{1} dx\, \sigma_0(x) = 1$, then $\sigma_0(dx) = \sigma_0(x)dx$ is the unique g-invariant probability measure absolutely continuous with respect to Lebesgue on $[-1, 1]$. In particular, $\sigma_0(dx)$ is ergodic.*

For continuous A on $[-1, 1]$ we have

$$\int_{-1}^{1} \sigma_0(dx)(A \circ g)(x) = \int_{-1}^{1} dx\, \sigma_0(x)A(g(x)) = \int_{-1}^{1} dx\, (\mathcal{L}\sigma_0)(x)A(x) = \int_{-1}^{1} \sigma_0(dx)A(x)$$

so that $\sigma_0(dx)$ is g-invariant. Let $\tilde{\sigma}(x)dx$ be another g-invariant probability measure absolutely invariant with respect to Lebesgue. Then, if $\tilde{\sigma} \neq \sigma_0$

$$\int_{-1}^{1} dx\, |\sigma_0(x) - \tilde{\sigma}(x)| = \int_{-1}^{1} dx\, |(\mathcal{L}(\sigma_0 - \tilde{\sigma}))(x)|$$

$$< \int_{-1}^{1} dx\,(\mathcal{L}|\sigma_0 - \tilde{\sigma}|)(x) = \int_{-1}^{1} dx\,|\sigma_0(x) - \tilde{\sigma}(x)|$$

by mixing: contradiction. ∎

Lemma 6. *Let $H_1 \subset H$ consist of those functions Φ with derivatives vanishing at ± 1: $\Phi'(-1) = \Phi'(1) = 0$. Then $\mathcal{L}H_1 \subset H_1$ and $\sigma_0 \in H_1$.*

$\mathcal{L}H_1 \subset H_1$ is an easy calculation using Lemma 1. Furthermore, by Lemma 4, $\sigma_0 = \lim_{n\to\infty} \mathcal{L}^n \frac{1}{2}$, and $\frac{1}{2} \in H_1$ implies $\sigma_0 \in H_1$. ∎

The image $\rho(dx) = \rho(x)dx$ of $\sigma_0(x)dx$ by ω is the unique f-invariant probability measure absolutely continuous with respect to Lebesgue on $[-1,1]$. We have

$$\rho(x) = \sigma_0(\varpi x)\varpi'(x)$$

Consider now the expression

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

where we assume that X extends to a holomorphic function in a neighborhood of $[-1,1]$ and $A \in \mathcal{C}^1[-1,1]$. For sufficiently small $|\lambda|$, the series defining $\Psi(\lambda)$ converges. Writing $B = A \circ \omega$ and $x = \omega y$ we have

$$X(x)\frac{d}{dx}A(f^n x) = X(\omega y)\frac{1}{\omega'(y)}\frac{d}{dy}B(g^n y)$$

hence

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} dy\,\sigma_0(y)\frac{X(\omega y)}{\omega'(y)}\frac{d}{dy}B(g^n y)$$

Defining $Y(y) = \sigma_0(y)X(\omega y)/\omega'(y)$, we see that Y extends to a function holomorphic in a neighborhood of $[-1,1]$, which we may take to be U, except for simple poles at -1 and 1. We may write

$$\int_{-1}^{1} dy\,\sigma_0(y)\frac{X(\omega y)}{\omega'(y)}\frac{d}{dy}B(g^n y) = \int_{-1}^{1} dy\,Y(y)g'(y)\cdots g'(g^{n-1}y)B'(g^n y)$$

$$= \int_{-1}^{1} ds\,(\mathcal{L}_0^n Y)(s)B'(s)$$

where

$$(\mathcal{L}_0\Phi)(s) = \sum_{j=1}^{m} (-1)^{j+1}\Phi(\psi_j s)$$

and we have thus

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} ds\,(\mathcal{L}_0^n Y)(s)B'(s)$$

Lemma 7. *Let $H_0 \subset H$ be the space of functions vanishing at -1 and 1. Then $\mathcal{L}_0 H_0 \subset H_0$.*

This follows readily from Lemma 1. $\square$

Lemma 8. *There are meromorphic functions $\Phi_\pm$ with Laurent series*

$$\Phi_\pm(z) = \frac{1}{z \mp 1} + O(z \mp 1)$$

at ± 1 and $\Phi_\pm(\mp 1) = 0$ such that

$$\mathcal{L}_0 \Phi_- = \sqrt{f'(-1)}\Phi_-$$

$$\begin{cases} \mathcal{L}_0 \Phi_+ = \sqrt{f'(1)}\Phi_+ & \textit{if } m \textit{ is odd} \\ \mathcal{L}_0(\Phi_+/\sqrt{|f'(1)|} + \Phi_-/\sqrt{f'(-1)}) = \tilde{Y} \in H_0 & \textit{if } m \textit{ is even} \end{cases}$$

Define

$$p_\pm(z) = \frac{1}{z \mp 1} - \frac{1}{4}(z \mp 1)$$

then Lemma 1 yields

$$(\mathcal{L}_0 - \sqrt{f'(-1)})p_- = u_- \in H_0$$

$$\begin{cases} (\mathcal{L}_0 - \sqrt{f'(1)})p_+ = u_+ \in H_0 & \text{if } m \text{ is odd} \\ \mathcal{L}_0 p_+ + \sqrt{|f'(1)|}p_- = u_0 \in H_0 & \text{if } m \text{ is even} \end{cases}$$

Since $f'(-1) > 1$, Lemma 4 shows that $\mathcal{L} - \sqrt{f'(-1)}$ is invertible on H, hence there is v_- such that

$$(\mathcal{L} - \sqrt{f'(-1)})v_- = u'_-$$

and since $\int_{-1}^1 dx\, u'_-(x) = 0$, also $\int_{-1}^1 dx\, v_-(x) = 0$ and we can take $w_- \in H_0$ such that $w'_- = v_-$. Then

$$((\mathcal{L}_0 - \sqrt{f'(-1)})w_-)' = (\mathcal{L} - \sqrt{f'(-1)})w'_- = (\mathcal{L} - \sqrt{f'(-1)})v_- = u'_-$$

so that

$$(\mathcal{L}_0 - \sqrt{f'(-1)})w_- = u_-$$

without additive constant because the left-hand side is in H_0 by Lemma 7. In conclusion

$$(\mathcal{L}_0 - \sqrt{f'(-1)})(p_- - w_-) = 0$$

and we may take $\Phi_- = p_- - w_-$.

If m is odd, Φ_+ is handled similarly. If m is even, taking $\Phi_+ = p_+$ and writing $\tilde{Y} = u_0/\sqrt{|f'(1)|} - w_-$ we obtain

$$\mathcal{L}_0\left(\frac{\Phi_+}{\sqrt{|f'(1)|}} + \frac{\Phi_-}{\sqrt{f'(-1)}}\right) = \tilde{Y} \in H_0$$

which completes the proof. ∎

We have $\sigma_0 \in H_1$ (Lemma 6), and $X \circ \omega \in H_1$ by our choice of ω. Also

$$\omega'(\pm(1-\xi)) = 2C\xi - 4D\xi^3 \ldots$$

so that

$$Y = \mathbf{C}\Phi_- + \mathbf{C}\Phi_+ + H_0$$

If m is odd let $Y = c_-\Phi_- + c_+\Phi_+ + Y_0$, with $Y_0 \in H_0$. Then

$$\Psi(\lambda) = \frac{c_-}{1-\lambda\sqrt{f'(-1)}}\int_{-1}^{1} ds\,\Phi_-(s)B'(s) + \frac{c_+}{1-\lambda\sqrt{f'(1)}}\int_{-1}^{1} ds\,\Phi_+(s)B'(s) + \Psi_0(\lambda)$$

where Ψ_0 is obtained from Ψ when Y is replaced by Y_0.

If m is even let $Y = c_-\Phi_- + \tilde{c}(\Phi_+/\sqrt{|f'(1)|} + \Phi_-/\sqrt{f'(-1)}) + Y_0$, with $Y_0 \in H_0$. Then

$$\Psi(\lambda) = \frac{c_-}{1-\lambda\sqrt{f'(-1)}}\int_{-1}^{1} ds\,\Phi_-(s)B'(s) + \tilde{c}\int_{-1}^{1} ds\,(\frac{\Phi_+}{\sqrt{|f'(1)|}} + \frac{\Phi_-}{\sqrt{f'(-1)}})B'(s)$$

$$+\lambda\tilde{\Psi}(\lambda) + \Psi_0(\lambda)$$

where $\tilde{\Psi}(\lambda)$ is obtained from Ψ when Y is replaced by $\tilde{Y}$.

Writing $\mu_\pm = \sqrt{f'(\pm 1)}$ we see that $\Psi(\lambda)$ has two poles at $\mu_\pm^{-1}$ if m is odd, and one pole at μ_-^{-1} if m is even; the other poles are those of $\Psi_0(\lambda)$ and possibly $\tilde{\Psi}(\lambda)$. Since $Y_0 \in H_0$ and $\mathcal{L}_0 H_0 \subset H_0$, we have

$$\Psi_0(\lambda) = \sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}_0^n Y_0)(s)B'(s) = -\sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}_0^n Y_0)'(s)B(s)$$

$$= -\sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}^n Y_0')(s)B(s)$$

It follows that $\Psi_0(\lambda)$ extends meromorphically to $\mathbf{C}$ with poles at the μ_k^{-1}. We want to show that the residue of the pole at $\mu_0^{-1} = 1$ vanishes . By Lemma 4, $\int_{-1}^{1} dx\,\sigma_k(x) = 0$ for $k \geq 1$. Thus, up to normalization, the coefficient of σ_0 in the expansion of Y_0' is

$$\int_{-1}^{1} dx\,Y_0'(x) = Y_0(1) - Y_0(-1) = 0$$

because $Y_0 \in H_0$. Therefore $\Psi_0(z)$ is holomorphic at $z = 1$, and the same argument applies to $\tilde{\Psi}(z)$, concluding the proof of the theorem. ∎

Discussion.

It can be argued that the *physical measure* describing a physical dynamical system is an SRB (Sinai-Ruelle-Bowen) measure ρ (see the recent reviews [11], [2] which contain a number of references), or an a.c.i.m. ρ in the case of a map of the interval. But, typically, physical systems depend on parameters, and it is desirable to know how ρ depends on the parameters (*i.e.*, on the dynamical system). The dependence is smooth for uniformly hyperbolic dynamical systems (see [5], [6] and references given there), but discontinuous in general.

The present note is devoted to an example in support of an idea put forward in [8]: that derivatives of $\rho(A)$ with respect to parameters can be meaningfully defined in spite of discontinuities. An ambitious project would be to have Taylor expansions on a large set Σ of parameter values and, using a theorem of Whitney [10], to connect these expansions by a function extrapolating $\rho(A)$ smoothly outside of Σ. In a different dynamical situation, that of KAM tori, a smooth extension à la Whitney has been achieved by Chierchia and Gallavotti [3], and Pöschel [4].

In our study we have considered only a rather special set Σ consisting of maps satisfying a Markov property. (Reference [1] should be consulted for a discussion of the poles encountered in the study of a Markovian map f). Note that the studies of a.c.i.m. for maps of the interval, and of SRB measures for Hénon-like maps, are typically based on perturbations of a map satisfying a Markov property (for the use of slightly more general Misiurewicz-type maps see [9], which also gives references to earlier work).

The function $\Psi(\lambda)$ that we have encountered is related to the *susceptibility* $\omega \mapsto \Psi(e^{i\omega})$ giving the response of a system to a periodic perturbation. The existence of a holomorphic extension of the susceptibility to the upper half complex plane is expected to follow from *causality* (causality says that cause preceeds effect, resulting in a *response function* κ having support on the positive half real axis, and its Fourier transform $\hat{\kappa}$ extending holomorphically to the upper half complex plane). A discussion of nonequilibrium statistical mechanics [7] shows that the expected support and holomorphy properties hold close to equilibrium, or if uniform hyperbolicity holds. In the example discussed in this note, κ has the right support property, but increases exponentially at infinity, and holomorphy in the upper half plane fails, corresponding the existence of a pole of Ψ at $\lambda = 1/\sqrt{f'(-1)}$. This might be expressed by saying that ρ is *not linearly stable*. The physically interesting situation of *large systems* (thermodynamic limit) remains quite unclear at this point.

Acknowledgments. For many discussions on the subject of this note, I am indebted to V. Baladi, M. Benedicks, G. Gallavotti, M. Viana, and L.-S. Young.

References.

[1] V. Baladi, Y. Jiang, H.H. Rugh. "Dynamical determinants via dynamical conjugacies for postcritically finite polynomials." J. Statist. Phys. **108**,973-993(2002).

[2] C. Bonatti, L. Diaz, and M. Viana. *Dynamics beyond uniform hyperbolicity: a global geometric and probabilistic approach.* Springer, to appear.

[3] L. Chierchia and G. Gallavotti. "Smooth prime integrals for quasi-integrable Hamiltonian systems." Nuovo Cim. **67B**,277-295(1982).

[4] J. Pöschel. "Integrability of Hamiltonian systems on Cantor sets." Commun. in Pure and Applied Math. **35**,653-696(1982).

[5] D. Ruelle. “Differentiation of SRB states.” Commun. Math. Phys. **187**,227-241(1997); “Correction and complements.” Commun. Math. Phys. **234**,185-190(2003).

[6] D. Ruelle. “Differentiation of SRB states for hyperbolic flows.” In preparation.

[7] D. Ruelle. “Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics.” J. Statist. Phys. **95**,393-468(1999).

[8] D. Ruelle. “Application of hyperbolic dynamics to physics: some problems and conjectures.” Bull. Amer. Math. Soc. (N.S.) **41**,275-278(2004).

[9] Q. Wang and L.-S. Young. “Towards a theory of rank one attractors.” Preprint

[10] H. Whitney. “Analytic expansions of differentiable functions defined in closed sets.” Trans. Amer. Math. Soc. **36**,63-89(1934).

[11] L.-S. Young. “What are SRB measures, and which dynamical systems have them?” J. Statist. Phys. **108**,733-754(2002).